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MODIFIED MANN AND VISCOSITY ALGORITHMS FOR ENRICHED NONEXPANSIVE MAPPINGS

By

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Abstract

This paper examines algorithms for solving fixed point problems for enriched nonexpansive mappings. We combine the inertial technique with our algorithms for a better rate of convergence. Strong convergence of proposed algorithms in a real Hilbert space is proved. We also provide a numerical example to demonstrate that our algorithm defined by equation (3.1) converges more quickly than Modified Kransnoselskii Mann Algorithm (*MKMA*) [4].
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1 Introduction

Fixed point theory provides a crucial tool for nonlinear analysis while investigating the existence and approximation of solutions of nonlinear functional equations. (integral equations, differential equations, integro-differential equations, etc) [1, 9, 11, 14]. The initial movable theorem in metric fixed point theory has been given by Banach [2] in the setting of complete normed space (what we now call Banach space), which is called the Banach contraction principle and further extended by Caccioppoli to complete metric space. It states that
"Let (X, d) be a complete metric space and let $U : X \to X$ be a contraction on X. Then, U has a unique fixed point."
Banach contraction principle has many applications in various fields, such as in biology, computer science, physics, and also in many branches of mathematics. The study of fixed points for nonexpansive mapping has risen in recent years due to its numerous applications in areas like compressed sensing, economics, and other applied sciences. Contrary to the Banach contraction principle, the nonexpansive mapping U on a nonempty, closed, and convex subset Q of a real Hilbert space H may not have a fixed point or if it has a fixed point then it may not be unique; additionally, it is possible that, if U has a single fixed point, the Picard iteration may not converge to that fixed point. This makes the problem of the existence and approximation of its fixed points crucial. Due to these aspects of nonexpansive mapping, the study of such mapping has become a significant area of research. Now for approximation(estimation) of fixed points of nonexpansive mappings, one way is to use another iteration scheme instead of Picard iteration. For instance, Mann [12] and Krasnoselskii [10] considered the following iteration:

$$x_{n+1} = (1 - \mu_n)x_n + \mu_n U x_n, \quad n \geq 0, \tag{1.1}$$

where the initial value $x_0 \in Q$ is arbitrary. In 2020, Tan et al.[15] introduced a modified inertial Mann algorithm for nonexpansive mappings and proved its strong convergence. In 2019, on the other hand, Berinde [3] proposed the notion of enriched nonexpansive mapping as an extension of the nonexpansive mappings and approximated its fixed points by Kransnoselskij iteration. In 2022, the same author [4] approximated fixed points of enriched nonexpansive mappings using a modified KransnoselskijMann algorithm.
Inspired and motivated by the all above results, our main contributions to this research are as follows:

- We introduce modified inertial Mann and viscosity algorithms for enriched nonexpansive mapping which is inspired and motivated by algorithms defined in [15] for nonexpansive mappings;

- We prove the strong convergence of proposed algorithms under some conditions;
- We also provide two numerical examples to compare Modified Inertial Mann Halpern Algorithm(*MIMHA*) with Modified Krasnoselskii Mann Algorithm(*MKMA*)[4] and show that *MIMHA* has a better rate of convergence than *MKMA*.

2 Preliminaries

Assume that H is a real Hilbert space during this study. We use the symbol $\to$ for strong convergence and for weak convergence, we use the symbol $\rightharpoonup$. We now provide a few definitions and lemmas that will be applied to prove our findings. Assume Q is a nonempty, closed and convex subset of H and $Fix(U)$ denotes the collection of all fixed points of a mapping U i.e., $Fix(U) = \{x \in Q : Ux = x\}$.

Definition 2.1 ([5]). *A mapping $U : Q \to Q$ is said to be nonexpansive if*

$$\|Ux - Uy\| \leq \|x - y\| \text{ for all } x, y \in Q.$$

Definition 2.2 ([5]). *A mapping $P_Q : H \to Q$ is called metric projection if*

$$\|x - P_Q x\| = \text{ inf. } \{\|x - y\| : y \in Q\} \text{ for all } x \in H.$$

Note that projection mapping is nonexpansive and it satisfies the following:
(i) $\langle x - P_Q(x), y - P_Q(x)\rangle \leq 0$ *for all* $x \in H, y \in Q$.
(ii) $\|P_Q(x) - P_Q(y)\|^2 \leq \langle P_Q(x) - P_Q(y), x - y\rangle$ *for all* $x, y \in H$.

