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Professor V. P. Saxena

on His 80th Birth Anniversary Celebrations

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Jñānābha

This Special Volume of

JÑĀNĀBHA

is Being Dedicated to

Professor Vinod Prakash Saxena

on His 80^{th} Birth Anniversary Celebrations

PROFESSOR VINOD PRAKASH SAXENA

(Born: December 11, 1943)

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PROFESSOR VINOD PRAKASH SAXENA (V.P. SAXENA) : A TOWERING AND LEADING MATHEMATICIAN

By

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On behalf of ***VIJÑĀNA PARISHAD OF INDIA*** and ***JÑĀNĀBHA*** Family, we ourselves feel great honored to publish Special Issue of *JÑĀNĀBHA* , Vol.53(2) (2023) *(Dedicated to Professor V.P. Saxena on His 80th Birth Anniversary Celebrations)*

Professor Vinod Prakash Saxena is an amazing man towering and leading mathematician, well known topmost eminent figure of Biomathematics and allied topics of Applied Mathematics. He has credit to introduce I-function (1982) as final generalization of earlier sequence of functions including H-function due to Charle's Fox (1962) [first Indian formula of Mathematics applied in the index of *American Math. Society Reviewers*].

Professor Saxena is well associated with me since 1964. Both of us as top classmates got *M.Sc.* (Mathematics) in 1966 with distinction from Jiwaji University, Gwalior, Madhya Pradesh, India. Then both of us worked as regular Research Scholar under Research Training Scheme of Ministry of Education, Government of India in S.A. Technological Institute, Vidisha, Madhya Pradesh [August 01,1966 - July 31, 1969] under Professor P.M. Gupta, and both got Ph.D. from Vikram University, Ujjain, Madhya Pradesh, India in 1970.

Professor Saxena is well associated with JÑĀNĀBHA since very inception 1971 as an author. He has credit to be Hon'ble member of Executive Council of VPI in 1988.

He was honored by ***DISTINGUISHED SERVICE AWARD*** during 6^{th} Annual Conference of VPI held at Bundelkhand Institute of Engineering and Technology, Jhansi, Uttar Pradesh, India (December 26-28,1996). Professor V.P. Saxena has credit to grace the chair of President of VPI (April 2005- March 2008). He was elected as Honorary Fellow of VPI and honored by title ***FVPI*** in 2007 during 12^{th} Annual Conference of VPI held at JNV University, Jodhpur, Rajasthan, India [October 25-27, 2007]. Professor Saxena was also honored by Highest Prestigious Award ***LIFE -LONG ACHIVEMENTS AWARD*** of VPI during 2^{nd} International Conference of VPI held at Bundelkhand University, Jhansi, Uttar Pradesh, India (March 09-11, 2018). Professor V.P. Saxena is recently honored by ***VPI GOLDEN JUBILEE AWARD*** during Fifth International Conference and Golden Jubilee Celebration of VPI held at Jawaharlal Nehru University, New Delhi (June 16-18, 2022). Professor Saxena is also active Hon'ble Senior Member on Editorial Board of JÑĀNĀBHA . On this great occasion of Professor Saxena's Birth Anniversary Celebrations, we wish him a happy, and long joyful life . May he continue to guide, encourage, and enlighten the global Mathematics community for decades to come.

AT A GLANCE:

PROFESSOR VINOD PRAKASH SAXENA, *FISMMCS, FRMS, FVPI*
(Ex-Vice-Chancellor Jiwaji University, Gwalior)
E-mail: vinodpsaxena@gmail.com
Mob. 9425109044, Ph: 0755-4055277
Resi: B-147, New Minal Residency,
J.K. Road, Bhopal-462023, India

1 PERSONAL DATA:

Date of Birth : December 11, 1943
Place of Birth : Shivpuri (M.P.), India

2 EDUCATIONAL DATA:

(A) *Basic*

Ph.D.	1970	Vikram University Ujjain	Faculty: Engineering Subject: Applied Mathematcs Topic : Integral Transforms and Their Technical Applications
M.Sc.	1966	Jiwaji Univeristy Gwalior	Faculty: Science Subject: Mathematcs Position : First Division with Distinction. Stood Second in the University Merit

(B) *Post-Doctoral Fellowships/Visitor ship/Training*

POSTDOCTORAL

VISITING SCIENTIST 'Worked as CSIR Senior Research Fellow and Post Doctor Research Fellow at SATI, Vidisha and M.A. College of Technology Bhopal from May 1969 to July 1971. Did research and taught post-graduate classes during this period.

Worked at the University of Cambridge, England as visiting scientist in 1978 under British Council UGC (India) exchange of Young Scientists programme and worked under Sir James Light hill, (Lucasian Professor) of the University.

SHORT-TERM TRAININGS

Participated in several advanced short term courses on Management, Mathematics and Computer Science at leading Institutions like I.I.M., Ahmedabad; I.I.T., Delhi; I.I.T., Kanpur and MANIT, Bhopal

3 PROFESSIONAL AND ADMINISTRATIVE ASSIGNMENTS

Lecturer :

Worked as Associate Lecturer/Lecturer of Mathematics at S.V. Regional College of Engineering and Technology, Surat during July 1971 to May 1980.

Associate Professor

Worked as Associate Professor of Mathematics at P.A. University, Ludhiana during May 1980 to March 1984

Professor

Professor of Mathematics at Jiwaji University, Gwalior during March 1984 to December, 2005.

Dean

Dean Faculty of Science, Jiwaji University, Gwalior during 1984-86 and 1990-92.

Vice-chancellor

Took over as emergency Vice-Chancellor, Jiwaji University under Section 52 during August 9, 2000.

Director/Principal

- Worked as Director/Principal, Yagyavalkya Institute of Technology, Jaipur from August, 2007 to August 2008.
- Director, Sagar Institute of Research, Technology and Science, Bhopal, from August 2008 to January 2012.

Advisor

- Appointed as Advisor, Research and Development, Sharda Group of Institutions (Anand Engineering College, Agra) in April, 2006.
- Appointed as Advisor and Coordinator (Research) Sagar Group of Institutions (SIRT, SIRTS, and SIRTE) in January 2012.

Additional Assignments/Positions

i) Worked as the Head, School of Mathematics and Allied Sciences, Jiwaji University, Gwalior since March 1984 till August 2000.
ii) Worked as Director, Computer Centre, Jiwaji University, Gwalior during 1998-99.
iii) Worked as Coordinator, M.Sc. Computer Science Programme of Jiwaji University, Gwalior during 1992-2000.
iv) Founder Head of Computer Centre, Jiwaji University, Gwalior from 1987 to 1994.

v) Appointed Proctor, Jiwaji University, Gwalior in 1994 for two years.
vi) Coordinator, M.P. Council of Science and Technology (Gwalior) for two terms 1987-91.

Supporting Positions:

i) Member, Executive Council, Jiwaji University, Gwalior, first during 1984-86 and second time in 1990-93.
ii) Chairman, Board of Studies in Mathematics, Jiwaji University, Gwalior during 1984 to 2001.
iii) Chairman, Board of Studies in Computer Science, Jiwaji University, Gwalior for two years.
iv) Member, Board of Governors, Madhav Institute of Technology and Science, Gwalior since 1994.
v) Member of several other University bodies like Academic Council, Standing Council Board of Studies of affiliated colleges etc.
vi) Chairman, UGC-NAAC peer team since March 2002 and evaluated an accredited more than 75 Universities and Colleges.

Academic Positions Held in Other Institutions:

i) Expert Member, Standing Committee, UGC COSIST program in Mathematics at Jodhpur University for two terms.
ii) Expert Member of Board of Studies of several Universities like Rajasthan, Agra, Vikram, Bhopal etc.
iii) Expert Member of Research Degree Committees of several Universities like Srinagar (Garhwal), Gurukul Kangri, Bhopal, Rewa, JAYPEE Univesity etc.
iv) Member, Advisory Board of UGC Centre on Mathematical Modelling in Jadavpur University, Calcutta.
v) Advisor, Ansal Institute of Technology, Gurgaon.

4 FOREIGN VISITS:

i) During 1978 visited Imperial College (London) (Prof. C.G. Caro), Brunel University (Uxbridge) (Prof. J.R. Whiteman), University of Glasgow (Prof. I.N. Sneddon) and University of Stratnelyde (Glasgow) (Prof. R.M. Keneddy). These visits have helped to establish collaboration on long term basis with eminent British Scientists.
ii) Also visited Xian-Jiaotong University, China in 1988 to deliver lecture and participate in International Conference on Biomathematics.
iii) Visited National University of Singapore, Singapore in 1996 to participate in International Conference.
iv) First ever Indian invited to deliver Plenary lecture in International Congress of Bio-mathematics, delivered three lectures in 8^{th} International Congress Bio-mathematics at Panama in 1997.
v) Delivered lectures in Argentina University and National Health Institute, Buenos Aires, Argentina in 1997.
vi) Delivered lectures in Beykent University and other Universities of Istanbul, Turkey in 2001.
vii) Visited Cyprus in 2001 to participate in Commonwealth Universities Vice-Chancellors' Conference.
viii) Visited and delivered lecture in the University of Cambridge (U.K.) IN 2006.
ix) Visited and delivered lecture at Fraunhaufer Institute, Kaiserslautern, Germany in December 2007.
x) Visited Isaac Newton Centre of applied mathematics at the University of Cambridge and also presented research tutorials at Second IASTED International conference in July 2011.
xi) Delivered a plenary lecture at American University in the Emirates in the International conference on Transnational Education and Cultural Effects, organized by Eurasian Universities Union in 2014 at Dubai (UAE).

5 TEACHING and RESEARCH:

Specializations in Teaching:

Teaching : Analysis, Biomathematics, Numerical Methods, Mathematical Methods, Neural Networks, Simulation, Mathematical Modelling, Air Pollution and other areas of Mathematics and theoretical computer science.

Research Areas:

i) Physiological Heat Transfer
ii) Mathematical Ecology (including Population Modelling)
iii) Atmospheric Pollution
iv) Epidemiology
v) Pharmaco-Kinetics
vi) Theoretical Computer Science

vii) Mathematical Finance
viii) Higher Transcendental Functions

Research Projects Completed:

i) "Temperature Distribution in Skin and Subcutaneous Tissues" supported by South Gujrat University, Surat under UGC unassigned grant (1974-76).
ii) "Mathematical and Numerical Approach to Physiological heat Flow Problems" supported by UGC, New Delhi (1985-88).
iii) "Quantitative Study of Effect of Growth of Population and Pollution on Rural Ecosystem" supported by M.P. Council of Science and Technology, Bhopal (1989-91).
iv) "Mathematical Approach to the Thermal Studies of Abnormal Growths and Heat Injuries in a Human Body" supported by UGC, New Delhi (1995-1998)
v) "Mathematical Modelling in the Study of Ecological Effects of Pollution on the Existence of Interacting Species Systems" supported by UGC, New Delhi (2002-onwards).
vi) Mathematical study of Wild Life Population with Special Reference to Shivpuri District of M.P. Supported by M.P. Council of Science and Technology since 2002.
vii) "A Study on the Ecological Determinants and Their Impact on Human Population in Industrial Complexes near Gwalior" supported by DST, New Delhi (1993-1996).

Most Significant Contribution to Mathematical Sciences

Introduced and Defined a New Formula/Function: "I-Function" (1982) which is final generalization of Hypergeometric functions in the sequence of E-Function (MacRobert, 1937), G-Function (Meijers, 1944) and H-Function (Fox, 1962) which is adopted as a research topic by many mathematicians throughout.
First Indian's formula of Mathematics appeared in the index of American Mathematical Society Reviewers.

Prizes and Honours

(A) Won following prizes / honours for research work.
- Nominated fellow of Indian Society for Mathematical Modelling and Simulation in 2022.
- Nominated fellow of Vijñāna Prishad of India in 2003.
- President of India cash prize for presenting the best research paper in the sixteenth Congress of Indian Society of Theoretical and Applies Mechanics (Allahabad 1972).
- Hariom Ashram, Bhai Kaka Prerit prize for the best research publication during 1972-73.
- Nominated Fellow of Ramanujan Mathematical Society in its 10th Annual Conference (1995).
- Elected fellow of Vijñāna Parishad of India in 2008.

(B) Won following prizes / honours for teaching.
- "Shikshak Samman" presented by Late Srimant Madhav Rao Ji Scindia (then Railway Minister) as ideal teacher on behalf of Sajag Nagrik Manch, Gwalior in 1989.
- "Adarsh Shikshak Samman" presented by Dr. P.S. Bisen (Former Vice-chancellor, Jiwaji University, Gwalior) on behalf of S.D.R. Shiksha Prasar Samiti, Gwalior
- Honoured by Ramanujan Mathematical Society in its 16th Annual Conference, 3-5 June, 2001 at Fergusson College, Pune
- "Rashtriya Shiksha Ratna Award - 2007" presented by National Education and Human Resource Development Organization, Pune.
- Honored by Chief Minister of Madhya Pradesh on the occasion of "Teachers' day" on 4th Sept. 2011 amongst five senior Professors of Madhya Pradesh.
- "Teacher of the Year" award by Gwalior Vikas Samiti in 2006.

(C) Won Following prizes/honours for others.
- Special felicitation and award by Hon'ble Governor of Madhya Pradesh for contribution in the development of Jiwaji University, Gwalior during Golden Jubilee celebration in 2014.
- First Asian invited to deliver Planery lecture in any International Congress of Biomathematics. The lecture was delivered in 8th Congress (Panama in Aug. 1997).
- Invited to deliver P.D. Verma Memorial Lecture by the University of Rajasthan, Jaipur in 2001.
- Only Indian invited in second IASTED International conference at Cambridge to present three hours research tutorial on Computational Physiology.
- Invited to deliver J.N. Kapur memorial lecture at Kanpur during 15th Annual conference of Vijñāna Parishad of India, 2011.

Position in Research Bodies:

i) President, Vijñāna Parishad of India, 2004-2007.
ii) President, Gwalior Academy of Mathematical Sciences 1994-98, 2001-2005; 2022 onwards.
iii) President, Sagar Society of Interdisciplinary Research and Technology, Bhopal 2013-2014.
iv) Vice-President of Indian Society of Theoretical and Applied Mechanics (1986-87).
v) Chairman, Computer Society of India, Gwalior Chapter.
vi) Vice President, Ramanujan Mathematical Society.
vii) Academic Secretary of Ramanujan Mathematical Society.
viii) General Secretary of "The Mathematics Consortium" since 01/01/2015
ix) Executive Committee member of:
   (a) Indian Society of Theo. and Appl. Mech.
   (b) National Society of Biomechanics,
   (c) Vijñāna Parisad of India
   (d) Ramanujan Mathematical Society
   (e) Indian Academy of Mathematics
   (f) Indian Society of Industrial and Applied Mathematics.
x) Member National Committee on Mathematics Educational Research DST

Position in Research Journals

i) Executive Editor of GAMS Journal of Mathematics and Mathematical Biosciences.
ii) Editor-In-Chief of SSIRT Journal of Engineering, Management and Pharmaceutical Science.
iii) Member, Editorial Board of Indian Academy of Mathematics.
iv) Publication Committee Member of Wurtz Publications Canada.
v) Member of Editorial Board of 'Jñānābha ' Published by VPI.
vi) Member of Editorial Board of 'Ganita Sandesh' of Rajasthan Ganita Parishad.
vii) Member, Editorial Advisory Board of JUET Research Journal of Science and Technology.

Research and Educational Programmes Organized

Following Research programmes have been organised as programme Director/Convener/Coordinator:

International

> Organised IVth International Conference on Physiological Fluid Dynamics (1995)
> Organised First International Conference of Gwalior Academy of Mathematical Science (2008)
> Organised First International Conference of Sagar Society of Interdisciplinary Research and Technology (2014)
> Organised Second International Conference of Sagar Society of Interdisciplinary Research and Technology (2015)
> Organised Third International Conference of Sagar Society of Interdisciplinary Research and Technology (2016)
> Organised Fourth International Conference of Sagar Society of Interdisciplinary Research and Technology (2017)
> Organised Fifth International Conference of Sagar Society of Interdisciplinary Research and Technology (2018)

National

i) All India Seminar on Finite Element Method and its Applications to Biology (1982), (sponsored by UGC, New Delhi).
ii) All India Symposium on system Theory and its Application to Biology (1985), (sponsored by UGC, New Delhi).
iii) Twenty Third Annual Congress of Indian Society of Theoretical and Applied Mechanics (1986).
iv) Orientation Programme for College Teachers-1 (sponsored by M.P. UGC) (1986).
v) Orientation Programme for College Teachers-11 (sponsored by M.P. UGC) (1987),
vi) Silver Jubilee All India Workshop for College Teacher (1988) (sponsored by UGC)
vii) Third M.P. Young Scientists Congress (1988) (sponsored by M.P. Council of Science and Technology).
viii) All India Continuing Programme in Forecasting Methodologies (1990) (sponsored by DST, New Delhi).
ix) Workshop and Camp on Future Studies and Human Population (1992) (sponsored by DST, New Delhi).
x) Seventh Annual Conference of Ramanujan Mathematical Society and All India Symposium on Mathematical Biology (1992).

xi) Instructional Conference on Mathematical Modelling in Biology and Medicine (1997).
xii) Indian Science Congress Symposium on Mathematical Ecology and Biomechanics (1998).
xiii) Thirteenth National Conference of Gwalior Academy of Mathematical Sciences (2008).
xiv) National Workshop on "Mathematical Modelling and Computation" sponsored by The Mathematics Consortium India, during 2019.

Research Talks/Lectures Delivered:

Time to time invited lectures and talks delivered in several institutions and Research Conferences/ Seminars/ Symposia of International/ National Level, including the following important ones:

(A) Institutions

University of Buenos Aires, Argentina; University of Cambridge, England; Imperial College, London; Glasgow University, Scotland; Beykent University, Istanbul; Indian Institute of Science, Bangalore; I.I.T. Madras; I.I.T. Delhi; I.I.T. Mumbai; I.I.T. Kanpur; Kurukshetra University; South Gujarat University; REC, Surat; Osmania University; REC, Warangal; Udaipur University; Aligarh Muslim University; University of Rajasthan, Jaipur; Thapar University Patiala; MANIT Bhopal, Kashmir University J and K; Katmandu University, Nepal; Pune University, Pune; DAVV, Indore; Jiwaji University, Gwalior; Central university, Sagar; National College, Tiruchirappalli; Kerala University, Thiruanantpuram; Calcutta Mathematical Society, Kolkata; Department of Higher Education Goa.

International Conferences

i) International Conference on Biomathematics, Xian, China, 1988.
ii) Participated and Presented Paper in 9th International Symposium on Transport Phenomena in Thermal Fluids Engineering, Singapore, 1996.
iii) Eighth International Congress of Biomathematics, Panama, 1997.
iv) First International Congress on Physiological Fluid Dynamics, Madras.
v) International Conference on Theory of Differential Equations and Application to Oceanography, Goa.
vi) International Workshop on Approximation Theory and Applications, Aligarh.
vii) Third International Conference on Physiological Fluid Dynamics, Madras.
viii) Fourth International Conference on Physiological Fluid Dynamics, Gwalior.
ix) International Conference on Mathematics at the University of Lucknow, Lucknow (India).
x) First and Second International Conferences of Indian Society of Industrial and Applied Mathematics (New Delhi).
xi) Third International conference of GAMS at SVNIT, Surat in 2014.
xii) International conference on Mathematics, SRM University, Channai, 2018.
xiii) International conference of The Mathematics Consortium at BHU, Varanasi, 2019.
xiv) International conference of VPI at JNU, New Delhi, 2022.

(B) National and Regional Level Conferences / Symposia / Seminars etc.

About sixty lectures have been delivered in almost all the conferences of National Societies concerning Mathematics and Allied subjects including those of:

i) Indian Mathematical Society (Jaipur, Muzzaffarpur, Ahmedanagar)
ii) Indian Society of Theoretical and Applied Mechanics (Pantnagar, Surat).
iii) Indian Sciene Congress (Madurai, Hyderabad, Chennai and Pune)
iv) Ramanujan Mathematical Society (Rajkot, Tirupati, Gwalior, Shimoga, Rishikesh Trivendrum).
v) Vijñāna Parishad of India (Hardwar, Gorakhpur, Jhansi, Lucknow).
vi) Banaras Mathematical Society (Varanasi).
vii) Rajasthan Ganita Parishad (Jodhpur, Kota)
viii) Indian Academy of Mathematics (Indore etc.)
ix) Bharat Ganita Parishad (Lucknow)
and several others..

Research Guidance:

(A) Guided Ph.D. candidates on the following titles:

i) Mathematical Study of Physiological Heat Transfer Problems (D. Arya).
ii) A Mathematical Study of Heat Transfer Problems in Cutaneous and Subcutaneous In-vivo tissues (J.S. Bindra).
iii) Mathematical Investigations on Human Physiological Heat Flow Problems with Special Relevance to

Cancerous Tumours (K.R. Pardasani).

iv) Mathematical Study of Diffusion Problems in In-vivo Human Skin and Other Peripheral Tissues (Praveen Miri).

v) Analytical Study of Physiological Heat Flow Problems with Special Relevance to Human limbs (M.P. Gupta).

vi) Computerised Solution of Bio-mathematical and Bio-statistical Problems Related to Dermatoglyphic and Genetic Studies in Sports (J. P. Verma).

vii) Finite Element Approach to Ecological Problems with Special Relevance to Environmental Pollution and Population Growth (Aishwarya Srivastava).

viii) Mathematical and Numerical Approach to Atmospheric Diffusion Problems with Application to Epidemics (A. Juneja)

ix) A Mathematical Study of Temperature Profiles in Human Dermal Parts with Burns and other Abnormalities (T. Varma).

x) Mathematical Study of Prey-Predator Population with Mutual Interaction (Poonam Sinha).

xi) Mathematical Study of Blood Flow Effect on Normal and Abnormal Heat Flow in Human Dermal Regions. (B.K. Tiwari).

xii) Mathematical Study of Heat Flow in Human Skin with Thermal Injury (Anoop Singh).

xiii) A Study of Hypergeometric Functions and its Application in Biology (Ram Kumar Gupta)

xiv) Mathematical Numerical Study of Diffusion Process in In-vivo Tissues (Vinod Kumar Gupta)

xv) Mathematical Study of Effect of Cold Environment on Temperature Distribution in Outer Parts of Human Body (Bharat Suman Gupta).

xvi) A study of Finite Element Method and Its Application to Pollution Problem (D.S. Kushwah).

xvii) Mathematical and Computational Study Environmental Pollution Problems (Hakim Singh).

xviii) The I-Function and Its Properties (Lily Agarwal).

xix) Mathematical Modeling of Analysis of Ecological Problems with Special Reference to Atmospheric Pollution Problems (Santosh Bharadwaj).

xx) Variational Finite Element Based Mathematical Study of Atmospheric Pollution Problems Based on Variable Diffusivity and Surface Deposition (Rajesh Deolia).

xxi) Mathematical Study of Thermal Injury in Human Subjects (D.B. Gurung).

xxii) Saxenas I-Function and Its Biological Applications (G.D. Vaishya).

xxiii) Mathematical and Numerical Study of Distribution and Diffusion of Wild Life Population (Shobha Agarwal).

xxiv) Data Mining and Artificial Neural Network Applications to Financial Management in the Indian Context (Nitin Merh).

xxv) Mathematical Study of Thermo regulation in Human Body Eposed to Cold Environment (Mukhtar Ahmad Khandey).

xxvi) Mathematical and Numerical Estimation of Financial Markets Using Black-Schole's Model (Jainendra Jain).

xxvii) Mathematical Study of Transdermal Drug Administration in Human Subjects (Archana Sharma).

xxviii) Mathematical Study of Air Pollution in Patchy Areas with Special Reference to Oil Refinery. (Amit Khandelwal)

xxix) Mathematical Investigations of Thermal Injuries in Protected and Unprotected Human Dermal Layers. (Arun Kumar Tripathi)

xxx) Certain Problems in Mathematical Ecology Pertaining to The Conservation and Migration of Animal Species. (Namreen Rasool)

xxxi) Mathematical And Numerical Study Of Solid Tumor (With or Without Malignancy) in Human Body. (Sushma Nema)

xxxii) Mathematical Study of Single and Two Interacting Species with Special Reference to Protected Wild Life. (V. K. Chaturvedi)

xxxiii) Mathematical Study of Migration of Different Animal Species with Special Reference to Marine Life. (Neeta Mazumdar)

xxxiv) Mathematical Study of Thermo-Regulation in Human Dermal Region Under Variable Metabolic Conditions. (Manoj Kumar Sharma)

xxxv) Mathematical and Numerical Study of Transdermal Drug Distribution in Human Body. (Vineeta

Gupta)

xxxvi) Some New Properties and Inter-Relations of Saxena's I-function. (Pankaj Jain)

xxxvii) Analytical Study of Saxenas I-function and Its Applications. (Vandana Jat)

xxxviii) Mathematical Modelling and Numetical Study of Heat Regulation in Human Body. (Padam Sharma)

xxxix) Study of Some Problems and Applications of I-Function. (Prachi Jain)

xl) Mathematical Moedliing of Finite Age Structured Populations and It's Applications in Wild Life (Lalita Dhurve)

(B) Guided more than thirty M.Phil dissertations on various areas of Bio-mathematics.

6 PUBLICATIONS

Research Papers:

Published more than hundred twenty research papers in reputed International and National Journals. Lists of selected papers and other papers are enclosed herewith (Enclosure-1).

*Presented about one hundred and fifty research papers in various International and National Research Conferences / Seminars / Symposia.

Research Articles Appeared in Books/Monographs:

Articles appeared in following advanced level books / monographs:-

i) Numerical Methods in thermal Problems, Pineridge Press, U.K. (Ed. K. Mogan) 1979.

ii) Bio-mechanics, Wiley Eastern Ltd. 1989 (Eds. K.B. Sabay and R.K. Saxena).

iii) 'Physiological Fluid Dynamics-II', NAROSA 1991 (Ed. N.V. Swamy and M. Singh).

iv) Physiological Fluid Dynamics-1 (Eds. M. Singh),

v) 'Theory of Differential Equations and Applictions to Oceanography' - EWP 1992 (Eds S.G. Deo and Y.S. Prahalad).

vi) "Lecture Notes on Research Methodology" Indra Pub. House, Bhopal (2013)

*Many other articles have been included partially in several other research books.

General Scientific Articles

Ten general articles on scientific topics of teaching and research have also appeared in standard journals.

Books

Published following books:

I) *The I-Function, Anamaya,* New Delhi, 2008.

II) *Advances in Physiological Fluid Dynamics,* NAROSA, (Jt. eds.), Narosa Publishing House, New Delhi, 1995.

III) *Mathematical Modelling of Real Life Problems*, Anamaya, 2006.

IV) *Real Analysis* (Joint Authorship), Allied Publishers, New Delhi, 2003.

V) *Introductory Topics in Biomathematics* (Hindi) Wiley Eastern Ltd., New Delhi, 1987.

VI) *Calculus of Two and More Variables* (in Joint Authorship), Wiley Eastern Ltd., New Delhi, 1986.

VII) Calculus of one Variable (Joint Authorship), Wiley Eastern Ltd., New Delhi (1987).

VIII) *Introduction to Biomathematics* (Hindi) M.P. Hindi Granth Rachna Academy, Bhopal, 1988.

IX) *Engineering Mathematics-I* (with Shishir Bhaskar), Deepak Prakashan, 2000.

X) *Engineering Mathematics-II* (with Praveen Miri and Shishir Bhaskar) 2001, Deepak Prakashan, 2001.

XI) *Engineering Mathematics-III* (with Praveen Miri), Deepak Prakashan, 2001.

XII) *Lecture Notes on Research Methodology*, Indra Publication, Bhopal, 2013.

7 PARTICIPATION IN SOCIAL ACTIVITIES:

Member Board of Governors/Advisory Boards/Executive Committee of L.I.C. (Central Zone), Family Planning Association of India, Rama Krishna Ashram and nominated as Vice-chairman of Red Cross Society, President Anjuman Tarrakki E' Urdu and Patron Bazme E' Urdu, Gwalior.

8 PUBLISHED FOLLOWING HINDI LITERATURE BOOKS:

(i) "*SHAHILON KE BEECH*" Indra Publication, Bhopal, 2018.

(ii) "*SATH HOTE TUM AGAR*" AISECT Publication, Bhopal, 2022.

(iii) "*PROFESSOR DHOTI PANDEY*" AISECT Publication, Bhopal, 2023.

Enclosure-1

LIST OF SELECTED RESEARCH PUBLICATIONS OF PROFESSOR VINOD P. SAXENA (EX V. C. JIWAJI UNIVERSITY)

1. V. P. Saxena, Mathematical modelling of biological populations with and without dispersion, *Research in Statistics, Taylor & Francis,* **1**(1) (2023), https://doi.org/10.1080/27684520.2023.2215638.
2. Padam Sharma, V. P. Saxena, Mathematical Study of Blood Circulation and Bio-Chemical Reaction Based Heat Distribution Problem in Human Dermal Region, *Jñānābha* , **53**(1) (2023), 308-314.
3. Padam Sharma, N.S. Lodhi and V.P. Saxena, Yoga Therapy and Heat Regulatory and Cardiovascular Systems of a Human Body. *International Journal on Emerging Technologies* **12**(2) (2021), 282-289.
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AN ANALYTICAL STUDY OF SPACE-TIME FRACTIONAL ORDER GAS DYNAMIC EQUATIONS

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Abstract

In this article, the Sumudu transform with iterative method is implemented to obtain approximate analytical solutions in series form to non-linear homogeneous and non-homogeneous space-time fractional gas dynamic equations. The fractional derivatives presented here are in the Caputo sense. Furthermore, the findings of this study are graphically represented and the solution graphs demonstrate a strong connection between the approximate and exact solutions.

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Keywords and Phrases: Gas dynamic equations, Sumudu transform, iterative method, Mittag-Leffler functions, Caputo fractional derivatives, fractional differential equations.

1 Introduction

Fractional calculus is a branch of applied mathematics that is extremely useful in a variety of fields of research [13, 21]. The fractional differential equations have sparked the interest of a vast scope of researchers working on a variety of applications [2, 3, 10, 28, 31]. Many efforts have been made to develop analytical and numerical approaches for solving differential equations of fractional order, such as the homotopy analysis method (*HAM*) [17], the q-homotopy analysis method (q-*HAM*) [14], the optimal q-homotopy analysis method (Oq-*HAM*) [32], the homotopy analysis transform method (*HATM*) [30], the adomian decomposition method (*ADM*) [12], the Laplace decomposition method (*LDM*) [18], the homotopy perturbation method (*HPM*) [19], the homotopy perturbation transform method (*HPTM*) [20, 23], and so on.

In 2006, Daftardar-Gejji and Jafari [8, 15] proposed an iterative method for numerically solving nonlinear functional equations. Since then, the iterative technique has been used to solve a wide variety of nonlinear differential equations of integer and fractional order [5] as well as fractional boundary value problems [7]. Recently, Wang and Liu [33] introduced the Sumudu transform iterative method (*STIM*) by combining the Sumudu transform with an iterative technique to determine approximate analytical solutions of time-fractional Cauchy reaction diffusion equations. The Sumudu transform iterative technique has been used successfully to solve a variety of time and space fractional partial differential equations and related systems [22], as well as the random component time-fractional Klein-Gordon equation [27].

In this work, we consider the non-linear homogeneous and non-homogeneous fractional gas dynamic equations with space and time fractional derivatives as follows
(i) The non-linear homogeneous space-time-fractional gas dynamic partial differential equation of the form

$$D_t^{\alpha}u(x,t)+\frac{1}{2}D_x^{\beta}u^2(x,t)-u(x,t)\big(1-u(x,t)\big)=0,\quad 0<\alpha,\beta\le 1, \tag{1.1}$$

$$u(x,0)=g(x), \tag{1.2}$$

(ii) The non-linear non-homogeneous space-time-fractional gas dynamic partial differential equation of the form

$$D_t^{\alpha}u(x,t)+\frac{1}{2}D_x^{\beta}u^2(x,t)-u(x,t)\big(1-u(x,t)\big)=f(x,t),\quad 0<\alpha,\beta\le 1, \tag{1.3}$$

$$u(x,0)=g(x), \tag{1.4}$$

where α and β are the parameters that describe the order of the time-fractional and space-fractional derivatives, respectively. Also, $u(x,t)$ is the probability density function and f is a known analytic function.

2 Preliminaries and Basic Definitions

This section introduces some fundamental definitions, notations, and properties of fractional calculus utilizing Sumudu transform theory, which will be applied later in this paper.

Definition 2.1. *The Caputo fractional derivative of a function $u(x,t)$ is defined as* [28, 31]

$$D_t^{\alpha}u(x,t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\eta)^{m-\alpha-1}u^{(m)}(x,\eta)d\eta,\ m-1<\alpha\le m, m\in\mathbb{N}. \tag{2.1}$$

Definition 2.2. *The Sumudu transform is defined over the set of functions*

$$\Big\{f(t)|\exists M,\rho_1>0,\rho_2>0,|f(t)|<Me^{|t|/\rho_j}\ \text{if}\ t\in(-1)^j\times[0,\infty),\ j=1,2\Big\}$$

by the following formula [4, 34]

$$S[f(t)] = F(\omega) = \int_0^{\infty} e^{-t}f(\omega t)dt\ ,\ \omega\in(-\rho_1,\rho_2). \tag{2.2}$$

Definition 2.3. *The Sumudu transform of Caputo fractional derivative is defined in the following manner* [9, 33]

$$S[D_t^{\alpha}u(x,t)] = \omega^{-\alpha}S[u(x,t)] - \sum_{k=0}^{m-1}\omega^{-\alpha+k}u^{(k)}(x,0),\ m-1<\alpha\le m\ ,\ m\in\mathbb{N}, \tag{2.3}$$

where $u^{(k)}(x,0)$ is the k-order derivative of $u(x,t)$ with respect to t at $t=0$.

Definition 2.4. *The Mittag-Leffler function, a generalization of the exponential function, is defined as follows* [28, 31]

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}\ ,\alpha\in\mathbb{C}, Re(\alpha)>0. \tag{2.4}$$

A further generalization of equation (2.4) is as follows [35]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}\ ,\alpha,\beta\in\mathbb{C}, Re(\alpha)>0, Re(\beta)>0, \tag{2.5}$$

where $\Gamma(.)$ is the well-known Gamma function.

3 Basic Idea of Sumudu Transform Iterative Method

To explain the basic idea of the Sumudu transform iterative technique [33], we take the following space and time general fractional partial differential equation having the prescribed initial conditions may be written in the form of an operator as

$$\begin{aligned} D_t^{\alpha}u(x,t) =& F\big[x,u(x,t),D_x^{\beta}u(x,t),...,D_x^{l\beta}u(x,t)\big], \\ & l-1<\alpha\le l,\ m-1<\beta\le m;\ l,m\in\mathbb{N} \end{aligned} \tag{3.1}$$

$$u^{(k)}(x,0) = h_k(x),\ k=0,1,2,...,n-1, \tag{3.2}$$

where $D_t^{\alpha}u(x,t)$ and $D_x^{\beta}u(x,t)$ are the Caputo fractional derivatives of order α, $l-1<\alpha\le l$ and β, $m-1<\beta\le m$, respectively, defined by the equation (2.1), $F\big[x,u,D_x^{\beta}u,...,D_x^{l\beta}u\big]$ is a linear/non-linear operator and $u=u(x,t)$ is the unknown function and fractional derivative $D_x^{l\beta}u(x,t), l\in\mathbb{N}$ is taken as the sequential fractional derivative [28] that is

$$D_x^{l\beta}u = D_x^{\beta}D_x^{\beta},...,D_x^{\beta}u\ \ (l\ times). \tag{3.3}$$

Applying the Sumudu transform on both sides of equation (3.1), we have

$$S\Big[D_t^{\alpha}u(x,t)\Big] = S\Big[F\big(x,u(x,t),D_x^{\beta}u(x,t),...,D_x^{l\beta}u(x,t)\big)\Big]. \tag{3.4}$$

Using the differentiation property of the Sumudu transform, we get

$$S[u(x,t)] = \omega^{\alpha}\sum_{k=0}^{m-1}\Big[\omega^{-\alpha+k}u^{(k)}(x,0)\Big] + \omega^{\alpha}S\Big[F\big(x,u,D_x^{\beta}u,...,D_x^{l\beta}u\big)\Big]. \tag{3.5}$$

On taking inverse Sumudu transform of equation (3.5), we have

$$u(x,t) = S^{-1}\Big[\omega^{\alpha}\sum_{k=0}^{m-1}\Big[\omega^{-\alpha+k}u^{(k)}(x,0)\Big]\Big] + S^{-1}\Big[\omega^{\alpha}S\Big[F\big(x,u,D_x^{\beta}u,...,D_x^{l\beta}u\big)\Big]\Big]. \tag{3.6}$$

Equation (3.6) may be written as

$$u(x,t) = f(x,t) + N\Big(x, u, D_x^{\beta}u, ..., D_x^{l\beta}u\Big), \tag{3.7}$$

where

$$f(x,t) = S^{-1}\Big[\omega^{\alpha}\sum_{k=0}^{m-1}\Big[\omega^{-\alpha+k}u^{(k)}(x,0)\Big]\Big], \tag{3.8}$$

$$N\Big(x, u, D_x^{\beta}u, ..., D_x^{l\beta}u\Big) = S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, u, D_x^{\beta}u, ..., D_x^{l\beta}u\Big)\Big]\Big]. \tag{3.9}$$

Here N is a linear/nonlinear operator and f is a known function .
Furthermore, we employ the iterative method proposed by Daftardar-Gejji and Jafari [8], which represents a solution in an infinite series of components as

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t). \tag{3.10}$$

The operator N is decomposed as follows

$$\begin{aligned} N\Big(x, \sum_{i=0}^{\infty}u_i, D_x^{\beta}\Big(\sum_{i=0}^{\infty}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{\infty}u_i\Big)\Big) &= N\Big(x, u_0, D_x^{\beta}u_0, ..., D_x^{l\beta}u_0\Big) \\ &+ \sum_{j=1}^{\infty}\Big[N\Big(x, \sum_{i=0}^{j}u_i, D_x^{\beta}\Big(\sum_{i=0}^{j}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{j}u_i\Big)\Big)\Big] \\ &- \sum_{j=1}^{\infty}\Big[N\Big(x, \sum_{i=0}^{j-1}u_i, D_x^{\beta}\Big(\sum_{i=0}^{j-1}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{j-1}u_i\Big)\Big)\Big], \end{aligned} \tag{3.11}$$

$$\begin{aligned} & S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, \sum_{i=0}^{\infty}u_i, D_x^{\beta}\Big(\sum_{i=0}^{\infty}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{\infty}u_i\Big)\Big)\Big]\Big] \\ &= S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, u_0, D_x^{\beta}u_0, ..., D_x^{l\beta}u_0\Big)\Big]\Big] \\ &+ \sum_{j=0}^{\infty}\Big[S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, \sum_{i=0}^{j}u_i, D_x^{\beta}\Big(\sum_{i=0}^{j}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{j}u_i\Big)\Big)\Big]\Big]\Big] \\ &- \sum_{j=0}^{\infty}\Big[S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, \sum_{i=0}^{j-1}u_i, D_x^{\beta}\Big(\sum_{i=0}^{j-1}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{j-1}u_i\Big)\Big)\Big]\Big]\Big]. \end{aligned} \tag{3.12}$$

Using equations (3.10) to (3.12) in equation (3.7), we obtain

$$\begin{aligned} \sum_{i=0}^{\infty}u_i(x,t) &= S^{-1}\Big[\omega^{\alpha}\sum_{k=0}^{m-1}\Big(\omega^{-\alpha+k}u^{(k)}(x,0)\Big)\Big] \\ &+ S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, u_0, D_x^{\beta}u_0, ..., D_x^{l\beta}u_0\Big)\Big]\Big] \\ &+ \sum_{j=0}^{\infty}\Big[S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, \sum_{i=0}^{j}u_i, D_x^{\beta}\Big(\sum_{i=0}^{j}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{j}u_i\Big)\Big)\Big]\Big]\Big] \\ &- \sum_{j=0}^{\infty}\Big[S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, \sum_{i=0}^{j-1}u_i, D_x^{\beta}\Big(\sum_{i=0}^{j-1}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{j-1}u_i\Big)\Big)\Big]\Big]\Big]. \end{aligned} \tag{3.13}$$

The recurrence relations have been defined as follows

$$u_0(x,t) = S^{-1}\Big[\omega^{\alpha}\sum_{k=0}^{m-1}\Big(\omega^{-\alpha+k}u^{(k)}(x,0)\Big)\Big], \tag{3.14}$$

$$u_1(x,t) = S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, u_0, D_x^{\beta}u_0, ..., D_x^{l\beta}u_0\Big)\Big]\Big], \tag{3.15}$$

$$u_{r+1}(x,t) = S^{-1}\Big[\omega^\alpha S\Big[F\Big(x, \sum_{i=0}^{r} u_i, D_x^\beta\Big(\sum_{i=0}^{r} u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{r} u_i\Big)\Big)\Big]\Big]$$
$$- S^{-1}\Big[\omega^\alpha S\Big[F\Big(x, \sum_{i=0}^{r-1} u_i, D_x^\beta\Big(\sum_{i=0}^{r-1} u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{r-1} u_i\Big)\Big)\Big]\Big], r \geq 1. \tag{3.16}$$

Therefore, the approximate analytical solution of equations (3.1) and (3.2) in truncated series form is given by

$$u(x,t) \cong \lim_{\mathbb{N}\to\infty} \sum_{m=0}^{\mathbb{N}} u_m(x,t). \tag{3.17}$$

In general, the solutions in the above series converge quickly. The classical approach to the convergence of this type of series has been presented by Bhalekar and Daftardar-Gejji [6] and Daftardar-Gejji and Jafari [8].