Definition 2.3 ([3]). *Assume $(E, \|.\|)$ is a normed linear space. A mapping $U : E \to E$ is called enriched nonexpansive if there exists a non-negative real number k such that*

$$\|k(x - y) + Ux - Uy\| \leq (k + 1)\|x - y\| \text{ for all } x, y \in E. \tag{2.1}$$

Definition 2.4 ([7]). *Assume that $U : Q \to Q$ is a mapping of a convex subset Q of a normed linear space E. Then, for all $\mu \in (0, 1]$, the mapping U_μ is said to be averaged mapping (a term coined in [8]) and is defined as*

$$U_\mu(x) = (1 - \mu)x + Ux \text{ for all } x \in Q$$

and the mapping U_μ satisfies $Fix(U_\mu) = Fix(U)$.

Lemma 2.1 ([15]). *For each $x, y \in H$,*
(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$
(ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$, *for all* $t \in \mathbb{R}$.

Lemma 2.2 ([6]). *Suppose that Q is a nonempty, convex and closed subset of H and $U : Q \to H$ is a nonexpansive mapping. Assume that $\{x_n\} \in Q$ and $x \in H$ such that $x_n \rightharpoonup x$ and $Ux_n - x_n \to 0$ as $n \to \infty$. Then $x \in Fix(U)$.*

Lemma 2.3 ([16]). *Suppose that $\{x_n\} \in [0, \infty)$ be a sequence such that*

$$x_{n+1} = (1 - \gamma_n)x_n + \gamma_n s_n, \quad n \geq 0,$$

where $\{\gamma_n\} \in (0, 1)$ and $\{s_n\}$ is a sequence of real numbers such that
(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} s_n \leq 0$ *or* $\limsup_{n\to\infty} \gamma_n s_n < \infty$.
Then, $lim_{n\to\infty} x_n = 0$.

Lemma 2.4 ([13]). *Let $\{x_n\} \subseteq [0, \infty)$ such that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} < x_{n_j+1}$. Then, there exists a sequence (nondecreasing) $\{n_k\}$ of natural numbers satisfying $\lim_{n\to\infty} n_k = \infty$, $x_{n_k} \leq x_{n_k+1}$ and*
$x_k \leq x_{n_k+1}$ for all $k \in \mathbb{N}$. Also, n_k is the greatest number n in the set $\{1, 2, ..., k\}$ satisfying $x_n < x_{n+1}$.

3 Main Results

3.1 Modified Inertial Mann Halpern Algorithm

Theorem 3.1. *Suppose that H is a real Hilbert space, Q is a nonempty, convex, closed subset of H and $U : Q \to Q$ is an enriched nonexpansive mapping with $Fix(U) \neq \phi$. For any given $u \in Q$ and two real sequences $\{\gamma_n\}, \{\beta_n\} \in (0,1)$ and $\{\alpha_n\}$ is a real sequence satisfying $\lim_{n\to\infty} \frac{\alpha_n}{\gamma_n} = 0$. Also, suppose $\theta > 0, \sum_{n=0}^{\infty} \gamma_n = \infty$ and $\lim_{n\to\infty} \gamma_n = 0$. Let $x_{-1}, x_0 \in Q$ be chosen arbitrarily. Assume that the sequence $\{x_n\}$ is defined by the following algorithm:*

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), where \quad \theta_n = \begin{cases} min\left\{\frac{\alpha_n}{\|x_n - x_{n-1}\|}, \theta\right\}, if x_n \neq x_{n-1} \\ \theta, otherwise \end{cases} \\ y_n = (1-\mu\beta_n)z_n + \mu\beta_n U z_n, \\ x_{n+1} = \gamma_n u + (1-\gamma_n)y_n, \end{cases} \tag{3.1}$$

where $n \geq 0$.

Then the sequence given by equation (3.1) strongly converges to fixed point ν of U, where $\nu = P_{Fix(U)}u$ and $\mu \in (0,1]$ is some constant.

Proof. First we prove that the mapping $U_\mu : Q \to Q$ defined by

$$U_\mu(x) = (1-\mu)x + \mu U x \text{ for all } x \in Q,$$

is nonexpansive.