4 Solution of the Space-Time Fractional Gas Dynamic Equations

In this section, we make an attempt to solve non-linear homogeneous and non-homogeneous space-time fractional gas dynamic equations by means of the Sumudu transform iterative method.

Example 4.1. Consider the following non-linear homogeneous space-time fractional gas dynamic equation [30, 32]

$$D_t^\alpha u(x,t) + \frac{1}{2} D_x^\beta u^2(x,t) - u(x,t)(1-u(x,t)) = 0, \quad t > 0, \ 0 < \alpha, \beta \leq 1, \tag{4.1}$$

with the initial condition

$$u(x,0) = e^{-x}. \tag{4.2}$$

Taking the Sumudu transform on the both sides of equation (4.1), and making use of the result given by equation (4.2), we have

$$S[u(x,t)] = e^{-x} + \omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta u^2(x,t)}{\partial x^\beta} + u(x,t)(1-u(x,t))\Big]. \tag{4.3}$$

On taking inverse Sumudu transform of equation (4.3), we get

$$u(x,t) = e^{-x} + S^{-1}\Big[\omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta u^2(x,t)}{\partial x^\beta} + u(x,t)(1-u(x,t))\Big]\Big]. \tag{4.4}$$

Substituting the results from equations (3.10) to (3.12) in the equation (4.4) and applying the equations (3.14) to (3.16), we determine the components of the solution as follows

$$u_0(x,t) = u(x,0) = e^{-x}, \tag{4.5}$$

$$u_1(x,t) = S^{-1}\Big[\omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta u_0^2}{\partial x^\beta} + u_0(1-u_0)\Big]\Big]$$
$$= -e^{-x}\Big[\frac{t^\alpha}{\Gamma(\alpha+1)}\Big]\Big(2^{-1+\beta}e^{-x+i\beta\pi} - 1 + e^{-x}\Big), \tag{4.6}$$

$$u_2(x,t) = S^{-1}\Big[\omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta (u_0+u_1)^2}{\partial x^\beta} + (u_0+u_1)\big(1-(u_0+u_1)\big)\Big]\Big]$$
$$- S^{-1}\Big[\omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta u_0^2}{\partial x^\beta} + u_0(1-u_0)\Big]\Big]$$
$$= \Big[-e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 2e^{-2x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big](2^{-1+\beta}e^{-x+i\beta\pi} - 1 + e^{-x})$$
$$+ \Big[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big](3^\beta e^{-3x+i\beta\pi} - 2^\beta e^{-2x+i\beta\pi} + 2^{-1+\beta}3^\beta e^{-3x+2i\beta\pi}), \tag{4.7}$$

and so on. The remaining components may be obtained in the same way.

Thus, the approximate analytical solution in the series form can be obtained as

$$u(x,t) \cong \lim_{\mathbb{N}\to\infty} \sum_{m=0}^{\mathbb{N}} u_m(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) +, ..., \tag{4.8}$$

$$
\begin{aligned}
&= e^{-x} - e^{-x}\Big[\frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big]\Big(2^{-1+\beta}e^{-x+i\beta\pi} - 1 + e^{-x}\Big)\\
&+ \Big[-e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 2e^{-2x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big](2^{-1+\beta}e^{-x+i\beta\pi} - 1 + e^{-x})\\
&+ \Big[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big](3^{\beta}e^{-3x+i\beta\pi} - 2^{\beta}e^{-2x+i\beta\pi} + 2^{-1+\beta}3^{\beta}e^{-3x+2i\beta\pi})+, ..., \ .
\end{aligned}
$$

The same result was obtained by Saad *et al.* [32] by using the method of optimal *q-HAM.*
If we put $\alpha = \beta = 1$, in equation (4.8), we have the result in simple form

$$u(x,t) = e^{t-x}, \tag{4.9}$$

which is the exactly the same solution obtained by earlier by Jafari *et al.* [19] by using *HPM* method.

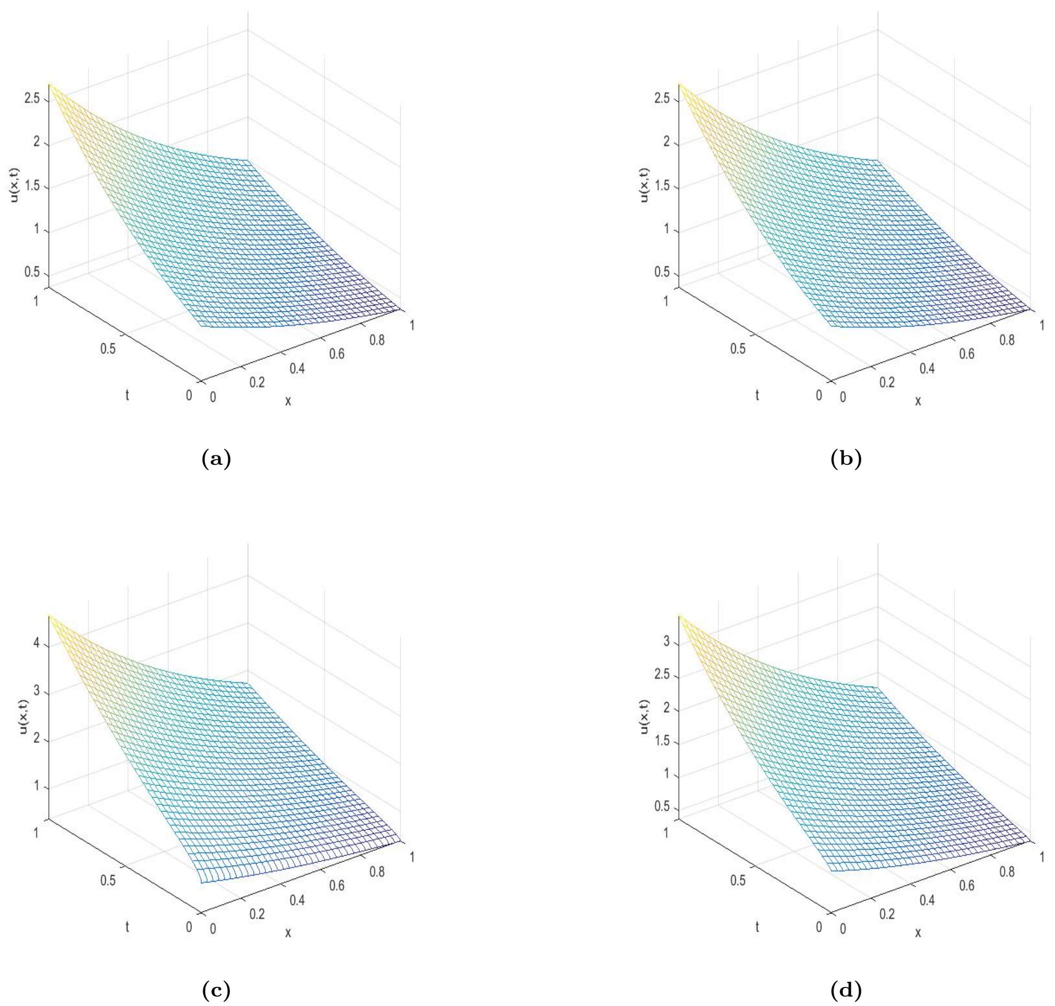

***Figure* 4.1:** Graph of the $u(x,t)$ for Example 4.1, when $\beta = 1$: (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.75$.

Example 4.2. Consider the following non-linear non-homogeneous space-time fractional gas dynamic equation [32]

$$D_t^\alpha u(x,t) + \frac{1}{2}D_x^\beta u^2(x,t) - u(x,t)(1-u(x,t)) = -e^{t-x}, \quad t>0,\ 0<\alpha,\beta\le 1, \tag{4.10}$$

with the initial condition

$$u(x,0) = 1 - e^{-x}. \tag{4.11}$$

Taking the Sumudu transform on the both sides of equation (4.10), and making use of the result given by equation (4.11), we have

$$S[u(x,t)] = 1 - e^{-x} + \omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta u^2(x,t)}{\partial x^\beta} + u(x,t)(1-u(x,t)) - e^{t-x}\Big]. \tag{4.12}$$

On taking inverse Sumudu transform of equation (4.12), we get

$$u(x,t) = 1 - e^{-x} + S^{-1}\Big[\omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta u^2(x,t)}{\partial x^\beta} + u(x,t)(1-u(x,t)) - e^{t-x}\Big]\Big]. \tag{4.13}$$

Substituting the results from equations (3.10) to (3.12) in the equation (4.13) and applying the equations (3.14) to (3.16), we determine the components of the solution as follows

$$u_0(x,t) = u(x,0) = 1 - e^{-x}, \tag{4.14}$$

$$\begin{aligned} u_1(x,t) &= S^{-1}\Big[\omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta u_0^2}{\partial x^\beta} + u_0(1-u_0) - e^{t-x}\Big]\Big] \\ &= -e^{-x}\Big[\frac{t^\alpha}{\Gamma(\alpha+1)}\Big]\Big(-e^{i\beta\pi} + 2^{-1+\beta}e^{-x+i\beta\pi} - 1 + e^{-x}\Big) - e^{-x}t^\alpha E_{1,\alpha+1}(t), \end{aligned} \tag{4.15}$$

$$\begin{aligned} u_2(x,t) &= S^{-1}\Big[\omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta (u_0+u_1)^2}{\partial x^\beta} + (u_0+u_1)\big(1-(u_0+u_1)\big) - e^{t-x}\Big]\Big] \\ &- S^{-1}\Big[\omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta u_0^2}{\partial x^\beta} + u_0(1-u_0) - e^{t-x}\Big]\Big] \\ &= t^{2\alpha}(-e^{i\pi\beta} + 2^{-1+\beta}e^{-x+i\pi\beta} - 1 + e^{-x})\Big(-\frac{2e^{-2x}}{\Gamma(2\alpha+1)} + \frac{e^{-x}t^{2\alpha}}{\Gamma(2\alpha+1)}\Big) \\ &- \frac{(3^\beta e^{-3x+i\pi\beta} - 2^\beta e^{-2x+i\pi\beta} + 2^{-1+\beta}3^\beta e^{-3x+2i\pi\beta} - 2^\beta e^{-2x+2i\pi\beta})t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ \frac{(2^\beta e^{-2x+i\pi\beta} - e^{-x+i\pi\beta} + 2^\beta e^{-2x+2i\pi\beta} - e^{-x+2i\pi\beta})t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ \big(-2e^{-2x}E_{1,2\alpha+1}(t) + e^{-x}E_{1,2\alpha+1}(t) \\ &\qquad + e^{-x+i\pi\beta}E_{1,2\alpha+1}(t) - 2^\beta e^{-2x+i\pi\beta}E_{1,2\alpha+1}(t)\big)t^{2\alpha}, \end{aligned} \tag{4.16}$$

and so on. The remaining components may be obtained in the same way.
Thus, the approximate analytical solution in the series form can be obtained as

$$\begin{aligned} u(x,t) &\cong \lim_{\mathbb{N}\to\infty}\sum_{m=0}^{\mathbb{N}} u_m(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) +, ..., \\ &= 1 - e^{-x} - e^{-x}\frac{t^\alpha}{\Gamma(\alpha+1)}\big(-e^{i\pi\beta} + 2^{-1+\beta}e^{-x+i\pi\beta} - 1 + e^{-x}\big) - e^{-x}\ t^\alpha E_{1,\alpha+1}(t) \\ &+ t^{2\alpha}(-e^{i\pi\beta} + 2^{-1+\beta}e^{-x+i\pi\beta} - 1 + e^{-x})\Big(-\frac{2e^{-2x}}{\Gamma(2\alpha+1)} + \frac{e^{-x}t^{2\alpha}}{\Gamma(2\alpha+1)}\Big) \\ &- \frac{(3^\beta e^{-3x+i\pi\beta} - 2^\beta e^{-2x+i\pi\beta} + 2^{-1+\beta}3^\beta e^{-3x+2i\pi\beta} - 2^\beta e^{-2x+2i\pi\beta})t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ \frac{(2^\beta e^{-2x+i\pi\beta} - e^{-x+i\pi\beta} + 2^\beta e^{-2x+2i\pi\beta} - e^{-x+2i\pi\beta})t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ \big(-2e^{-2x}E_{1,2\alpha+1}(t) + e^{-x}E_{1,2\alpha+1}(t) \\ &\qquad + e^{-x+i\pi\beta}E_{1,2\alpha+1}(t) - 2^\beta e^{-2x+i\pi\beta}E_{1,2\alpha+1}(t)\big)t^{2\alpha} +, ..., \quad . \end{aligned} \tag{4.17}$$

The same result was obtained by Saad *et al.* [32] by using the method of optimal *q-HAM*.
If we put $\alpha = \beta = 1$, in equation (4.17), we have the result in simple form

$$u(x,t) = 1 - e^{t-x}, \tag{4.18}$$

which is the exactly the same solution obtained by earlier by Jafari *et al.* [16] by using two-dimensional *DTM* method.

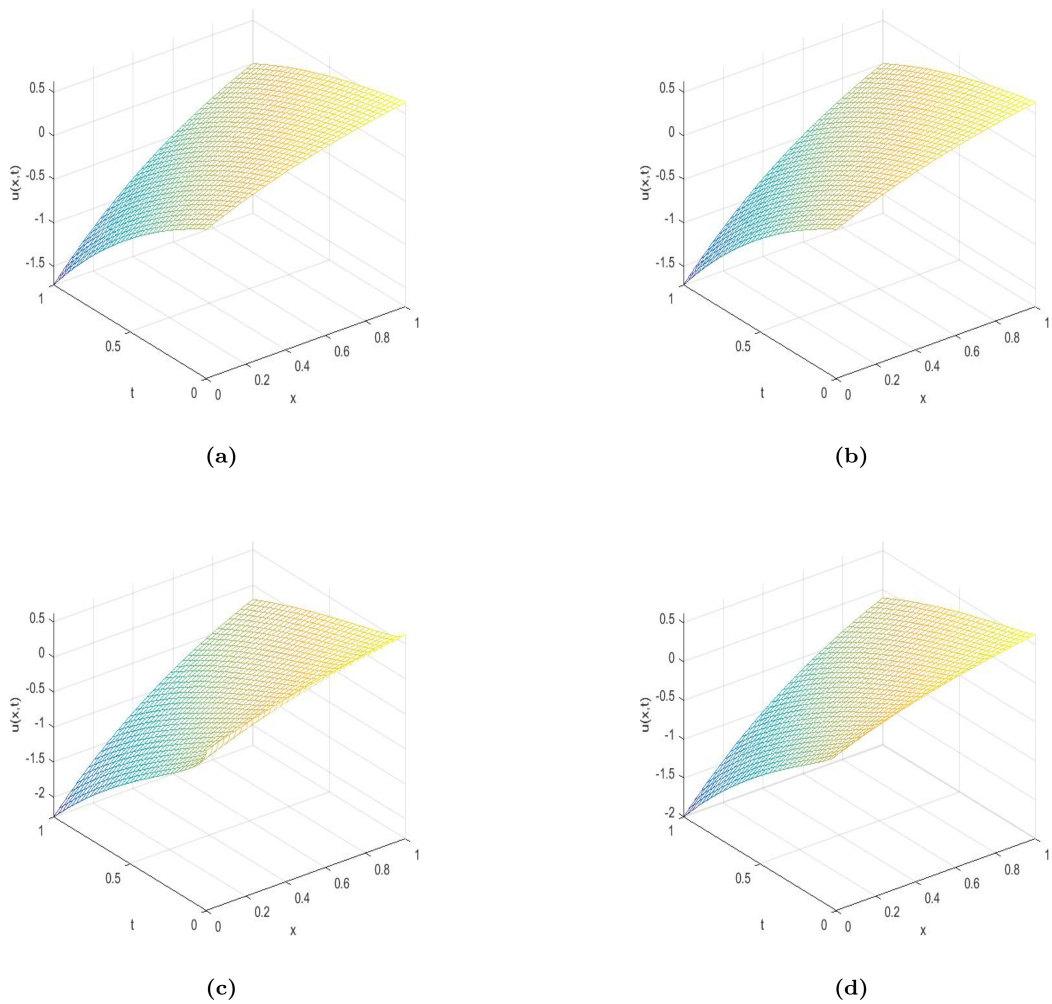

***Figure* 4.2:** Graph of the $u(x,t)$ for Example 4.2, when $\beta = 1$: (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.75$.

5 Conclusion

In this paper, we have successfully and efficiently applied the Sumudu transform iterative method (*STIM*) to derive the approximate analytical solutions of the non-linear homogeneous and non-homogeneous space-time fractional gas dynamic equations with Caputo fractional derivatives. *STIM* is a hybrid approach of the Sumudu transform and the iterative method. The graphical representation of the obtained solutions was completed successfully by the *MATLAB* software. The analytical results derived from the proposed approach indicate that the method is simple to use and precise.

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SOME RESULTS ON COUPLED FIXED POINT BY DARBO EXTENSION THEOREM

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Abstract

This paper is based on extension of some basic results on coupled fixed point using extended Kuratowski measure. In the first part of the paper we have extended coupled fixed point results on product Banach space by measure of non compactness defined in the paper of Banas (1980). The second part of the paper contains the results on existence of coupled fixed point based on generalized Theorem 3.2 of Samih et.al. (2016) to get coupled fixed point of set valued co set contraction map on complete metric type of space.
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1 Introduction

In nonlinear functional analysis many operator equation has the expression $T(x,y) = (f(x,y), f(y,x))$, where f is a map from $X \times X$ to X and T is map from $X \times X$ to $X \times X$ and X is a Banach space.In such cases the fixed the point of T is nothing but coupled fixed point of f. That is $f(x,y) = x, f(y,x) = y$. Bhaskar and Lakshmikantham [6] introduced the concept of coupled fixed point of function of several variable. He gave results on fixed point of each partial function of several variables in product space. He also generalized Banach contraction principle for such maps. It plays an important role in nonlinear functional analysis. Latter some authors [1,17] proved some results on fixed point and coupled fixed point on closed bounded convex set of Banach space of different nonlinear maps.

Every continuous map on a closed ball of R has a fixed point. Tarski [19] and Kanster [12] extended this result to complete lattice, that every monotone function on complete lattice has a fixed point. Brouwer [5] extended it to R^n to get fixed point and further Schauder [18] extended this result to topological vector space. He proved that every continuous map on compact set has a fixed point.The main drawback of Schuader fixed point result is that, it is not applicable when a set losses compactness and the loss of compactness always occurs on boundary. There are several problems related to this area like Integral equation with singular kernel, differential equation over unbounded domain, embedding theorem between Sobolev spaces. Therefore, three important measures of non compactness were developed
(i) Housdorff measure of non compactness,
(ii) Kuratowski measure of non compactness,
(iii) Istratescu measure of non compactness.
Darbo [8] gave a nice result on compact convex bounded subset of Banach space, known as Darbo fixed point theorem.

Banach gave fixed point results on metric space but some space like l^p, for $0 < p < 1$ is not a metric space. Some author gave fixed point results on those space using some weaker topological condition. In 1989 Baktin [4] has first introduced a new topological space called b-metric space almost similar to metric space in order to generalize Banach contraction principle. Some more fixed point results are added in distance space by Kirk [13].In this work we give some results on coupled fixed point of KKM type map on b-metric space.

Mathematical Preliminaries.

Definition 1.1 ([6]). *Let F be a map on X, then x is called fixed point of F, if $F(x) = x$.*

Definition 1.2 ([6]). *Let F be a map from $X \times X$ to X, then (x,y) is called coupled fixed point, if $F(x, y)$ $= x$ and $F(y, x) = y$.*

Definition 1.3 ([17]). *Let F be a map from X to 2^X, then f is called set valued map, where 2^X is power set of X.*

Definition 1.4 ([17]). *Let F be a set valued map from X to 2^X, then x is called fixed point of F if $x \in$ F(x).*

Definition 1.5 ([1]). *Let F be a set map from $X \times X$ to 2^X, then (x,y) is called coupled fixed point of F if $x \in F(x, y)$ and $y \in F(y, x)$.*

Definition 1.6 ([20]). *Let X and Y be two topological space and T be a set valued map from X to 2^Y, then T is called :*
(i) closed if graph $G_T = \{(x, y) : y \in T(x)\}$ is closed,
(ii) compact if closure of $T(X) = \bigcup_{x\in X} T(x)$ is compact,
(iii) lower semi continuous if for every open subset B of Y, the set $T^{-1}(B) = \{x \in X : T(x) \bigcap B \neq \phi\}$ is open.

Definition 1.7 ([14]). *Let S be a bounded subset of metric space X, then*
$\delta(S) = inf\{\epsilon > 0 :$ A can be covered by finitely many sets of diameter less than or equal to $\epsilon\}$, where diameter of $S = sup\{d(x, y) : x, y \in S\}$, is called measure of noncompactness in X.

Definition 1.8 ([9]). *Let (X, $\| \ . \ \|$) be a Banach space and β be the family of bounded subset of X. The function δ from β to R+ defined for every $B \in \beta$ by*
$\delta(B) = inf\{\epsilon > 0 :$ B can be covered by finitely many sets of diameter less than or equal to $\epsilon\}$, where diameter of $S = sup\{\|x - y : x, y \in S\|\}$, is called measure of noncompactness in X.

2 Some Basic definitions and results

In this section we give some definitions and results on coupled fixed point by contraction with Kuratowski measure on Banach space.

Theorem 2.1 ([18]). *(Schauder fixed point theorem) Everey continuous and compact map on closed bounded convex set has fixed point.*

Theorem 2.2 ([8]). *(Darbo fixed point theorem) Let E be closed bounded convex subset of a Banach space X and $F : E \to E$ be continuous such that $\mu(F(E)) \leq k\mu(E)$, where $k \in (0, 1)$ and μ is the Kuratowski measure of non-compactness in X, then F has fixed point in E.*

Definition 2.1 ([21]). *Let X be a real Banach space, Y be the set of all bounded subsets of X and let $\mu : Y \to R$ be a map satisfying:*
(i) $Ker\mu$ (zero set of μ) is a non empty subset of Y,
(ii) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$,
(iii) $\mu(C) = \mu(\overline{C})$,
(iv) $\mu(C) = \mu(coC)$,
(v) $\mu(rA + (1 - r)(B)) \leq r\mu(A) + (1 - r)\mu(B))$, for all $A, B \in Y$, for all $r \in (0, 1)$,
(vi) If (A_n) be a sequence of sets in Y with $A_{n+1} \subset A_n$ and $lim_{n\to\infty}\mu(A_n) = 0$, then $\bigcap A_n$ is non empty, μ is called Kuratowski measure of non-compactness.

Now we give our first result on coupled fixed point based on Kuratowski measure on product space.

Theorem 2.3. *Suppose X be a real Banach space. Let μ_1, μ_2 be two Kuratowski measures of noncompactness on X. Let the measures μ and $\mu^{'}$ be defined on the product space $X \times X$ by*
(i) $\mu(Y) = \mu_1(Y_1) + \mu_2(Y_2)$,
(ii) $\mu^{'}(Y) = max(\mu_1(Y_1), \mu_2(Y_2))$, where
Y_1 and Y_2 are natural projections of Y on X and Y be a arbitrary subset of $X \times X$.
Let f be a continuous map from $X \times X$ to X such that for every bounded closed convex subset C of $X \times X$ satisfies the following set valued contraction:
(1) *$\mu_1(f(C)) \leq a\mu(C)$ and $\mu_2(f^{'}(C)) \leq b\mu(C)$, for $a + b \in (0, 1)$,*
(2) *$\mu_1(f(C)) \leq \mu^{'}(C)$ and $\mu_2(f^{'}(C)) \leq \mu^{'}(C)$,*
where $f^{'} : X \times X \to X$ and $f^{'}(x, y) = f(y, x)$,
then f has a coupled fixed point on $X \times X$.

Proof. From [21] we can conclude μ and μ' are Kuratowski measure. Since X is a real Banach space, then $X \times X$ is also a real Banach space. In product space $X \times X$ we define
$T(x, y) = (f(x, y), f^{'}(x, y))$, where $f^{'}(x, y) = f(y, x)$. Hence T is a map from $X \times X$ to $X \times X$.
Let C be a arbitrary subset of $X \times X$. Now we claim that

$$T(C) \subset f(C) \times f^{'}(C).$$

Let $(z, w) \in T(C) \Rightarrow$ there exist $(x, y) \in C$ such that $T(x, y) = (z, w)$. Then it follows that $(z, w) = T(x, y) = (f(x, y), f^{'}(x, y))$, then $z = f(x, y)$ and $w = f(y, x)$
$\Rightarrow (z, w) \in f(C) \times f^{'}(C)$
$\Rightarrow T(C) \subset f(C) \times f^{'}(C)$.
Now
$\mu(T(C)) \leq \mu(f(C) \times f^{'}(C))$
$= (\mu_1(f(C) + \mu_2(f^{'}(C)))$
$\leq a\mu(C) + b\mu(C)$
$= (a + b)\mu(C)$. Therefore, T has a fixed point in $X \times X$.

By Darbos theorem there exist a point $(x, y) \in X \times X$ such that $T(x, y) = (x, y)$
$\Rightarrow f(x, y) = x$ and $f(y, x) = y$. So f has a coupled fixed point.
Further using condition (2), we get

$$\mu^{'}(T(C)) \leq \mu^{'}(f(C) \times f^{'}(C)) = \max(\mu_1(f(C), \mu_2(f^{'}(C))) \leq \mu^{'}(C).$$

Proceding similarly as done under condition (1), we get f has a coupled fixed point in X.
Remark 2.1. *Theorem2.3 generalizes Darbo fixed point theorem on product space by Kuratowski measures μ and μ'.*

3 b-metric space

Definition 3.1 ([20]). *Let X be a non empty set equipped with a map d from $X \times X$ to $\mathbb{R}$ is called b-metric space or metric type space, if it satisfies the following conditions:*
(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq k\,(d(x, z) + d(z, y))$, for all $k > 1$, for all $x, y, z \in X$.

Note:We have used the word d-topology as synonym of b-metric.

Definition 3.2 ([20]). *Let (X, d) be a b-metric space, then we define*
(i) a sequence(x_n) in X converges to x, if $lim_{n\to\infty} d(x_n, x) = 0$,
(ii) a sequence(x_n) in X is called Cauchy, if $lim_{n\to\infty} lim_{m\to\infty} d(x_n, x_m) = 0$,
(iii) The space (X, d) is complete, if every Cauchy sequence is convergent.

Definition 3.3 ([20]). *Let (X, d) be a b-metric space and $A \subset X$, then we define*
(i) $\bar{A}$ = Intersection of all closed set containing A.
(ii) co(A)= Intersection of all closed ball containing A.

Theorem 3.1 ([20]). *Let (X, d) be a b-metric space, then following results hold:*
(i) $A \subset X$ is closed $\Leftrightarrow$ every sequence $(x_n) \in A$ converges to x, then $x \in A$,
(ii) For $x \in \overline{A}$, we have $B(x, r) \bigcap A \neq \phi$,
for every $r > 0$, where $\overline{A}$ is the intersection of all closed sets containing A,
(iii) A is called totally bounded, if for every $r > 0$ there exist $x_1, x_2, ..., x_n \in A$ such that,

$$A \subset B(x_1, r) \bigcup B(x_2, r) \bigcup ... \bigcup B(x_n, r),$$

(iv) Every compact set is sequentially compact but the converse is not true.

Definition 3.4 ([20]). *Let (X, d) be a b-metric space. Let $A \subset X$ is called admissible if $co(A) = A$ and it is called sub admissible, if for every finite subset B of A, $co(B) \subset A$.*

Definition 3.5 ([20]). *Let (X, d) be a b-metric space $A \subset X$ is called nearly sub-admissible, if for each compact subset B of A and each $r \geq 0$, there exists a function $f : B \to A$ such that $x \in B(f(x), r)$, for each $x \in B$ and $co(f(B)) \subset A$.*

Definition 3.6 ([20]). *Let(X,d) be a b-metric space. A set valued map $T : A \to 2^X$ is called KKM map, if for every finite subset B of A the $co(B) \subset T(B) = \bigcup_{x \in B} T(x)$.*

Theorem 3.2 ([20]). *Let (X,d) be a b-metric space and A be a nonempty subset of X. Suppose Y be a topological space, then the following properties of KKM map hold:*
(i) if $T \in KKM(A,Y)$ and $F \in C(Y,X)$, then $FoT \in KKM(A,X)$,
(ii) if B is a nonempty subset of A, then $T|B \in KKM(B,Y)$.

Definition 3.7 ([20]). *Let (X,d) be a $b-metricspace$.*
$A \subset X$, then $diam(A) = sup\{d(x,y) : x, y \in A\}$.

In 2016, Samih et al. [17] extended Kuratowski measure on b-metric space. He proved results on existence of fixed point on b-metric space using generalized Kuratowski measure. We are looking extension of generalized Kuratowski measure on product of b-metric space to get coupled fixed point of KKM type function of double variable map.

Definition 3.8 ([20]). *Let (X,d) be a b-metric space with coefficient k and A be a subset of X, then Kuratowski measure of A denoted as $\alpha(A)$, is given by*
$\alpha(A) = inf\{\epsilon > 0 : A$ can be covered by finitely many sets of diameter less than or equal to $\epsilon\}$.

Then the following properties hold:
(i) $\alpha(A) = 0 \Leftrightarrow A$ is totally bounded,
(ii) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$,
(iii) If B is finite subset of X, then $\alpha(A \bigcup B) = \alpha(A)$,
(iv) $\alpha(A) \leq \alpha(\overline{A}) \leq k^2\alpha(A)$,

Definition 3.9 ([20]). *Let (X,d) be a b-metric space. Let $A \subset X$ and f be a map on A, is said to be co contraction on A, if for every bounded subset $B \subset A$ with $f(B)$ bounded, then $\alpha(f(coB)) \leq k\alpha(B))$, for $0 < k < 1$.*

Theorem 3.3 ([20]). *Let (X,d) be a complete b-metric space. Let A be a nonempty bounded nearly sub-admissible subset of X and F be a map from A to 2^A with closed, co set contraction and KKM on A and $\overline{F(A)} \subset A$, then F has a fixed point on A.*

4 Main results

Theorem 4.1. *Let (X,d) be a b-metric space. Let D be a map from $X' \times X'$ to $[0,\infty)$ defined by $D((x_1,y_1),(x_2,y_2)) = max\{d((x_1,x_2),(y_1,y_2)\}$, then (X',D) is a b-metric space, where $X' = X \times X$.*

Proof. Let $p_1 = (x_1,y_1)$, $p_2 = (x_2,y_2)$, $p_3 = (x_3,y_3)$, where $p_i \in X'$, $i = 1,2,3$, then we have
(i) $D(p_1,p_2) = D(p_2,p_1)$ is obvious,
(ii) Symmetric property is obvious,
(iii) we wish to show k-triangular property.
From definition

$$d(x_1,x_2) \leq k[d(x_1,x_3) + d(x_3,x_2)]$$

and

$$d(y_1,y_2) \leq k[d(y_1,y_3) + d(y_3,y_2)].$$

Now

$$\begin{aligned}
&D(p_1,p_2) = max\,[d((x_1,x_2),d(y_1,y_2)]\\
&\leq k([d(x_1,x_3) + d(x_3,x_2)] + [d(y_1,y_3) + d(y_3,y_2)])\\
&\leq k([d(x_1,x_3) + d(y_1,y_3)] + [d(x_2,x_3) + d(y_2,y_3)])\\
&\leq 2k\ max[d(x_1,x_3),d(y_1,y_3)] + 2k\ max[d(x_2,x_3),d(y_2,y_3)]\\
&= 2k[D(p_1,p_3) + D(p_3,p_2)].
\end{aligned}$$

Taking $K = 2k$, we get the result.
Note: Since both X and $X^{'}$ are b-metric space, therefore, we have taken the notation d-topology for X and D-topology for product space $X^{'}$ in this section.

Theorem 4.2. *Let (X,d) be a b-metric space, then the following results hold:*
(i) If A and B be two closed subsets of X with respect to d-topology, then $A \times B$ is closed in $X \times X$ with respect to D-topology,
(ii) If A and B are two subsets of X, then $\overline{A \times B} \subset \overline{A} \times \overline{B}$,
(iii) $B_d(a,r) \times B_d(b,r) = B_D((a,b),r)$, for every $a,b \in X$ and every $r > 0$,
(iv) if X is complete w.r.t d-topology, then $X \times X$ is complete w.r.t D-topology.

Proof. (i) Let A and B be two closed subset of X with respect to d-topology we wish to show $A \times B$ is closed with respect to D-topology. Let $(z_n) = (x_n, y_n)$ converge to some (a,b) with respect to D-topology. Now

$$
\begin{aligned}
&lim_{n\to\infty} D(((x_n, y_n), (a,b)) = 0\\
&\Rightarrow lim_{n\to\infty}\ max[d(x_n, a), d(y_n, b)] = 0,\\
&\Rightarrow lim_{n\to\infty} d(x_n, a) = 0 \ and \ lim_{n\to\infty} d(y_n, b) = 0
\end{aligned}
$$

$\Rightarrow (x_n)$ converges to a and (y_n) converges to b.
Since A and B are closed in X, then $a \in A$ and $b \in B \Rightarrow (a,b) \in A \times B \Rightarrow A \times B$ is closed in $X \times X$.
(ii) Let A and B be two subsets of X, then

$$A \subset \overline{A} \text{ and } B \subset \overline{B} \Rightarrow A \times B \subset \overline{A} \times \overline{B}.$$

Since $\overline{A} \times \overline{B}$ is closed in X' with respect to D-topology and $\overline{A \times B}$ is smallest closed set in $X \times X$, then $\overline{A \times B} \subset \overline{A}\ \times \overline{B}$.
(iii) Let

$$
\begin{aligned}
&(x,y) \in B_D((a,b),r)\\
&\Rightarrow D((a,b),(x,y)) \leq r\\
&\Rightarrow max[d(a,x), d(b,y)] \leq r\\
&\Rightarrow d(a,x) \leq r \text{ and } d(b,y) \leq r\\
&\Rightarrow x \in B_d(a,r) \text{ and } y \in B_d(b,r)\\
&\Rightarrow (x,y) \in B_d(a,r) \times B_d(b,r)\\
&\Rightarrow B_D((a,b),r) \subset B_d(a,r) \times B_d(b,r).
\end{aligned}
$$

(iv) Proof is easy so we omit it.
For reverse inclusion the proof is similar.
Hence, $B_D((a,b),r) = B_d(a,r) \times B_d(b,r)$.

Theorem 4.3. *Let (X,d) be metric a type of space, which induces D-topology on $X \times X$. Suppose α and β are Kurtoswki measures on X and $X \times X$ respectively, then following conclusion hold:*
(i) for a subset A of X and $x \in X, \beta(A \times \{x\}) = \alpha(A)$,
(ii) $\beta(A_1 \times A_2) \leq max[\alpha(A_1), \alpha(A_2)]$, for all A_1, A_2 subsets of X,
(iii) $\beta(A) = \beta(A')$, where $A' = \{(x,y) : (y,x) \in A\}$.

Proof. (i) Let $\epsilon > 0$, then there exist $\bigcup A_i$ $(i = 1,2,...,n)$, such that

$$
\begin{aligned}
&A \subset \bigcup A_i \text{ and } \alpha(A) \leq\ diam(A_i) \leq \alpha(A) + \epsilon.\\
&Now\\
&A_i \times \{x\} \text{ covers } A \times \{x\}, \text{ for } i = 1,2,..,n.\\
&diam[A_i \times \{x\}] = sup\{D[(a_1,x),(a_2,x)] : a_1, a_2 \in A_i\}\\
&= sup\{d(a_1,a_2) : a_1, a_2 \in A_i\}\\
&= diam A_i \leq \alpha(A) + \epsilon\\
&\Rightarrow \beta(A \times \{x\}) \leq \alpha(A) + \epsilon.\\
&\text{Since } \epsilon \text{ is arbitary, so we get } \beta(A \times \{x\}) \leq \alpha(A).
\end{aligned}
$$

Now we the show reverse inequality.
Take ϵ (> 0). Then there exist $\bigcup C_i \subset X \times X$, $i = 1,2,...,n$ such that $A \times \{x\} \subset \bigcup C_i$ and

$$\beta(A \times \{x\}) \leq diam(C_i), \quad i = 1,2,...,n$$

$\leq \beta(A \times \{x\}) + \epsilon.$

Let

$C_i{}^x = \{c : (c,x) \in C_i\}.$

$\bigcup C_i^x$ covers A.

Therefore,

$$
\begin{aligned}
& diam C_i^x \\
& = sup\{d(r,s) : r,s \in C_i{}^x\} \\
& \leq sup\{D((r_1,s_1),(r_2,s_2)) : (r_1,s_1),(r_2,s_2) \in C_i\} \\
& \leq \beta(A \times \{x\}) + \epsilon \\
& \Rightarrow \alpha(A) \leq \beta(A \times \{x\}) + \epsilon,
\end{aligned}
$$

since ϵ is arbitrary we get $\alpha(A) \leq \beta(A \times \{x\})$. Hence $\alpha(A) = \beta(A \times \{x\})$.
(ii) Let $\epsilon > 0$, then there exist $\bigcup P_i$ and $\bigcup Q_j$, $i = 1,2,...,n$ and $j = 1,2,...,m$ such that

$$A_1 \subset \bigcup P_i,\ A_2 \subset \bigcup Q_j.$$

So we have

$$\alpha(A_1) \leq diam(P_i) \leq \alpha(A_1) + \epsilon$$

and

$$\alpha(A_2) \leq diam(Q_j) \leq \alpha(A_2) + \epsilon.$$

Take $R_{ij} = P_i \times Q_j$ clearly we can see

$$A_1 \times A_2 \subset \bigcup (P_i \times Q_j).$$

Now,

$$
\begin{aligned}
& diam((P_i \times Q_j)) = sup\{D[(x_1,y_1),(x_2,y_2)] : x_1,x_2 \in P_i \ and\ y_1,y_2 \in Q_j\} \\
& = sup\{max(d(x_1,x_2),d(y_1,y_2)) : x_1,x_2 \in P_i \ and\ y_1,y_2 \in Q_j\} \\
& \leq sup[diam(P_i), diam(Q_j)] \\
& \leq sup[\alpha(A_1) + \epsilon, \alpha(A_2) + \epsilon].
\end{aligned}
$$

Since ϵ is arbitrary, we obtain

$$\beta(A_1 \times A_2) \leq max[\alpha(A_1), \alpha(A_2)].$$

(iii) The proof is obvious, so we omit it.

Theorem 4.4. *Let (X,d) be a complete b-metric space and A be a nonempty bounded subset of X such that $A \times A$ is nearly sub admissible with respect to D-topology on $X \times X$ and f be a map from $A \times A$ to $2^{A \times A}$ satisfying:*
(i) For every finite subset E of $A \times A$, we have $co(E) \subset f(A) \times f^{'}(A)$,where $f^{'}(x,y) = f(y,x)$,
(ii)$f(A \times A) \subset A$,
(iii) $co(f(C) \times f^{'}(C)) = co(f(C)) \times co(f^{'}(C))$ and $\alpha(co(f(C))) \leq k\beta(C)$, for $0 < k < 1$, for every bounded subset C of $A \times A$ with $f(C)$ bounded,
(iv) the graph f and graph $f^{'}$ are closed, where$f^{'}(x,y) = f(y,x)$, then f has a coupled fixed point on $A \times A$.

*Proof***.** From Theorem 4.2(iv) $X \times X$ is complete w.r.t D-topology.
Let T be a map from $A \times A$ to $2^{A \times A}$ defined by $T(x,y) = f(x,y) \times f(y,x)$.
Now we claim that for every subset C of $A \times A$,
$T(C) = f(C) \times f^{'}(C)$.
Now

$$
\begin{aligned}
T(C) &= \bigcup_{(x,y) \in C} T(x,y) \\
&= \bigcup_{(x,y) \in C} f(x,y) \times f^{'}(x,y)
\end{aligned}
$$

$$= \bigcup_{(x,y)\in C} f(x,y) \times \bigcup_{(x,y)\in C} f^{'}(x,y).$$

Hence, $T(C) = f(C) \times f^{'}(C)$.
Form condition (i), we conclude that T is KKM on $A \times A$.
From Theorem 3.3, we can conclude

$$\begin{aligned} &\overline{T(A \times A)} = \overline{f(A \times A) \times f'(A \times A)} \\ &\subset \overline{f(A \times A)} \times \overline{f'(A \times A)} \\ &\subset A \times A \end{aligned}$$

From condition-(iv), we can conclude T is closed.
Next we show that T has set valued co contraction on every bounded subset of $A \times A$.
Let C be arbitrary bounded subset of $A \times A$ with $f(C)$ bounded, then

$$\beta(coT(C) = \beta(co(f(C) \times f^{'}(C))) = \beta(co(f(C)) \times co(f^{'}(C))).$$

and $T(C)$ is bounded. From theorem 4.3 we can conclude that

$$\begin{aligned} &\beta(coT(C)) \\ &= \beta(co(f(C) \times f^{'}(C))) \\ &\leq max\{\alpha((co(f(C)), \alpha(co(f^{'}(C))\} \\ &\Rightarrow \beta(coT(C) \leq k\beta(C), \end{aligned}$$

Using Theorem 3.3 we obtain T has a fixed point in $A \times A$.
So there exist (x, y) such that

$$\begin{aligned} &(x,y) \in T(x,y) \Rightarrow (x,y) \in f(x,y) \times f(y,x) \\ &\Rightarrow x \in f(x,y) \text{ and } y \in f(y,x). \end{aligned}$$

Then f has a coupled fixed point on $A \times A$.

5 Conclusion

Theorem 4.4 generalizes Theorem 3.3 of the results of Samih et.al on the product space of fixed point of set valued map.

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

ON WEIGHTED PÁL TYPE (0,2) – INTERPOLATION ON THE UNIT CIRCLE

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Abstract

In this paper, we study the explicit representation of weighted Pál - type (0,2) - interpolation on two pairwise disjoint sets of nodes on the unit circle, which are obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ and $P_n^{''}(x)$ respectively, where $P_n(x)$ stands for n^{th} Legendre polynomial.

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1 Introduction

In 1979, Turán [12] studied the (0,2) – Interpolation for getting an approximate solution of differential equation $y''+fy=0$. Balázs [8] introduced the weighted (0, 2)– Interpolation on the zeros of Ultraspherical polynomial $P_n^{(\alpha)}(x), \alpha > -1$. In 1960, Kiš [10] initiated the Lacunary interpolation on the unit circle. He considered (0,2)- Interpolation on the unit circle and established the convergence theorem. After that several mathematician have considered (0,2) – Interpolation viz. on the unit circle, infinite interval and on the real line. In 1996, Xie [13] considered $(0,1,3)^*$- interpolation on the vertically projected nodes onto the unit circle. He claimed the regularity, explicit representation and convergence of (0,1,3)* - Interpolation. In 2003, Dikshit [9] considered the Pál – type Interpolation on non uniformly distributed nodes on the unit circle. After that author and Mathur [1] considered the weighted (0,2)* – Interpolation on the set of nodes obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ on the unit circle and established a convergence theorem for that interpolatory polynomial. In 2012, she [2,3] considered weighted (0;0,2) and (0,2;0) – Interpolation on projected nodes onto the unit circle, obtained the regularity, fundamental polynomial and established a convergence theorem. In 2017, authors [4] considerd the regularity and explicit forms of weighted (0,2;0)- interpolation on the unit circle on two pairwise disjoint sets of nodes obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ and $P_n^{''}(x)$ respectively onto the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial. After that the auhors [5] also established convergence for the above said interpolatory polynomials. Recently, authors [6] considered weighted Lacunary interpolation on the nodes, which are obtained by projecting vertically the zeros of the $(1-x^2)P_n^{'}(x)$ onto the unit circle and established a convergence theorem for the same. Recently, author with Iqram [7] considered generalized Hermite-Fejér interpolation on the nodes, which are obtained by vertically projected zeros of the $(1+x)P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial established the convergence theorem. These have motivated us to consider (0;0,2) interpolation on two pairwise disjoint sets of nodes on the unit circle. Let

$$(1.1) \qquad Z_n = \begin{cases} z_k = cos\theta_k + i\ sin\theta_k \\ z_{n+k} = \overline{z_k}\ , \quad k = 1\,(1)\,n, \end{cases}$$

$$(1.2) \qquad T_n = \begin{cases} t_k = cos\varphi_k + i\ sin\varphi_k, \\ t_{(n-2)+k} = \overline{t_k}\ , \quad k = 1\,(1)\,n-2, \end{cases}$$

be two set of nodes. In which the Lagrange data is prescribed on the first set of nodes whereas Lacunary data on the other one.We obtained regularity, explicit forms and established a convergence theorem of the interpolatory polynomials. In Section 2, we give some preliminaries, in Section 3, we describe the problem and regularity, in Section 4 and Section 5, we present the explicit forms and convergence of weighted Pál – type (0,2) – interpolation on the unit circle respectively.

2 Preliminaries

The differential equation satisfied by $P_n(x)$ is

(2.1) $$\left(1-x^2\right) P_n''(x) - 2x\ P_n'(x) + n(n+1) P_n(x) = 0.$$

(2.2) $$W(z) = \prod_{k=1}^{2n} (z - z_k) = K_n\ P_n\left(\frac{1+z^2}{2z}\right) z^n,$$

(2.3) $$R(z) = \left(z^2 - 1\right) W(z),$$

(2.4) $$H(z) = \prod_{k=1}^{2n-4} (z - t_k) = K_n^{**}\ \ P_n^{''}\left(\frac{1+z^2}{2z}\right) z^{n-2}.$$

We shall require the following fundamental polynomials of Lagrange interpolation based on the zeros of $W(z)$ and $R(z)$, are respectively defined as

(2.5) $$L_{1k}(z) = \frac{W(z)}{(z-z_k) W'(z_k)}, \quad k = 1(1)2n,$$

(2.6) $$L_k(z) = \frac{R(z)}{(z-z_k) R'(z_k)}, \quad k = 0(1)2n+1,$$

(2.7) $$l_{2k}(z) = \frac{H(z)}{(z-t_k)\ H^{'}(t_k)}, \quad k = 1(1)2n-4,$$

(2.8) $$J_k(z) = \int_0^z t\ \ l_{2k}(t)\ dt,$$

(2.9) $$J(z) = \int_0^z H(t)\ dt,$$

which satisfies

(2.10) $$J(-z) = -J(z).$$

We shall also use the following results in our investigations :

(2.11) $$W'(z_k) = \frac{K_n}{2}\left(z_k^{2} - 1\right) P_n'(x_k)\, z_k^{n-2}, \quad k = 1(1)2n-2,$$

(2.12) $$W^{''}(z_k) = K_n\ \left[(n-1)\left(z_k^2\ - 1\right) - 1\right] z_k^{n-3} P_n^{'}(x_k), \ k = 1(1)2n,$$

(2.13) $$H^{'}(t_k) = \frac{K_n^{**}}{2}\left(t_k^2\ - 1\right) t_k^{n-4}\ P_n^{'''}(x_k^*), \ \ k = 1(1)2n-4,$$

(2.14) $$W^{'}(t_k) = \ K_n \frac{n\left\{(n+3)\left(t_k^2\ - 1\right) + 4\right\}}{2\left(t_k^2\ + 1\right)} t_k^{n-1} P_n(x_k^*),$$

(2.15) $$W^{''}(t_k) = K_n \frac{n(n-1)\left\{(n-1)\left(t_k^2\ - 1\right) - 1\right\}}{2\left(t_k^2\ + 1\right)} t_k^{n-2} P_n(x_k^*),$$

(2.16) $$R^{'}(t_k) = \left(\mathrm{z}_\mathrm{k}^2 - 1\right) \mathrm{W}^{'}(\mathrm{z}_\mathrm{k}),$$

(2.17) $$R^{''}(z_k) = 4 z_k\ W^{'}(z_k) + \left(z_k^2\ - 1\right) W^{''}(z_k),$$

(2.18) $$R^{'}(t_k) = \left(t_k^2\ - 1\right) W^{'}(t_k),$$

$$R^{''}(t_k) = 4t_k W^{'}(t_k) + \left(t_k^2 - 1\right) W^{''}(t_k) + 2W(t_k), \tag{2.19}$$

$$H^{''}(t_k) = K_n^* \left\{(n-5)\left(t_k^2 - 1\right) - 5\right\} t_k^{n-5} P_n^{'''}(x_k^*) . \tag{2.20}$$

We shall also use the following well known inequalities:
For $-1 < x < 1$

$$|P_n(x)| \leq 1, \tag{2.21}$$

$$\left(1-x^2\right)^{1/4} |P_n(x)| \leq \sqrt{\frac{2}{\pi}} n^{-1/2}, \tag{2.22}$$

$$\left(1-x^2\right)^{3/4} |P_n'(x)| \leq \sqrt{2}\, n^{1/2}, \tag{2.23}$$

$$\left(1-x^2\right) \left|P_n^{''}(x)\right| \sim n^2 . \tag{2.24}$$

Let $x_k = cos\theta_k, \quad k = 1(1)n$ are the zeros of n^{th} Legendre polynomial $P_n(x)$, with $1 > x_1 > x_2 > \cdots > x_n > -1$, then

$$\left(1-x_k^2\right)^{-1} \sim \left(\frac{k}{n}\right)^{-2}, \tag{2.25}$$

$$\left|P_n^{(s)}(x_k)\right| \sim k^{-s-\frac{1}{2}} \quad n^{2s} , \quad s = 0, 1, 2, 3 . \tag{2.26}$$

For more details one can refer to [11].

3 The Problem and Regularity

Let $Z_n \cup \{-1, 1\}$ and T_n be two disjoint set of nodes obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ and $P_n^{''}(x)$ onto the unit circle respectively, where $P_n(x)$ stands for n^{th} Legendre polynomial, Z_n and T_n are defined in (1.1) and (1.2), we take here $z_0 = 1, z_{2n+1} = -1$.
Here we are interested to determine the following polynomial $Q_{6n-7}(z)$ of degree $\leq 6n-7$ satisfying the conditions:

$$\begin{cases} Q_{6n-7}(z_k) & = \alpha_k, & k = 0\,(1)\,2n+1 \\ Q_{6n-7}(t_k) & = \beta_k, & k = 1\,(1)\,2n-4 \\ \left[p(z)\,Q_{6n-7}(z)\right]^{''}_{z=t_k} & = \gamma_k, & k = 1\,(1)\,2n-4, \end{cases} \tag{3.1}$$

where $\alpha_k's$, $\beta_k's$ and $\gamma_k's$ are arbitrary complex constants and

$$p(z) = z^{n(n-3)/2}\left(z^2-1\right)^{7/2}\left(z^2+1\right)^{-n(n+1)/2}$$

is a weight function.

Theorem 3.1. *$Q_{6n-7}(z)$ is regular on $Z_n \cup \{-1, 1\}$ and T_n.*

Proof. It is sufficient, if we show that the unique solution of (3.1) is

$$Q_{6n-7}(z) \equiv 0,$$

when all data $\alpha_k = \beta_k = \gamma_k = 0$.
In this case, we have

$$Q_{6n-7}(z) = W(z)\, H(z)\, q(z)\,,$$

where $q(z)$ is a polynomial of degree $\leq 2n-3$, $W(z)$ and $H(z)$ are defined in (2.2) and (2.4) respectively. Obviously

$$Q_{6n-7}(z_k) = 0, \qquad k = 1\,(1)\,2n,$$

$$Q_{6n-7}(t_k) = 0, \qquad k = 1\,(1)\,2n-4.$$

From

$$[p(z)\,Q_{6n-7}(z)]''_{z=t_k}=0,$$

using (2.13) - (2.15) and (2.20), we get

$$q'(t_k)=0. \tag{3.2}$$

Therefore, we have

$$q'(z)=\ a\ H(z), \tag{3.3}$$

where a is an arbitrary constant.
Thus, we get

$$q(z)=a\ J(z)+b, \tag{3.4}$$

where

$$J(z)=\int_0^z H(t)\ dt. \tag{3.5}$$

For $q(\pm 1)=0$, we have

$$\begin{cases} a\ \ J(1)+b=0 \\ a\ J(-1)+b=0. \end{cases} \tag{3.6}$$

Since

$$J(-z)=-J(z), \tag{3.7}$$

therefore, using (3.7) in (3.6), we get $a=b=0$.
Hence the theorem follows. □

4 Explicit Representation of Interpolatory Polynomials

We shall write $Q_{6n-7}(z)$ satisfying (3.1) as

$$Q_{6n-7}(z)=\sum_{k=0}^{2n+1}\alpha_k B^*_{0k}(z)\ +\sum_{k=1}^{2n-4}\beta_k B_{0k}(z)+\sum_{k=1}^{2n-4}\gamma_k B_{2k}(z), \tag{4.1}$$

where $B^*_{0k},\ B_{0k}$ and B_{2k} are unique polynomials, each of degree at most $6n-7$ satisfying the conditions:

For $k=0\,(1)\,2n+1$

$$\begin{cases} B^*_{0k}(z_j) &= \ \delta_{jk}, & j=0\,(1)\,2n+1 \\ B^*_{0k}(t_j) &= \ 0, & j=1\,(1)\,2n-4 \\ [p(z)\,B^*_{0k}(z)]''_{z=t_j} &= \ 0, & j=1\,(1)\,2n-4. \end{cases} \tag{4.2}$$

For $k=1\,(1)\,2n-4$

$$\begin{cases} B_{0k}(z_j) &=0, & j=0\,(1)\,2n+1 \\ B_{0k}(t_j) &=\delta_{jk}, & j=1\,(1)\,2n-4 \\ [p(z)\,B_{0k}(z)]''_{z=t_j} &=0, & j=1\,(1)\,2n-4. \end{cases} \tag{4.3}$$

For $k=1\,(1)\,2n-4$

$$\begin{cases} B_{2k}(z_j) &=0, & j=0\,(1)\,2n+1 \\ B_{2k}(t_j) &=0, & j=1\,(1)\,2n-4 \\ [p(z)\,B_{2k}(z)]''_{z=t_j} &=\delta_{jk}, & j=1\,(1)\,2n-4. \end{cases} \tag{4.4}$$

Theorem 4.1. *For $k=1\,(1)\,2n-4$, we have*

$$B_{2k}(z)=W(z)\,H(z)\left\{c_k\ J_k(z)+c^*_k\ J(z)+c^{**}_k\right\}, \tag{4.5}$$

where $J_k(z)$ is defined in (2.8)

$$c_k\ \ =\ \ \frac{1}{2t_k\ p(t_k)\,W(t_k)\,H'(t_k)}\ , \tag{4.6}$$

$$c^*_k\ \ =\ -\ c_k\ \frac{\{J_k(1)-J_k(-1)\}}{2\ J(1)}\ , \tag{4.7}$$

$$c^{**}_k=-\ c_k\frac{\{J_k(1)+J_k(-1)\}}{2}, \tag{4.8}$$

and $J(z)$ is defined in (2.9) .

From (4.5), we have

$$B_{2k}(z_j) = 0, \qquad j = 1\,(1)\,2n,$$
$$B_{2k}(t_j) = 0, \qquad j = 1\,(1)\,2n-4.$$

For $j = 1\,(1)\,2n-4$, we get

$$[p(z)\,B_{2k}(z)]''_{z=t_j} = 0, \quad \text{for} \quad j \neq k\ .$$

For $j = k$, we get (4.6).
From $B_{2k}(z_j) = 0$, for $j = 0$ and $2n+1$, we get (4.7) –(4.8).

Theorem 4.2. *For $k = 1\,(1)\,2n-4$, we have*

(4.9) $$B_{0k}(z) = \frac{\left(z^2-1\right)W(z)}{(t_k^2-1)\,W(t_k)}\, l_{2k}^2(z) + \frac{W(z)\,H(z)}{(t_k^2-1)\,W(t_k)\,H'(t_k)}\left\{S_k(z) + b_k^*\ J(z) + b_k^{**}\right\} + b_k B_{2k}(z)$$

where

(4.10) $$S_k(z) = -\int_0^z \left(t^2-1\right)\frac{[l'_{2k}(t) - l'_{2k}(t_k)\,l_{2k}(t)]}{(t-t_k)}\,dt,$$

(4.11) $$b_k = -4\{l'_{2k}(t_k)\}^2 p(t_k) - \frac{\left\{p(z)\left(z^2-1\right)W(z)\right\}''_{z=t_k}}{(t_k^2-1)\,W(t_k)} - 4\ l'_{2k}(t_k)\,\frac{\left\{p(z)\left(z^2-1\right)W(z)\right\}'_{z=t_k}}{(t_k^2-1)\,W(t_k)}$$

(4.12) $$b_k^* = -\frac{\{S_k(1) - S_k(-1)\}}{2\ \ J(1)},$$

(4.13) $$b_k^{**} = -\frac{\{S_k(1) + S_k(-1)\}}{2}\ .$$

From (4.9) one can see

$$B_{0k}(z_j) = 0, \qquad j = 1\,(1)\,2n\ .$$
$$B_{0k}(t_j) = \delta_{jk}\ , \qquad j = 1\,(1)\,2n-4.$$

Now from

$$[p(z)\,B_{0k}(z)]''_{z=t_j} = 0, \qquad \text{for} \qquad j \neq k,$$

we get

$$S'_k(t_j) = -\frac{\left(t_j^2-1\right)}{\left(t_j-t_k\right)}\,l'_{2k}(t_j)\,.$$

Owing to third condition of (4.3), we derive

$$S'_k(z) = \left(z^2-1\right)\frac{\left[l'_{2k}(z) - l'_{2k}(t_k)\ \ l_{2k}(z)\right]}{(z-t_k)}.$$

On solving it we obtain (4.10).
From

$$[p(z)\,B_{0k}(z)]''_{z=t_j} = 0, \qquad \text{for} \qquad j = k,$$

we establish (4.11).
From (4.9), for

$$B_{0k}(z_j) = 0, \qquad j = 0 \text{ and } 2n+1,$$

we derive (4.12) - (4.13).

Theorem 4.3. *For $k = 1\,(1)\,2n$, we have*

$$B_{0k}^{*}(z) = \frac{\left(z^2-1\right) H^2(z)}{\left(z_k^2-1\right) H^2(z_k)}\; L_{1k}(z) + \frac{W(z)\, H(z)}{\left(z_k^2-1\right) W'(z_k)\, H^3(z_k)} \left\{M_k(z) + a_k^{*}\; J(z) + a_k^{**}\right\}, \tag{4.14}$$

where

$$M_k(z) = -\int_0^z \frac{\left[\left(t^2-1\right) H'(t)\, H(z_k) - \left(z_k^2-1\right) H'(z_k)\, H(t)\right]}{(t-z_k)}\; dt, \tag{4.15}$$

$$a_k^{*} = -\frac{\{M_k(1) - M_k(-1)\}}{2\; J(1)} \tag{4.16}$$

$$a_k^{**} = -\frac{\{M_k(1) + M_k(-1)\}}{2}. \tag{4.17}$$

For $k = 0$ and $2n+1$, we have

$$B_{0k}^{*}(z) = W(z)\, H(z) \left\{a_{1k}^{*}\; J(z) + a_{2k}^{*}\right\}, \tag{4.18}$$

where

$$a_{1k}^{*} = \frac{1}{2\; W(z_k)\, H(z_k)\, J(z_k)}, \tag{4.19}$$

$$a_{2k}^{*} = \frac{1}{2\; W(z_k)\, H(z_k)}. \tag{4.20}$$

From (4.14)

$$B_{0k}^{*}(z_j) = \delta_{jk}, \qquad j = 1\,(1)\,2n,$$
$$B_{0k}^{*}(t_j) = 0, \qquad j = 1\,(1)\,2n-4.$$

From

$$\left[p(z)\, B_{0k}^{*}(z)\right]_{z=t_j}^{''} = 0, \qquad j = 1\,(1)\,2n-4,$$

we derive

$$M_k^{'}(t_j) = -H(z_k)\, \frac{\left(t_j^2-1\right) H^{'}(t_j)}{(t_j - z_k)},$$

Employing to third condition of (4.2), we establish

$$M_k^{'}(z) = -\frac{\left[\left(z^2-1\right) H^{'}(z)\, H(z_k) - \left(z_k^2-1\right) H^{'}(z_k)\, H(z)\right]}{(z-z_k)}.$$

On solving it we get (4.15).
From (4.14), for

$$B_{0k}^{*}(z_j) = 0, \qquad j = 0 \text{ and } 2n+1,$$

we derive (4.16) and (4.17).
For $k = 0$ and $2n+1$, from (4.18), we have

$$B_{0k}^{*}(z_j) = 0, \qquad j = 1\,(1)\,2n,$$
$$B_{0k}^{*}(t_j) = 0, \qquad j = 1\,(1)\,2n-4.$$
$$\left[p(z)\, B_{0k}^{*}(z)\right]_{z=t_j}^{''} = 0, \qquad j = 1\,(1)\,2n-4.$$

For

$$B_{0k}^{*}(z_j) = \delta_{jk}, \quad j = 0 \text{ and } 2n+1,$$

we get (4.19) and (4.20).

5 Estimation of Fundamental Polynomials

Lemma 5.1. *For $z = e^{i\theta}$, $(0 \le \theta < 2\pi)$, we have*

$$\sum_{k=1}^{2n-4} |p(z)\ B_{2k}(z)| \le c\ \log n\ , \tag{5.1}$$

where $B_{2k}(z)$ be defined in Theorem 4.1 and c is a constant independent of n and z.

Lemma 5.2. *For $z = e^{i\theta}$, $(0 \le \theta < 2\pi)$, we have*

$$\left|p(z)\, B^*_{0,0}(z)\right| \le c, \qquad \left|p(z)\, B^*_{0,2n+1}(z)\right| \le c, \tag{5.2}$$

and

$$\sum_{k=1}^{2n} |p(z)\, B^*_{0k}(z)| \le cn^2 \log n\ , \tag{5.3}$$

Lemma 5.3. *For $z = e^{i\theta}$, $(0 \le \theta < 2\pi)$, we have*

$$\sum_{k=1}^{2n-4} |p(z)\, B_{0k}(z)| \le cn^2 \log n\ , \tag{5.4}$$

where $B_{0k}(z)$ be defined in Theorem 4.2 and c is a constant independent of n and z.

Proof. Using the conditions from (2.21) – (2.26), we get the result. □

6 Convergence

Theorem 6.1. *Let $f(z)$ be continuous for $|z| \le 1$ and analytic for $|z| < 1$. Let the arbitrary numbers γ_k's be such that*

$$|\gamma_k| = O\left(n^2\ \omega_3\left(f, \frac{1}{n}\right)\right), \qquad k = 1\,(1)\,2n-4. \tag{6.1}$$

Then $\{Q_{6n-7}(z)\}$ defined by

$$Q_{6n-7}(z) = \sum_{k=0}^{2n+1} f(z_k)\, B^*_{0k}(z) + \sum_{k=1}^{2n-4} f(t_k)\, B_{0k}(z) + \sum_{k=1}^{2n-4} \gamma_k\ B_{2k}(z), \tag{6.2}$$

satisfies the relation

$$\left|p(z)\left\{\ Q_{6n-7}(z) - f(z)\right\}\right| = O\left(\omega_3\left(f, n^{-1}\right)\log n\ \right), \tag{6.3}$$

where $\omega_3\left(f, n^{-1}\right)$ be the third modulus of continuity of $f(z)$.

To prove the Theorem 6.1, we shall need the followings:

Remark 6.1. Let $f(z)$ be continuous for $|z| \le 1$ and analytic for $|z| < 1$ and $f'' \in Lip\ \alpha,\ \alpha > 0$, then the sequence $\{Q_{6n-7}(z)\}$ converges uniformly to $f(z)$ in $|z| \le 1$, which follows from (6.3) provided

$$\omega_3\left(f, n^{-1}\right) = O\left(n^{-2-\alpha}\right). \tag{6.4}$$

There exists a polynomial $F_n(z)$ of degree $\le 6n-7$, satisfying Jackson's inequality

$$|f(z) - F_n(z)| \le c\ \omega_3\left(f, n^{-1}\right),\quad z = e^{i\theta}\,(0 \le \theta < 2\pi), \tag{6.5}$$

and the inequality due to Kiš [10],

$$\left|F_n^{(m)}(z)\right| \le c\ n^m\ \omega_3\left(f, n^{-1}\right)\ ,\quad m \in I^+. \tag{6.6}$$

Proof. Since $Q_{6n-7}(z)$ be a uniquely determined polynomial of degree $\le 6n-7$ and the polynomial $F_n(z)$ of degree $\le 6n-7$ satisfying (6.5) and (6.6) can be expressed as

$$F_n(z) = \sum_{k=0}^{2n+1} F_n(z_k)\ B^*_{0k}(z) + \sum_{k=1}^{2n-4} F_n(t_k)\, B_{0k}(z) + \sum_{k=1}^{2n-4} F_n''(t_k)\, B_{2k}(z).$$

Then

$$|p(z)\{Q_{6n-7}(z) - f(z)\}| \le |p(z)\{Q_{6n-7}(z) - F_n(z)\}| + |p(z)\{F_n(z) - f(z)\}|$$

$$
\begin{aligned}
&\leq \sum_{k=0}^{2n+1} \left| f\left(z_k\right) - \quad F_n\left(z_k\right)\right| \left| p\left(z\right) \ B_{0k}^{*}\left(z\right)\right| \\
&+ \sum_{k=1}^{2n-4} \left| f\left(t_k\right) - \quad F_n\left(t_k\right)\right| \left| p\left(z\right) \ B_{0k}\left(z\right)\right| \\
&+ \sum_{k=1}^{2n-4} \left\{ \left|\gamma_k\right| + \left| F_n^{''}\left(t_k\right)\right| \right\} \left| p\left(z\right) \ B_{2k}\left(z\right)\right| \\
&\qquad + \ |p(z)| \left| F_n\left(z\right) - f\left(z\right)\right| .
\end{aligned}
$$

Using (6.1), (6.2), (6.4) – (6.6) and Lemmas 5.1 – 5.3, we get (6.3) . □

7 Conclusion

In this paper, we defined the weighted Pal - type (0,2) - interpolation on two pairwise disjoint sets of nodes on the unit circle, which converges uniformly to $f^{''} \in Lip\ \alpha,\ \alpha > 0$.

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SOLUTIONS OF PELL'S EQUATION INVOLVING SOPHIE GERMAIN PRIMES

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Abstract

We bring forth one of the most sought after and intriguing space pertaining to the magical world of Number Theory; and our attempts to uncover the continuing research and developments to find solutions for different aspects of the Pells equation. As indicated in this research paper, we attempt to find the possible solutions for the Pells equation $x^2 = 41y^2 - 5^m$ for all choice of $m \in \mathbb{N}$. In this paper, we focused primarily on Pells equations involving the Sophie Germain primes and present to you another mysterious series and pattern typically associated with the Pells equation. As we proceed through the research, we will bring to the fore the recurrence relations among the identified solutions.

2020 Mathematical Sciences Classification: 11D09.
Keywords and Phrases: Pell's equation, Diophantine equations, Integer solutions, Recurrence relation, Sophie Germain Primes.

1 Introduction

Pells equation, the prime object in this research. It is a representation of Diophantine equation $x^2 - dy^2 = 1$, where a non-square positive integer d is given and will search for integer solutions in x and y. As an illustration, for d having value 5; one of the integer solutions is $x = 9, y = 4$. One thing to note about is that with d not a perfect square, Pells equation will certainly have infinitely many distinct integer solutions. For initial literature, we may refer to [2, 4, 6, 7, 9, 10, 11, 14, 15]. It has multiple references to various forms of Diophantine equations, which provide us the base knowledge to go about learning more about these equations. For additional references, we may also refer to another book [16]. We imbibed the problem identification as applied for exponential Diophantine equations. Further investigation and approach techniques can be referred to [15]. Assimilating and conceptualizing these learnings enabled to look ahead and ensure the concrete steps towards our research. To dive into the crux of the problem, major ideas were incorporated from the literatures due to [8, 12, 13]. Using these inputs, we develop our solution appropriately.

The focus of discussion in this paper is a negative Pells equation given as $x^2 - dy^2 = -N$, to be solved in positive integers x and y. As indicated here forth, we are using the Sophie Germain prime in negative Pells equation in finding the positive integer solutions. In number theory, a prime number p is a Sophie Germain prime if $2p + 1$ is also prime. A safe prime indicates the number $2p + 1$ associated with a Sophie Germain prime. In the Pells equation $x^2 = 41y^2 - 5^m$, $m \in \mathbb{N}$; we are using the Sophie Germain primes 41 and 5 and will attempt to search for its non-trivial integer solutions. To derive the solutions, we approached the quest with the case of choices of m generalized in all even and odd integers. We initiated the proof by involving the odd integers 1, 3, 5.

Applying Brahma Gupta lemma [1], we obtained the sequence of non-zero distinct integer solutions. This solution addresses the many positive integer solutions obtained thence. A few research driven relations with respect to the solutions are presented. Furthermore, the process is taken a bit ahead to derive the recurrence relations that addresses such types of Pells equations.

The references that we had indicated above were just a stepping stone for us to take us to the next level. The objective we have in mind is to enable our study to put across the outcomes for understanding the concepts and usage of cryptography. As is understood, that Cryptograph is an acknowledged area of application that involves the protection of information in the huge network of computing world. This concept goes a long way to ensure that only authorized personnel are enabled to read the concerned information and process it accordingly. It is also well understood that mathematics and mathematical concepts are the building blocks of cryptography and has gifted the world of computers a large set of algorithms and concepts to implement this protective logic involving cryptography. In the whole process, Pell's equation has been a major contributor in the science of cryptography. To generalize our research purpose, it is also come to the fore that the Sophie Germain primes are one of the leading contributors to this field. Since negative Pell equations are mostly unsolvable; it presents a complex method to undermine a strong security algorithm for cryptography. In due course, we intend to finetune our research to also generate an application to showcase the usage of Sophie Germain primes to devise a cryptographic solution.

2 Preliminaries

Theorem 2.1. *If x_1, y_1 is considered as the fundamental solution of $x^2 - dy^2 = 1$. Then to be noted is that every positive solution of the equation is given by x_n, y_n where x_n and y_n are the integers determined from $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n, for\ n = 1, 2, 3, ...$*

2.1 Solubility of the negative Pell equation - Our test approach

We assume that D is a positive integer, and considered not a perfect square. Then the negative Pell equation $x^2 - Dy^2 = -1$ is considered soluble if and only if D is expressed as $D = a^2 + b^2,\ gcd(a, b) = 1, a$ and b are positive, b is odd and the Diophantine equation, $-bV^2 + 2aVW + bW^2 = 1$ has a solution. (We highlight this as the case of solubility that occurs for exactly one such (a, b)). The solubility concepts were derived from article [3].

The Algorithm followed by us is illustrated below

(i) We will first find all expressions of D considered as a sum of two relatively prime squares using Cornacchia's method [5]. If none exists - the negative Pell equation is not soluble.

(ii) For each representation $D = a^2 + b^2, gcd(a, b) = 1, a$ and b positive, b odd, we will test the solubility of $-bV^2 + 2aVW + bW^2 = 1$ using the Lagrange-Matthews algorithm [3]. If soluble and it exists - the negative Pell equation is soluble.

(iii) If each representation yields no probable solution, then the negative Pell equation is insoluble.

Theorem 2.2. *Let us consider p to be a prime. The negative Pell equation $x^2 - py^2 = -1$ is considered solvable if and only if $p = 2$ or $p \equiv 1(mod4)$.*

Proof. This paper focusses on a negative Pell equation $x^2 = 41y^2 - 5^m$, $m \in \mathbb{N}$. For the negative Pell equation, we will consider the prime $p = 41$, which satisfies the identified conditions of Theorem 2.2. Therefore, we can substantiate with certainty the proof that the negative Pells equation $x^2 = 41y^2 - 5^m$, $m \in \mathbb{N}$ is solvable and prevalent in integers.

Using the Algorithm as illustrated in Theorem 2.1 and testing for $(a, b) = (4, 5) : -bV^2 + 2aVW + bW^2 = 1$ has a solution $(V, W) = (2, 1)$, so $x^2 - 41y^2 = -1$ is soluble. □

3 Method of Analysis

Choice 1: $m = 1$

The Pell equation in focus is

$$x^2 = 41y^2 - 5. \tag{3.1}$$

Let (x_0, y_0) be the initial solution of (3.1) given by $x_0 = 6; y_0 = 1$.

In our quest to find the other solutions of (3.1), consider the generalized form of the Pell equation

$$x^2 = 41y^2 + 1. \tag{3.2}$$

The initial solution of (3.2) is (2049,320) and the general solution $(\tilde{x_n},\ \tilde{y_n})$ given by Theorem 2.1 as $\tilde{x_n} = \frac{1}{2}f_n$, $\tilde{y_n} = \frac{1}{2\sqrt{41}}g_n$, where $f_n = (2049 + 320\sqrt{41})^{(n+1)} + (2049 - 320\sqrt{41})^{(n+1)}$, $g_n = (2049 + 320\sqrt{41})^{(n+1)} - (2049 - 320\sqrt{41})^{(n+1)}$, $n = 0, 1, 2 \cdots$.

By applying Brahma Gupta lemma [1] between (x_0, y_0) and $(\tilde{x_n},\ \tilde{y_n})$ the possible sequence of non-zero distinct integer solutions to (3.1) are obtained as given below

$$x_{n+1} = x_0\tilde{x_n} + dy_0\tilde{y_n},\ y_{n+1} = x_0\tilde{y_n} + dy_0\tilde{x_n}, \tag{3.3}$$

$$x_{n+1} = \frac{1}{2}[6f_n + \sqrt{41}g_n],\ y_{n+1} = \frac{1}{2\sqrt{41}}[\sqrt{41}f_n + 6g_n]. \tag{3.4}$$

Also to be noted is the recurrence relation satisfied by the solution of (3.1) given by

$$x_{n+2} - 4098\ x_{n+1} + x_n = 0,\ y_{n+2} - 4098\ y_{n+1} + y_n = 0. \tag{3.5}$$

Choice 2: $m = 3$

The Pell equation is

$$x^2 = 41y^2 - 125. \tag{3.6}$$

Let (x_0, y_0) be the initial solution of (3.6) given by $x_0 = 30; y_0 = 5$. Applying Brahma Gupta lemma [1] between (x_0, y_0) and $(\tilde{x_n},\ \tilde{y_n})$ the possible sequence of non-zero distinct integer solutions to (3.6) are obtained by equation (3.3) as given below

$$x_{n+1} = \frac{1}{2}[30f_n + 5\sqrt{41}g_n],\ y_{n+1} = \frac{1}{2\sqrt{41}}[5\sqrt{41}f_n + 30g_n]. \tag{3.7}$$

The recurrence relation satisfied by the solution of (3.6) are given by the equations below

$$x_{n+2} - 4098\ x_{n+1} + x_n = 0,\ y_{n+2} - 4098\ y_{n+1} + y_n = 0. \tag{3.8}$$

Choice 3: $m = 5$

The Pell equation in focus is

$$x^2 = 41y^2 - 3125. \tag{3.9}$$

Let (x_0, y_0) be the initial solution of (3.9) given by $x_0 = 14; y_0 = 9$.

Applying Brahma Gupta lemma [1] between (x_0, y_0) and $(\tilde{x_n},\ \tilde{y_n})$ the possible sequence of non-zero distinct integer solutions to (3.9) obtained by equation (3.3) as

$$x_{n+1} = \frac{1}{2}[14f_n + 9\sqrt{41}g_n],\ y_{n+1} = \frac{1}{2\sqrt{41}}[9\sqrt{41}f_n + 14g_n]. \tag{3.10}$$

The recurrence relation satisfied by the solution of (3.9) are given by the equations below

$$x_{n+2} - 4098\ x_{n+1} + x_n = 0,\ y_{n+2} - 4098\ y_{n+1} + y_n = 0. \tag{3.11}$$

Choice 4: $m = 2k\ , k \in \mathbb{N}$

The Pell equation is

$$x^2 = 41y^2 - 5^{2k},\ k \in \mathbb{N}. \tag{3.12}$$

Let (x_0, y_0) be the initial solution of equation (3.12) given by $x_0 = 32\ (5)^k;\ y_0 = 5\ (5)^k$.