Since U is given to be enriched nonexpansive, so according to definition of enriched nonexpansive mapping, there exists a nonnegative constant k such that

$$\|k(x-y) + Ux - Uy\| \leq (k+1)\|x-y\| \text{ for all } x, y \in Q. \tag{3.2}$$

Substituting $k = \frac{1}{\mu} - 1$, we have $\mu \in (0,1]$ and equation (3.2) becomes,

$$\|(1-\mu)(x-y) + \mu Ux - \mu Uy\| \leq \|x-y\| \text{ for all } x, y \in Q,$$

which implies

$$\|U_\mu x - U_\mu y\| \leq \|x-y\| \text{ for all } x, y \in Q.$$

So, the mapping U_μ is nonexpansive.

Now, we prove that the sequence $\{x_n\}$ defined by equation (3.1) is bounded. Let $\nu \in Fix(U) = Fix(U_\mu)$. Consider,

$$\begin{aligned} \|x_{n+1} - \nu\| &= \|\gamma_n u + (1-\gamma_n)y_n - \nu\| \\ &= \|\gamma_n(u-\nu) + (1-\gamma_n)(y_n - \nu)\|, \end{aligned}$$

which implies

$$\|x_{n+1} - \nu\| \leq \gamma_n\|u-\nu\| + (1-\gamma_n)\|y_n - \nu\|. \tag{3.3}$$

From equation (3.1),

$$\begin{aligned} \|y_n - \nu\| &= \|(1-\mu\beta_n)z_n + \mu\beta_n U z_n - \nu\| \\ &= \|(1-\beta_n)z_n + \beta_n U_\mu z_n - \nu\| \\ &= \|(z_n - \nu) - \beta_n(z_n - U_\mu z_n)\|. \end{aligned}$$

Now,

$$\|y_n - \nu\|^2 = \|z_n - \nu - \beta_n(z_n - U_\mu z_n)\|^2,$$

which gives

$$\begin{aligned} \|y_n - \nu\|^2 &= \|z_n - \nu\|^2 + \|\beta_n(z_n - U_\mu z_n)\|^2 - 2\langle z_n - \nu, \beta_n(z_n - U_\mu z_n)\rangle \\ &= \|z_n - \nu\|^2 + \|\beta_n(z_n - U_\mu z_n)\|^2 - 2\beta_n\langle z_n - \nu, z_n - U_\mu z_n\rangle. \end{aligned} \tag{3.4}$$

Since, U_μ is nonexpansive, so for all $x \in Q$, we have

$$\begin{aligned} \langle U_\mu x - \nu, U_\mu x - \nu\rangle &= \|U_\mu x - \nu\|^2 \\ &\leq \|x - \nu\|^2 \\ &= \langle x - \nu, x - \nu\rangle \end{aligned}$$

$$= \langle x-\nu, x-U_\mu x\rangle + \langle x-\nu, U_\mu x-\nu\rangle,$$

which implies

$$\langle U_\mu x-\nu, U_\mu x-\nu\rangle - \langle x-\nu, U_\mu x-\nu\rangle \le \langle x-\nu, x-U_\mu x\rangle.$$

Therefore

$$\langle U_\mu x-x, U_\mu x-\nu\rangle \le \langle x-\nu, x-U_\mu x\rangle$$
$$\langle U_\mu x-x, U_\mu x-x\rangle + \langle U_\mu x-x, x-\nu\rangle \le \langle x-\nu, x-U_\mu x\rangle,$$

which implies

$$\langle U_\mu x-x, U_\mu x-x\rangle \le \langle x-\nu, x-U_\mu x\rangle - \langle U_\mu x-x, x-\nu\rangle.$$

Hence

$$\|U_\mu x-x\|^2 \le 2\langle x-\nu, x-U_\mu x\rangle \text{ for all } x\in Q. \tag{3.5}$$

Since, equation (3.5) holds for all $x \in Q$, therefore we have

$$\|U_\mu z_n - z_n\|^2 \le 2\langle z_n-\nu, z_n-U_\mu z_n\rangle. \tag{3.6}$$

Using equation (3.6) in equation (3.4), we have

$$\|y_n-\nu\|^2 \le \|z_n-\nu\|^2 + \|\beta_n(z_n-U_\mu z_n)\|^2 - \beta_n\|U_\mu z_n - z_n\|^2,$$