Applying Brahma Gupta lemma [1] between (x_0, y_0) and $(\tilde{x_n},\ \tilde{y_n})$ the possible sequence of non-zero distinct integer solutions to (3.12) are obtained by equation (3.3) as given below

$$x_{n+1} = \frac{5^k}{2}[32f_n + 5\sqrt{41}g_n],\ y_{n+1} = \frac{5^k}{2\sqrt{41}}[5\sqrt{41}f_n + 32g_n]. \tag{3.13}$$

The recurrence relation satisfied by the solution of (3.12) are given by the equations below

$$x_{n+2} - 4098\ x_{n+1} + x_n = 0,\ y_{n+2} - 4098\ y_{n+1} + y_n = 0. \tag{3.14}$$

Choice 5: $m = 2k + 5\ , k \in \mathbb{N}$

The Pell equation is

$$x^2 = 41y^2 - 5^{2k+5},\ k \in \mathbb{N}. \tag{3.15}$$

Let (x_0, y_0) be the initial solution of equation (3.15) given by $x_0 = 70\ (5)^{k-1};\ y_0 = 45\ (5)^{k-1}$.

Applying Brahma Gupta lemma [1] between (x_0, y_0) and $(\tilde{x_n}, \tilde{y_n})$ the sequence of non-zero distinct integer solutions to (3.15) obtained by equation (3.3) as

$$x_{n+1} = \frac{5^{k-1}}{2}[70f_n + 45\sqrt{41}g_n],\ y_{n+1} = \frac{5^{k-1}}{2\sqrt{41}}[45\sqrt{41}f_n + 70g_n]. \tag{3.16}$$

The recurrence relation satisfied by the solution of (3.15) are given by the equations below

$$x_{n+2} - 4098\ x_{n+1} + x_n = 0,\ y_{n+2} - 4098\ y_{n+1} + y_n = 0. \tag{3.17}$$

4 Conclusion

As seen and proved with the research put forth, solving a negative Pells equation that involves the Sophie Germain primes has in fact provided a more intrinsic and dynamic interpretation for finding solutions to equations satisfying occurrences of the similar nature. In due course, our research will be one of the pointers going ahead to conceptualize the effort to making/ creating a security encryption model.

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STUDY OF VARYING CONCENTRATION AND SIZE OF NANOPARTICLES IN A CATHETERIZED ARTERY WITH CLOT AND STENOSIS*

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Abstract

This mathematical study comprising of a catheterized artery with clot and stenosis is conducted to highlight the usage of nanoparticles in treatment of thrombosis. Catheter coated with silver nanoparticles is inserted in the lumen of artery having clot and stenosis. The behavior of blood with nanoparticles is described using nanofluid. Navier-Stokes equation and diffusion equation for temperature as well as concentration are used to model the flow problem. Our prime intention is to study how concentration and nanoparticle size effect nanofluid flow considering the influence of various thermal features like thermal conductivity, specific heat capacity and thermal expansion. Solution has been obtained for concentration, temperature and velocity is obtained using finite difference method. The effects of radius of nanoparticle, Brownian motion parameter, stenosis depth, Grashof number and Darcy number have been examined graphically using *MATLAB*. It has been concluded that nanoparticles highly concentrate on the clot and stenosis and thus point to possible significant use of nanoparticles in antithrombotic therapy. This model can be, thus, utilized in thrombolytic therapies by proper optimization of concentration of nanoparticles as well as their geometries.
2020 Mathematical Sciences Classification: 76A05, 76D05, 35A08, 35A24, 9210, 92C10.
Keywords and Phrases: Nanofluids, Thermal conductivity, Viscosity, Concentration, Brownian motion.

1 Introduction

Nanoparticles have emerged as a promising technology that has revolutionized every field of science [11]. Recent years have witnessed an extensive attention of scientific researchers and clinicians in the field of nanomedicine or the use of nanoparticles in medicine. Nanoparticles provide enhanced treatment efficiency due to their convertible geometries and physiochemical properties because they mimic platelets by moving rapidly towards clots. Many nanoparticles-based drug delivery system have been used in medication and therapy of cardiovascular diseases and cancer. The application of nanoparticle in the therapeutics of thrombosis have exhibited amplified treatment efficiency [19]. In this paper we seek to understand the behavior of nanoparticles at the clot by controlling their concentration and size.

Thrombosis is the buildup of malignant clot in the blood vessels. It is a global health issue. The flow conditions of blood are affected by thrombus formation because clotted arteries have higher shear rates than healthy arteries. The thrombus or the malignant clot can be dissolved or reduced with the help of antiplatelet and anticoagulant agents like heparin, recombinant tPA *(rtPA)*, urokinase plasminogen activator *(uPA)* and streptokinase *(SK)* [19]. These agents are protein-based and have lesser bio-availability, thus, lesser therapeutic effect. Thus, it is important to develop such therapeutics that have higher bio-availability and efficiency. Here, nanoparticles have proven useful as their geometry and physio-chemical properties can be suitably controlled. Thus, nanoparticles have growing appeals in the treatment of clots.

Nanofluids are advanced fluids containing nanometer size particles suspended in a standard fluid like alcohol, water etc. Nanofluids hold an aptitude for heat transfer owing to its enhanced thermophysical properties. Thus, nanofluids are advantageous due to their better stability and better viscosity and dispersion properties.

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Saleem et al. [20] analysed nanofluid in an artery with a catheter having stenosis and clot. Rathore and Srikanth [15] worked on an artery with stenosis, clot and catheter whose outer surface is layered with nanoparticles. Guan and Dou [5] outlined the recent advances in the use of nanoparticles as thrombus-targeting agents. Shah and Kumar [16] studied blood with nanoparticles in a tapered artery having a blood clot. By the above literature survey, it is clear that the effect of nanoparticle concentration present in blood along with their temperature of nanofluid has not been inspected much. Thus, in the current mathematical analysis, we have developed a model for an artery with a clot including a catheter layered with nanoparticles and probed into influence of varying concentrations of nanoparticles and temperature of nanofluid.

Primary properties of nanoparticles depend on their thermal conductivity [17]. In return viscosity and thermal conductivity of nanofluid rely on Reynolds number and Prandtl number because of convections arising in them. Saito [21] gave a model for viscosity of nanofluids containing very small spherical nanoparticles with pronounced Brownian motion, as

$$\mu_{nf} = \mu_f \left(1 + \frac{2.5\phi}{1 - \frac{\phi}{0.87}}\right), \tag{1.1}$$

where ϕ is volume fraction of nanoparticles, μ_{nf} describes viscosity of nanofluid while μ_f is viscosity of base fluid. The interactions of nanoparticles caused by Brownian motion produces effects similar to convection at the nanoscale level. Thus, we have used this model to describe the viscosity of nanofluid.

The Navier-Stokes equation and temperature diffusion equation show that nanoparticle dispersion is elevated under strong Brownian forces. Jang and Choi [9] fabricated a model to define thermal conductivity accounting for contribution of nanoparticle Brownian motion in nanofluid, given as

$$k_{nf} = k_f(1-\phi) + k_p\phi + 3s\frac{r_0}{r_p}k_f Re^2 Pr\phi, \tag{1.2}$$

where ϕ is volume fraction of nanoparticles, k_{nf} describes thermal conductivity of nanofluid while k_f is thermal conductivity of base fluid and $k_p$is thermal conductivity of nanoparticles. Pr is Prandtl number and Re is Reynolds number. r_0 is radius of base fluid particles and $r_p$is radius of nanoparticles. s is an empirical constant. The vital role of Brownian motion is thus considered in our problem as we have used this model to describe the thermal conductivity of nanofluid.

Volume fraction of a solute present in a solvent is a measure of concentration of solute. The volume fraction is same as the concentration in an ideal solution i.e. where there is no reaction between the solute and solvent particles. In our case, the blood cells do not react with the nanoparticles in the nanofluid but accumulate only at the clot and stenosis. Thus, we have considered volume fraction of nanoparticles as concentration of nanoparticles in the nanofluid. The formulations have been carried out following the same.

When nanoparticles are administered in systemic circulation, they have their first encounter with blood cells. Nanoparticles are schemed specifically to deal with diseased cells to treat thrombosis. The compatibility of administered nanoparticles depends on their concentrations. Thus, to fine tune the nanoparticles before they are used in nanomedicine, it is important to understand their mathematical modelling. Hence, in this paper we have made an attempt to study blood flow in an artery with a clot in presence of a catheter coated with nanoparticles. The mathematical equations are modelled using Navier-Stokes equation, temperature and concentration diffusion equation in cylindrical co-ordinates. The concentration, temperature and velocity of nanofluid is found using finite difference method. The effects of nanoparticle concentration, temperature and velocity of nanofluid has been observed on parameters like radius of nanoparticle, Brownian motion parameter, stenosis depth, Grashof number and Darcy number. Outcomes have been discussed through graphs plotted using *MATLAB*. This study could act as a prototype in bio-medicine for the use of nanoparticles in treating thrombosis.

2 Mathematical Formulation

The incompressible, steady and laminar blood flow is assumed in an artery of length L and radius R_0 with a clot $\varepsilon'(z')$ and stenosis $R(z)$ (Fig 2.1). Silver nanoparticles are coated on the catheter of radius R_c. Cylindrical co-ordinates (r', θ', z') are taken into consideration. Equation of continuity, Navier-Stokes equation and diffusion equations for temperature and concentration are employed to frame the mathematical model.

The clot $\varepsilon'(z')$ [20] is defined as

$$\varepsilon'(z') = \begin{cases} R_0(1 + e^(-\pi^2(z'-0.5)^2)) & a' \le z' \le a' + b', \\ R_c & otherwise. \end{cases} \tag{2.1}$$

The geometry of the stenosis $R'(z')$[23] is given as:-

$$R'(z') = \begin{cases} R_0 - \delta' e^{\left(-\frac{m^2 z'^2}{L'^2}\right)} & a' \le z' \le a' + b', \\ R_0 & otherwise. \end{cases} \tag{2.2}$$

where δ' is the depth of stenosis and m is a parametric constant

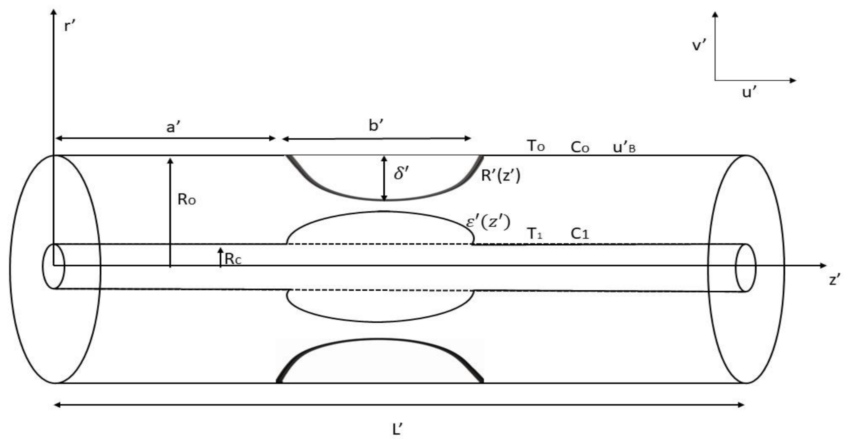

***Figure* 2.1:** Geometrical representation.

The governing equations are given as: Equation of continuity in cylindrical co-ordinates

$$\frac{\partial \rho_{nf}}{\partial t'} = \frac{1}{r'}\frac{\partial (r\ \rho_{nf} v')}{\partial r'} + \frac{1}{r'}\frac{\partial \rho_{nf} w'}{\partial \theta'} + \frac{\partial \rho_{nf} u'}{\partial z'} = 0, \tag{2.3}$$

Navier-Stokes equation in cylindrical co-ordinates

$$\begin{aligned} &\rho_{nf}\left(\frac{\partial v'}{\partial t'} + v'\frac{\partial v'}{\partial r'} + \frac{u'}{r'}\frac{\partial v'}{\partial \theta'} - \frac{u'^2}{r'} + u'\frac{\partial v'}{\partial z'}\right) \\ &= F_{r'} - \frac{\partial p'}{\partial r'} + \mu_{nf}\left(-\frac{v'}{r^2} + \frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial v'}{\partial r'}\right) + \frac{1}{r'^2}\frac{\partial^2 v'}{\partial \theta'^2} + \frac{\partial^2 v'}{\partial z'^2} - \frac{2}{r'^2}\frac{\partial w'}{\partial \theta'}\right), \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\rho_{nf}\left(\frac{\partial w'}{\partial t'} + v'\frac{\partial w'}{\partial r'} + \frac{u'}{r'}\frac{\partial w'}{\partial \theta'} - \frac{v'w'}{r'} + u'\frac{\partial w'}{\partial z'}\right) \\ &= F_{\theta'} - \frac{\partial p'}{\partial \theta'} + \mu_{nf}\left(-\frac{w'}{r^2} + \frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial w'}{\partial r'}\right) + \frac{1}{r'^2}\frac{\partial^2 w'}{\partial \theta'^2} + \frac{\partial^2 w'}{\partial z'^2} + \frac{2}{r'^2}\frac{\partial v'}{\partial \theta'}\right), \end{aligned} \tag{2.5}$$

$$\begin{aligned} &\rho_{nf}\left(\frac{\partial u'}{\partial t'} + v'\frac{\partial u'}{\partial r'} + \frac{u'}{r'}\frac{\partial u'}{\partial \theta'} + u'\frac{\partial u'}{\partial z'}\right) \\ &= F_{z'} - \frac{\partial p'}{\partial z'} + \mu_{nf}\left(\frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial u}{\partial r'}\right) + \frac{1}{r'^2}\frac{\partial^2 u'}{\partial \theta'^2} + \frac{\partial^2 u'}{\partial z'^2}\right), \end{aligned} \tag{2.6}$$

where F' in different indices stands for body forces in different co-ordinates and ρ_{nf} is density of nanofluid.

Diffusion equation for temperature T' of nanofluid in cylindrical co-ordinates

$$\begin{aligned} &\left(v'\frac{\partial T'}{\partial r'} + u'\frac{\partial T'}{\partial z'}\right) \\ &= \frac{k_{nf}}{\rho_{nf} c_{p_{nf}}}\left(\frac{\partial^2 T'}{\partial r'^2} + \frac{1}{r'}\frac{\partial T'}{\partial r'} + \frac{\partial^2 T'}{\partial z'^2}\right) + \frac{D_B}{\rho_{nf} c_{p_{nf}}}\left(\frac{\partial c'}{\partial r'}\frac{\partial T'}{\partial r'} + \frac{\partial c'}{\partial z'}\frac{\partial T'}{\partial z'}\right), \end{aligned} \tag{2.7}$$

where $c_{p_{nf}}$ is specific heat capacity of nanofluid, k_{nf} is thermal conductivity of nanofluid and ρ_{nf} is density of the nanofluid. D_B is Brownian diffusion coefficient. c' is concentration of nanoparticles. Temperature sensitive silver nanoparticles are coated on the catheter inserted in the lumen of artery [18]. The temperature is provided on the catheter to release nanoparticles for treating the clot.

Diffusion equation for concentration c' of nanoparticles in cylindrical co-ordinates

$$\frac{\partial c'}{\partial t'} + u'\frac{\partial c'}{\partial z'} + \ v'\frac{\partial c'}{\partial r'} \ + w'\frac{\partial c'}{\partial \theta'} = D_B(\frac{\partial^2 c'}{\partial r'^2} + \frac{1}{r'}\ \frac{\partial c'}{\partial r'} + \frac{1}{r'^2}\frac{\partial^2 c'}{\partial \theta'^2} + \frac{\partial^2 c'}{\partial z'^2}), \tag{2.8}$$

where D_B is Brownian diffusion coefficient. The silver nanoparticles are highly concentrated on the surface of catheter.

The governing equations (2.3) - (2.8) are solved under the following assumptions

1. Catheter has been inserted at the center of the clot in the artery,
2. Flow is considered two dimensional,
3. Flow is steady,
4. Flow is axisymmetric,
5. The azimuthal component of fluid velocity is zero,
6. The cross-section area is very small; thus, flow is described by low Reynolds number,
7. Free convection effects are ignored,
8. Nanoparticles and blood are in thermal equilibrium,
9. No chemical reaction takes place in the blood,
10. There is no heat transfer due to radiation.

Nanofluids are highly developed colloidal fluids attained by dispersing 1-100 nm nanoparticles in standard fluid. Studies over the time have proven that nanofluids hold outstanding thermophysical properties as compared to base fluids. The parameters like volume fraction, size of base fluid particles, their thermal conductivity, hold significance in defining thermal characteristics of nanofluids like viscosity, thermal conductivity, and specific heat capacity.

The better thermal characteristics of nanofluids is because of the small sized nanoparticles dispersed in it. Viscosity is an important thermal property in this momentum because it is caused by interparticle interactions. It has been observed that viscosity of a base fluid enhances when nanoparticles are suspended in it. Viscosity is thus a governing factor of the behaviour of nanofluids which is described by the dynamics of nanoparticles in it. Brownian motion of nanoparticles controls their thermal motion which is responsible for defining the viscosity. Saito [21] gave the model for describing viscosity of nanofluids by accounting for Brownian motion of spherical nanoparticles described as:

$$\mu_{nf} = \mu_f(1 + \frac{2.5c'}{1 - c'/0.87}), \tag{2.9}$$

where c' is concentration of nanoparticles; μ_{nf} is viscosity of nanofluid and μ_f is viscosity of blood.

Thermal conductivity is a relevant property of nanofluids as it is influenced by nanoparticle geometry, concentration and viscosity of base fluid. Thermal conductivity of nanofluids is evolved than their respective base fluids. The significant mechanism thar effects thermal conductivity of nanofluid is Brownian motion. Jang and Choi [9] gave the formula for thermal conductivity of nanofluid considering vital role of Brownian motion in thermal conduction. It has been reported by Gupta and Kumar [6] that Brownian motion enhances the thermal conductivity to 6 percent than their base fluids. Nanoparticles have a high random diffusion because of Brownian motion owing to their small dimensions. Thus, to study thermal conductivity of nanofluids, we use the formulation by Jang and Choi [9],

$$k_{nf} = k_f\,(1 - c') + k_p c' + 3s\frac{r_0}{r_p}k_f Re^2 Pr c', \tag{2.10}$$

where c' is concentration of nanoparticles; k_{nf} is thermal conductivity of nanofluid, k_f is thermal conductivity of blood and k_p is thermal conductivity of nanoparticles; Pr is Prandtl number and Re is Reynolds number; r_0 is radius of blood particles (taken average), $r_p$is radius of nanoparticles and s is an empirical constant.

Specific heat capacity is also one of the relevant parameters for stating the thermal characteristics of nanofluids. Specific heat capacity dictates transfer of heat. It has been proved that specific heat capacity of nanofluids is lesser compared to their base fluid. Xuan et al. [25] modelled specific heat capacity for thermal equilibrium in nanoparticles and its base fluid which is given as,

$$c_{p_{nf}} = \frac{(1 - c')\,\rho_f c_{p_f} + c'\rho_p c_{p_p}}{(1 - c')\,\rho_f + c'\rho_p}, \tag{2.11}$$

where c' is concentration of nanoparticles; $c_{p_{nf}}$ is specific heat capacity of nanofluid, c_{p_f} is specific heat capacity of blood and c_{p_p} is specific heat capacity of nanoparticles; ρ_f density of blood and $\rho_p$is density of nanoparticles.

The thermal expansion of the nanofluid is modelled using a simple formula based on mixture rule as

$$(\rho\gamma)_{nf} = (1 - c')\,\rho_f\gamma_f + c'\rho_p\gamma_p, \tag{2.12}$$

where c' is concentration of nanoparticles; $(\rho\gamma)_{nf}$ is thermal expansion of nanofluid, γ_f is specific thermal expansion of blood and $\gamma_p$is specific thermal expansion of nanoparticles; ρ_f density of blood and ρ_p is density of nanoparticles.

The modified equations using the assumptions and equations (2.9), (2.10), (2.11) and (2.12), along with their boundary conditions are given henceforth.

The equation of continuity

$$\frac{\partial u'}{\partial z'} = 0, \tag{2.13}$$

The equation of motion in the catheterized artery with clot at the center

$$g\,(\rho\gamma)_{nf}\,(T' - T_0) + g\,(\rho\gamma)_{nf}\,(c' - c_0) - (1/\rho_{nf})\frac{\partial p'}{\partial z'} + (\mu_{nf}\ /\rho_{nf})(\frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial u}{\partial r'}\right)) = 0, \tag{2.14}$$

No-slip at the boundary of the catheter is assumed.

$$u' = 0\ \ at\ \ r' = \varepsilon'(z'), \tag{2.15}$$

Using Beavers and Joseph condition [1] at the boundary of the artery, we get

$$u' = u'_B\ \ and\ \ \frac{\partial u'}{\partial r'} = \frac{\sigma'}{\sqrt{Da}}(u'_B - u'_p)\ \ at\ \ r' = R'(z'), \tag{2.16}$$

where

$$u'_p = -\frac{Da}{\mu_{nf}}\frac{\partial p'}{\partial z'}, \tag{2.17}$$

is velocity at the permeable boundary where u'_B is slip velocity, σ' is slip parameter, Da is Darcy number

The diffusion equation for temperature of the catheterized artery with clot at the center

$$\frac{k_{nf}}{\rho_{nf}c_{p_{nf}}}\left(\frac{\partial^2 T'}{\partial r'^2} + \frac{1}{r'}\frac{\partial T'}{\partial r'}\right) + \frac{D_B}{\rho_{nf}c_{p_{nf}}}\left(\frac{\partial c'}{\partial r'}\frac{\partial T'}{\partial r'}\right) = 0. \tag{2.18}$$

Temperature T_1 is prescribed over the catheter and clot for releasing nanodrug

$$T' = T_1\ \ at\ \ r' = \varepsilon'(z'). \tag{2.19}$$

Temperature at the boundary of the artery is T_o

$$T' = T_1\ \ at\ \ r' = R'(z'). \tag{2.20}$$

The diffusion equation for concentration of nanoparticles in the catheterized artery with clot at the center

$$D_B\left(\frac{\partial^2 c'}{\partial r'^2} + \frac{1}{r'}\ \frac{\partial c'}{\partial r'}\right) = 0. \tag{2.21}$$

Concentration c_1 of nanoparticles on the catheter and clot

$$c' = c_1\ \ at\ \ r' = \varepsilon'(z'). \tag{2.22}$$

Concentration of nanoparticles at the boundary of the artery is c_o

$$c' = c_0\ \ at\ \ r' = R'(z'). \tag{2.23}$$

Non-dimensional scheme
is given below as

$$\begin{cases} r = \frac{r'}{R_0}, z = \frac{z'}{R_0}, u = \frac{u'}{u_{avg}}, P = \frac{P'}{\rho_f u_{avg}^2} Re = \frac{R_o\ u_{avg}\rho_f}{\mu_f}, Da = \frac{k_f}{R_0^2}, Pr = \frac{\mu_f}{D_B}, \\ N_b = \frac{\rho_f c_{p_f} D_B(c_1 - c_2)}{k_f}, \theta = \frac{T' - T_0}{T_1 - T_0}, c = \frac{c' - c_0}{c_1 - c_0}, Gr = \frac{g(\rho\gamma)_f R_0^2 (T_1 - T_0)}{u_{avg}\mu_f}, \\ Br = \frac{g(\rho\gamma)_f R_0^2 (c_1 - c_0)}{u_{avg}\mu_f}, \sigma' = \frac{\sigma}{R_0}, \delta' = \frac{\delta}{R_0}, \end{cases} \tag{2.24}$$

where u_{avg} is average reference velocity, Re is Reynolds number, Da is Darcy number, Pr is Prandtl Number, Gr is Grashof number and Br is solutary Grashof number, N_b is Brownian motion parameter.

Non-dimensional equation for clot
is given below as

$$\varepsilon(z) = \begin{cases} 1 + e^{-\pi^2 (z-0.5)^2} & a \le z \le a+b, \\ 0.1 \quad otherwise. \end{cases} \tag{2.25}$$

Non-dimensional equation for stenosis
is given below as

$$\varepsilon(z) = \begin{cases} 1 - \frac{\delta}{R_0} e^{-m^2 z^2 / L^2} & a \le z \le a+b, \\ 1 \quad otherwise. \end{cases} \tag{2.26}$$

Non-dimensional equations
are given below as

$$\frac{\partial u}{\partial z} = 0, \tag{2.27}$$

$$\frac{\partial P}{\partial z} = \mu_f \left(1 + \frac{2.5c}{1 - c/0.87}\right) \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + \theta Gr \left((1-c) + c\frac{(\rho\gamma)_p}{(\rho\gamma)_f}\right) + cBr \left((1-c) + c\frac{(\rho\gamma)_p}{(\rho\gamma)_f}\right), \tag{2.28}$$

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{\partial \theta}{\partial r}\frac{\partial c}{\partial r} N_b \left((1-c) + c\frac{k_p}{k_f} + 3s\frac{r_0}{r_p} Re^2 Prc\right) \frac{\left(\frac{(1-c)}{c_{p\,f}} + c\frac{\rho_p}{\rho_f c_{p\,f}}\right)}{\left((1-c) + c\frac{\rho_p c_{p\,p}}{\rho_f c_{p\,f}}\right)} = 0, \tag{2.29}$$

$$\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r} = 0, \tag{2.30}$$

Non-dimensional boundary conditions
are given below as

$$c = 0 \;\; at \;\; r = R(z), \tag{2.31}$$

$$\theta = 0 \;\; at \;\; r = R(z), \tag{2.32}$$

$$u = u_B \;\; and \;\; \frac{\partial u}{\partial r} = \frac{\sigma}{\sqrt{Da}}(u_B - u_p) \;\; at \;\; r = R(z), \tag{2.33}$$

$$u = 0 \;\; at \;\; r = \varepsilon(z), \tag{2.34}$$

$$\theta = 1 \;\; at \;\; r = \varepsilon(z), \tag{2.35}$$

$$c = 1 \;\; at \;\; r = \varepsilon(z). \tag{2.36}$$

3 Solution

Mathematical solution for equations (2.25) to (2,30) employing the boundary conditions (2.31) to (2.36) is calculated numerically using *MATLAB* version 9.1R2016b.

Finite difference method

Denote c_i^k or Θ_{i+1}^kas the value of c or Θ at node r_i or z_i. In this notation, the finite difference formulation of various partial derivatives are given as

$$\frac{\partial c}{\partial r} \cong \frac{c_{i+1}^k - c_{i-1}^k}{2\Delta r} = c_r, \tag{3.1}$$

$$\frac{\partial^2 c}{\partial r^2} \cong \frac{c_{i+1}^k - 2c_i^k + c_{i-1}^k}{(\Delta r)^2} = c_{rr}, \tag{3.2}$$

$$\frac{\partial \Theta}{\partial r} \cong \frac{\Theta_{i+1}^k - \Theta_{I+1}^k}{2\Delta r} = \Theta_r, \tag{3.3}$$

$$\frac{\partial^2\Theta}{\partial r^2} \cong \frac{\Theta_{i+1}^k - 2\Theta_i^k + \Theta_{i-1}^k}{(\Delta r)^2} = \Theta_{rr}, \tag{3.4}$$

$$\frac{\partial u}{\partial r} \cong \frac{u_{i+1}^k - u_{i-1}^k}{2\Delta r} = u_r, \tag{3.5}$$

$$\frac{\partial^2 u}{\partial r^2} \cong \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta r)^2} = u_{rr}, \tag{3.6}$$

The governing equations (2.28), (2.29) and (2.30) are as follows

$$\frac{c_{i+1}^k - c_{i-1}^k}{2\Delta z} + \frac{1}{r}\frac{c_{i+1}^k - 2c_i^k + c_{i-1}^k}{(\Delta r)^2} = 0, \tag{3.7}$$

$$\frac{\Theta_{i+1}^k - 2\Theta_i^k + \Theta_{i-1}^k}{(\Delta r)^2}\frac{1}{r}\frac{\Theta_{i+1}^k - \Theta_{I+1}^k}{2\Delta r} + \frac{\Theta_{i+1}^k - \Theta_{I+1}^k}{2\Delta r}\frac{c_{i+1}^k - c_{i-1}^k}{2\Delta z}$$

$$N_b((1-c_i^k) + \frac{k_p}{k_f}c_i^k 3s\frac{r_0}{r_p}Re^2 Prc_i^k)\frac{(1-c_i^k) + \frac{k_p}{k_f}c_i^k 3s\frac{r_0}{r_p}Re^2 Prc_i^k}{((1-c_i^k) + \frac{k_p}{k_f}c_i^k 3s\frac{r_0}{r_p}Re^2 Prc_i^k)} = 0, \tag{3.8}$$

$$\frac{\Delta P}{\Delta z} = \mu_f(1 + \frac{2.5c_i}{1 - c_i/0.87})(\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta r)^2} + \frac{1}{r}\frac{u_{i+1}^k - u_{I+1}^k}{2\Delta r})$$

$$+\theta_i^k Gr((1-c_i^k) + c_i^k\frac{(\rho\gamma)_p}{(\rho\gamma)_f}) + c_i^k Br((1-c_i^k) + c_i^k\frac{(\rho\gamma)_p}{(\rho\gamma)_f}), \tag{3.9}$$

$$c_i^k = 1 \ \ at \ \ r_i = \varepsilon(z_i), \tag{3.10}$$

$$c_i^k = 0 \ \ at \ \ r_i = R(z_i), \tag{3.11}$$

$$\Theta_i^k = 1 \ \ at \ \ r_i = \varepsilon(z_i), \tag{3.12}$$

$$\Theta_i^k = 0 \ \ at \ \ r_i = R(z_i), \tag{3.13}$$

$$u_i^k = 0 \ \ at \ \ r_i = \varepsilon(z_i), \tag{3.14}$$

$$u_i^k = u_{iB} \ \ and \ \ u_r = \frac{\sigma}{\sqrt{Da}}(u_{iB} - u_p) \ \ at \ \ r_i = R(z_i). \tag{3.15}$$

The algorithm for solving the equations is given as

1. The radial domain is represented by a mesh of $(n+1)$ grid points $0 = r_0 < r_1 < \ldots < r_{n-1} < r_n = 1$.
2. We seek the solution for c, θ and u at the mesh points for their respective regions.
3. The difference equations (3.7) to (3.9) and boundary conditions (3.10) to (3.15) are solved using *bvp4c* solver to obtain the values at each grid point applying Thomas algorithm for tridiagonal system of matrices

The value of concentration c in the thrombolytic and non-thrombolytic regions against radial direction r is given by Table 3.1 as $Rc = 0.1, m = 1$, $\delta = 0.01$

***Table* 3.1**

Radius	**Concentration in non-thrombolytic region**	**Concentration in thrombolytic region**
0.1	1	19.0743
0.2	0.6989	15.7190
0.3	0.5228	12.4227
0.4	0.3979	9.5837
0.5	0.3010	7.2690
0.6	0.2218	5.3592
0.7	0.1549	3.7422
0.8	0.0969	2.3142
0.9	0.4575	1.1054
1.0	0	0

The value of temperature of nanofluid θ against radial direction r is given by Table 3.2 as $Rc = 0.1, \delta = 0.01, rp = 30nm, Nb = 1.5$

***Table* 3.2**

Radius	**Temperature of nanofluid**
0.1	0
0.2	0.6225
0.3	0.4359
0.4	0.3168
0.5	0.2313
0.6	0.1657
0.7	0.1129
0.8	0.0692
0.9	0.3210
1.0	0

The value of velocity of nanofluid u against radial direction r is given by Table 3.3 as $Rc = 0.1, \delta = 0.01, rp = 30nm, Nb = 1.5, Gr = 0.2, Br = 0.1, Da = 0.1$

***Table* 3.3**

Radius	**Velocity of nanofluid**
0.1	0
0.2	2.6577
0.3	3.8602
0.4	4.3308
0.5	4.3001
0.6	3.8707
0.7	3.0970
0.8	2.0117
0.9	0.6354
1.0	-1.0176

4 Graphical results and discussions

This article gives theoretical research about the effects of treating clot in an artery using nanoparticles with respect to concentration of nanoparticles, radius of nanoparticles, Brownian motion parameter, Grashof number and Darcy number on velocity and temperature of nanofluids. Figures 4.1-4.11 show the graphs of results obtained.

Fig 4.1 shows graph of concentration of nanoparticles (c) against radial direction (r) for thrombolytic and non-thrombolytic regions. The graph shows that concentration is decreasing with the increase in radial direction. This is because the concentration of nanoparticles is highest at the catheter and clot as compared to the wall of the artery and stenosis. It can be concluded from the graph that concentration of nanoparticles is greater in thrombolytic region than in non-thrombolytic region. This result highlights the applications of nanoparticles in the treatment of clot. Khurshid et al. [12] gave identical conclusions in their experimental study.

Fig 4.2 shows graph of concentration of nanoparticles (c) against radial direction (r) for different values of catheter radius (Rc). The plots show that greater the value of catheter radius greater the concentration of nanoparticles in the artery. This is directly related to the fact that greater radius would accommodate greater number of nanoparticles on the surface of catheter. However, the radius of catheter should be optimized depending upon the severity of the clot. Comparative results have also been given by Karami et al. [13].

Fig 4.3 depicts graph of temperature of nanofluid (θ) against radial direction (r) for different values of radius of nanoparticles (r_p). Graph of temperature of decreases with increasing radial distance. This happens because nanoparticles present towards wall of catheter are at a higher temperature which causes them to migrate to walls of the artery which is at a lower temperature. The trend shows that the increase in radius of nanoparticle brings about a rise in the temperature of nanofluid. Qu et al. [14] gave this result in their experimental study. This happens because the increase in radius enhances the size of nanoparticles which causes greater interparticle collision owing to reduction in interparticle space. Hoshyar et al. [7] summarized similar results in their review on effect of nanoparticle size on their cellular interactions. They reported that larger diameter nanoparticles offer decreased cellular uptake. The optimal size of nanoparticle should be 30nm- 60 nm for effective delivery of drug.

Fig 4.4 displays graph of temperature of nanofluid (θ) against radial direction (r) for different values of Brownian motion parameter (N_b). The graph shows that temperature of nanofluid increases with increase in Brownian motion parameter. Nanoparticle motion increases with rise in Brownian motion, thus, temperature increases. Experimental validation was given by Jiang et al. [10].

Fig 4.5 shows graph of temperature of nanofluid (θ) against radial direction (r) for different values of stenosis depth (δ). Greater the stenosis depth, lesser the temperature. Xinting et al. [26] presented comparable experimental result for the effect of stenosis depth on the temperature of nanofluid.

Fig 4.6 depicts graph of velocity of nanofluid (u) against radial direction (r) for different values of radius of nanoparticles (r_p). Graph shows a parabolic variation similar to Hagen-Poiseuille flow. This is because the velocity is affected by zero acceleration because of constant pressure drop in the artery. Graph also shows that greater radius of nanoparticle lesser the velocity. Larger sized nanoparticles aggregate to increase flow resistance, hence velocity decreases. Hu et al. [8] analyzed similar result in their experimental study of effect of nanoparticle size on viscosity.

Fig 4.7 displays graph of velocity of nanofluid (u) against radial direction (r) for different values of Brownian motion parameter (N_b). It is seen that velocity decreases with increase in value of Brownian motion parameter. Brownian motion parameter is directly related to size of nanoparticles. Thus, larger the size, greater the Brownian motion parameter, lesser is the velocity. Saghir and Rahman [22] proved analogous experimental results.

Fig 4.8 shows graph of velocity of nanofluid (u) against radial direction (r) for different values of stenosis depth (δ). The results show that velocity increases with increase in stenosis depth. It follows from Bernoullis law for incompressible fluids, that reduction in cross-section area increases the velocity of fluid. This can also be supported by the fact that arteriosclerotic and thrombolytic arteries have higher blood pressure as compared to normal arteries [2].

Fig 4.9 shows graph of velocity of nanofluid (u) against radial direction (r) for different values of Grashof number (Gr). Grashof number stands for ratio of buoyancy force to viscous force. Thus, increase in its value increases the velocity of nanofluid because of the increase in temperature due to reduction in viscous forces [24].

Fig 4.10 depicts graph of velocity of nanofluid (u) against radial direction (r) for different values of solutary Grashof number (Br). Solutary Grashof number Br defines ratio between buoyant force and viscous hydrodynamic forces [18]. The trend observed is similar to Grashof number. It is because as the concentration increases the flow increases, thus increasing velocity.

Fig 4.11 displays graph of velocity of nanofluid (u) against radial direction (r) for different values of

Darcy number (Da). It is seen that increase in Darcy number increases velocity. Darcy number physically represents permeability at the arterial wall. Enhancing its value reduces flow resistance at the wall thus increasing velocity at arterial wall. Such experimental investigation was given by Boettcher et al. [3].

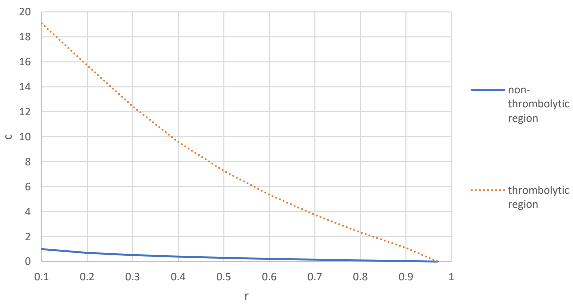

***Fig* 4.1** Variation of concentration of nanoparticles c against radial direction r in thrombolytic and non-thrombolytic regions
Rc =0.1

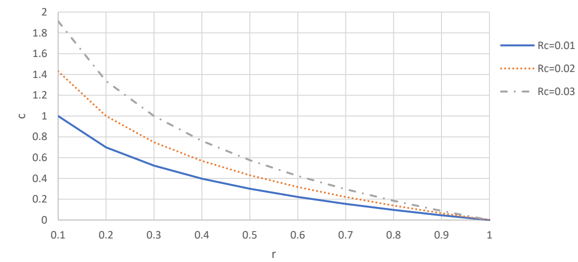

***Fig* 4.2** Variation of concentration of nanoparticles c against radial direction r for different values of catheter radius

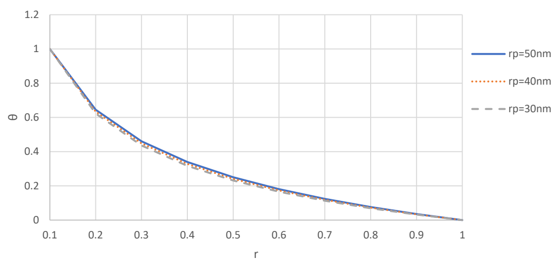

***Fig* 4.3** Variation of temperature of nanofluid θ against radial direction r for different values of nanoparticle radius rp
Rc=0.1, Nb=1.0, δ=0.01

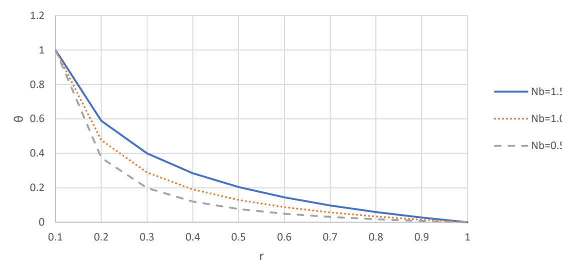

***Fig* 4.4** Variation of tempertaure of nanofluid θ against radial direction r for different values of Brownian motion parameter Nb
Rc=0.1, rp=30nm, δ=0.01

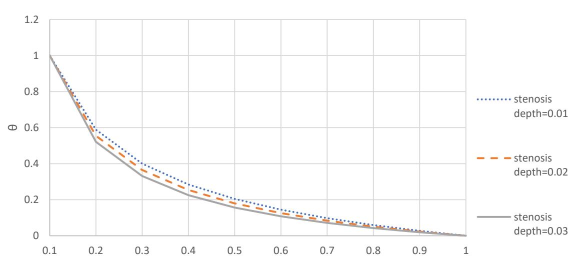

***Fig* 4.5** Variation of tempertaure of nanofluid θ against radial direction r for different values of stenosis depth δ
Rc=0.1, Nb=1.0, rp=30 nm

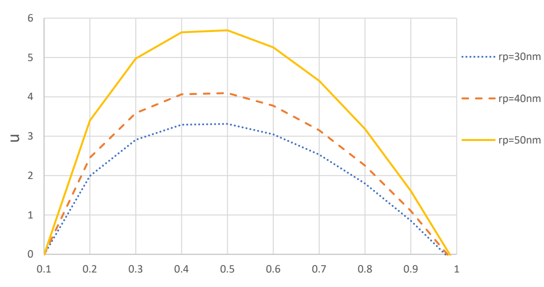

***Fig* 4.6** Variation of velocity of nanofluid u against radial direction r for different values of nanoparticle radius rp
Nb=0.1, δ=0.01, Gr=2.0, Br=2.0, Da=0.1

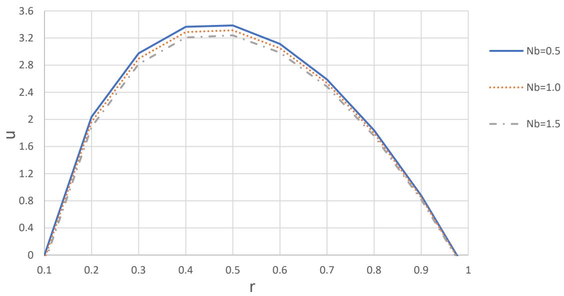

***Fig* 4.7** Variation of velocity of nanofluid *u* against radial direction *r* for different values of Brownian motion parameter *Nb*
rp=30nm, δ=0.01, *Gr*=2.0, *Br*=2.0, *Da*=0.1

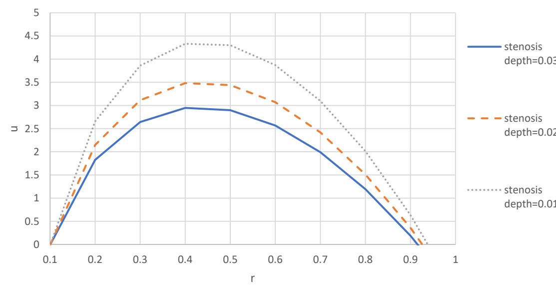

***Fig* 4.8** Variation of velocity *u* against radial direction *r* for different values of stenosis depth δ
Rc=0.1, *Nb*=1.0, *rp*=30nm, *Gr*=2.0, *Br*=3.0, *Da*=0.1

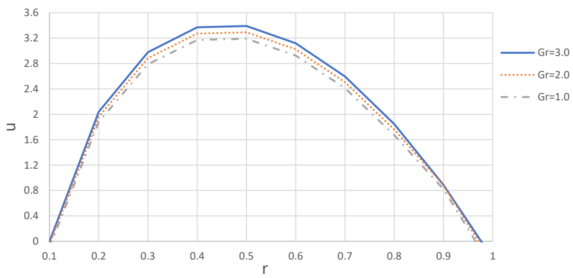

***Fig* 4.9** Variation of velocity of nanofluid *u* against radial direction *r* for different values of Grashof number *Gr*
Nb=0.1, δ=0.01, *rp*=30 nm, *Br*=2.0, *Da*=0.1

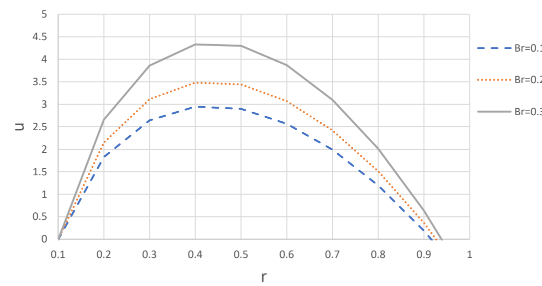

***Fig* 4.10** Variation of velocity of nanofluid *u* against radial direction *r* for different values of solutary Grashof number *Br*
Nb=0.1, δ=0.01, *rp*=30nm, *Gr*=2.0, *Da*=0.1

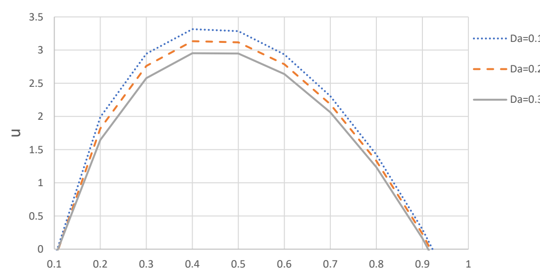

***Fig* 4.11** Variation of velocity of nanofluid *u* against radial direction *r* for different values of Darcy number *Da*
Nb=0.1, δ=0.01, *rp*=30nm, *Gr*=2.0, *Br*=2.0

5 Conclusion

This study focuses on the influence of nanoparticle concentration, temperature and velocity of nanofluid in a catheterized artery with clot and stenosis. The study contributes to the understanding and use of nanoparticles as anti-thrombolytic agents. The outcomes are encapsulated as

1. The concentration of nanoparticles is higher at the clot compared to other regions.
2. The temperature of nanofluid increases with increase in nanoparticle radius, Brownian motion parameter and decreases with increase in stenosis depth.
3. The velocity of nanofluid decreases with increase in nanoparticle radius and Brownian motion parameter.
4. The velocity of nanofluid increases with increase in stenosis depth, Grashof number, solutary Grashof number and Darcy number.

The above model has useful application in the treatment of cardiovascular diseases.

6 Appendix

The thermophysical properties of blood are

***Table* 6.1**

Physical properties	**Blood**
Heat Capacitance (c_p)	$3594 J/KgK$
Thermal Conductivity (k)	$0.492 W/mK$
Density (ρ)	$1060 Kg/m^3$
Thermal expansion coefficient (γ)	$0.18X10^-5K^-1$

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

HIGHER DIMENSIONAL TOPOLOGICAL DEFECT SOLUTIONS WITH MASSIVE SCALAR FIELD IN GENERAL RELATIVITY

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Abstract

In this paper, we investigate a five dimensional Locally Rotationally Symmetric *(LRS)* Bianchi type-V string cosmological model with massive scalar field in general relativity. In order to obtain an exact solution of the field equations, we used the following conditions: (i) the shear scalar is proportional to the expansion scalar, resulting in a relationship between metric potentials, and (ii) the average scale factor is proportional to the massive scalar field, resulting in a power law relationship. In addition, the physical and kinematical parameters are discussed in detail.

2020 Mathematical Sciences Classification: 83-3, 83-08, 83C15, 83E15, 83E30, 83F05.
Keywords and Phrases: Five dimensional LRS Bianchi -V model, cosmic string model, massive scalar field.

1 Introduction

General relativity *(GR)* a is geometric theory that describes gravitational phenomena. It is also useful in constructing mathematical models in cosmology which deals with the large scale structure of the universe. A phase transition in the early universe took place when the temperature dropped and symmetry of the universe broken spontaneously leading to topologically stable defects called vacuum domain walls, strings, and monopoles [7]. As cosmic strings are key parts of the description of the universe in the early stages of its evolution and give rise to density perturbations that lead to the formation of galaxies [7, 22]. It has attracted considerable interest among cosmologists to study cosmic strings within the framework of general relativity. A scalar meson field can be classified into two types, namely, zero mass scalar field and massive scalar field. Massive scalar fields describe short-range interactions while zero mass scalar fields describe long-range interactions. Cosmological models with massive scalar fields have been discussed by several authors in general and in modified theories of gravitation. Mete et al. [9] explored Bianchi type-V magnetized cosmological model with wet dark fluid in *GR*. Reddy [16] discussed the Bianchi type-V dark energy model with a scalar meson field in *GR*. Using a modified holographic Ricci dark energy model with an attractive massive scalar field, Naidu [11] studied Bianchi type-II modified holographic Ricci dark energy model. Naidu et al. [12] developed an anisotropic and spatially homogeneous Bianchi type-V dark energy cosmological model in the presence of an attractive massive scalar field in *GR*. A study by Aditya et al. [1] examined Kaluza-Klein dark energy models in Lyra manifolds with massive scalar fields. A spatially homogeneous and anisotropic Kantowski-Sachs cosmological model with anisotropic dark energy *(DE)* fluid and massive scalar fluid is presented by Raju et al. [17].

Rao et al. [18] constructed LRS Bianchi type-II cosmological models based on a mixture of a cosmic string cloud and anisotropic dark energy fluid as the source of gravitation. Aditya et al. [2] explored a spatially homogeneous and anisotropic Bianchi type- VI_0 cosmological model with dark energy fluid. An attractive massive scalar field with Bianchi type- I cosmological model with perfect fluid and attractive scalar fields in Lyra manifold has been discussed by Naidu et al. [13].Aditya et al. [3] investigated the solution of Einstein field equations using some physically relevant conditions in order to obtain an exact plane-symmetric dark energy cosmological model in the presence of an attractive massive scalar field. Poonia et al. [15] examined a Bianchi type-VI inflationary cosmological model with massive string source in general relativity. Recently Keerti Acharya et al.[4] discussed some Bianchi type-III string cosmological models for perfect fluid distribution with an alternate approach.

An Extra dimension is a concept in cosmology aimed at unifying gravity with other forces through higher-dimensional space-time. We are living in a four-dimensional stage of the universe, which may have been preceded by a multi-dimensional stage. Higher-dimensional cosmological models play an important role in studying the evolution of the universe in its early stages after the big bang due to their ability to study the early stages of the universe in a more detailed way. Kaluza Klein minimally interacting dark energy model in the presence of massive scalar field has been investigated by Naidu et al. [14]. Mohanty et al. [10] constructed a five dimensional string cosmological models in Lyra manifold when a massive string is the source of the gravitational field with $\rho = (1+\omega)\lambda$ (Takabayasi string). Very recently, a spatially homogeneous and anisotropic Bianchi type-V cosmological model coupled with a massive scalar meson field in presence of cosmic string has been studied by Raju et al. [19].

In light of the above discussion, we have investigated the higher dimensional LRS Bianchi type-V string cosmological model with massive scalar field. Some physical and kinematical properties of the model are discussed in detail.

2 Metric and field equations

Here we consider the space-time represented by five dimensional LRS Bianchi type-V metric in the form

$$ds^2 = dt^2 - A^2dx^2 - B^2e^{2x}\left(dy^2 + dz^2\right) - C^2d\psi^2, \tag{2.1}$$

where metric coefficients A, B, C are the functions of time t.

The Einstein field equation is

$$R_{ij} - \frac{1}{2}g_{ij}R = -\left(T_{ij}^{(s)} + T_{ij}^{(\varphi)}\right), \tag{2.2}$$

where R_{ij} is the Ricci tensor, R is Ricci scalar and $T_{ij}^{(s)}$ is the energy momentum tensor corresponding to massive string defined as

$$T_{ij}^{(s)} = \rho u_i u_j - \lambda x_i x_j, \tag{2.3}$$

where ρ is the energy density, λ is the string tension density, u^i is five velocity and x_i is string direction.

$T_{ij}^{((\phi)}$ is the energy momentum tensor for attractive massive scalar field defined as,

$$T_{ij}^{(\phi)} = \varphi_{,i}\varphi_{,j} - \frac{1}{2}\left(\varphi_{,k}\varphi^{,k} - M^2\varphi^2\right), \tag{2.4}$$

where M is mass of scalar field φ which satisfies Klein-Gordan equation

$$g^{ij}\varphi_{,ij} + M^2\varphi = 0 \tag{2.5}$$

and a comma (,) and a semicolon (;) denote ordinary and covariant differentiation respectively and $\varphi = \varphi(t)$.

In a co-moving coordinates system, the velocity vector u^i and direction of string x^i satisfy the conditions

$$u^iu_i = -x^ix_i = 1. \tag{2.6}$$

$$u^ix_i = 0. \tag{2.7}$$

$$\rho = \rho_p + \lambda, \tag{2.8}$$

where ρ_p is the rest energy density of the particles attached to the string (λ) which may be negative or positive [8].

Using commoving coordinates the field equations (2.2), for the metric (2.1) with the help of equations (2.3) to (2.8) can be written as

$$2\frac{\dot{A}\dot{B}}{AB} + 2\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}^2}{B^2} - \frac{3}{A^2} = \rho + \frac{\dot{\varphi}^2}{2} + \frac{M^2\varphi^2}{2}, \tag{2.9}$$

$$2\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}^2}{B^2} + 2\frac{\dot{B}\dot{C}}{BC} - \frac{1}{A^2} = -\frac{\dot{\varphi}^2}{2} + \frac{M^2\varphi^2}{2}, \tag{2.10}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{1}{A^2} = -\frac{\dot{\varphi}^2}{2} + \frac{M^2\varphi^2}{2}, \tag{2.11}$$

$$\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B} + 2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} - \frac{3}{A^2} = \lambda - \frac{\dot{\varphi}^2}{2} + \frac{M^2\varphi^2}{2}, \tag{2.12}$$

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = 0. \tag{2.13}$$

From equation (2.13) we get

$$A = lB, \tag{2.14}$$

where l is a constant of integration. Thus, without loss of generality we take $l = 1$.

$$A = lB. \tag{2.15}$$

Using equation (2.15), equations (2.9) to (2.12) reduce to

$$3\frac{\dot{A}^2}{A^2} + 3\frac{\dot{A}\dot{C}}{AC} - \frac{3}{A^2} = \rho + \frac{\dot{\varphi}^2}{2} + \frac{M^2\varphi^2}{2}, \tag{2.16}$$

$$2\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}^2}{A^2} + 2\frac{\dot{A}\dot{C}}{AC} - \frac{1}{A^2} = -\frac{\dot{\varphi}^2}{2} + \frac{M^2\varphi^2}{2}, \tag{2.17}$$

$$3\frac{\ddot{A}}{A} + 3\frac{\dot{A}^2}{A^2} - \frac{3}{A^2} = \lambda - \frac{\dot{\varphi}^2}{2} + \frac{M^2\varphi^2}{2}, \tag{2.18}$$

where overhead dot(.) represents differentiation with respect to the cosmic time t. The conservation law for matter energy tensor gives us

$$\dot{\rho} + \rho\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) - \lambda\frac{\dot{C}}{C} = 0 \tag{2.19}$$

and the Klein- Gordon equation for the metric (2.1) takes the form

$$\ddot{\varphi} + \dot{\varphi}\left(3\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) + M^2\varphi = 0. \tag{2.20}$$

Now, we define the following physical parameters that are useful for solving the above field equations.

The spatial volume V and the scale factor $R(t)$ are given by

$$V = AB^2C = R^4. \tag{2.21}$$

The expansion scalar (θ), the Hubble parameter (H) and the deceleration parameter (q), for the metric (2.1) are

$$\theta = 4H = \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \tag{2.22}$$

$$H = \frac{1}{4}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \tag{2.23}$$

$$q = -\frac{R\ddot{R}}{R^2}. \tag{2.24}$$

The shear scalar $\left(\sigma^2\right)$ and the anisotropy parameter (Δ) are respectively given by

$$\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{4} H_i^2 - 4H^2\right). \tag{2.25}$$

$$\Delta = \frac{1}{4}\left(\sum_{i=1}^{4}\frac{H_i - H}{H}\right)^2, \tag{2.26}$$

where H_i denotes the directional Hubble parameters in x, y, z and ψ directions.

3 Cosmological solutions of the field equations

Now the system of equations (2.16)-(2.18) are three independent equations in five unknowns A, C, ρ, λ and φ. However, equation (2.19) being conservation equation. To find a deterministic solution we use the following physically relevant conditions:

(i) First, using well known fact that the shear scalar is proportional to scalar expansion we assume a relation between the metric potentials as follows [6].

$$A = C^n, \tag{3.1}$$

where $n \neq 1$ is a positive constant.
(ii) Also in order to solve the highly non-linear field equations we use the following mathematical condition which several researchers have studied from different aspects of the scalar field φ. [5,20,21],

$$(3n+1)\frac{\dot{C}}{C} = -\frac{\dot{\varphi}}{\varphi}, \tag{3.2}$$

which simplifies the mathematical complexity of the field equations.

From equation (2.20), (3.1) and (3.2), we obtain

$$\varphi = \exp\left(\varphi_0 t - \frac{M^2 t^2}{2} + \varphi_1\right), \tag{3.3}$$

where φ_0 and φ_1 are constants of integration.

Now using equation (3.1), (3.2) and (3.3), we obtain

$$A = B = \exp n\left(\frac{\frac{M^2t^2}{2} - \varphi_0 t - \varphi_1}{3n+1}\right), \tag{3.4}$$

$$C = \exp\left(\frac{\frac{M^2t^2}{2} - \varphi_0 t - \varphi_1}{3n+1}\right). \tag{3.5}$$

Using equations (3.4) to (3.5) in equation (2.1), we obtain five dimensional LRS Bianchi typeV model in the presence of string source and with massive scalar field given by equation (3.3).

$$\begin{aligned} ds^2 = dt^2 &- \exp 2n\left(\frac{\frac{M^2t^2}{2} - \varphi_0 t - \varphi_1}{3n+1}\right)dx^2 - \left[\exp 2n\left(\frac{\frac{M^2t^2}{2} - \varphi_0 t - \varphi_1}{3n+1}\right)\right]\left(e^{2x}dy^2 + e^{2x}dz^2\right) \\ &- \exp 2\left(\frac{\frac{M^2t^2}{2} - \varphi_0 t - \varphi_1}{3n+1}\right)d\psi^2. \end{aligned} \tag{3.6}$$

4 Cosmological Parameters

In this section, we obtained the following kinematical and physical parameters for the model (3.6) which are important in discussion of cosmology.

The average Hubble parameter is given as

$$H = \frac{1}{4}\left(M^2 t - \varphi_0\right). \tag{4.1}$$

The expansion scalar is

$$\theta = \left(M^2 t - \varphi_0\right). \tag{4.2}$$

The deceleration parameter for our model is given as

$$q = -\left[1 + \frac{4M^2}{(M^2 t - \varphi_0)}\right]. \tag{4.3}$$

The spatial volume is

$$V = \exp\left(\frac{M^2t^2}{2} - \varphi_0 t - \varphi_1\right). \tag{4.4}$$

The shear scalar of the model is

$$\sigma^2 = \frac{3}{8}\left(\frac{n-1}{3n+1}\right)^2\left(M^2 t - \varphi_0\right)^2. \tag{4.5}$$

The average anisotropy parameter is

$$\Delta = 3\left(\frac{n-1}{3n+1}\right)^2. \tag{4.6}$$

From equation (2.16) the energy density (ρ) for the model (3.6) is given by

$$\rho = \frac{3n(n+1)\left(M^2 t - \varphi_0\right)^2}{(3n+1)^2} - 3e^{\frac{n\left(2\varphi_0 t - M^2t^2 + 2\varphi_1\right)}{(3n+1)}} - \left[\frac{\left(\varphi_0 - M^2 t\right)^2 + M^2}{2}\right]e^{\left(2\varphi_0 t - M^2t^2 + 2\varphi_1\right)}. \tag{4.7}$$

From equation (2.18) the string density (λ) for the model (3.6) is obtained as

$$\lambda = \frac{3nM^2}{3n+1} + \frac{6n^2\left(M^2 t - \varphi_0\right)^2}{(3n+1)^2} - 3e^{\frac{n\left(2\varphi_0 t - M^2t^2 + 2\varphi_1\right)}{(3n+1)}} + \left[\frac{\left(\varphi_0 - M^2 t\right)^2 - M^2}{2}\right]e^{\left(2\varphi_0 - M^2t^2 + 2\varphi_1\right)}.$$

From equations (2.8), (4.7) and (4.8) the particle density (ρ_p) for the model (3.6) is given by

$$\rho_p = \frac{3n(1-n)\left(M^2 t - \varphi_0\right)^2}{(3n+1)^2} - \frac{3nM^2}{3n+1} - \left(\varphi_0 - M^2 t\right)^2 e^{\left(2\varphi_0 t - M^2t^2 + 2\varphi_1\right)}. \tag{4.8}$$

5 Physical discussion of the model

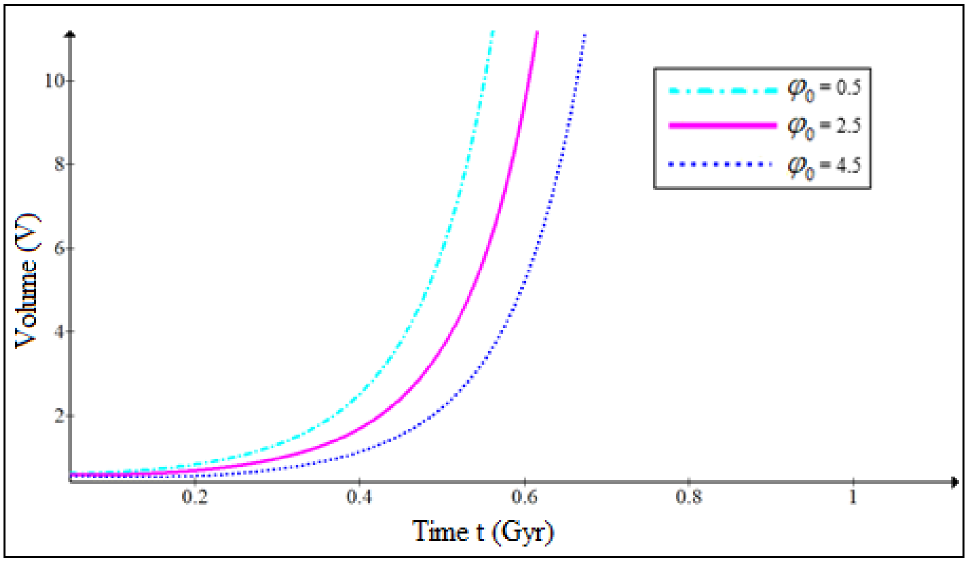

***Figure* 5.1:** Plot of volume versus time for $\varphi_1 = 0.5, n = 0.9$ and $M = 4.5$

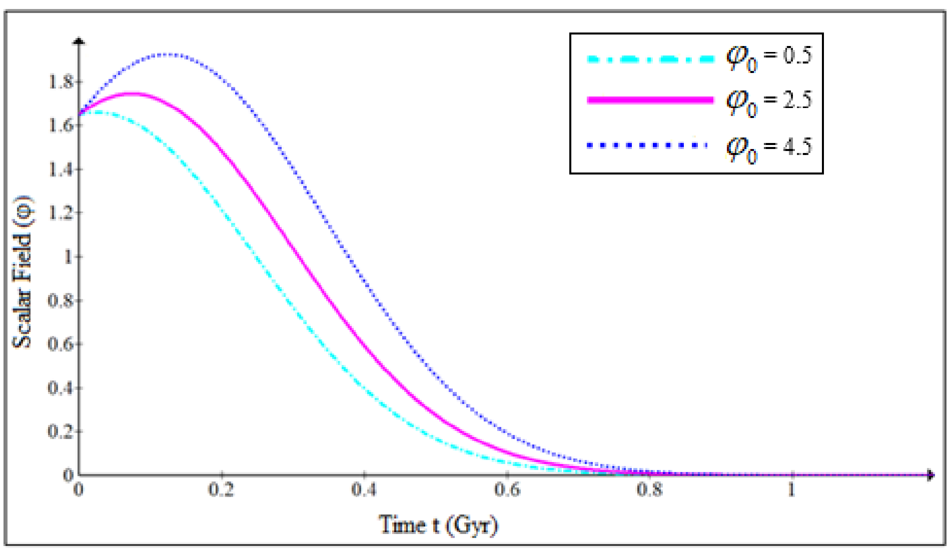

***Figure* 5.2:** Plot of scalar field versus time for $\varphi_1 = 0.5, n = 0.9$ and $M = 4.5$

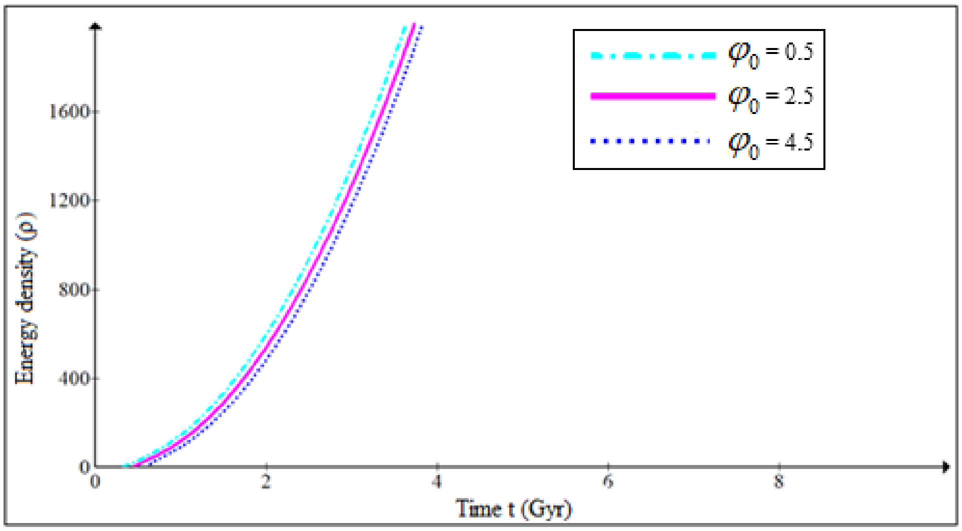

***Figure* 5.3:** Plot of energy density versus time for $\varphi_1 = 0.5, n = 0.9$ and $M = 4.5$

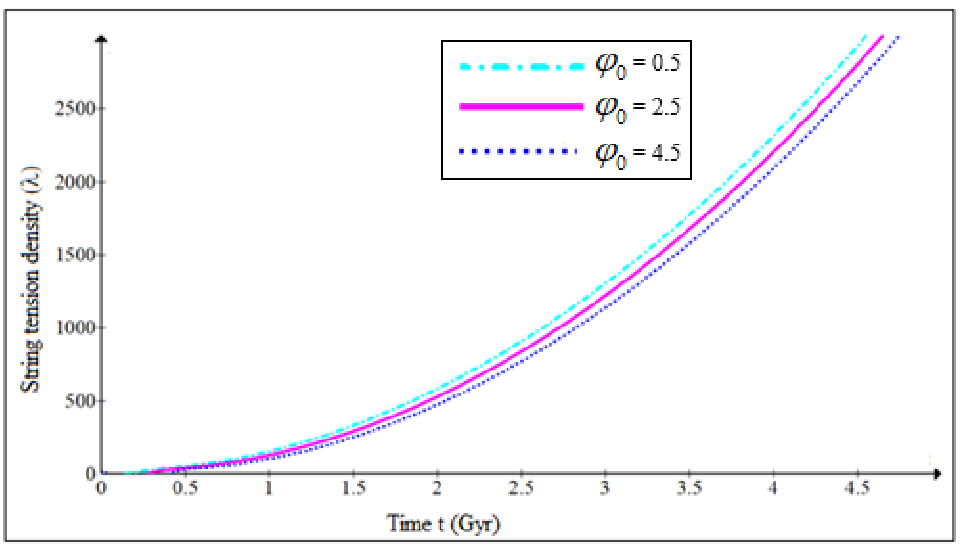

***Figure* 5.4:** Plot of string tension density versus time for $\varphi_1 = 0.5, n = 0.9$ and $M = 4.5$

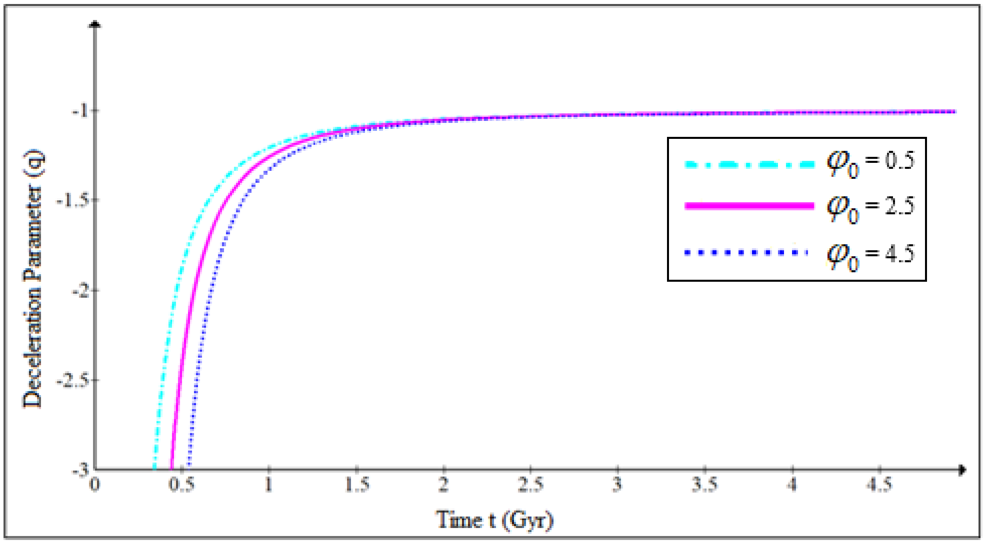

***Figure* 5.5:** Plot of deceleration parameter versus time for $\varphi_1 = 0.5$ and $M = 4.5$

Fig.5.1 depicts the behavior of the volume versus cosmic timet. For our model, it has been found that the spatial volume increases exponentially with time from a finite volume and attains infinite value as time $t \to \infty$.

Fig. 5.2 describes the behavior of scalar field versus time. It can be seen that scalar field is positive and decreasing function of cosmic timet. The behavior of scalar field of our model is quite similar to the scalar field shown in the model constructed by [18, 19].

The behavior of energy density versus cosmic time for various values of φ_0 is depicted in Fig.5.3. It is observed that the energy density is always positive throughout the evolution and is increasing function of cosmic time t. The realistic energy conditions, $\rho \geq 0$ and $\rho_p \geq 0$ are satisfied in our model.

Fig.5.4 exhibits the behavior of string tension density versus cosmic timet . The string tension density λ is positive throughout the evolution of the model and increases with cosmic time t. This behavior of string tension density is quite different from the behavior of string tension density obtained in string cosmological model by Raju et al. [19].

In Fig.5.5 we depicted the behavior of deceleration parameter versus cosmic timet . One of the important physical quantity is deceleration parameter which shows whether the universe is accelerating or decelerating. For our model, we observe that initially since q is less than -1 , hence we obtain a universe with super exponential expansion and finally it approaches to $q = -1$, hence we obtain a universe with exponential expansion.

6 Conclusion

In this paper, we have discussed the dynamical aspects of the five-dimensional LRS Bianchi type-V string cosmological model with massive scalar field. It is noteworthy that the results obtained for our model resemble with the result obtained by Raju et al.[19] except with the behavior of string tension density (λ). Here we can observe that for the specific case $n = 1$, the anisotropy parameter and the shear scalar vanish and therefore the universe is isotropic and shear free for the model. In addition, because the average anisotropic parameter is constants, the universe is anisotropic throughout the evolution, except when $n = 1$.

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

OBSERVATION ON THE BIQUADRATIC EQUATION WITH FIVE UNKNOWNS $2(x-y)(x^3+y^3)+x^4-y^4=2(z^2-w^2)p^2$

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Abstract

This paper focuses on obtaining non-zero integer quintuples (x, y, z, w, p) satisfying the bi-quadratic equation with five unknowns given by $2(x-y)(x^3+y^3)+x^4-y^4 = 2(z^2-w^2)p^2$. Various patterns of solutions are obtained by reducing the given bi-quadratic equation to solvable ternary quadratic equation through employing linear transformations.
2020 Mathematical Sciences Classification: 11D25.
Keywords and Phrases: homogeneous bi-quadratic, quinary bi-quadratic, integer solutions.

1 Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular biquadratic Diophantine equation, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians, since antiquity. In this context , one may refer [1-31] for various problems on biquadratic equations with three ,four and five variables. This paper concerns with yet another problem of determining non-trivial integral solutions on the biquadratic equation with five unknowns given by $2(x-y)(x^3+y^3)+x^4-y^4 = 2(z^2-w^2)p^2$.

2 Method of Analysis

The Diophantine equation representing the biquadratic equation under consideration with five unknowns is given by

$$2(x-y)(x^3+y^3)+x^4-y^4 = 2(z^2-w^2)p^2. \tag{2.1}$$

Introducing the linear transformations

$$x = u+v, y = u-v, z = 2u+v, w = 2u-v \tag{2.2}$$

in (2.1) , we get

$$u^2+2v^2 = p^2. \tag{2.3}$$

The above equation (2.3) is solved through different methods and thus, one obtains different patterns of distinct integer solutions to (2.1).

3 Patterns

Pattern 3.1

The most cited solutions to (2.3) are

$$v = 2rs, u = 2r^2-s^2 \tag{3.1}$$

and

$$p = 2r^2+s^2. \tag{3.2}$$

Using (3.1) in (2.2) , we get

$$x = 2r^2-s^2+2rs, y = 2r^2-s^2-2rs, z = 4r^2-2s^2+2rs, w = 4r^2-2s^2-2rs. \tag{3.3}$$

Thus,(3.2) and (3.3) give the integer solutions to (2.1).

Pattern 3.2
Rewrite (2.3) as

$$u^2 + 2v^2 = p^2 * 1. \tag{3.4}$$

Assume

$$p = a^2 + 2b^2. \tag{3.5}$$

Express integer 1 on the R.H.S of (3.4) as the product of complex conjugates as

$$1 = \frac{(1 + i2\sqrt{2})(1 - i2\sqrt{2})}{9}. \tag{3.6}$$

Substituting (3.5) and (3.6) in (3.4) and employing the method of factorization we define

$$u + i\sqrt{2}v = \frac{(a + i\sqrt{2}b)^2(1 + i2\sqrt{2})}{3},$$

from which, on equating the real and imaginary parts, we obtain

$$u = \frac{(a^2 - 2b^2 - 8ab)}{3}, v = \frac{(2a^2 - 4b^2 + 2ab)}{3}. \tag{3.7}$$

As our interest is on finding integer solutions, replacing a by $3A$ and b by $3B$ in (3.5) and (3.7) and in view of (2.2) , the corresponding integer solutions to (2.1) are given by

$$x = 3(3A^2 - 6B^2 - 6AB), y = 3(-A^2 + 2B^2 - 10AB),$$
$$z = 3(4A^2 - 8B^2 - 14AB), w = -54AB, p = 9(A^2 + 2B^2).$$

Observation 3.2.1
Apart from (3.6), the integer 1 on the R.H.S. of (3.4) is expressed as

$$1 = \frac{(7 + i6\sqrt{2})(7 - i6\sqrt{2})}{121}.$$

For this choice ,the corresponding integer solutions to (2.1) are obtained as

$$x = 11(13A^2 - 26B^2 - 10AB), y = 11(A^2 - 2B^2 - 38AB)$$
$$z = 11(20A^2 - 40B^2 - 34AB), w = 11(8A^2 - 16B^2 - 62AB), p = 11(11A^2 + 22B^2).$$

Observation 3.2.2
It is worth to mention here that the integer 1 on the R.H.S. of (3.4) may be written in the general form as

$$1 = \frac{(2p^2 - q^2 + i\sqrt{22}pq)(2p^2 - q^2 - i\sqrt{22}pq)}{(2p^2 + q^2)^2}.$$

The repetition of the above process leads to a set of integer solutions to (2.1).

Pattern 3.3
Consider (2.3) as

$$p^2 - 2v^2 = u^2 * 1. \tag{3.8}$$

Assume

$$u = a^2 - 2b^2. \tag{3.9}$$

Express integer 1 on the R.H.S of (3.8) as the product of irrational pairs as

$$1 = (3 + 2\sqrt{2})(3 - 2\sqrt{2}). \tag{3.10}$$

Substituting (3.9) and (3.10) in (3.4) and employing the method of factorization, define

$$p + \sqrt{2}v = (a + \sqrt{2}b)^2(3 + 2\sqrt{2}).$$

from which, we get

$$v = 2a^2 + 4b^2 + 6ab \tag{3.11}$$

and

$$p = 3a^2 + 6b^2 + 8ab. \tag{3.12}$$

Using (3.9) and (3.11) in (2.2) , we write

$$x = 3a^2 + 2b^2 + 6ab, y = -a^2 - 6b^2 - 6ab, z = 4a^2 + 6ab, w = -8b^2 - 6ab. \tag{3.13}$$

Thus,(3.12) and (3.13) give the integer solutions to (2.1).

Observation 3.3.1
Apart from (3.10), the integer 1 on the R.H.S. of (3.8) is expressed as

$$1 = \frac{(11+6\sqrt{2})(11-6\sqrt{2})}{49}.$$

For this choice, the corresponding integer solutions to (2.1) are obtained as

$$x = 7(13A^2 - 2B^2 + 22AB), y = 7(A^2 - 26B^2 - 22AB),$$
$$z = 7(20A^2 - 16B^2 + 22AB), w = 7(8A^2 - 40B^2 - 22AB), p = 7(11A^2 + 22B^2 + 24AB).$$

Observation 3.3.2
It is worth to mention here that the integer 1 on the R.H.S. of (3.8) may be written in the general form as

$$1 = \frac{(2p^2 + q^2 + \sqrt{22}pq)(2p^2 + q^2 - \sqrt{22}pq)}{(2p^2 - q^2)^2}.$$

The repetition of the above process leads to a set of integer solutions to (2.1).

4 Conclusion

An attempt has been made to determine many non-zero distinct integer solutions to the considered fourth degree equation with five unknowns in the title. It is worth to mention that, in addition to the transformations presented in (2.2) ,one may also consider the transformations represented by

(4.1) $$x = u + v, y = u - v, z = u + 2v, w = u - 2v,$$

(4.2) $$x = u + v, y = u - v, z = 2uv + 1, w = 2uv - 1.$$

To conclude, the readers of this paper may search for integer solutions to other forms of homogeneous or non-homogeneous quinary bi-quadratic Diophantine equations.

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INTEGER SOLUTION ANALYSIS FOR A DIOPHANTINE EQUATION WITH EXPONENTIALS

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Abstract

The exponential Diophantine equation is one of the distinctive types of Diophantine equations where the variables are expressed as exponents. For these equations, considerable excellent research has already been done. In this study, we try to solve the equations $3^\lambda + 103^\mu = \xi^2$, $3^\lambda + 181^\mu = \xi^2$, $3^\lambda + 193^\mu = \xi^2$.
2020 Mathematical Sciences Classification: 11D09
Keywords and Phrases: Pell equation, Integer Solutions, Diophantine equations, Algebraic Number Theory.Pell equation, Integer Solutions, Diophantine equations, Algebraic Number Theory.

1 Introduction

For Elementry Number Theory, we may refer to [4, 6, 11].
A Diophantine equation is a polynomial equation with two or more unknowns and integer coefficients, with the only interesting solutions being those with integer coefficients. A Diophantine equation is called an exponential Diophantine equation if it contains an additional variable or variables that occur as exponents. There has been done some interested research work on these equation so far [1, 2, 3, 7].

The Exponential Diophantine equation of the form $p^x + q^y = z^2$ where p and q are distinct primes and $x, y$and z are non-negative integers has been the focus of a lot of research in recent years [5,10,12].

Recently, Pandichelvi and Vanaja [8] studied generating diophantine triples relating to figurate numbers with Thought-Provoking property. Also, Pandichelvi and Umamaheshwari [9] studied perceiving solutions for an exponential Diophantine equation linking safe and Sophie Germain primes. In the present paper, we shall discuss Integer solution Analysis for a Diophantine equation with exponentials.

2 Preliminaries

Proposition 2.1 ([6])**.** (3,2,2,3) *is a unique solution* (a, b, λ, μ) *for the Diophantine equation* $a^\lambda - b^\mu = \xi^2$ *where* a, b, λ, μ *are integers such that min* $(a, b, \lambda, \mu) > 1$.