which gives

$$\|y_n-\nu\|^2 \le \|z_n-\nu\|^2 - \beta_n(1-\beta_n)\|U_\mu z_n - z_n\|^2,$$

and

$$\|y_n-\nu\|^2 \le \|z_n-\nu\|^2. \tag{3.7}$$

Therefore

$$\|y_n-\nu\| \le \|z_n-\nu\|. \tag{3.8}$$

From equations (3.1), (3.3) and (3.8), we have

$$\begin{aligned}
\|x_{n+1}-\nu\| &\le \gamma_n\|u-\nu\| + (1-\gamma_n)\|y_n-\nu\| \\
&\le \gamma_n\|u-\nu\| + (1-\gamma_n)\|z_n-\nu\| \\
&\le \gamma_n\|u-\nu\| + (1-\gamma_n)\|x_n+\theta_n(x_n-x_{n-1})-\nu\| \\
&\le \gamma_n\|u-\nu\| + (1-\gamma_n)[\|x_n-\nu\| + \|\theta_n(x_n-x_{n-1})\|] \\
&\le (1-\gamma_n)\|x_n-\nu\| + \gamma_n\|u-\nu\| + (1-\gamma_n)\theta_n\|x_n-x_{n-1}\|.
\end{aligned}$$

Let $K = 2 \quad \max\{\|u-\nu\|,\ \sup \frac{(1-\gamma_n)\theta_n}{\gamma_n}\|x_n-x_{n-1}\|\}$, therefore we have

$$\begin{aligned}
\|x_{n+1}-\nu\| &\le (1-\gamma_n)\|x_n-\nu\| + \gamma_n K \\
&\le \max\{\|x_n-\nu\|, K\}.
\end{aligned}$$

Continuing in this way, we have

$$\|x_{n+1}-\nu\| \le \max\{\|x_0-\nu\|, K\}.$$

This shows that the sequence $\{x_n\}$ is bounded and consequently the sequence $\{z_n\}$ and $\{y_n\}$ are also bounded. From equation (3.1) and Lemma (2.1.1) , we obtain

$$\begin{aligned}
\|x_{n+1}-\nu\|^2 &= \|\gamma_n(u-\nu) + (1-\gamma_n)(y_n-\nu)\|^2 \\
&\le (1-\gamma_n)^2\|y_n-\nu\|^2 + 2\gamma_n\langle u-\nu, x_{n+1}-\nu\rangle \\
&\le (1-\gamma_n)\|y_n-\nu\|^2 + 2\gamma_n\langle u-\nu, x_{n+1}-\nu\rangle.
\end{aligned}$$

Using equation (3.7), we have

$$\|x_{n+1}-\nu\|^2 \le (1-\gamma_n)\{\|z_n-\nu\|^2 - \beta_n(1-\beta_n)\|U_\mu z_n - z_n\|^2\} + 2\gamma_n\langle u-\nu, x_{n+1}-\nu\rangle \tag{3.9}$$

$$\begin{aligned}
&\le (1-\gamma_n)\{\|x_n-\nu\|^2 + \theta_n^2\|x_n-x_{n-1}\|^2 + 2\theta_n\langle x_n-\nu, x_n-x_{n-1}\rangle - \beta_n(1-\beta_n)\|U_\mu z_n - z_n\|^2\} \\
&\quad + 2\gamma_n\langle u-\nu, x_{n+1}-\nu\rangle, \\
&\le (1-\gamma_n)\{\|x_n-\nu\|^2 + \theta_n^2\|x_n-x_{n-1}\|^2 + 2\theta_n\langle x_n-\nu, x_n-x_{n-1}\rangle\} + 2\gamma_n\langle u-\nu, x_{n+1}-\nu\rangle.
\end{aligned} \tag{3.10}$$

Now, on assuming

$$t_n = \|x_n - \nu\|^2,$$
$$s_n = \frac{\theta_n^2(1-\gamma_n)}{\gamma_n}\|x_n - x_{n-1}\|^2 + \frac{2\theta_n}{\gamma_n}(1-\gamma_n)\langle x_n - \nu, x_n - x_{n-1}\rangle + 2\langle u - \nu, x_{n+1} - \nu\rangle.$$

So, the inequality (3.8) becomes,

$$t_{n+1} = (1-\gamma_n)t_n + \gamma_n s_n. \tag{3.11}$$

From equation (3.1), we have

$$\lim_{n\to\infty}\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\| \le \lim_{n\to\infty}\frac{\alpha_n}{\gamma_n} = 0.$$

Therefore

$$\lim_{n\to\infty}\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\| = 0. \tag{3.12}$$

Now, we look at two situations.