Lemma 2.1. (1,0,2) *is a unique solution for the Diophantine equation* $3^\lambda + 1 = \xi^2$*, where* λ *and* ξ *are the non-negative integers.*

Proof. Let $\lambda, \xi \in \mathbb{N} \cup 0$. If $\lambda = 0$, then $\xi^2 = 2$, is not an integer solution. So consider $\lambda \geq 1$. Then $\xi^2 = 3^\lambda + 1 \geq 4 \Rightarrow \xi \geq 2$.

Consider $\xi^2 - 3^\lambda = 1$ then by proposition 2.1, λ must be equal to 1.Hence $\xi^2 = 4 \Rightarrow \xi = 2$.Therefore $(1, 0, 2)$ is a unique solution for $3^\lambda + 1 = \xi^2$. □

Lemma 2.2. *The Diophantine equation* $1 + (103)^\mu = \xi^2$ *has no non-negative integer solution.*

Proof. If $\mu = 0$, we obtain an irrational solution. So consider $\mu \geq 0$, then $\xi^2 = 1 + 103^\mu \geq 104 \Rightarrow \xi \geq 11$. Therefore the equation $\xi^2 - 103^\mu = 1$ is solvable only if $\mu = 1$, by proposition 2.1. But when $\mu = 1, \xi^2 = 104$,which is not a square number. Therefore there is no non-negative integer solution for the Diophantine equation $1 + 103^\mu = \xi^2$. □

Lemma 2.3. *The Diophantine equation* $1 + 181^\mu = \xi^2$ *has no non-negative integer solution.*

Proof. If $\mu = 0$, we obtain an irrational solution. So consider $\mu \geq 0$, then $\xi^2 = 1 + 181^\mu \geq 182 \Rightarrow \xi \geq 14$. Therefore the equation $\xi^2 - 181^\mu = 1$ is solvable only if $\mu = 1$, by proposition 2.1.But when $\mu = 1, \xi^2 = 182$,which is not a square number. Therefore there is no non-negative integer solution for the Diophantine equation $1 + 181^\mu = \xi^2$. □

Lemma 2.4. *The Diophantine equation* $1 + 193^\mu = \xi^2$ *has no non-negative integer solution.*

Proof. If $\mu = 0$, we obtain an irrational solution. So consider $\mu \geq 0$, then $\xi^2 = 1 + 193^\mu \geq 194 \Rightarrow \xi \geq 14$. Therefore the equation $\xi^2 - 193^\mu = 1$ is solvable only if $\mu = 1$, by proposition 2.1.But when $\mu = 1, \xi^2 = 194$,which is not a square number. Therefore there is no non-negative integer solution for the Diophantine equation $1 + 193^\mu = \xi^2$. □

3 Main Results

Theorem 3.1. $(1, 0, 2)$ *is a unique solution for the Diophantine equation* $3^\lambda + 103^\mu = \xi^2$*, where* λ, μ *and* $\xi \in \mathbb{N} \cup 0$ *.*

Proof. Consider the integral value of μ into two cases as (i) μ is even and (ii) μ is odd. □

Case (i): Let μ be even. If $\mu = 0$, then by Lemma 2.1 $(1, 0, 2)$ is a solution. If $\mu = 2n, n \in \mathbb{N}$, the equation becomes $3^\lambda + 103^{2n} = \xi^2$.This can be written as $\xi^2 - 103^{2n} = 3^\lambda$.

$$\Rightarrow (\xi + 103^n)(\xi - 103^n) = 3^{\alpha+\beta}, \text{ where } \alpha + \beta = \lambda$$
$$\Rightarrow (\xi + 103^n) - (\xi - 103^n) = 3^\beta - 3^\alpha, \beta > \alpha$$
$$\Rightarrow 2(103^n) = 3^\alpha(3^{\beta-\alpha} - 1) .$$

$\alpha = 0$ is the only possible value. Therefore the equation becomes $2(103)^n = 3^\lambda - 1$. Adding -2 on both sides we obtain $-2 + 2(103^n) = 3^\lambda - 3$.This gives that $\lambda = 2$.
Thus $103^n = 4$, which is impossible.
Case (ii): Let μ be odd. Then $\mu = 2n + 1, n \in \mathbb{N} \cup 0$.Therefore the equation becomes $3^\lambda + 103^{2n+1} = \xi^2$.

$$\Rightarrow 3^\lambda + 103(103)^{2n} = \xi^2.$$
$$\Rightarrow 3^\lambda + 3(103)^{2n} = \xi^2 - 10^2(103)^{2n}.$$
$$\Rightarrow 3(3^{(\lambda-1)} + (103)^{2n}) = (\xi + 10(103)^n))(\xi - 10(103)^n).$$

We observe that ξ is even. i.e. the equation above can be written as

$$3(3^{\lambda-1} + ((103)^{2n}) = (2m + 10(103)^n))(2m - 10(103)^n)),$$
$$= 4(m + 5(103)^n)(m - 5(103)^n).$$

Now we have two possibilities: (i) $m = 3 + 5(103)^n$ (ii) $m = 3 - 5(103)^n$.
Subcase (i): $m = 3 + 5(103)^n$.
Then $3(3^{\lambda-1} + (103)^{2n})) = 4(3 + 5(103)^n + 5(103)^n)(3 + 5(103)^n - 5(103)^n)$.
$= 4(3)[3 + 10(103)^n)]$
$\Rightarrow 3^{\lambda-1} + 103^{2n} = 4(3 + 10(103)^n)$
$\Rightarrow 3^{\lambda-1} - 12 = 40(103^n) - 103^{2n}$
$\Rightarrow 3(3^{\lambda-2} - 4) = 103^n(40 - 103^n)$.
$n = 0$ is the only possible value. But $3^{\lambda-1} = 51$ is not solvable.
Subcase (ii): $m = 3 - 5(103)^n$.
Then $3(3^{\lambda-1} + 103^{2n}) = 4(3 - 5(103)^n + 5(103)^n)(3 - 5(103)^n - 5(103)^n)$
$= 4(3)[3 - 10(103)^n]$.
$\Rightarrow 3^{\lambda-1} + 103^{2n} = 4(3 - 10(103)^n)$
$\Rightarrow 3^{\lambda-1} - 12 = -40(103^n) - (103^{2n})$
$\Rightarrow 3(3^{\lambda-2} - 4) = -103^n(40 + (103^n))$.
$n = 0$ is the only possible value. But $3^{\lambda-1} = -29$ is not solvable.
Thus in any cases, we are never able to come up with a non-negative integral solution. Therefore $(1, 0, 2)$ is a unique solution for the Diophantine equation $3^\lambda + 103^\mu = \xi^2$.

Theorem 3.2.$(1,0,2)$ *is a unique solution for the Diophantine equation* $3^\lambda + 181^\mu = \xi^2$*, where* λ, μ *and* $\xi \in \mathbb{N} \cup 0$.

Proof. Consider the integral value of μ into two cases as (i) μ is even and (ii) μ is odd.

Case (i): Let μ be even. If $\mu = 0$, then by Lemma 2.2 $(1,0,2)$ is a solution. If $\mu = 2n, n \in \mathbb{N}$, the equation becomes $3^\lambda + 181^{2n} = \xi^2$.This can be written as $\xi^2 - 181^{2n} = 3^\lambda$.

$$\Rightarrow (\xi + 181^n)(\xi - 181^n) = 3^{\alpha+\beta}, \text{ where } \alpha + \beta = \lambda$$
$$\Rightarrow (\xi + 181^n) - (\xi - 103^n) = 3^\beta - 3^\alpha, \beta > \alpha$$
$$\Rightarrow 2(181^n) = 3^\alpha(3^{\beta-\alpha} - 1) \ .$$

$\alpha = 0$ is the only possible value. Therefore the equation becomes $2(181)^n = 3^{\lambda-1}$. Adding -2 on both sides we obtain $-2 + 2(181^n) = 3^\lambda - 3$.This gives that $\lambda = 2$.
Thus $181^n = 4$, which is impossible.

Case (ii): Let μ be odd. Then $\mu = 2n+1, n \in \mathbb{N} \cup 0$.Therefore the equation becomes $3^\lambda + 181^{2n+1} = \xi^2$.

$$\Rightarrow 3^\lambda + 181(181)^{2n} = \xi^2.$$
$$\Rightarrow 3^\lambda + 3^4(181)^{2n} = \xi^2 - 10^2(181)^{2n}.$$
$$\Rightarrow 3^u(3^{\lambda-u} + 3^{4-u}(181)^{2n}) = (\xi + 10(181)^n))(\xi - 10(181)^n).$$

We observe that ξ is even. i.e. the equation above can be written as

$$3^u(3^{\lambda-u} + 3^{4-u}(181)^{2n}) = (2m + 10(181)^n))(2m - 10(181)^n)).$$
$$= 4(m + 5(181)^n)(m - 5(181)^n).$$

Now we have two possibilities: (i) $m = 3^u + 5(181)^n$ (ii) $m = 3^u - 5(181)^n$.

Subcase (i): $m = 3^u + 5(181)^n$.
Then $3^u(3^{\lambda-u} + 3^{4-u}(181)^{2n}) = 4(3^u + 5(181)^n + 5(181)^n)(3^u + 5(181)^n - 5(181)^n)$.
$= 4(3^u)[3^u + 10(181^n)]$.
$\Rightarrow (3^{\lambda-u} + 3^{4-u}(181^{2n}) = 4(3^u + 10(181^n))$.
$\Rightarrow 3^{4-u}(181^{2n}) - 40(181^n) = 3^u(4 - 3^{\lambda-2u})$
$\Rightarrow 181^n(3^{4-u}(181^n) - 40) = 4.3^u - 3^{\lambda-u}$.
$n = 0$ is the only possible value.
$\Rightarrow 3^{4-u} - 40 = 4.3^u - 3^{\lambda-u}$
$\Rightarrow 3^4 - 40(3^u) = 3^{2u}(4 - 3^{\lambda-2u})$
$\Rightarrow 4((3^{2u} + 10(3^u) - 20) = 3^\lambda + 1$,which is not a viable solution.

Subcase (ii): $m = 3^u - 5(181)^n$.
Then $3^u(3^{\lambda-u} + 3^{4-u}(181)^{2n}) = 4(3^u - 5(181)^n + 5(181)^n)(3^u - 5(181)^n - 5(181)^n)$.
$= 4(3^u)[3^u - 10(181^n)]$.
$\Rightarrow (3^{\lambda-u} + 3^{4-u}(181^{2n}) = 4(3^u - 10(181^n))$
$\Rightarrow 3^{4-u}(181^{2n}) + 40(181^n) = 3^u(4 - 3^{\lambda-2u})$
$\Rightarrow 181^n(3^{4-u}(181^n) + 40) = 4.3^u - 3^{\lambda-u}$.
$n = 0$ is the only possible value.
$\Rightarrow 3^{4-u} + 40 = 4.3^u - 3^{\lambda-u}$
$\Rightarrow 3^4 + 40(3^u) = 3^{2u}(4 - 3^{\lambda-2u})$
$\Rightarrow 4((3^{2u} - 10(3^u) - 20) = 3^\lambda + 1$,which is not a viable solution.

Thus in any cases, we are never able to come up with a non-negative integral solution. Therefore $(1,0,2)$ is a unique solution for the Diophantine equation $3^\lambda + 181^\mu = \xi^2$.

Theorem 3.2. $(1,0,2)$ *is a unique solution for the Diophantine equation* $3^\lambda + 193^\mu = \xi^2$*, where* λ, μ *and* $\xi \in \mathbb{N} \cup 0$.

Proof. Consider the integral value of μ into two cases as (i) μ is even and (ii) μ is odd. □

Case (i): Let μ be even. If $\mu = 0$, then by Lemma 2.1 $(1,0,2)$ is a solution. If $\mu = 2n, n \in \mathbb{N}$, the equation becomes $3^\lambda + 193^{2n} = \xi^2$.This can be written as $\xi^2 - 193^{2n} = 3^\lambda$.

$$\Rightarrow (\xi + 193^n)(\xi - 193^n) = 3^{\alpha+\beta}, \text{ where } \alpha + \beta = \lambda$$
$$\Rightarrow (\xi + 193^n) - (\xi - 193^n) = 3^\beta - 3^\alpha, \beta > \alpha$$
$$\Rightarrow 2(193^n) = 3^\alpha(3^{\beta-\alpha} - 1) \ .$$

$\alpha = 0$ is the only possible value. Therefore the equation becomes $2(193)^n = 3^{\lambda-1}$. Adding -2 on both sides we obtain $-2 + 2(193^n) = 3^\lambda - 3$.This gives that $\lambda = 2$.
Thus $193^n = 4$, which is impossible.
Case (ii): Let μ be odd. Then $\mu = 2n+1, n \in \mathbb{N} \cup 0$.Therefore the equation becomes $3^\lambda + 193^{2n+1} = \xi^2$.

$$\Rightarrow 3^\lambda + 193(193)^{2n} = \xi^2.$$
$$\Rightarrow 3^\lambda + 3(193)^{2n} = \xi^2 - 14^2(103)^{2n}.$$
$$\Rightarrow 3(3^{(\lambda-1)} + (193)^{2n}) = (\xi + 14(103)^n))(\xi - 14(103)^n).$$

We observe that ξ is even. i.e. the equation above can be written as

$$3(3^{\lambda-1} + ((193)^{2n}) = (2m + 14(193)^n))(2m - 14(193)^n)).$$
$$= 4(m + 7(193)^n)(m - 7(193)^n).$$

Now we have two possibilities: (i) $m = 3 + 7(193)^n$ (ii) $m = 3 - 7(193)^n$.
Subcase (i): $m = 3 + 7(193)^n$.
Then $3(3^{\lambda-1} + (193)^{2n})) = 4(3 + 7(193)^n + 7(193)^n)(3 + 7(193)^n - 7(193)^n)$.
$= 4(3)[3 + 14(193)^n)]$
$\Rightarrow 3^{\lambda-1} + 193^{2n} = 4(3 + 14(193)^n)$
$\Rightarrow 3^{\lambda-1} - 12 = 56(193^n) - 193^{2n}$
$\Rightarrow 3(3^{\lambda-2} - 4) = 193^n(56 - 193^n)$.
$n = 0$ is the only possible value. But $3^{\lambda-1} = 69$ is not solvable.
Subcase (ii): $m = 3 - 7(193)^n$.
Then $3(3^{\lambda-1} + 193^{2n}) = 4(3 - 7(193)^n + 7(193)^n)(3 - 7(193)^n - 7(193)^n)$.
$= 4(3)[3 - 14(193)^n]$.
$\Rightarrow 3^{\lambda-1} + 193^{2n} = 4(3 - 14(193)^n)$
$\Rightarrow 3^{\lambda-1} - 12 = -56(193^n) - (193^{2n})$
$\Rightarrow 3(3^{\lambda-2} - 4) = 193^n(-56 + (193^n))$.
$n = 0$ is the only possible value. But $3^{\lambda-1} = -43$ is not solvable.
Thus in any cases, we are never able to come up with a non-negative integral solution. Therefore $(1, 0, 2)$ is a unique solution for the Diophantine equation $3^\lambda + 193^\mu = \xi^2$.

Corollary 3.1. *The Diophantine equation* $3^\lambda + 103^\mu = \psi^4$*has no non-negative integer solution.*

Proof. λ, μ and ψ be non-negative integers such that $3^\lambda + 103^\mu = \psi^4$. Let $\xi = \psi^2$. Then by theorem 3.1, $3^\lambda + 103^\mu = \xi^2$ has a unique solution $(1, 0, 2)$.
That is $\psi^2 = 2 \Rightarrow \psi = \sqrt{2}$, which is impossible as $\psi \in \mathbb{N} \cup 0$. Therefore $3^\lambda + 103^\mu = \psi^4$ has no non-negative integer solution. ☐

Corollary 3.2. Corollary 3.2. *The Diophantine equation* $9^\psi + 103^\mu = \xi^2$ *has no non-negative integer solution.*

Proof. μ, ψ, and ξ be non-negative integers such that $9^\psi + 103^\mu = \xi^2$. Let $\lambda = 2\psi$. Then by theorem 3.1, $3^\lambda + 103\mu = \xi^2$ has a unique solution $(1, 0, 2)$, which shows that $\lambda = 2 \Rightarrow \psi = \frac{1}{2} \in \mathbb{Q}$. Therefore $9^\psi + 103^\mu = \xi^2$ has no non-negative integer solution.
Corollary 3.3.*The Diophantine equation* $3^\lambda + 181^\mu = \psi^4$*has no non-negative integer solution.*
Proof. λ, μ and ψ be non-negative integers such that $3^\lambda + 181^\mu = \psi^4$. Let $\xi = \psi^2$. Then by theorem 3.2, $3^\lambda + 181^\mu = \xi^2$ has a unique solution $(1, 0, 2)$.
That is $\psi^2 = 2 \Rightarrow \psi = \sqrt{2}$, which is impossible as $\psi \in \mathbb{N} \cup 0$. Therefore $3^\lambda + 181^\mu = \psi^4$ has no non-negative integer solution. ☐

Corollary 3.3. *The Diophantine equation* $9^\psi + 181^\mu = \xi^2$ *has no non-negative integer solution.*

Proof. μ, ψ, and ξ be non-negative integers such that $9^\psi + 181^\mu = \xi^2$. Let $\lambda = 2\psi$. Then by theorem 3.2, $3^\lambda + 181\mu = \xi^2$ has a unique solution $(1, 0, 2)$, which shows that $\lambda = 2 \Rightarrow \psi = \frac{1}{2} \in \mathbb{Q}$. Therefore $9^\psi + 181^\mu = \xi^2$ has no non-negative integer solution.
Corollary 3.5.*The Diophantine equation* $3^\lambda + 193^\mu = \psi^4$*has no non-negative integer solution.*

Proof. λ, μ and ψ be non-negative integers such that $3^\lambda + 193^\mu = \psi^4$. Let $\xi = \psi^2$. Then by theorem 3.3, $3^\lambda + 193^\mu = \xi^2$ has a unique solution $(1, 0, 2)$.
That is $\psi^2 = 2 \Rightarrow \psi = \sqrt{2}$, which is impossible as $\psi \in \mathbb{N} \cup 0$. Therefore $3^\lambda + 193^\mu = \psi^4$ has no non-negative integer solution. $\square$

Corollary 3.4. *The Diophantine equation* $9^\psi + 193^\mu = \xi^2$ *has no non-negative integer solution.*

Proof. μ, ψ, and ξ be non-negative integers such that $9^\psi + 193^\mu = \xi^2$. Let $\lambda = 2\psi$. Then by theorem 3.3, $3^\lambda + 193\mu = \xi^2$ has a unique solution $(1, 0, 2)$, which shows that $\lambda = 2 \Rightarrow \psi = \frac{1}{2} \in \mathbb{Q}$. Therefore $9^\psi + 193^\mu = \xi^2$ has no non-negative integer solution. $\square$

4 Conclusion

. In this paper, we have shown the solutions of the Diophantine equations of several primes. One can find the solutions of the Exponential Diophantine equations using other primes.

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SECURITY OF PUBLIC KEY ENCRYPTION USING DICKSON POLYNOMIALS OVER FINITE FIELD WITH 2^k

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Abstract

The application of Dickson polynomial in public key cryptography is observed due to its permutation behaviors and semi-group property under composition. Here we have mostly concentrated on checking the one-wayness and semantic security of our scheme. The proposed scheme is based on Dickson polynomial over a finite field with 2^k, whose security depends on the Integer Factorization Problem(IFP) and the Discrete Dickson Problem(DDP), which is as difficult as solving discrete logarithmic Problem (DLP). Our proposed cryptosystem is computationally secured with one-wayness and semantic security, it also reduces the complexity of many other proposed schemes.
2020 Mathematical Sciences Classification: 94A60, 11T06
Keywords and Phrases: Dickson Polynomial, Integer Factorization Problem, Discrete Dickson Problem, Discrete logarithm Problem, Encryption Scheme.

1 Introduction

Diffie and Hellman [5] in 1976 firstly proposed a cryptosystem, where transmission of messages takes place in an open network, known as Public Key Cryptography or asymmetric cryptosystem. In symmetric cryptosystem, the transmission of the secret key is done over an insecure channel and hence it is of higher insecurity. However, in asymmetric cryptosystem, separate keys are being used for encryption and decryption and hence it overcomes the insecurity problem. Security is of key importance in cryptography, as it is on which the proposed cryptosystem depends on.

Various parameters including number theory, group theory, field theory, braid group[21] and many others were involved to propose numerous cryptosystem to improve the security & efficiency and hence also came the involvement of Dickson polynomials too for the preparation of a more computationally secured and efficient cryptosystem. The application of Dickson polynomial in public key cryptography[10, 11, 12, 13] was involved due to its permutation behaviors and semi-group property under composition. It gave the researchers a new direction in cryptography. Dickson polynomial was firstly introduced by Dickson [4] in 1896, but it was later named by Schur[23] as Dickson polynomial. Lidl [13], in his paper have also surveyed the algebraic properties of Dickson polynomial over $\mathbb{F}_q$ and over the integers $\mathbb{Z}_n$, which helped to found the better way of its application in public key cryptography.

If prior knowledge of the hard problems is known only then it can be solved both ways, based on which most of the cryptographic schemes are being developed. Discrete logarithm and factoring of a large composite number in terms of primes, taken only one hard problem at a time were initially the hard problems that were being used includes for the propose of schemes. In 1988, two different number theoretic assumption were involved in the development of a single key distribution protocol by McCurley [17]. Numerous cryptosystem were proposed in the later year by [2, 6, 7, 8, 9, 18, 20, 24, 25, 26] which were based on the merging of two hard problems such as Discrete logarithm and factoring of a large composite number, Elliptic Curve discrete Logarithm, knapsack problem, and many more.

Here Discrete Logarithm and Integer Factorization is operated on Discrete Dickson Problem(DDP) over the finite field with cardinality 2^k and proposed a cryptosystem whose security is based on the hardness of solving *IFP* and *DDP*. Our work is mainly based on checking the one-wayness and semantic security of the scheme.

The rest of the paper is started with defining dickson polynomial, then followed by the security of our proposed cryptosystem which involved the One-way security and semantic security and finally the conclusion.

2 Dickson Polynomial

In 1896, Dickson [4] introduced a type of polynomial of the form

$$y^k + k\sum_{i=1}^{(k-1)/2} \frac{(k-i-1)...(k-2i+1)}{2.3...i} a^i y^{k-2i},\ k \text{ is odd},$$

over finite field F_q, which later came to be known as Dickson polynomial by Schur[23].

Definition 2.1. *(Dickson polynomial of first kind)* ([27]). *Let N be a positive integer and $a \in \mathbb{F}_q$,then the Dickson polynomial $D_N(y,a)$ of the first kind over any finite field F_q is defined by*

$$D_N(y,a) = \sum_{i\ =\ 0}^{\lfloor \frac{N}{2} \rfloor} \frac{N}{N-i}\binom{N-i}{i}(-a)^i y^{N-2i},$$

where $\lfloor \frac{N}{2} \rfloor$ is the largest integer less than or equal to $\frac{n}{2}$.

The Dickson polynomial satisfy the resurrence relation : $D_N(y,a) = yD_{N-1}(y,a) - aD_{N-2}(y,a)$, $N \geq 2$. under the initial condition $D_0(y,a) = 2$ and $D_1(y,a) = y$ and few initial polynomial are given below:

$D_2(y,a)= y^2 - 2a$,
$D_3(y,a)=y^3 - 3ay$,
$D_4(y,a)=y^4 - 4ay^2 + 2a^2$,
$D_5(y,a)=y^5 - 5ay^3 + 5a^2y$.

Commutativity under composition is of considerable importance satisfied by Dickson polynomial for $a = 0$ or 1 [19] *and hence it satisfies the semi-group property under composition:*

$$D_{MN}(y,1) = D_M(D_N(y,1),1) = D_M(y,1) \circ D_N(y,1) = D_N(y,1) \circ D_M(y,1) = D_{NM}(y,1)$$

Definition 2.2 (Modified Dickson Polynomial)**.** *Let us define a map, $D_P : \mathbb{Z}_N \to \mathbb{Z}_N$ defined as $z = D_P(y)(modN)$, where y and N are positive integers. Here, we call $z = D_P(y)(modN)$ as the modified Dickson polynomial. Below are few properties satisfied by modified dickson polynomial.*

1. *Modified Dickson polynomial is commutative under composition, that is*

$$D_P(D_Q(y)(modN)) = D_{PQ}(y)(modN) = D_Q(D_P(y)(modN)).$$

2. *Let Q be an odd prime and let $y \in \mathbb{Z}$ such that $0 \leq y < Q$. Then the period of the sequence $D_N(y)(mod\ Q)$ for $N = 0, 1, 2, 3, 4, ...$ is a divisor of $Q^2 - 1$.*

Müller and Nöbauer [19] *in* 1981, *firstly introduced the first key exchange cryptosystem which was based on Dickson polynomial, where the power functuions of the RSA system, introduced by Rivest et al.*[22] *in* 1978, *was replaced by Dickson ploynomials $D_n(x,a)$ with parameter $a = -1, 0, 1$. It was also observed the RSA cryptosystem was equivalent to Dickson system for parameter $a = 0$*[19] . *In* 2011, *Wei* [27] *introduced in his paper that Dickson polynomial $D_n(x,1)$ over a finite field 2^m is a permutation polynomial if and only if n is odd and proved that solving a discrete Dickson problem(DDP) is as difficult as solving discrete logarithmic problem (DLP). Note that, The hardness of DLP was also observed by McCurley* [16] *in his paper. It is also observed that, computable groups where DLP is hard to solve* [1, 3, 13] *are of very importance in cryptography.*

Definition 2.3 (Discrete Dickson Problem)**.** *Let R be a commutative ring with unity, for any $n \in \mathbb{Z}^+$, and given y and x , the problem of calculating the value of n such that $y = D_n(x,1)$ is called the Discrete Dickson Problem(DDP).*

It is observed throughout the paper that we have used for $a = 1$, $D_n(x,1) = D_n(x)$.

3 Security of the proposed public key encryption scheme

For the completeness of our work, here we have included our proposed public key encryption scheme. The scheme consists of three parts, that includes key generation, encryption and decryption.

Key Generation

1. Choose two random large primes P and Q of the same size, such that $2^P - 1$ and $2^Q - 1$ is prime.

2. Using the above P and Q compute $N = 2^k$, where $k = P \times Q$.
3. For the value of N, find $\phi(N)$, where $\phi(N) = (2^{2P} - 2^{P+1}) \times (2^{2Q} - 2^{Q+1})$.
4. Choose f, such that $1 < f < \phi(N)$ and $gcd(f, \phi(N)) = 1$.
5. Find c, such that $cf \equiv 1 \pmod{\phi(N)}$, where c is the modular inverse of f.
6. Choose h, such that $0 \leq h \leq \phi(N)$ - 1.
7. Choose a random $\alpha \in \mathbb{Z}_N^*$ and compute $y = \frac{1}{2} D_h(2\alpha) \pmod{N}$.

- ***PUBLIC KEY***: (N, f, y, α),
- ***PRIVATE KEY***: (P, Q, h, c).

Encryption

Here the process of encrypting the simple plain text into cipher text is permormed, so that an intruder doesn't get to read the message. For the message $M \in \mathbb{Z}_N$,

1. Select a random $s \in \mathbb{Z}_N^*$ and for the selected s, Compute $p_1 = \frac{1}{2} D_f(2s) \pmod{N}$.
2. Simillarly select $t \in \mathbb{Z}_N^*$ and for the selected t, compute $p_2 = \frac{1}{2} D_t(2\alpha) \pmod{N}$.
3. Now finally compute p_3 using selected s and the given y, where $p_3 = \frac{M}{4} D_t(2y) D_f(2s) \pmod{N}$.

For the plain text message 'M', the encrypted ciphertext is (p_1, p_2, p_3), which will be received by the decoder to generate the message.

Decryption

On receiving the encrypted message (p_1, p_2, p_3), the receiver performs the below given steps:

1. Firstly, he/she deals with obtaining the value of s, by computing $\frac{1}{2} D_c(2p_1) \pmod{N}$.
2. Followed by computing U, where $U = p_1^{-1} \pmod{N}$.
3. Compute V, where $V = p_3 U \pmod{N}$.
4. Then compute T, where $T = \frac{1}{2} D_{h^{\phi(N)+1}}(2p_2) \pmod{N} = \frac{1}{2} D_t(2y) \pmod{N}$.
5. Finally obtain the plain-text message $M = VT^{-1} \pmod{N}$.

The security of the proposed cryptosystem is found to be completely build upon Integer Factorization Problem(*IFP*) and Discrete Dickson Problem(*DDP*). Here we have observed few cases of common attacks, one-wayness and semantic security, where the proposed cryptosystem was found to be computationally secured.

As the encrypted message can be assessed by an intruder, he/she can have assess to (p_1, p_2, p_3). Now, for him/her to generate the message M, he/she have to obtain the value of P and Q of k and so the value of c and followed by finding h from $\frac{1}{2} D_h(2\alpha)(\text{mod } N)$. And this can only be achieved if Integer factorization problem and Discrete Dickson problem can be solved. The value of P and Q is chosen in such a way that the size of k is 1024-bit and above, so no known algorithm can be used to factor k. And also to find h from $\frac{1}{2} D_h(2\alpha)(\text{mod } N)$, the intruder have to solve *DDP*. Also the value of α and s should be large enough to prevent exhaustive search attack. It should be kept in mind that to encrypt different messages different values of s and t should be used. Because if a sender uses same parameters for the encryption of two different messages M_1 and M_2, then the intruder can obtain $p_3 = \frac{M_1}{4} D_t(2y) D_f(2s)(\text{mod} N)$ and $p_3' = \frac{M_2}{4} D_t(2y) D_f(2s)(\text{mod} N)$. And hence from the relation $M_2 = p_3' p_3^{-1} M_1$, the intruder can have the message M_2 on knowing M_1. So on choosing different values of s and t, the message M_2 cannot be known even on knowing M_1.

Suppose the intruder somehow manages to find the value of P and Q and then computes $s = \frac{1}{2} D_c(2p_1)(\text{mod} N)$ and $V = p_3 U(\text{mod} N) = p_3 p_1^{-1}(\text{mod} N) = \frac{M}{2} D_t(2y)(\text{mod} N)$. To find the message M from above, one has to know t, which is computationally impossible assumption of Discrete Dickson Problem which is equivalent to solving DLP.

3.1 One-wayness

Here we check the one wayness of our proposed cryptosystem mentioned above.

Theorem 3.1. *Our proposed cryptosystem is one-way secured if both Integer Factorization Problem(IFP) and Discrete Dickson Problem(DDP) holds.*

Proof. Let us suppose that both integer factorization problem and discrete dickson problem is simple i.e., given a composite integer X, finding integers p and q such that $p.q = X$ is effortless and under a commutative ring R with unity, with given y and z, the task of obtaining the value of N, such that $z = D_N(y, 1)$ is also effortless, where $N \in \mathbb{Z}^+$, which means that , there exist a PPT algorithm $\mathcal{A}$ which can solve both integer factorization problem and discrete dickson problem. Our motive is to break the one-wayness of our proposed scheme by using the algorithm $\mathcal{A}$ and hence recover the plain text message m.

Let the challenging ciphertext be (p_1, p_2, p_3), $p_1 = \frac{1}{2}D_f(2s) \pmod N$, $p_2 = \frac{1}{2}D_t(2\alpha) \pmod N$ and $p_3 = \frac{M}{4}D_t(2y)D_f(2s) \pmod N$ and the public key be (N, f, y, α), where $y = \frac{1}{2}D_h(2\alpha) \pmod N$. Now we commence with aquiring the value of M. From the given value of p_1 and f, we obtain the value of s, followed by using the algorithm $\mathcal{A}$ we obtain the value of t from $p_2 = \frac{1}{2}D_t(2\alpha) \pmod N$, followed by obtaining the value of M, as $M = 4p_3(D_t(2y))^{-1}(D_f(2s))^{-1} \pmod N$. □

3.2 Semantic Security

In this section we are involved with checking the semantic security of our proposed cryptosystem. In semantic security the challenger generates the public key and the private key, *pk* and *sk* respectively. Keeps the private key to himself/herself and sends the public key to the adversary. Next the adversary selects two distinct messages m_0 and $m_1 \in M$ of same length and send it to the challenger. Here, the challenger selects any one of m_0 or m_1 and encrypts the corresponding ciphertext to it and send it to the adversary. On receiving the ciphertext from the challenger, the adversary objective is to identify which message was encrypted. If it can be achieved then the encryption scheme is not semantically secured else not, then semantically secured.

Discrete Dickson Assumption

Under the Discrete Dickson assumption, we assume that it is computationally hard to obtain the value of $N \in \mathbb{Z}^+$, given the value of z and y, where $z = D_N(y, 1)$.

Computational Discrete Dickson Assumption

The Computational Discrete Dickson Assumption states that, given the value of y and z, it is computationally hard to obtain the value of N from $z = D_N(y, 1)$.

Theorem 3.2. *If Computational Discrete Dickson Assumption holds, then the scheme presented in section* 3, is semantically secured.

Proof. Let us presume that the scheme proposed in section 3 is not semantically secured for the purpose of contradiction. Which speaks about the existence of a polynomial time algorithm $\mathcal{A}$, which can break the semantic security of our proposed scheme. With this, our objective is that to, given $\mathcal{G} = (y, z, w)$, with the help of algorithm $\mathcal{A}$, it is to decide whether it is conjugacy search problem of a random one (i.e $p = ab$ or not). Where $y = \frac{1}{2}D_a(2\alpha) \pmod N$, $z = \frac{1}{2}D_b(2\alpha) \pmod N$ and $w = \frac{1}{2}D_{ab}(2\alpha) \pmod N$. We first set the public key (N, f, y, α), where $y = \frac{1}{2}D_p(2\alpha) \pmod N$ and $\alpha \in \mathbb{Z}_N^*$; then once the adversary has chosen the messages m_0 and m_1, we overturn a bit q and we encrypt m_q as follows: $\mathrm{E}(m_q) = (p_1, p_2, p_3)$ where $p_1 = \frac{1}{2}D_f(2s) \pmod N$, $p_2 = \frac{1}{2}D_t(2\alpha) \pmod N$ and $p_3 = \frac{m_q}{4}D_t(2y)D_f(2s) \pmod N$.

Seemingly if $\mathcal{G}$ is a discrete dickson assumption, the above is an authentic encryption of m_q and algorithm $\mathcal{A}$ will deliver the accurate output with non negligible gain. On the contrary, if $\mathcal{G}$ is not a discrete dickson assumption, we assert that even a polynomially unbounded adversary gains no information about m_q from $\mathrm{E}(m_q)$ in a strong information-theoretic sense.

Let $p = ab$, and then the information received by the adversary is of the form $p_1 = \frac{1}{2}D_f(2s) \pmod N$, $p_2 = \frac{1}{2}D_t(2\alpha) \pmod N$ and $p_3 = \frac{m_q}{4}D_t(2y)D_f(2s) \pmod N = \frac{m_q}{4}D_t(D_{ab}(2\alpha))D_f(2s) \pmod N = \frac{m_q}{4}D_{ab}(D_t(2\alpha))D_f(2s) \pmod N \implies m_q = 4p_3(D_{ab}(2p_2))^{-1}(D_f(2s))^{-1} \pmod N$, and hence making the value of m_q infeasible which is completely hidden. And thus $\mathcal{A}$ cannot guess q better than at random. □

4 Conclusion

In this paper we have proved the one-way security and semantic security of our proposed public key cryptosystem based on *IFP* and *DDP*. Satisfying the one-wayness and semantic security by our proposed cryptosystem has proven that, it is computationally well secured against any known attack.

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BIANCHI TYPE V MAGNETIZED STRING DUST BULK VISCOUS FLUID COSMOLOGICAL MODELS WITH VARIABLE DECELERATION PARAMETER

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Abstract

We have investigated Bianchi type V viscous fluid cosmological models with string dust universe in general relativity. Exact solution of Einstein field equations have been obtained by choosing deceleration parameter is function of cosmic time. Solutions for exponential and polynomial form are obtained. Some geometrical and physical aspects of models are also discussed.

2020 Mathematical Sciences Classification: 83C10, 83C20, 83E30.
Keywords and Phrases: Deceleration parameter, Bianchi type- V, Cosmological term Λ.

1 Introduction

Bianchi type-V cosmological models plays an important role in the investigation of origin and evaluation of universe and the study is more interesting as these models contain isotropic special cases and permit arbitrary small anisotropy levels at some point of time. The string theory of cosmology plays a significant role in the investigation of physical situation at the very early stages of the formation of the universe. It is generally assumed that after the big bang, the universe may have undergone a series of phase transitions as its temperature was lowered down below some critical temperature as predicted by grand unified theories [5, 6, 7, 26, 28, 29, 30]. At the very early stages of evolution of the universe, during phase transition, it is believed that the symmetry of the universe is broken spontaneously. It can give rise to topologically stable defects such as domain walls, strings and monopoles. In these three cosmological structures, cosmic strings are the most interesting [27] because they are believed to give rise to density perturbations which lead to the formation of galaxies [30]. These cosmic strings can be closed like loops or opened like a hair which move through time and trace out a tube or a sheet, according to whether it is closed or open. The string is free to vibrate and its different vibration modes present different types of particles carrying the force of gravitation. Hence, investigation of universe is very interesting to study the gravitational effect that arises from strings using Einstein's field equations.

Bulk viscosity is useful for the study of early stages of evolution of the universe. Bulk viscosity driven inflation is primarily due to the negative bulk viscous pressure giving rise to a total negative effective pressure which may overcome the pressure due to the usual gravity of matter distribution in the universe and provides an impetus for rapid expansion of the universe. Thus many workers have been study bulk viscous string cosmological model in the context of Einstein theory or modified theories of gravity. Bulk viscous cosmological models in general relativity of material distribution have been investigated by a number of workers [1, 8, 12, 13, 14, 16, 18, 21]. Singh and Kale [22] has examined anisotropic bulk viscous cosmological models with particle creation. Rao and Sireesha [19] investigated the Bianchi types II, VIII, and IX string cosmological models with bulk viscosity in Brans-Dicke theory of gravitation. Banerjee and Banerjee [2] investigated stationary distribution of dust and electromagnetic fields in general relativity. Banerjee et al. [3] studied an axially symmetric Bianchi Type I string dust cosmological model in presence and absence of magnetic field. Recently, Bali and Upadhaya [4] examined LRS Bianchi Type I strings dust-magnetized cosmological models.

Motivated by the above discussion, we have constructed Bianchi type- V magnetized string dust dust bulk viscous fluid cosmological models with variable Λ and deceleration parameter. The main reason to explore the Bianchi type- V model is that the standard FLRW models are contained as special cases of the Bianchi models. The Bianchi type- V model generalizes the open $(k=-1)$ Friedmann model and represents a model in which the fluid flow is not necessarily orthogonal to the three surfaces of homogeneity. At early stage of evolution, the universe was not so smooth as it looks in present time. Therefore anisotropic cosmological models have taken considerable interest of researcher workers.

In this paper, we have study the role of variable deceleration parameter in Bianchi type- V space time with magnetized string dust bulk viscous fluid.