Case 1. The sequence $\|x_n - \nu\|^2$ is decreasing. Then the sequence $\|x_n - \nu\|^2$ is convergent and therefore we obtain $\|x_{n+1} - \nu\|^2 - \|x_n - \nu\|^2 \to 0$ as $n \to \infty$. Equation (3.9) can be written as

$$\|x_{n+1} - \nu\|^2 \le \|x_n - \nu\|^2 + \frac{\theta_n^2}{\gamma_n^2}\gamma_n^2\|x_n - x_{n-1}\|^2 + \frac{2\theta_n}{\gamma_n}\gamma_n\|x_n - \nu\|\|x_n - x_{n-1}\| + 2\gamma_n\langle u - \nu, x_{n+1} - \nu\rangle$$
$$- \beta_n(1-\beta_n)\|U_\mu z_n - z_n\|^2. \tag{3.13}$$

Taking limit $n \to \infty$ in equation (3.13) and using condition (3.12) and $\lim_{n\to\infty}\gamma_n = 0$, we obtain

$$\lim_{n\to\infty}\|U_\mu z_n - z_n\| = 0.$$

Also, we have

$$\|z_n - x_n\| = \gamma_n\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\|.$$

Taking limit $n \to \infty$ in above equation and using equation (3.12), we get

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.14}$$

Since the sequence $\{x_n\}$ is bounded, so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q$ as $k \to \infty$ and from equation (3.14), $z_{n_k} \rightharpoonup q$ as $k \to \infty$. From Lemma 2.2, we obtain $q \in Fix(U_\mu) = Fix(U)$. Now,

$$\begin{aligned}\|y_n - z_n\| &= \|(1-\mu\beta_n)z_n + \mu\beta_n U z_n - z_n\|\\ &= \|(1-\beta_n)z_n + \beta_n U_\mu z_n - z_n\|\\ &= \beta_n\|U_\mu z_n - z_n\|.\end{aligned}$$

Taking limit $n \to \infty$ in above equation, we derive

$$\lim_{n\to\infty}\|y_n - z_n\| = 0. \tag{3.15}$$

Using equations (3.14), (3.15) and triangle inequality, we obtain

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.16}$$

Also by using equation (3.1) and condition $\lim_{n\to\infty}\gamma_n = 0$, we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\gamma_n u + (1-\gamma_n)y_n - x_n\|\\ &\le \gamma_n\|u - x_n\| + (1-\gamma_n)\|y_n - x_n\|.\end{aligned}$$

Using equation (3.16), $\lim_{n\to\infty}\gamma_n = 0$ and taking limit $n \to \infty$ in above equation, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.17}$$

Now by combining the projection property and using the fact that $\nu = P_{Fix(U_\mu)}u$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u - \nu, x_n - \nu\rangle &= \lim_{k\to\infty}\langle u - \nu, x_{n_k} - \nu\rangle\\ &= \langle u - \nu, q - \nu\rangle\end{aligned}$$

$$\leq 0,$$

which implies

$$\limsup_{n\to\infty}\langle u-\nu, x_n-\nu\rangle \leq 0. \tag{3.18}$$

Combining equations (3.17) and (3.18), we have

$$\limsup_{n\to\infty}\langle u-\nu, x_{n+1}-\nu\rangle \leq 0,$$

Using above condition and condition (3.12), we obtain that $\limsup_{n\to\infty} \quad s_n \leq 0$.

Thus, by using Lemma 2.3, $\lim_{n\to\infty} t_n = 0$ and hence $x_n \to \nu$ as $n \to \infty$.

Case 2. The sequence $\|x_n-\nu\|^2$ is not decreasing.

Then, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that

$$t_{n_k} \leq t_{n_k+1} \text{ for any } k \in \mathbb{N}. \tag{3.19}$$

Using Lemma 2.4, there exists a nondecreasing sequence $\{n_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} n_k = \infty$ which satisfies

$$t_{n_k} \leq t_{n_k+1}. \tag{3.20}$$

and

$$t_k \leq t_{n_k+1}. \tag{3.21}$$

Similarly as in Case 1,

$$\begin{cases} \lim_{k\to\infty}\|U_\mu z_{n_k}-z_{n_k}\| & = 0 \\ \lim_{k\to\infty}\|z_{n_k}-x_{n_k}\| & = 0 \\ \lim_{k\to\infty}\|y_{n_k}-z_{n_k}\| & = 0 \\ \lim_{k\to\infty}\|y_{n_k}-x_{n_k}\| & = 0 \\ \lim_{k\to\infty}\|x_{n_{k+1}}-x_{n_k}\| & = 0. \end{cases}$$

Using the similar proof as in Case 1, we get $\nu \in Fix(U) = Fix(U_\mu)$ and

$$\limsup_{k\to\infty}\langle u-\nu, x_{n_{k+1}}-\nu\rangle \leq 0. \tag{3.22}$$

From equation (3.11), we get

$$t_{n_k+1} \leq (1-\gamma_{n_k})t_{n_k} + \gamma_{n_k}s_{n_k}. \tag{3.23}$$

Using equation (3.20) in equation (3.23), we obtain

$$\gamma_{n_k}t_{n_k} \leq \gamma_{n_k}s_{n_k}.$$

Since $\gamma_{n_k} > 0$, we obtain $t_{n_k} \leq s_{n_k}$. Therefore, we have

$$\|x_{n_k}-\nu\|^2 \leq \frac{\theta_{n_k}^2(1-\gamma_{n_k})}{\gamma_{n_k}}\|x_{n_k}-x_{n_{k-1}}\|^2 + \frac{2\theta_{n_k}}{\gamma_{n_k}}(1-\gamma_{n_k})\langle x_{n_k}-\nu, x_{n_k}-x_{n_{k-1}}\rangle + 2\langle u-\nu, x_{n_{k+1}}-\nu\rangle,$$

which implies

$$\|x_{n_k}-\nu\|^2 \leq \frac{\theta_{n_k}^2(1-\gamma_{n_k})}{\gamma_{n_k}}\|x_{n_k}-x_{n_{k-1}}\|^2 + \frac{2\theta_{n_k}}{\gamma_{n_k}}(1-\gamma_{n_k})\|x_{n_k}-\nu\|\|x_{n_k}-x_{n_{k-1}}\| + 2\langle u-\nu, x_{n_{k+1}}-\nu\rangle.$$

Using condition (3.12) and from equation (3.22), we obtain $t_{n_k} \to 0$ as $k \to \infty$.

By using equation (3.23), we obtain $\lim_{k\to\infty} t_{n_k+1} = 0$ and from equation (3.21), we have $\lim_{k\to\infty} t_k = 0$. That is, $\|x_n-\nu\|^2 \to 0$ as $n \to \infty$. Hence, $x_n \to 0$ as $n \to \infty$. This completes the proof.

3.2 Modified Inertial Viscosity Algorithm

Theorem 3.2. *Let Q be a nonempty, convex and closed subset of H and $U : Q \to Q$ be enriched nonexpansive mapping and $Fix(U) \neq \phi$. Assume that $f : Q \to Q$ is λ- contraction mapping with $\lambda \in [0,1)$. For two real sequences $\{\gamma_n\}, \{\beta_n\} \in (0,1)$ and $\{\alpha_n\}$ is a real sequence satisfying $\lim_{n\to\infty}\frac{\alpha_n}{\gamma_n} = 0$. Also, suppose that $\theta > 0, \sum_{n=0}^{\infty}\gamma_n = \infty$ and $\lim_{n\to\infty}\gamma_n = 0$. Let $x_{-1}, x_0 \in Q$ be chosen arbitrarily. Assume that the sequence $\{x_n\}$ is defined by the following algorithm:*

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), where \quad \theta_n = \begin{cases} min\left\{\frac{\alpha_n}{\|x_n-x_{n-1}\|}, \theta\right\}, if x_n \neq x_{n-1}; \\ \theta, otherwise, \end{cases} \\ y_n = (1-\mu\beta_n)z_n + \mu\beta_n U z_n; \\ x_{n+1} = \gamma_n f(x_n) + (1-\gamma_n)y_n; \end{cases} \tag{3.24}$$

where $n \geq 0$.

Then the sequence given by equation (3.24) strongly converges to fixed point ν of U, where $\nu = P_{Fix(U)}f(\nu)$.

Proof. If we replace u by $f(x_n)$ in theorem (3.1), then we obtain the desired result.

4 Numerical Example

This section includes two numerical examples which indicate that Modified Inertial Mann Halpern Algorithm($MIMHA$) has better rate of convergence than Modified Kransnoselskii Mann Algorithm($MKMA$) [4]. In both examples, we show that Modified Inertial Mann Halpern Algorithm ($MIMHA$) converges faster than Modified Kransnoselskii Mann Algorithm ($MKMA$) [4]. We performed all numerical experiments on a Dell computer equipped with a 3.20 GHz Intel (R) Core (TM) i5-3470 CPU and 8 GB of memory. The MATLAB R 2022 a platform was used as the implementation environment.