2 Metric and Field Equations

We consider the Bianchi type V space-time in orthogonal form represented by the line-element

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 e^{2x} dy^2 + C^2 e^{2x} dz^2. \tag{2.1}$$

where A and B are the metric potentials considered as function of cosmic time only. The energy-momentum tensor $\left(T_i^j\right)$ for cloud of string dust is given by Letelier [9] with bulk viscous fluid and electromagnetic field $\left(E_i^j\right)$ given by Lichnerowicz [10] as

$$T_i^j = \rho v_i v^j - \lambda x_i x^j - \varepsilon\theta\left(g_i^j + v_i v^j\right) + E_i^j. \tag{2.2}$$

Where ρ is the rest energy density of the cloud of strings with particles attached to them, $\rho = \rho_{\mathrm{p}} + \lambda$ with ρ_{p} being the rest energy density of particles, λ the tension density of the cloud of strings, $\theta = v^i_{;i}$ is the scalar of expansion and ε the coefficient of bulk viscosity. The vector $v^i = (0,0,0,1)$ is the four-velocity of the particles and x^i is a unit space-like vector representing the direction of string.

The vector v^i and x^i satisfy the conditions

$$v_i v^i = -x_i x^i = -1, v^i x_i = 0. \tag{2.3}$$

Choosing x^i parallel to $\frac{\partial}{\partial x}$, we have

$$x^i = \left(A^{-1}, 0, 0, 0\right). \tag{2.4}$$

The electromagnetic field E_i^j is given by

$$E_i^j = \bar{\mu}\left[|h|^2\left(v_i v^j + \frac{1}{2} g_i^j\right) - h_i h^j\right], \tag{2.5}$$

where h_i is the magnetic flux vector given by

$$h_1 = \frac{\sqrt{-g}}{2\bar{\mu}} \in_{ijkl} F^{kl} v^j. \tag{2.6}$$

Here $\bar{\mu}$ is the magnetic permeability and ϵ_{ijkl} the Levi-Civita tensor. We assume that current is flowing along x-axis.

Thus F_{23} is the only non-vanishing component of F_{ij}.

Maxwell's equations

$$F[ij;k] = 0 \tag{2.7}$$

and

$$F_{\mathrm{ij}} = 0 \tag{2.8}$$

are satisfied by

$$F_{23} = \text{ constant } = K(\text{ say }). \tag{2.9}$$

Thus

$$h_1 \neq 0, h_2 = 0 = h_3 = h_4. \tag{2.10}$$

Thus equation (2.6) lead to

$$h_1 = \frac{AK}{\bar{\mu} B C e^{2x}}. \tag{2.11}$$

$F_{14} = 0 = F_{24} = F_{34}$ due to assumption of infinite electrical conductivity (Maartens[15]).

We assume that magnetic permeability ($\bar{\mu}$) is a variable and consider

$$\bar{\mu} = \mathrm{e}^{-4x} \text{ i.e. when } x \to \infty, \text{ then } \bar{\mu} \to 0.$$

Thus, from (2.5) and (2.11), we have

$$E_1^1 = -\frac{K^2}{2B^2C^2} = -E_2^2 = -E_3^3 = E_4^4. \tag{2.12}$$

The Einstein field equation (in gravitational units $c = 1, 8\ \Lambda\ G = 1$) with time varying cosmological term $\Lambda(t)$ are given by

$$R_i^j - \frac{1}{2}Rg_i^j = -T_i^j + \Lambda g_i^j. \tag{2.13}$$

The field equation (2.13) with subsequently lead to the following system of equation

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{1}{A^2} = \lambda + \varepsilon\theta + \frac{K^2}{2B^2C^2} + \Lambda, \tag{2.14}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\ddot{A}\dot{C}}{AC} - \frac{1}{A^2} = \frac{-K^2}{2B^2C^2} + \varepsilon\theta + \Lambda, \tag{2.15}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{1}{A^2} = -\frac{K^2}{2B^2C^2} + \varepsilon\theta + \Lambda, \tag{2.16}$$

$$\frac{\ddot{A}\dot{B}}{AB} + \frac{\ddot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{3}{A^2} = \rho + \frac{K^2}{B^2C^2} + \Lambda, \tag{2.17}$$

$$2\frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0. \tag{2.18}$$

In the above and elsewhere overhead dot stands for ordinary time-derivative of the concerned quantity. We define the average scale factor R as

$$R^3 = ABC. \tag{2.19}$$

Integrating (2.18), we get

$$A^2 = LBC = BC. \tag{2.20}$$

where $L = 1$ is constant of integration.

In analogy with FRW universe, we define a generalized Hubble parameter H and generalized deceleration parameter q as

$$H = \frac{\dot{R}}{R} = \frac{1}{3}\left(H_1 + H_2 + H_3\right), \tag{2.21}$$

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = \frac{-\dot{H}}{H^2} - 1, \tag{2.22}$$

where $H_1 = \frac{\dot{A}}{A}, H_2 = \frac{\dot{B}}{B}$ and $H_3 = \frac{\dot{C}}{C}$ are directional Hubble's factor in the x, y and z directional respectively.

The anisotropy parameter

$$\bar{A} = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2. \tag{2.23}$$

The physical quantities of observational interest in cosmology

$$\theta = \frac{3\dot{R}}{R}. \tag{2.24}$$

The components of shear tensor $\left(\sigma_j^i\right)$

$$\sigma_1^1 = \frac{1}{3}\left(\frac{\dot{2A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right), \tag{2.25}$$

$$\sigma_2^2 = \frac{1}{3}\left(\frac{2\dot{B}}{B} - \frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right), \tag{2.26}$$

$$\sigma_3^3 = \frac{1}{3}\left(\frac{2\dot{C}}{C} - \frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right), \tag{2.27}$$

$$\sigma_4^4 = 0. \tag{2.28}$$

Therefore,

$$\sigma^2 = \frac{1}{2}\left[\left(\sigma_1^1\right)^2 + \left(\sigma_2^2\right)^2 + \left(\sigma_3^3\right)^2 + \left(\sigma_4^4\right)^2\right]. \tag{2.29}$$

3 Solution of the Filed Equation

Observation of type Ia supernovae [20] allow to probe the expansion history of universe. In literature it is common to use a constant deceleration parameter, as it duly gives a power law for metric function or corresponding quantity. But at present the expansion of universe is accelerating and decelerating in the past. Also the transition redshift from deceleration phase to accelerating phase is about 0.5. Now for the universe which was decelerating in the past and accelerating at present time, the deceleration parameter must show signature flipping [17, 23, 24, 25]. So, in general, deceleration parameter is not constant but variable. On basis of supernovae searches, we consider the deceleration parameter to the variable i.e.

$$-\frac{R\ddot{R}}{\dot{R}^2} = q \text{ (variable)}, \tag{3.1}$$

where R is a average scale factor.

We assume

$$\varepsilon = \varepsilon_0 + \varepsilon_1 H, \tag{3.2}$$

where ε_0 and ε_1 are positive constant.

In this paper, we show how the variable deceleration parameter models with metric (2.1) behave in presence of string fluid as a source of matter.

From equation (3.1), we have

$$\frac{\ddot{R}}{R} + q\frac{\dot{R}^2}{R^2} = 0. \tag{3.3}$$

In order to solve equation (3.2), we have to assume $q = q(R)$. It is important to note here that one can assume $q = q(t) = q(R(t))$, as R is also a time dependent function. But this is possible only when one avoid singularity like big-bang or big rip because both t and R are increasing function.

Thus the general solution of equation (3.1) with assumption $q = q(R)$ is given by

$$\int e^{\int \frac{q}{R} dR} dR = t + t_0, \tag{3.4}$$

where t_0 is the constant of integration.

Without loss of generality, we chose

$$\int \frac{q}{R} dR = \log L(R). \tag{3.5}$$

Therefore,

$$\int L(R) dR = t + t_0. \tag{3.6}$$

The choice of $L(R)$ in equation (3.5), quite arbitrary but, since we are looking for a physically viable models of universe consistent with observations. We assume the following two cases.

4 Solution in Exponential Form

Let us consider $L(R) = \frac{1}{aR+b}$, where a & b are constant on integration, equation (3.6), we get

$$R = \frac{1}{a}\left(e^{aT} - b\right), \tag{4.1}$$

where $T = t + t_0$. In this case, the expression for the proper energy density (ρ), the string tension (λ), the cosmological constant (Λ) and particle density (ρ_p) are given by

$$\rho = \frac{-a^2 - 2a^2e^{-2aT} + 2a^2be^{-aT} + 3a^2\varepsilon_1}{\left(1 - be^{-aT}\right)^2} - \frac{a^6k_1^2e^{-6a}}{2\left(1 - be^{-aT}\right)} - \frac{K^2a^4e^{-4aT}}{\left(1 - be^{-aT}\right)^4} + \frac{3a^2\varepsilon_0}{\left(1 - be^{-aT}\right)}, \tag{4.2}$$

$$\lambda = \frac{-K^2a^4e^{-4aT}}{\left(1 - be^{-aT}\right)^4}, \tag{4.3}$$

$$\Lambda = \frac{-2a^2be^{-aT} + 3a^2 - a^2e^{-2aT} - 3a^2\varepsilon_1}{\left(1 - be^{-aT}\right)^2} + \frac{a^6k_1^2e^{-6aT}}{4\left(1 - be^{-aT}\right)^6} + \frac{K^2e^{-4aT}a^4}{2\left(1 - be^{-aT}\right)^4} - \frac{3a\varepsilon_0}{\left(1 - be^{-aT}\right)}, \tag{4.4}$$

$$\rho_p = \frac{-a^2 - 2a^2e^{-2aT} + 2a^2be^{-aT} + 3a^2\varepsilon_1}{\left(1 - be^{-aT}\right)^2} - \frac{a^6k_1^2e^{-6aT}}{2\left(1 - be^{-aT}\right)} + \frac{3a\varepsilon_0}{\left(1 - be^{-aT}\right)}. \tag{4.5}$$

The rate of expansion in the direction of x, y and z are given by

$$H_x = \frac{\dot{A}}{A} = \frac{a}{1-be^{-aT}}, \tag{4.6}$$

$$H_y = \frac{\dot{B}}{B} = \frac{a}{1-be^{-aT}} - \frac{a^3k_1e^{-3aT}}{2\left(1-be^{-aT}\right)^3}, \tag{4.7}$$

$$H_z = \frac{\dot{C}}{C} = \frac{a}{1-be^{-aT}} + \frac{a^3k_1e^{-3aT}}{2\left(1-be^{-aT}\right)^3}. \tag{4.8}$$

Expansion θ, shear σ^2, deceleration parameter q, spatial volume V, bulk viscosity ε and anisotropy parameter $\bar{A}$ of the model take the form

$$\theta = \frac{3a}{1-be^{-aT}}, \tag{4.9}$$

$$\sigma^2 = \frac{a^6k_1^2e^{-6aT}}{4\left(1-be^{-aT}\right)^6}, \tag{4.10}$$

$$q = -1 + be^{-aT} \tag{4.11}$$

$$V = \frac{1}{a^3}\left(e^{aT}-b\right)^3, \tag{4.12}$$

$$\varepsilon = \varepsilon_0 + \frac{\varepsilon_1 a}{1-be^{-aT}}, \tag{4.13}$$

$$\bar{A} = \frac{1}{6}\frac{a^4k_1^2e^{-6aT}}{\left(1-be^{-aT}\right)^4}. \tag{4.14}$$

We observe that model has singularity at $T = \frac{\log b}{a} = T_0$ (say). The model starts expanding with a big bang at $T = T_0$ and the expansion in the model decreases as time T increases. Expansion in the model becomes finite at $T = \infty$. Singularity in the model is of point type. Since scale factor cannot be negative, we find $R(T)$ is positive, if $a > 0$. Therefore, in the early stage of the universe i.e. near $T = T_0$, the scale factor of the universe had been approximately constant and had increased very slowly. Sometime later, the universe has exploded suddenly and expanded to a large scale. This picture is consistent with big-bang scenario. We observe that $q = 0$ for $T = T_0$ and as $T \to \infty, q = -1$. Thus the model represents an accelerating universe at late times. The physical quantities $\rho, \sigma, \Lambda, \varepsilon$ and ρ_p all diverge at $T = T_0$. In the limit of large times i.e. $T \to \infty, \rho \to 3a^2\varepsilon_1 + 3a^2\varepsilon_0 - a^2, \sigma \to 0, \Lambda \to 3a^2 - 3a^2\varepsilon_1 - 3a\varepsilon_0, \varepsilon \to \varepsilon_0 + \varepsilon_1 a$ and $\rho_p \to 3a^2\varepsilon_1 + 3a\varepsilon_0 - a^2$.

From (4.3) it is found that tension density λ is negative. It is pointed out by Letelier[11] that λ may be positive or negative. When $\lambda < 0$, the string phase of the universe disappears i.e. we have an anisotropic fluid of particles. The mean anisotropic parameter is decreasing function of time.

For the model

$$\frac{\sigma}{\theta} = \frac{a^2k_1e^{-3aT}}{6\left(1-be^{-aT}\right)^2}. \tag{4.15}$$

For the large value of $t, \frac{\sigma}{\theta} \to 0$ implying that the model approaches isotropy at late time. We observe that the pressure of shear viscosity accelerates the process of isotropization.

5 Solution in polynomial form

Let $L(R) = \frac{1}{2a\sqrt{R+b}}$, where a and b are constant.

On integration, equation (3.6), we get

$$R = a^2T^2 - b \text{ where } t + t_0 = T. \tag{5.1}$$

In this case, the expansion for the proper energy density (ρ), the string tension (λ), the cosmological constant (Λ) and particle density (ρ_p) are given by

$$\rho = \frac{-4a^2b - 2 + 12\varepsilon_1a^4T^2}{\left(a^2T^2-b\right)^2} - \frac{k_1}{2\left(a^2T^2-b\right)^6} + \frac{6\varepsilon_0a^2T}{a^2T^2-b} - \frac{K^2}{\left(a^2T^2-b\right)^4}, \tag{5.2}$$

$$\lambda = \frac{-K^2}{\left(a^2T^2-b\right)^4}, \tag{5.3}$$

$$\Lambda = \frac{a^4T^2 - 4a^2b - 1 - 12\varepsilon_1 a^4T^2}{\left(a^2T^2 - b\right)^2} + \frac{k_1}{4\left(a^2T^2 - b\right)^6} - \frac{6\varepsilon_0 a^2T}{a^2T^2 - b} + \frac{K^2}{2\left(a^2T^2 - b\right)}, \tag{5.4}$$

$$\rho_p = \frac{-4a^2b - 2 + 12\varepsilon_1 a^4T^2}{\left(a^2T^2 - b\right)^2} - \frac{k_1}{2\left(a^2T^2 - b\right)^6} + \frac{6\varepsilon_0 a^2T}{\left(a^2T^2 - b\right)}. \tag{5.5}$$

The rate of expansion in the direction of x, y and z are given by

$$H_x = \frac{\dot{A}}{A} = \frac{2a^2T}{a^2T^2 - b}, \tag{5.6}$$

$$H_y = \frac{\dot{B}}{B} = \frac{2a^2T}{a^2T^2 - b} - \frac{k_1}{2\left(a^2T^2 - b\right)^3}, \tag{5.7}$$

$$H_z = \frac{\dot{C}}{C} = \frac{2a^2T}{a^2T^2 - b} + \frac{k_1}{2\left(a^2T^2 - b\right)^3}. \tag{5.8}$$

Expansion θ, shear σ^2, deceleration parameter q, spatial volume V, bulk viscosity ε and anisotropy parameter $\bar{A}$ of the model take the form

$$\theta = \frac{6a^2T}{a^2T^2 - b}, \tag{5.9}$$

$$\sigma^2 = \frac{1}{4}\frac{k_1^2}{\left(a^2T^2 - b\right)^6}, \tag{5.10}$$

$$q = -\frac{1}{2} + \frac{b}{2a^2T^2}, \tag{5.11}$$

$$V = \left(a^2T^2 - b\right)^3, \tag{5.12}$$

$$\varepsilon = \varepsilon_0 + \frac{2\varepsilon_0 a^2T}{a^2T^2 - b}, \tag{5.13}$$

$$\bar{A} = \frac{k_1^2}{12\left(a^2T^2 - b\right)^2 a^4T^2}. \tag{5.14}$$

We observe that model has singularity at $T = \sqrt{b}/a = T_1$ (say). The model starts with a big bang at $T = T_1$ and the expansion in the model decreases as time increases. Expansion in the model stops at $T = \infty$.

In the early stage of the universe i.e. near $T = T_1$, the scale factor of the universe had been approximately constant and had increased very slowly. Sometime later, the universe had exploded suddenly and expanded to large scale. This picture is consistent with big-bang scenario. At $T = T_1$, $q = 0$ and $q \to -1/2$ as $T \to \infty$. Thus the model represents an accelerating universe at late times. The physical quantity $\rho, \sigma, \Lambda, \varepsilon$ and ρ_p all diverge at $T = T_1$. In the limit of large times i.e. $T \to \infty, \rho, \sigma, \Lambda, \rho_p$ are negligible and $\varepsilon \to \varepsilon_0$. From (5.3), it is found that tension density λ is negative. Therefore the string phase of the universe disappears i.e. we have an anisotropic fluid of particles. The mean anisotropic parameter is decreasing function of time. For the model

$$\frac{\sigma}{\theta} = \frac{k_1}{12a^2T\left(a^2T - b\right)^2}. \tag{5.15}$$

For large value of T, $\frac{\sigma}{\theta} \to 0$ implying that the model approaches isotropy at late times. In the absence of magnetic field, the model represents an isotropic universe.

6 Conclusion

In this paper, we have studied Bianchi type-V magnetized string dust bulk viscous fluid cosmological models with variable deceleration parameter q in the context of general relativity. The Einstein's field equation have been solved exactly by considering a deceleration parameter $q =$ variable which yields time dependent scale factor. We have found that cosmological term Λ being very large at initial times relaxes to genuine cosmological constant at late times. The models are found to be compatible with the results of recent observations.

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IDENTITIES OF A GENERAL MULTIPLE HURWITZ-LERCH ZETA FUNCTION AND APPLICATIONS

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Abstract

In this article we introduce a general multiple Hurwitz-Lerch Zeta function. Then its convergence conditions and identities are obtained under certain conditions. We also derive some of connections to the multiple Hurwitz-Lerch Zeta function based upon Srivastava-Daoust hypergeometric series in several variables and other related functions of one and more variables found in the literature. Further, we study its integral representations and find their applications for deriving generating relations and solving the non-homogeneous fractional differential equation.
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1 Introduction

Recently, Srivastava et al. [16] extended the double Hurwitz-Lerch Zeta function due to (1.1) - (1.3) into the multiple Hurwitz-Lerch Zeta function [16, Eqn. (4.1)] in terms of Srivastava-Daoust hypergeometric series in several variables [20, p.37] and defined in the form

$$\text{(1.1)}\quad {}_{\omega}^{\sigma}F\begin{matrix} A:B^{(1)};...;B^{(n)} \\ C:D^{(1)};...;D^{(n)}\end{matrix}\left(\begin{matrix}[(a);\theta^{(1)},...,\theta^{(n)}]:[(b^{(1)}:\psi^{(1)})];...;[(b^{(n)}):\psi^{(n)}]; \\ [(c);\delta^{(1)},...,\delta^{(n)}]:[(d^{(1)}:\phi^{(1)})];...;[(d^{(n)}):\phi^{(n)}];\end{matrix} z_1,...,z_n\right)$$
$$= \sum_{m_1,...,m_n=0}^{\infty} \mathcal{H}\begin{matrix} A:B^{(1)};\dots;B^{(n)} \\ C:D^{(1)};...;D^{(n)}\end{matrix}(m_1,...,m_n)\frac{z_1^{m_1}\dots z_n^{m_n}}{m_1!\dots m_n!\,(m_1+\cdots+m_n+\omega)^{\sigma}},$$

where, $\sigma\in\mathbb{C},\omega\in\mathbb{C}\backslash\mathbb{Z}_0,\mathbb{C}=\{z:z=x+iy,i=\sqrt{(-1)},x,y\in\mathbb{R}\},\mathbb{R}=(-\infty,\infty),\mathbb{Z}_0=\{0,-1,-2,-3,\dots\}$. Additionally, as usual $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}^{+}$denotes the set of positive real numbers and for convenience

(1.2)
$$\mathcal{H}\begin{matrix} A:B^{(1)};\dots;B^{(n)} \\ C:D^{(1)};\dots;D^{(n)}\end{matrix}(m_1,...,m_n)=\frac{\prod_{j=1}^{A}(a_j)_{\theta_j^{(1)}m_1+\ldots+\theta_j^{(n)}m_n}\prod_{j=1}^{B^{(1)}}\left(b_j^{(1)}\right)_{\psi_j^{(1)}m_1}\cdots\prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}\right)_{\psi_j^{(n)}m_n}}{\prod_{j=1}^{C}(c_j)_{\delta_j^{(1)}m_1+\ldots+\delta_j^{(n)}m_n}\prod_{j=1}^{D^{(1)}}\left(d_j^{(1)}\right)_{\phi_j^{(1)}m_1}\cdots\prod_{j=1}^{D^{(n)}}\left(d_j^{(n)}\right)_{\phi_j^{(n)}m_n}}.$$

The coefficients

$$\text{(1.3)}\quad \begin{cases}\theta_j^{(l)},j=1,\dots,\ A;\ \psi_j^{(l)},j=1,\dots,\ B^{(l)};\delta_j^{(l)},j=1,\dots,C; \\ \qquad\phi_j^{(l)},j=1,\dots,D^{(l)};\ \forall\, l\in\{1,2,3,\dots,n\},\end{cases}$$

are the members of the set $\mathbb{R}^{+}$ and (a) abbreviates the array of A parameters $a_1,\dots,a_A$; $\left(b^{(l)}\right)$ abbreviates the array of $B^{(l)}$ parameters $b_1^{(l)},\dots,b_{B^{(l)}}^{(l)},\quad\forall\, l\in\{1,2,3,\dots,n\}$, with similar interpretations for (c) and $\left(d^{(l)}\right),\ l=1,2,\dots,n$; are complex numbers *et cetera*.

The multiple series (1.1) with (1.2) and (1.3) converges due to [3, 7] for

$$\text{(1.4)}\qquad |z_1|<\infty,\dots,|z_n|<\infty,$$

if

$$\sum_{j=1}^{C}\delta_j^{(l)}+\sum_{j=1}^{D^{(l)}}\phi_j^{(l)}-\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{B^{(l)}}\psi_j^{(l)}+1>0 \quad (\forall\, l=1,2,3,\ldots,n). \tag{1.5}$$

Further if $\sum_{j=1}^{C}\delta_j^{(l)}+\sum_{j=1}^{D^{(l)}}\phi_j^{(l)}-\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{B^{(l)}}\psi_j^{(l)}+1=0 \quad (\forall\, l=1,2,3,\ldots,n)$, then the multiple series (1.1) with (1.2) and (1.3) converges for

$$(|z_1|)^{\frac{1}{\mathfrak{H}_1}}+\ldots+(|z_n|)^{\frac{1}{\mathfrak{H}_n}}<1, \text{ when } \mathfrak{H}_l=\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{C}\delta_j^{(l)} \ (\forall\, l=1,2,3,\ldots,n),\ \mathfrak{H}_l>0; \text{ and for} \tag{1.6}$$

$$\{|z_1|\,(z_1\neq 1),\ldots,|z_n|\,(z_n\neq 1)\}<1, \text{ when } \mathfrak{H}_l<0 \ (\forall\, l=1,2,3,\ldots,n). \tag{1.7}$$

But when $z_1=1,\ldots,z_n=1$, the multiple Srivastava-Daoust-Hurwitz-Lerch Zeta function (1.1) - (1.3), with the aid of the formulae, $\lim_{n\to\infty}\Gamma(z+n+1)=\lim_{n\to\infty}\Gamma(n+1)n^z$, is written by (see in [5, 6])

$$\begin{aligned}
&{}_{\omega}^{\sigma}F\,{}^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta^{(1)},\ldots,\theta^{(n)}]:[(b^{(1)}):\psi^{(1)}];\ldots;[(b^{(n)}):\psi^{(n)}];\\ [(c):\delta^{(1)},\ldots,\delta^{(n)}]:[(d^{(1)}):\phi^{(1)}];\ldots;[(d^{(n)}):\phi^{(n)}];\end{matrix}1,\ldots,1\right)\\
&=\sum_{m_1,\ldots,m_n=0}^{m_1=M_1-1,\ldots,m_n=M_n-1}\mathcal{H}\,{}^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}}(m_1,\ldots,m_n)\frac{1}{m_1!\ldots m_n!\,(m_1+\ldots+m_n+\omega)^{\sigma}}\\
&+\sum_{m_1,\ldots,m_n=0}^{\infty}\mathcal{H}\,{}^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}}(m_1+M_1,\ldots,m_n+M_n)\\
&\times\frac{\Gamma(m_1+M_1+\ldots+m_n+M_n+\omega+1)}{(m_1+M_1)!\ldots(m_n+M_n)!\Gamma(m_1+M_1+\ldots+m_n+M_n+\sigma+\omega+1)}.
\end{aligned}\tag{1.8}$$

Here in (1.8), the first series which is finite and the second infinite series converges for

$$\sum_{j=1}^{C}\delta_j^{(l)}+\sum_{j=1}^{D^{(l)}}\phi_j^{(l)}-\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{B^{(l)}}\psi_j^{(l)}+1=0 \quad (\forall\, l=1,2,3,\ldots,n) \tag{1.9}$$

and if $\mathfrak{H}_l=0$, where, $\mathfrak{H}_l=\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{C}\delta_j^{(l)} \ (\forall\, l=1,2,3,\ldots,n)$, then there exists

$$\Re\left(\sum_{j=1}^{C}c_j+\sum_{j=1}^{D^{(l)}}d_j^{(l)}+\sigma-\sum_{j=1}^{A}a_j-\sum_{j=1}^{B^{(l)}}b_j^{(l)}\right)>0 \ (\forall\, l=1,2,3,\ldots,n).$$

It is remarked that to obtain the condition (1.9), we make an appeal to the formula $\frac{\Gamma(n+a)}{\Gamma(n+b)}\sim n^{(a-b)}\ (n\to\infty)$, and apply the techniques due to Hái, Marichev and Srivastava [3].

Here in this research article, in order to explore new ideas for studying various known and unknown Hurwitz-Lerch Zeta functions of one, two and multiple Hurwitz-Lerch Zeta functions (see in [5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19]), we introduce a general multiple Hurwitz-Lerch Zeta function given by

$$ {}_{\eta}^{s}K(\chi;\rho_1,\ldots,\rho_n;z_1,\ldots,z_n)=\sum_{m_1=0,\ldots,m_n=0}^{\infty}\frac{\chi(m_1,\ldots,m_n)(\rho_1)_{m_1}\ldots(\rho_n)_{m_n}}{m_1!\ldots m_n!}\frac{(z_1)^{m_1}\ldots(z_n)^{m_n}}{(m_1+\ldots+m_n+\eta)^{s}}, \tag{1.10}$$

provided that the multiple sequence of function $\chi(m_1,\ldots,m_n)$ $(\forall\ m_1\geq 0,\ldots,m_n\geq 0)$ is convergent under certain restrictions, all $\rho_1,\ldots,\rho_n;x_1,\ldots,x_n;$ $s\in\mathbb{C}$ and $\eta\in\mathbb{C}\backslash\mathbb{Z}_0$.

2 Identities of the general multiple Hurwitz-Lerch Zeta function (1.10)

In this section, we show that the general multiple Hurwitz-Lerch Zeta function (1.10) gives some identities under certain conditions.

Theorem 2.1. *If in* (1.10), $\chi(m_1,\ldots,m_n)=\chi(m_1+\ldots+m_n)$ *and* $z_1=\ldots=z_n=z$, *then there exists an identity*

$$ {}_{\eta}^{s}K(\chi;\rho_1,\ldots,\rho_n;z,\ldots,z)=\sum_{k=0}^{\infty}\frac{\chi(k)(\rho_1+\ldots+\rho_n)_k}{k!}\frac{z^k}{(k+\eta)^s}, \tag{2.1}$$

provided that the series involved converges absolutely.

Proof. In the formula (1.10), setting $\chi(m_1,\ldots,m_n)=\chi(m_1+\ldots+m_n)$ and $z_1=\ldots=z_n=z$, we find (2.2)

$$ {}^{s}_{\eta}K(\chi;\rho_1,\ldots,\rho_n;z,\ldots,z)=\sum_{m_1=0,\ldots,m_n=0}^{\infty}\frac{\chi(m_1+\ldots+m_n)(\rho_1)_{m_1}\ldots(\rho_n)_{m_n}}{m_1!\ldots m_n!}\frac{(z)^{m_1+\ldots+m_n}}{(m_1+\ldots+m_n+\eta)^{s}}. $$

Now in the equality (2.1) applying the formula [21, pp. 61-62]

$$ \sum_{m_1=0,\ldots,m_n=0}^{\infty}F(m_1+\ldots+m_n)(\rho_1)_{m_1}\ldots(\rho_n)_{m_n}\frac{x^{m_1+\ldots+m_n}}{m_1!\ldots m_n!}=\sum_{k=0}^{\infty}F(k)(\rho_1+\ldots+\rho_n)_k\frac{x^k}{k!}, \tag{2.3} $$

provided that the series involved in (2.3) converges absolutely, we get the result (2.1). □

Corollary 2.1. *If in the Theorem* 2.1, *for all* $p,q\in\mathbb{N}_0$, *where* $\mathbb{N}_0=\mathbb{N}\cup\{0\}$; $a_j\in\mathbb{C}$ $(j=1,\ldots,p)$, $b_j\in\mathbb{C}\backslash\mathbb{Z}_0$ $(j=1,..,q)$; $\lambda_j\in\mathbb{R}^+$ $(j=1,\ldots,p)$, $\mu_j\in\mathbb{R}^+$ $(j=1,\ldots,q)$; *set*

$$ \chi(m_1,\ldots,m_n)=\frac{\prod_{j=1}^{p}(a_j)_{(m_1+\ldots+m_n)\lambda_j}}{\prod_{j=1}^{q}(b_j)_{(m_1+\ldots+m_n)\mu_j}}, $$

and make an appeal to the Zeta function due to [15, Eqn. (16)], *then there exists following identities*

$$ \begin{aligned} {}^{s}_{\eta}K(\chi;\rho_1,\ldots,\rho_n;z,\ldots,z)&=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{k\lambda_j}(\rho_1+\ldots+\rho_n)_k}{\prod_{j=1}^{q}(b_j)_{k\mu_j}\,k!}\frac{z^k}{(k+\eta)^s}\\ &=\Phi^{(\lambda_1,\ldots\lambda_p,1;\mu_1,\ldots,\mu_q)}_{(a_1,\ldots a_p,\rho_1+\ldots+\rho_n;b_1,\ldots,b_q)}(z,s,\eta). \end{aligned} \tag{2.4} $$

Now make an appeal to the conditions given in (1.4)-(1.7) *and* (1.9), *the series in* (2.4) *converges due to the conditions*
$\sum_{j=1}^{q}\mu_j-\sum_{j=1}^{p}\lambda_j>0$, *if* $|z|<\infty$; $\sum_{j=1}^{q}\mu_j-\sum_{j=1}^{p}\lambda_j=0$, *if* $|z|<1$; *again for* $\sum_{j=1}^{q}\mu_j-\sum_{j=1}^{p}\lambda_j=0$, *along with* $z=1$, *if* $\sum_{j=1}^{q}b_j+s-\sum_{j=1}^{p}a_j-\rho_1-\ldots-\rho_n>0$.

3 Various connected known and unknown multiple Hurwitz-Lerch Zeta functions to the general multiple Hurwitz-Lerch Zeta function (1.10)

In this section, we derive various known and unknown multiple Hurwitz-Lerch Zeta functions as on manipulation of the multiple sequence of function $\chi(m_1,\ldots,m_n)$ $\forall$ $m_1\geq 0,\ldots,m_n\geq 0$ and on application of the Theorem 2.1.

Theorem 3.1. *In Eqn.* (1.10), *if* $s\in\mathbb{C}$, $\eta\in\mathbb{C}\backslash\mathbb{Z}_0$, $max\{|z_1|(z_1\neq 1),\ldots,|z_n|(z_n\neq 1)\}<1$, *and*

$$ \sum_{j=1}^{C}\delta_j^{(l)}+\sum_{j=1}^{D^{(l)}}\phi_j^{(l)}-\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{B^{(l)}}\psi_j^{(l)}=0,\mathfrak{H}_l=\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{C}\delta_j^{(l)}\leq 0. \tag{3.1} $$

Again when $z_1=1,\ldots,$ $z_n=1$, $(\forall\, l=1,2,3,\ldots,n)$ *and there are*

$$ \sum_{j=1}^{C}\delta_j^{(l)}+\sum_{j=1}^{D^{(l)}}\phi_j^{(l)}-\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{B^{(l)}}\psi_j^{(l)}=0,\mathfrak{H}_l=\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{C}\delta_j^{(l)}=0, $$

along with

$$ \Re\left(\sum_{j=1}^{C}c_j+\sum_{j=1}^{D^{(l)}}d_j^{(l)}+s-\sum_{j=1}^{A}a_j-\sum_{j=1}^{B^{(l)}}b_j^{(l)}-\rho_l\right)>0\quad(\forall\, l=1,2,3,\ldots,n). $$

Then by the coefficient

$$ \chi(m_1,\ldots,m_n)=\mathcal{H}^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}}(m_1,\ldots,m_n), \tag{3.2} $$

the formula (1.10) *is connected by a multiple Hurwitz-Lerch Zeta function* [16, Eqn.(4.1)] *based upon the Srivastava-Daoust hypergeometric series in several variables* [20, P. 37] *as*

$$ \begin{aligned} {}^{s}_{\eta}K\left(\mathcal{H}^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}};\rho_1,\ldots,\rho_n;z_1,\ldots,z_n\right)&={}^{s}_{\eta}F^{A:B^{(1)}+1;\ldots;B^{(n)}+1}_{C:D^{(1)};\ldots;D^{(n)}}\\ &\times\left(\begin{matrix}[(a):\theta^{(1)},\ldots,\theta^{(n)}]:[(b^{(1)}):\psi^{(1)}],\ [\rho_1:1];\ldots;[(b^{(n)}):\psi^{(n)}],[\rho_n:1];\\ [(c):\delta^{(1)},\ldots,\delta^{(n)}]:[(d^{(1)}):\phi^{(1)}];\ldots;[(d^{(n)}):\phi^{(n)}];\end{matrix}z_1,\ldots,z_n\right). \end{aligned} \tag{3.3} $$

Proof. Making an appeal to the formulae (1.10) and ((3.2), we get the series

$$(3.4)\quad {}_{\eta}^{s}K\left(\mathcal{H}_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}};\rho_1,\ldots,\rho_n;z_1,\ldots,z_n\right)$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\mathcal{H}_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}}(m_1,\ldots,m_n)\frac{(\rho_1)_{m_1}\ldots(\rho_n)_{m_n}\ z_1^{m_1}\ldots z_n^{m_n}}{m_1!\ldots m_n!\,(m_1+\ldots+m_n+\omega)^{\sigma}}.$$

Then in the series of (3.4), define (1.2) and the multiple hypergeometric function under the conditions given in the Theorem 3.1, we get the multiple Hurwitz-Lerch Zeta function (1.1)-(1.3) based upon the Srivastava-Daoust hypergeometric series in several variables [20, p.37] as given in (3.3). □

Special cases of the multiple Hurwitz-Lerch Zeta function based upon the Srivastava-Daoust hypergeometric series in several variables (3.3)

In (3.3) set $\delta_j^{(l)}=1,\ \phi_j^{(l)}=1,\ \theta_j^{(l)}=1,\ \psi_j^{(l)}=1,\ (\ \forall\ l=1,2,3,\ldots,n)$ and then for $s\in\mathbb{C}$, $\eta\in\mathbb{C}\backslash\mathbb{Z}_0$, we find $C+D^{(l)}-A-B^{(l)}=0, A-C\le 0\ \forall\ l=1,2,3,\ldots,n, \max\{|z_1|\,(z_1\neq1),\ldots,|z_n|\,(z_n\neq1)\}<1$.

But when $z_1=1,\ldots,\ z_n=1$, there exists $C+D^{(l)}-A-B^{(l)}=0,\ A-C=0$, along with

$$\Re\left(\sum_{j=1}^{C}c_j+\sum_{j=1}^{D^{(l)}}d_j^{(l)}+s-\sum_{j=1}^{A}a_j-\sum_{j=1}^{B^{(l)}}b_j^{(l)}-\rho_l\right)>0\quad(\ \forall\ l=1,2,3,\ldots,n),$$

then we get the multiple Hurwitz-Lerch Zeta function based upon the Srivastava-Panda hypergeometric series in several variables [18] as

$$(3.5)\qquad {}_{\eta}^{s}K\left(\mathcal{H}_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}};\rho_1,\ldots,\rho_n;z_1,\ldots,z_n\right)$$

$$={}_{\eta}^{s}F_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)}+1;\ldots;B^{(n)}+1}\left(\begin{matrix}(a):\left(b^{(1)}\right),\ \rho_1;\ldots;\left(b^{(n)}\right),\rho_n;\\ (c):\left(d^{(1)}\right);\ldots;\left(d^{(n)}\right);\end{matrix}z_1,\ldots,z_n\right).$$

Further in cite (3.5), set $A=1,\ C=1,\ \ B^{(l)}=0,\ D^{(l)}=0,\ \left(b^{(l)}\right)=\left(d^{(l)}\right)\ (\ \forall\ \ l=1,2,3,\ldots,n),\ a,s\in\mathbb{C}$ and $\eta,\quad \mathrm{c}\in\mathbb{C}\backslash\mathbb{Z}_0$, we find a multiple Hurwitz-Lerch Zeta function based upon the Lauricella's hypergeometric series in several variables [9] as

$$(3.6)\qquad {}_{\eta}^{s}K\left(\mathcal{H}_{1:0;\ldots;0}^{1:0;\ldots;0};\rho_1,\ldots,\rho_n;z_1,\ldots,z_n\right)={}_{\eta}^{s}F_{1:0;\ldots;0}^{1:1;\ldots;1}\left(\begin{matrix}a:\ \rho_1;\ldots;\rho_n;\\ c:0;\ldots;0;\end{matrix}z_1,\ldots,z_n\right)$$

$$=F_D^{(n)}\left(a,\rho_1,\ldots,\rho_n;c;z_1,\ldots,z_n\right),$$

provided that $\max\left\{|z_1|\left(z_1\neq1\right),\ldots,|z_n|\left(z_n\neq1\right)\right\}<1$; as well as with aid of Theorem 2.1, for

$z_1=\ldots=z_n=z$, it converges for $|z|<1\ (z\neq1)$, and

$\Re\left(c+s-a-\rho_1-\ldots-\rho_n\right)>0$ with $z=1$.

Again, in (3.5) put $n=2$, we find the double Hurwitz-Lerch Zeta function based upon the Kampé de Fériet hypergeometric series in two variables [21, p.63, Eqn.(16)] as

$$(3.7)\quad {}_{\eta}^{s}K\left(\mathcal{H}_{C:D^{(1)};D^{(2)}}^{A:B^{(1)};B^{(2)}};\rho_1,\rho_2;z_1,z_2\right)={}_{\eta}^{s}F_{C:D^{(1)};D^{(2)}}^{A:B^{(1)}+1;B^{(2)}+1}\left(\begin{matrix}(a):\left(b^{(1)}\right),\ \rho_1;\left(b^{(2)}\right),\rho_2;\\ (c):\left(d^{(1)}\right);\left(d^{(2)}\right);\end{matrix}z_1,z_2\right).$$

It is provided that $s\in\mathbb{C}$, $\eta\in\mathbb{C}\backslash\mathbb{Z}_0$, we find $C+D^{(l)}-A-B^{(l)}=0$ and $\mathfrak{H}_l=A-C\le0,\ \forall\ \ l=1,2$; $\max\left\{|z_1|\left(z_1\neq1\right),|z_2|\left(z_2\neq1\right)\right\}<1$.