Example 4.1. Let $H = \mathbb{R}$ and $Q = [0, 1]$. Suppose that $U : Q \to Q$ is defined by $U(x) = 2 - x$ for all $x \in Q$. Clearly, U is nonexpansive (0-enriched nonexpansive) mapping. Assume that $\alpha_n = \frac{1}{n^3}, \beta_n = \frac{1}{n}, \gamma_n = \frac{1}{n+6}, u = 0, \quad \theta = 0.999, \quad \mu = 0.5$ and we select $\|x_n - x_{n-1}\| \leq 10^{-5}$ as the stopping criterion. The values of iteration number(n) for Modified Inertial Halpern Algorithm ($MIMHA$) and Modified Kransnoselskii Mann Algorithm ($MKMA$) are given in Table 4.1 and a comparison of both algorithms is shown in Figure 4.1.

***Table* 4.1**

Algorithm	Iteration Number(n)	cpu time(in seconds)
MIMHA	494	0.076035
MKMA	587	0.078332

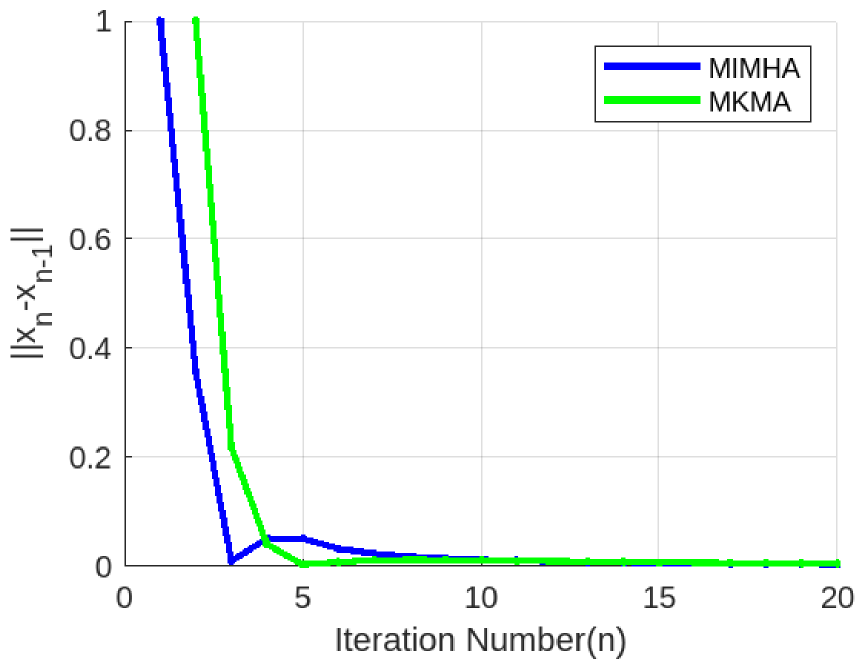

***Figure* 4.1** *Comparison of MIMHA with MKMA for* $U(x) = 2 - x$

Example 4.2 (Example 2.1, [3]). Let $Q = [\frac{1}{2}, 2] \subset \mathbb{R}$ and suppose that $U : Q \to Q$ is defined by $U(x) = \frac{1}{x}$ for all $x \in Q$. Then U is $\frac{3}{2}$-enriched nonexpansive mapping. We take $\alpha_n = \frac{1}{n^3}, \beta_n = \frac{1}{n}, \gamma_n = \frac{1}{n+6}, u = 0, \quad \theta = 0.999, \quad \mu = 0.5$ and we select $\|x_n - x_{n-1}\| \leq 10^{-5}$ as the stopping criterion which is the same as we take for Example 4.1. The values of iteration number(n) for Modified Inertial Halpern Algorithm ($MIMHA$) and Modified Kransnoselskii Mann Algorithm ($MKMA$) are given in Table 4.2 and a comparison of both the algorithms is shown in Figure 4.2.

Table **4.2**

Algorithm	Iteration Number(n)	cpu time(in seconds)
MIMHA	379	0.069946
MKMA	480	0.072002

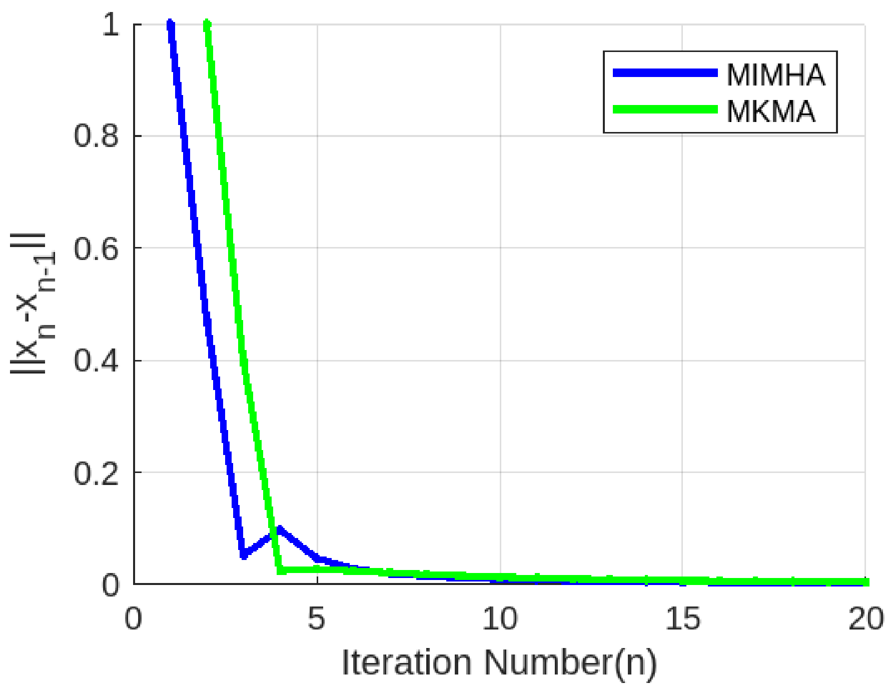

***Figure* 4.2** *Comparison of MIMHA with MKMA for* $U(x) = \frac{1}{x}$

5 Conclusion

In this study, the strong convergence of modified inertial Mann Halpern and viscosity algorithms for approximating fixed points of enriched nonexpansive mappings is demonstrated. Additionally, we numerically compared the rates of convergence of the modified inertial Mann Halpern algorithm with the modified Kransnoselskii Mann algorithm [4].

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References

[1] E. Blum and W. Oettli, Variational principles for equilibrium problems, *Parametric Optimization and Related Topics III,* Peter Lang Frankfurt am Main, Germany, (1993), 79-88.

[2] S. Banach, Sur les operations dans les ensembles abstraits et leurs applications aux equations integrales, *Fundamenta Mathematicae,* **3** (1922), 133181.

[3] V. Berinde, Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces, *Carpathian Journal of Mathematics,* **35** (2019), 293-304.

[4] V. Berinde, A modified KrasnoselskiiMann iterative algorithm for approximating fixed points of enriched nonexpansive mappings, *Symmetry,* **14** (2022), 1-12.

[5] H. H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Review,* **38** (1996), 367- 426.

[6] H. H. Bauschke and P. L. Combettes, Convex analysis and monotone operator theory in Hilbert spaces, *Springer,* **408** (2011), 1-978.
[7] V. Berinde and M. Pcurar, Fixed points theorems for unsaturated and saturated classes of contractive mappings in Banach spaces, *Symmetry*, **13** (2021), 1-15.
[8] J. B. Bailion, R. E. Bruck and S. Reich, On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces, *Houston journal of Mathematics*, **4** (1978), 1-9.
[9] Y. Censor and T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numerical Algorithms*, **8** (1994), 221-239.
[10] M. A. Krasnoselski, Two remarks on the method of successive approximations, *Uspekhi Matematicheskikh Nauk*, **10** (1955), 123-127.
[11] L. Muu and W. Oettli, Convergence of an adaptive penalty scheme for finding constrained equilibria, *Nonlinear Analysis Theory, Methods and Applications*, **18** (1992), 1159-1166.
[12] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506-510.
[13] P. E. Maing, A hybrid extragradient-viscosity method for monotone operators and fixed point problems, *SIAM Journal on Control and Optimization*, **47** (2008), 1499-1515.
[14] G. Stampacchia, Formes bilineaires coercivites sur les ensembles convexes, *Comptes rendus de l'Acadmie des Sciences Paris*, **258** (1964), 44134416.
[15] B. Tan, Z. Zhou and S. Li, Strong convergence of modified inertial Mann algorithms for nonexpansive mappings, *Mathematics*, **8** (2020), 1-11.
[16] H. K. Xu, Iterative algorithms for nonlinear operators, *Journal of the London Mathematical Society*, **66** (2002), 240-256.