But when $z_1=1,\ z_2=1$, there exists $C+D^{(l)}-A-B^{(l)}=0$ and $\mathfrak{H}_l=A-C=0\quad\forall\ l=1,2$; along with

$$\Re\left(\sum_{j=1}^{C}c_j+\sum_{j=1}^{D^{(l)}}d_j^{(l)}+s-\sum_{j=1}^{A}a_j-\sum_{j=1}^{B^{(l)}}b_j^{(l)}-\rho_l\right)>0\quad(\forall\ l=1,2).$$

Obviously, by (3.7) we get a double zeta function due to Choi and Parmar [1]

$$(3.8)\qquad {}_{\eta}^{s}K\left(\mathcal{H}_{1:0;0}^{1:0;0};\rho_1,\rho_2;z_1,z_2\right)={}_{\eta}^{s}F_{1:0;0}^{1:1;1}\left(\begin{matrix}a:\ \rho_1;\rho_2;\\ c:-;-;\end{matrix}z_1,z_2\right)=\phi_{a,\rho_1,\rho_2,c}\left(z_1,z_2,s,\eta\right).$$

provided that $\max\left\{|z_1|\left(z_1\neq1\right),|z_2|\left(z_2\neq1\right)\right\}<1,\ s\in\mathbb{C},\ \ \eta,c\in\mathbb{C}\backslash\mathbb{Z}_0$.

But when $\left(z_1=1\right),\left(z_2=1\right),\Re\left(c+s-a-\rho_1-\rho_2\right)>0,(\forall\ l=1,2)$ along with $\eta,c\in\mathbb{C}\backslash\mathbb{Z}_0$.

4 Integral representations of the multiple Hurwitz-Lerch Zeta function (1.10)

We use the Eulerian integral formula [8, 10, 11] and find an integral representation of the multiple Hurwitz-Lerch Zeta function (1.10) as

$$(4.1)\qquad {}_{\eta}^{s}K\left(\chi;\rho_1,\ldots,\rho_n;z_1,\ldots,z_n\right)$$
$$=\frac{1}{\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}\left\{\sum_{k_1=0,\ldots,k_n=0}^{\infty}\chi\left(k_1,\ldots,k_n\right)\prod_{j=1}^{n}\frac{(\rho_j)_{k_j}}{k_j!}\left(z_je^{-t}\right)^{k_j}\right\}dt,$$

provided that $\eta, s\in\mathbb{C}$ such that $\Re(s)>0$, $\Re(\eta)>0$.

Theorem 4.1. *If* $\zeta(u)=u\left(1+\zeta(u)\right)^{\beta+1}$, $\zeta(0)=0$ *and* $M=m_1k_1+\ldots+m_nk_n\le N$*; and*

$$(4.2)\qquad \Delta_N^{(\alpha,\beta)}\left[\chi;m_1,\ldots,m_n;z_1,\ldots,z_n,t\right]=\sum_{k_1,\ldots,k_n=0}^{M\le N}\frac{(-N)_M}{(\alpha+\beta N+1)_M}\chi\left(k_1,\ldots,k_n\right)\prod_{j=1}^{n}\frac{(\rho_j)_{k_j}}{k_j!}\left(z_je^{-t}\right)^{k_j},$$

then the exists an integral formula

$$(4.3)\qquad \frac{1}{\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}\left\{\sum_{N=0}^{\infty}u^N\binom{\alpha+(\beta+1)N}{N}\Delta_N^{(\alpha,\beta)}\left[\chi;m_1,\ldots,m_n;z_1,\ldots,z_n,t\right]\right\}dt$$
$$=\frac{\left(1+\zeta(u)\right)^{\alpha+1}}{\{1-\beta\zeta(u)\}}{}_{\eta}^{s}K\left(\chi;\rho_1,\ldots,\rho_n;z_1\{-\zeta\}^{m_1},\ldots,z_n\{-\zeta\}^{m_n}\right).$$

Proof. Consider left hand side of (4.3) and make an appeal to the formula (4.2) to get

$$(4.4)\qquad \frac{1}{\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}\sum_{N=0}^{\infty}u^N\binom{\alpha+(\beta+1)N}{N}\sum_{k_1,\ldots,k_n=0}^{m_1k_1+\ldots+m_nk_n\le N}\frac{(-N)_{m_1k_1+\ldots+m_nk_n}}{(\alpha+\beta N+1)_{m_1k_1+\ldots+m_nk_n}}$$
$$\times\chi\left(k_1,\ldots,k_n\right)\frac{(\rho_1)_{k_1}}{k_1!}\ldots\frac{(\rho_n)_{k_n}}{k_n!}\left(z_1e^{-t}\right)^{k_1}\ldots\left(z_ne^{-t}\right)^{k_n}dt.$$

Now in the integrand of (4.4) use generalized formula due to Carlitz (see in Srivastava and Manocha [21, p.360, Eqn.(1)]), given by

$$(4.5)\qquad \sum_{N=0}^{\infty}u^N\binom{\alpha+(\beta+1)N}{N}\sum_{k=0}^{\frac{N}{m}}\frac{(-N)_{\mathrm{mk}}}{(\alpha+\beta N+1)_{\mathrm{mk}}}\frac{\gamma_kx^k}{k!}=\frac{\left(1+\zeta(u)\right)^{\alpha+1}}{\{1-\beta\zeta(u)\}}\sum_{k=0}^{\infty}\frac{\gamma_k}{k!}\left(x\{-\zeta(u)\}^m\right)^k,$$

and then we obtain

$$(4.6)\qquad \frac{\left(1+\zeta(u)\right)^{\alpha+1}}{\{1-\beta\zeta(u)\}\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}\sum_{k_1,\ldots,k_n=0}^{\infty}\chi\left(k_1,\ldots,k_n\right)\prod_{j=1}^{n}\frac{(\rho_j)_{k_j}}{k_j!}\left(z_je^{-t}\{-\zeta(u)\}^{m_j}\right)^{k_j}\mathrm{dt}.$$

In Eqn. (4.6) we use the formula (4.1) and derive right hand side of (4.3). □

Corollary 4.1. *If all conditions of the Theorem* 4.1 *are satisfied and if*

$$(4.7)\qquad \chi=\mathcal{H}_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}},\ \text{(given in (1.2))},$$

then by the definition *of Srivastava-Panda hypergeometric series in several variables* [18]*, there exists*

$$(4.8)\qquad \Delta_N^{(\alpha,\beta)}\left[\mathcal{H}_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}};m_1,\ldots,m_n;z_1,\ldots,z_n,t\right]$$
$$=F_{C+1:D^{(1)};\ldots;D^{(n)}}^{A+1:B^{(1)}+1;\ldots;B^{(n)}+1}\left(\begin{matrix}\left[(a):\theta^{(1)},\ldots,\theta^{(n)}\right],\ [-N:m_1,\ldots,m_n]:\\ \left[(c):\delta^{(1)},\ldots,\delta^{(n)}\right],[\alpha+\beta N+1:m_1,\ldots,m_n]:\end{matrix}\right.$$
$$\left.\begin{matrix}\left[\left(b^{(1)}\right):\psi^{(1)}\right],\ [\rho_1:1];\ldots;\left[\left(b^{(n)}\right):\psi^{(n)}\right],[\rho_n:1];\\ \left[\left(d^{(1)}\right):\phi^{(1)}\right];\ldots;\left[\left(d^{(n)}\right):\phi^{(n)}\right];\end{matrix}z_1e^{-t},\ldots,z_ne^{-t}\right).$$

and the integral representation

$$(4.9)\qquad \frac{\left(1+\zeta(u)\right)^{\alpha+1}}{\{1-\beta\zeta(u)\}}{}_{\eta}^{s}K\left(\mathcal{H}_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}};\rho_1,\ldots,\rho_n;z_1\{-\zeta(u)\}^{m_1},\ldots,z_n\{-\zeta(u)\}^{m_n}\right)$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty e^{-\eta t} t^{s-1} \left\{ \sum_{N=0}^{\infty} u^N \binom{\alpha + (\beta+1)N}{N} \right.$$

$$\times F^{A+1:B^{(1)}+1;\ldots;B^{(n)}+1}_{C+1:D^{(1)};\ldots;D^{(n)}} \left(\begin{matrix} [(a):\theta^{(1)},\ldots,\theta^{(n)}],\ [-N:m_1,\ldots,m_n]: \\ [(c):\delta^{(1)},\ldots,\delta^{(n)}],[\alpha+\beta N+1:m_1,\ldots,m_n]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b^{(1)}):\psi^{(1)}],\ [\rho_1:1];\ldots;[(b^{(n)}):\psi^{(n)}],[\rho_n:1]; \\ [(d^{(1)}):\phi^{(1)}];\ldots;[(d^{(n)}):\phi^{(n)}]; \end{matrix} z_1e^{-t},\ldots,z_ne^{-t} \right) dt$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty e^{-\eta t} t^{s-1} \left\{ \sum_{N=0}^{\infty} u^N \binom{\alpha + (\beta+1)N}{N} \quad \Delta_N^{(\alpha,\beta)} \left[\mathcal{H}^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}}; m_1,\ldots,m_n; z_1,\ldots,z_n,t \right] \right\} dt.$$

Corollary 4.2. *If all conditions of the Theorem 4.1 are satisfied and in the coefficients*

$$\chi = \mathcal{H}^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}},$$

we consider

(4.10) $$A = C = 0; B^{(1)} = D^{(1)},\ldots,\ B^{(n)} = D^{(n)}; \theta^{(1)} = \ldots = \theta^{(n)} = 1; \delta^{(1)} = \ldots = \delta^{(n)} = 1;$$
$$\psi^{(1)} = \ldots = \psi^{(n)} = 1; \phi^{(1)} = \ldots = \phi^{(n)} = 1; \left(b^{(1)}\right) = \left(d^{(1)}\right);\ \ldots; \left(b^{(n)}\right) = \left(d^{(n)}\right).$$

Also let $m_1 = \ldots = m_n = 1$.

Then by the definition *of Lauricella's hypergeometric series in several variables* [9],*following results hold*

(4.11) $$\Delta_N^{(\alpha,\beta)} \left[\mathcal{H}^{0:B^{(1)};\ldots;B^{(n)}}_{0:B^{(1)};\ldots;B^{(n)}}; 1,\ldots,1; z_1,\ldots,z_n,t \right]$$
$$= F_D^{(n)}(-N,\rho_1,\ldots,\rho_n;\alpha+\beta N+1; z_1e^{-t},\ldots,z_ne^{-t}).$$

(4.12) $$\frac{(1+\zeta(u))^{\alpha+1}}{\{1-\beta\zeta(u)\}} {}^s_\eta K \left(\mathcal{H}^{0:B^{(1)};\ldots;B^{(n)}}_{0:B^{(1)};\ldots;B^{(n)}}; \rho_1,\ldots,\rho_n; z_1\{-\zeta(u)\},\ldots,z_n\{-\zeta(u)\} \right)$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty e^{-\eta t} t^{s-1} \left\{ \sum_{N=0}^{\infty} u^N \binom{\alpha + (\beta+1)N}{N} \quad \Delta_N^{(\alpha,\beta)} \left[\mathcal{H}^{0:B^{(1)};\ldots;B^{(n)}}_{0:B^{(1)};\ldots;B^{(n)}}; 1,\ldots,1; z_1,\ldots,z_n,t \right] \right\} dt.$$

In the Eqns. (4.11) *and* (4.12), *set* $z_1 = \ldots = z_n = z$ *to get*

(4.13) $$\Delta_N^{(\alpha,\beta)} \left[\mathcal{H}^{0:B^{(1)};\ldots;B^{(n)}}_{0:B^{(1)};\ldots;B^{(n)}}; 1,\ldots,1; z,\ldots,z,t \right] = {}_2F_1(-N,\rho_1+\ldots+\rho_n;\alpha+\beta N+1; ze^{-t}).$$

(4.14) $${}^s_\eta K \left(\mathcal{H}^{0:B^{(1)};\ldots;B^{(n)}}_{0:B^{(1)};\ldots;B^{(n)}}; \rho_1,\ldots,\rho_n; z\{-\zeta(u)\},\ldots,z\{-\zeta(u)\} \right)$$
$$= \frac{(1+\zeta(u))^{\alpha+1}}{\{1-\beta\zeta(u)\}} \int_0^\infty \frac{e^{-\eta t}}{(1-z\{-\zeta(u)\}e^{-t})^{\rho_1+\ldots+\rho_n}} \frac{t^{s-1}}{\Gamma(s)} dt$$
$$= \frac{(1+\zeta(u))^{\alpha+1}}{\{1-\beta\zeta(u)\}} \int_0^\infty \frac{e^{-(\eta-\rho_1-\ldots-\rho_n)t}}{(e^t-z\{-\zeta(u)\})^{\rho_1+\ldots+\rho_n}} \frac{t^{s-1}}{\Gamma(s)} dt,$$

provided that $\Re(\eta - \rho_1 - \ldots - \rho_n) > 0$.

Again since we know that a relation of Lauricella's multiple hypergeometric function with the Appell's double hypergeometric function given by

(4.15) $$F_D^{(n)} = F_1,$$

and hence on setting $n = 2$ *in* (4.11) *and* (4.12), *we get*

(4.16) $$\Delta_N^{(\alpha,\beta)} \left[\mathcal{H}^{0:B^{(1)};B^{(2)}}_{0:B^{(1)};B^{(2)}}; 1,1; z_1,z_2,t \right] = F_1(-N,\rho_1,\rho_2;\alpha+\beta N+1; z_1e^{-t}, z_2e^{-t}).$$

(4.17) $$\frac{(1+\zeta(u))^{\alpha+1}}{\{1-\beta\zeta(u)\}} {}^s_\eta K \left(\mathcal{H}^{0:B^{(1)};B^{(2)}}_{0:B^{(1)};B^{(2)}}; \rho_1,\rho_2; z_1\{-\zeta(u)\}, z_2\{-\zeta(u)\} \right)$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty e^{-\eta t} t^{s-1} \left\{ \sum_{N=0}^{\infty} u^N \binom{\alpha + (\beta+1)N}{N} \quad \Delta_N^{(\alpha,\beta)} \left[\mathcal{H}^{0:B^{(1)};B^{(2)}}_{0:B^{(1)};B^{(2)}}; 1,1; z_1,z_2,t \right] \right\} dt$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty e^{-\eta t} t^{s-1} \left\{ \sum_{N=0}^{\infty} u^N \binom{\alpha + (\beta+1)N}{N} \quad F_1(-N,\rho_1,\rho_2;\alpha+\beta N+1; z_1e^{-t}, z_2e^{-t}) \right\} dt.$$

5 Application in non-homogeneous initial value fractional differential equation

It is familiar that the Caputo fractional differential operator $\left({}^{C}D^{\alpha}_{a+}y\right)(x)$ is defined on a finite interval $[a,b]$ where $y(x)\in \mathrm{AC}^{n}[a,b]$, $\Re(\alpha)\geqq 0$ and for $n=[\Re(\alpha)]+1$ for $\alpha\notin\mathbb{N}_0$; $\mathbb{N}_0=\mathbb{N}\cup\{0\}$; $n=\alpha$ for $\alpha\in\mathbb{N}_0$. $\mathrm{AC}^{n}[a,b]=\left\{y:[a,b]\to\mathbb{C}\quad\text{and}\quad\left(D^{n-1}y\right)(x)\in AC[a,b]\ \left(D=\frac{d}{\mathrm{dx}}\right)\right\}$; again if $\varphi(t)\in L(a,b)$, then for $y(x)\in AC[a,b]\Leftrightarrow y(x)=c+\int_a^x\varphi(t)dt\Longrightarrow\frac{d}{\mathrm{dx}}y(x)=\varphi(x),\ \frac{d}{\mathrm{dx}}y(x)\Big|_{x\,=\,a}=c$; then there exists

$$\left({}^{C}D^{\alpha}_{a+}y\right)(x)=\frac{1}{\Gamma\left((n-\alpha)\right)}\int_0^x\frac{y^{(n)}(t)dt}{(x-t)^{\alpha+1-n}}\quad\left(x\in\mathbb{R}^{+}\right). \tag{5.1}$$

(See in [5, p.97]).

Again if $0<\alpha\leqq 1$, then by [4, p.98] where $(\mathcal{L}y)(\eta)=\frac{1}{\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}y(t)dt,\quad \eta\in\mathbb{C},\ \Re(\eta)>0$, there exists

$$\left(\mathcal{L}^{C}D^{\alpha}_{0+}y\right)(\eta)=\eta^{\alpha}\left(\mathcal{L}y\right)(\eta)-\eta^{\alpha-1}y(0). \tag{5.2}$$

Theorem 5.1. *In reference of* (5.1), *for* $t\in\mathbb{R}^{+}=(0,\infty)$, *if we introduce a non-homogeneous initial value fractional differential equation as*

$$\left({}^{C}D^{\alpha}_{a+}y\right)(t)+y(t)=\frac{t^{s-1}}{\Gamma(s)}\left\{\sum_{k_1=0,\ldots,k_n=0}^{\infty}\chi\left(k_1,\ldots,k_n\right)\prod_{j=1}^{n}\frac{\left(\rho_j\right)_{k_j}}{k_j!}\left(z_je^{-t}\right)^{k_j}\right\};y(0)=0; \tag{5.3}$$

provided that $s\in\mathbb{C}$, $\Re(s)>0$.

Then for $s\in\mathbb{C}$, $\Re(s)>0,\quad 0<\alpha\leqq 1$, *there exists*

$$y(t)=\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}e^{t\eta}\frac{{}^{s}_{\eta}K(\chi;\rho_1,...,\rho_n;z_1,...,z_n)}{(\eta^{\alpha}+1)}d\eta\qquad\left(\tau=\Re(\eta)>0,\quad i=\sqrt{(-1)}\right). \tag{5.4}$$

Proof. Consider $\eta\in\mathbb{C},\Re(\eta)>0$, and take *Laplace transformation* of both sides of the Eqn. (5.3) and then use formulae (4.1) and (5.2) with initial value given in (5.3), we get

$$\left(\eta^{\alpha}+1\right)\left(\mathcal{L}y\right)(\eta)={}^{s}_{\eta}K\left(\chi;\rho_1,\ldots,\rho_n;z_1,\ldots,z_n\right), \tag{5.5}$$

which on taking *inverse Laplace transformation* gives a solution

$$y(t)=\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}e^{t\eta}\frac{{}^{s}_{\eta}K(\chi;\rho_1,...,\rho_n;z_1,...,z_n)}{(\eta^{\alpha}+1)}d\eta\qquad\left(\tau=\Re(\eta)>0,\quad i=\sqrt{(-1)}\right) \tag{5.6}$$

provided that $s\in\mathbb{C}$, $\Re(s)>0,\ \ 0<\alpha\leqq 1$. ☐

6 Concluding remarks

In this section, in the Corollary 2.2, set $p=1,\ q=1,\ a_1=\rho_1+\ldots+\rho_n-\frac{1}{2},\ b_1=2\left(\rho_1+\ldots+\rho_n\right)\in\mathbb{C}\backslash\mathbb{Z}_0$, $\lambda_1=1$, $\mu_1=1$ and then apply the result [12, p.70].Thus we find an interesting Zeta function for $\Re(s)>0,\Re(\eta)>0$ as

$$\begin{aligned}{}^{s}_{\eta}K\left(\chi;\rho_1,\ldots,\rho_n;z,\ldots,z\right)&=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\ \left(\rho_1+\ldots+\rho_n\right)_k}{\left(b_1\right)_k\quad k!}\frac{z^k}{(k+\eta)^s}=\Phi^{(1,1;1)}_{(a_1,\rho_1+\ldots+\rho_n;b_1)}(z,s,\eta)\\&=\frac{1}{\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}\left\{{}_2F_1\left(\begin{matrix}\rho_1+\ldots+\rho_n-\ \frac{1}{2},\rho_1+\ldots+\rho_n;\\2\left(\rho_1+\ldots+\rho_n\right);\end{matrix}ze^{-t}\right)\right\}dt\\&=\frac{1}{\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}\left\{\left(\frac{2}{1+\sqrt{(1-ze^{-t})}}\right)^{2(\rho_1+\ldots+\rho_n)\ -1}\right\}dt\\&=\frac{1}{\Gamma(s)}\int_0^\infty e^{-\left(\eta\ +\ \ \frac{1}{2}\ -\rho_1-\ldots-\rho_n\right)t}t^{s-1}\left\{\left(\frac{2}{e^{\frac{t}{2}}\ +\sqrt{(e^t-z)}}\right)^{2(\rho_1+\ldots+\rho_n)\ -1}\right\}dt,\end{aligned} \tag{6.1}$$

provided $\Re\left(\eta\ +\ \ \frac{1}{2}\ -\rho_1-\ldots-\rho_n\right)>0$.

Again, in the Corollary 2.2, set $p=1,\ q=1,\ a_1=\rho_1+\ldots+\rho_n+\frac{1}{2},\ b_1=2\left(\rho_1+\ldots+\rho_n\right)\in\mathbb{C}\backslash\mathbb{Z}_0$,

$\lambda_1 = 1$, $\mu_1 = 1$, and then on application of the result [12, p.70], we find another interesting Zeta function for $\Re(s) > 0, \Re(\eta) > 0$ as

$$(6.2) \qquad {}^{s}_{\eta}K\left(\chi; \rho_1, \ldots, \rho_n; z, \ldots, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ \ (\rho_1 + \ldots + \rho_n)_k}{(b_1)_k \quad k!} \frac{z^k}{(k+\eta)^s} = \Phi^{(1,1;1)}_{(a_1, \rho_1 + \ldots + \rho_n; b_1)}(z, s, \eta)$$

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-\eta t} t^{s-1} \left\{ {}_2F_1 \left(\begin{matrix} \rho_1 + \ldots + \rho_n + \ \frac{1}{2}, \rho_1 + \ldots + \rho_n; \\ 2\left(\rho_1 + \ldots + \rho_n\right); \end{matrix} ze^{-t} \right) \right\} dt$$

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-(\eta \ - \rho_1 - \ldots - \rho_n)t} t^{s-1} \left(\frac{1}{\sqrt{(e^t - z)}} \right) \left\{ \left(\frac{2}{e^{\frac{t}{2}} \ + \sqrt{(e^t - z)}} \right)^{2(\rho_1 + \ldots + \rho_n) \ -1} \right\} dt,$$

provided $\Re\left(\eta \ + \ \ \frac{1}{2} \ - \rho_1 - \ldots - \rho_n\right) > 0$.

Various generating relations and integral representations may be found by applying results obtained in the *Sections* 3 to 5.

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

FIXED POINT RESULTS IN ORDERED EXTENDED RECTANGULAR $b-$ METRIC SPACE WITH GERAGHTY-WEAK CONTRACTION

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Abstract

In this paper, we prove some ordered-theoretic fixed point results for a Geraghty-weak contraction on an ordered extended rectangular $b-$metric spaces. Our results generalize several core results of the existing literature especially involving Geraghty-weak contractions and the results proved in extended rectangular $b-$metric space. Some examples are also furnished to exhibits the utility of our main results.

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1 Introduction

In 1992, Banach [10] introduced the classical fixed point theorem which is known as Banach contraction principle. The concept of generalized metric space has been increased by adding new generalized metrics one after another. The class of $b-$metric spaces [12] is generalized by the classes of extended $b-$metric spaces [21] as well as rectangular $b-$metric spaces [15] and so on. Now days, it is not only the metric spaces that are generalized by time to time but mappings are also. For example contraction mapping is generalized by weak contractions Geraghty contractions [14] and many others. The importance of fixed theory is also increasing day by day. In 2008, George et al. [15] introduced rectangular $b-$metric with the combination of rectangular and $b-$metric. In 2019, Asim et al. [4] introduced extended rectangular $b-$metric space and prove some fixed points. Recently in 2021, sharma and Tiwari [31] established some fixed-point theorems for three functions on contraction and expansive mappings in rectangular $b-$metric spaces. Also, very recently in 2022, Joshi [22] established some common fixed-point theorems for generalized multi-valued contraction in $b-$metric and dislocated $b-$metric spaces. Now, we apply the concept of ordered on extended rectangular $b-$metric space by using the mapping Geraghty-weak contraction. We recall the Definition of extended rectangular $b-$metric space as follow:

2 Preliminaries

Definition 2.1. ([4]). *Let U be non-empty set. Also $\theta : U\times U \to [1,\infty)$. Let a mapping $r_\theta : U\times U \to \mathbb{R}^+$ will be extended rectangular $b-$metric on U if it satisfy following properties ($\forall\ u, v \in U$ and $a, b \in U \setminus \{a, b\},\ a \neq b$):*

(a) $r_\theta(u, v) = 0 \iff u = v$,
(b) $r_\theta(u, v) = r_\theta(v, u)$,
(c) $r_\theta(u, v) \leq \theta(u, v)[r_\theta(u, a) + r_\theta(a, b) + r_\theta(b, v)]$.

The pair (U, r_θ) is said to be extended rectangular $b-$metric space.

Example 2.1. ([4]). Let $U = \{1, 2, 3, 4, 5\}$. A mapping $\theta : U \times U \to [1,\infty)$ such that $\theta(u, v) = u + v + 1\ \forall\ u, v \in U$. Also $r_\theta : U \times U \to \mathbb{R}^+$. Now, we can see that '$r_\theta$' is an extended $b-$metric space.

Definition 2.2. ([4]). *Let (U, r_θ) be an extended rectangular $b-$metric space and consider a sequence $\{u_n\}$ of U. We say that*

(a) $\{u_n\}$ *is said to be Cauchy if for each $\epsilon > 0$ there exists a natural number N such that $r_\theta(u_n, u_m) < \epsilon\ \forall\ n > m > N$.*
(b) $\{u_n\}$ *is said to be convergent if for each $\epsilon > 0$ there exists a natural number N such that $r_\theta(u_n, u) < \epsilon\ \forall\ n > N$.*

(c) (U, r_θ) *is said to be complete if every Cauchy sequence is convergent in* (U, r_θ).

Remark 2.1 ([4]). *If we replace the function 'θ' with a variable $s \geq 1$ then it will result in rectangular $b-$metric space. We can conclude that extended rectangular $b-$metric space $\Longrightarrow$ rectangular $b-$metric space. Before chalking out our main result, we give the following definitions, notations and results for the setting of ordered relation in the framework of extended rectangular $b-$metric spaces. So, lets move to the some definition which is needed in our forthcoming discussion.*

Definition 2.3. *Let $(U, \preceq)$ be an ordered set and (U, r_θ) an extended rectangular $b-$metric space. Then a triplet $(U, r_\theta, \preceq)$ is called an ordered extended rectangular $b-$metric space.*

Definition 2.4. *Let $(U, r_\theta, \preceq)$ be an ordered extended rectangular $b-$metric space. Let T be a self-mapping. Then*

(a) $(U, r_\theta, \preceq)$ *is said to follow the property increasing-convergence-comparable (in short ICC-property) if each terms of $\{u_{n_k}\}$, any subsequence of increasing convergent sequence $\{u_n\}$ in U is comparable with the limit of $\{u_n\}$. In other words,* $u_n \uparrow u$, *there exists $\{u_{n_k}\}$ a subsequence of $\{u_n\}$ also $u_n \prec\succ u \;\forall\; k \in \mathbb{N}$.*

(b) $(U, r_\theta, \preceq)$ *is said to follow the property of decreasing-convergence-comparable (in short DCC-property) if each terms of $\{u_{n_k}\}$, any subsequence of decreasing convergent sequence $\{u_n\}$ in U is comparable with the limit of $\{u_n\}$. In other words, $u_n \downarrow u$, there exists $\{u_{n_k}\}$ a subsequence of $\{u_n\}$ also $u_n \prec\succ u \;\forall\; k \in \mathbb{N}$.*

(c) $(U, r_\theta, \preceq)$ *is said to the property of follow monotone-convergence-comparable (in short MCC-property) if each terms of $\{u_{n_k}\}$, any subsequence of monotone convergent sequence $\{u_n\}$ in U is comparable with the limit of $\{u_n\}$. In other words, $u_n \uparrow\downarrow u$, there exists $\{u_{n_k}\}$ a subsequence of $\{u_n\}$ also $u_n \prec\succ u \;\forall\; k \in \mathbb{N}$.*

Definition 2.5. *Let $(U, r_\theta, \preceq)$ be an ordered extended rectangular $b-$metric space and T be a self-mapping on U. Then T is called $\overline{O} - r_\theta$-continuous (resp. $\underline{O} - r_\theta$-continuous, $O - r_\theta$-continuous) at point $u \in U$ if $T(u_n) \xrightarrow{r_\theta} T(u)$ $u_n \uparrow u$ (resp. $u_n \downarrow u, u_n \uparrow\downarrow u$) for any sequence $\{u_n\} \subset U$. Also, T is said to be $\overline{O} - r_\theta$-continuous (resp. $\underline{O} - r_\theta$-continuous, $O - r_\theta$-continuous) if T is $\overline{O} - r_\theta$-continuous (resp. $\underline{O} - r_\theta$-continuous, $O - r_\theta$-continuous) at each point of U.*

Remark 2.2. *In $(U, r_\theta, \preceq)$, continuity $\Longrightarrow$ $O - r_\theta$-continuity $\Longrightarrow$ $\overline{O} - r_\theta$-continuity also $\underline{O} - r_\theta$-continuity.*

Definition 2.6. *Let $\{u_n\}$ be a sequence in $(U, r_\theta, \preceq)$. Then $\{u_n\}$ will be $\overline{O} - r_\theta$-Cauchy (resp. $\underline{O} - r_\theta$-Cauchy, $O - r_\theta$-Cauchy) at point $u \in U$ if $\{u_n\}$ is an increasing sequence (resp. decreasing and monotone) and r_θ-Cauchy. Moreover, $\{u_n\}$ is called $\overline{O} - r_\theta$-convergent (resp. $\underline{O} - r_\theta$-convergent, $O - r_\theta$-convergent) at point $u \in U$ if $\{u_n\}$ is an increasing (resp. decreasing and monotone) r_θ-convergent sequence, abbreviated by $u_n \uparrow u$ (resp. $u_n \downarrow u, u_n \uparrow\downarrow u$).*

Definition 2.7. *Let $\{u_n\}$ be any sequence in $(U, r_\theta, \preceq)$. Then $\{u_n\}$ is said to be $\overline{O} - r_\theta$-complete (resp. $\underline{O} - r_\theta$-complete, $O - r_\theta$-complete) at point $u \in U$ if each $\overline{O} - r_\theta$-Cauchy (resp. $\underline{O} - r_\theta$-Cauchy, $O - r_\theta$-Cauchy) sequence in U if it converges to any point $u \in U$.*

Remark 2.3. *In ordered extended rectangular $b-$metric space, completeness $\Longrightarrow$ $O - r_\theta$-completeness $\Longrightarrow$ $\overline{O} - r_\theta$-completeness also $\underline{O} - r_\theta$-completeness.*
Now, we have all the definition regarding the topic in our minds. The first classic fixed point theory was given by S. Banach [10] known as Banach contraction principle. But as few decades passed away, it has been generalized number of ways one of them is Geraghty-weak contraction. Geraghty principle came into existence in 1973 when Geraghty generalized the Banach contraction principle. Later, in 2016 Roshan et al. by using Geraghty-weak contraction proved fixed point results in the b-metric space. Also in 2021, fixed point results in ordered partial rectangular $b-$metric space was proved by Asim et al. [5] with Geraghty-weak contraction theory. Now a day, many researchers are utilizing of this mapping in their research. In this chapter we are using Geraghty-weak contraction principle to prove fixed point results for ordered extended rectangular $b-$metric space by employing suitable conditions.

3 Main Results

Definition 3.1. *Let* $\lambda : [0,\infty) \to [0, \frac{1}{\theta})$ $(\theta : U \times U \to [1,\infty))$ *which satisfy the given condition for* $u_n \in [0,\infty)$*, any sequence:*

$$\lim_{n\to\infty} \sup \lambda(u_n) = \frac{1}{\theta} \Longrightarrow \lim_{n\to\infty} (u_n) = 0.$$

The collection of such functions of λ *is denoted by* Λ*.*

Definition 3.2. *Suppose* $(U, r_\theta, \preceq)$ *is an ordered extended rectangular* $b-$*metric space. Let* T *be a self-mapping, is called Geraghty-weak contraction if* $\exists \;\; \lambda \in \Lambda$ *we have* $u \preceq v \;\; \forall\, u, v \in U)$ *such that*

$$r_\theta(T(u, T(v))) \leq \lambda(r_\theta(u,v)) M(r_\theta(u,v)), \tag{3.1}$$

and

$$M(r_\theta(u,v)) = \max\left\{(r_\theta(u,v)), \frac{r_\theta(u,T(u))r_\theta(v,T(v))}{1+r_\theta(T(u),T(v))}, \frac{r_\theta(u,T(u))r_\theta(v,T(v))}{1+r_\theta(u,v)}, \frac{r_\theta(u,T(u))r_\theta(u,T(v))}{1+r_\theta(u,T(v))+r_\theta(v,T(u))}\right\}.$$

Theorem 3.1. *Let* $(U, r_\theta, \preceq)$ *be an ordered extended rectangular b-metric space and* $T : U \to U$ *an increasing mapping. Suppose these conditions holds:*

(i) there exists an $u_0 \in U$ *such that* $u_0 \preceq T(u_0)$*,*
(ii) T *is Geraghty-weak contraction,*
(iii) $(U, r_\theta, \preceq)$ *is* $\overline{O} - r_\theta$*-complete,*
(iv) either
  (a) T *is* $\overline{O} - r_\theta$*-continuous or*
  (b) $(U, r_\theta, \preceq)$ *have the ICC-property.*

Then we assure that T *has a fixed point.*

Proof. Let $u_0 \in U$ such that $u_0 \preceq T(u_0)$. As we know the mapping T is an increasing hence, we can construct an increasing sequence $\{u_n\}$, then we have for all $n \in \mathbb{N}_0$

$$u_1 = T(u_0),\; u_2 = T(u_1),\; u_3 = T(u_2), \cdots, u_{n+1} = T(u_n).$$

If we have $r_\theta(u_n, u_{n+1}) = 0$ for some $n \in \mathbb{N}_0$, then we can say that $\{u_n\}$ is a fixed point of T and we get our required result. Now, we have to suppose that $r_\theta(u_n, u_{n+1}) > 0$ for all $n \in \mathbb{N}_0$. We assert that $\lim_{n\to\infty} r_\theta(u_n, u_{n+1}) = 0$. By placing $u = u_{n-1}$ with $v = u_n$ in (3.1), we have result

$$\begin{aligned} r_\theta(u_n, u_{n+1}) &= r_\theta(T(u_{n-1}), T(u_n)) \\ &\leq \lambda(r_\theta(u_{n-1}, u_n)) M(r_\theta(u_{n-1}, u_n)) \\ &< \frac{1}{\theta} M(r_\theta(u_{n-1}, u_n)) \leq M(r_\theta(u_{n-1}, u_n)) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} M(r_\theta(u_{n-1}, u_n)) &= \max\left\{r_\theta(u_{n-1}, u_n), \frac{r_\theta(u_{n-1}, T(u_{n-1}))r_\theta(u_n, T(u_n))}{1+r_\theta(T(u_{n-1}), T(u_n))},\right. \\ &\qquad \frac{r_\theta(u_{n-1}, T(u_{n-1}))r_\theta(u_n, T(u_n))}{1+r_\theta(u_{n-1}, u_n)}, \\ &\qquad \left.\frac{r_\theta(u_{n-1}, T(u_{n-1}))r_\theta(u_{n-1}, T(u_n))}{1+r_\theta(u_{n-1}, T(u_n))+r_\theta(u_n, T(u_{n-1}))}\right\} \\ &= \max\left\{r_\theta(u_{n-1}, u_n), \frac{r_\theta(u_{n-1}, u_n)r_\theta(u_n, u_{n+1})}{1+r_\theta(u_n, u_{n+1})},\right. \\ &\qquad \left.\frac{r_\theta(u_{n-1}, u_n)r_\theta(u_n, u_{n+1})}{1+r_\theta(u_{n-1}, u_n)}, \frac{r_\theta(u_{n-1}, u_n)r_\theta(u_{n-1}, u_{n+1})}{1+r_\theta(u_{n-1}, u_{n+1})+r_\theta(u_n, u_n)}\right\} \\ &\leq \max\{r_\theta(u_{n-1}, u_n), r_\theta(u_{n-1}, u_n), r_\theta(u_n, u_{n+1}), r_\theta(u_{n-1}, u_n)\} \end{aligned}$$

$$= \max\{r_\theta(u_{n-1},u_n), r_\theta(u_n,u_{n+1})\}.$$

Now suppose that, $\max\{r_\theta(u_{n-1},u_n), r_\theta(u_n,u_{n+1})\} = r_\theta(u_n,u_{n+1})$, by using (3.2) we get,

$$r_\theta(u_n,u_{n+1}) < \frac{1}{\theta} M(r_\theta(u_{n-1},u_n)) \leq r_\theta(u_n,u_{n+1}).$$

which is a contradiction. Hence, $\max\{r_\theta(u_n,u_{n+1}, r_\theta(u_n,u_{n+1})\} = r_\theta(u_{n-1},u_n)$. Therefore, by using (3.2) we have,

$$r_\theta(u_n,u_{n+1}) < r_\theta(u_{n-1},u_n). \tag{3.3}$$

Thus $\{r_\theta(u_n,u_{n+1})\}$ is the decreasing sequence of non-negative real numbers. Hence, there must exists $b \geq 0$ such that

$$\lim_{n\to\infty} r_\theta(u_n,u_{n+1}) = b.$$

Assume that $b > 0$. Then from (3.2), we get

$$\lim_{n\to\infty} r_\theta(u_n,u_{n+1}) \leq \lim_{n\to\infty} [\lambda(r_\theta(u_{n-1},u_n))M(r_\theta(u_{n-1},u_n)].$$

By the definition of λ we get $b < \frac{1}{\theta} b$, a contraction. Thus,

$$\lim_{n\to\infty} r_\theta(u_n,u_{n+1}) = 0. \tag{3.4}$$

Now, by taking $u = u_{n-1}$ with that we take $v = u_{n+1}$ in (3.1), we get

$$\begin{aligned} r_\theta(u_n,u_{n+2}) &= r_\theta(T(u_{n-1}),T(u_{n+1})) \leq \lambda(r_\theta(u_{n-1},u_{n+1})M(r_\theta(u_{n-1},u_{n+1})) \\ &< \frac{1}{\theta} M(r_\theta(u_{n-1},u_{n+1})) \leq M(r_\theta(u_{n-1},u_{n+1})), \end{aligned} \tag{3.5}$$

where,

$$\begin{aligned} M(r_\theta(u_{n-1},u_{n+1})) &= \max\Bigg\{ r_\theta(u_{n-1},u_{n+1}), \frac{r_\theta(u_{n-1},T(u_{n-1}))r_\theta(u_{n+1},T(u_{n+1}))}{1 + r_\theta(T(u_{n-1}),T(u_{n+1}))}, \\ &\quad \frac{r_\theta(u_{n-1},T(u_{n-1}))r_\theta(u_{n+1},T(u_{n+1}))}{1 + r_\theta(u_{n-1},u_{n+1})}, \\ &\quad \frac{r_\theta(u_{n-1},T(u_{n-1}))r_\theta(u_{n-1},T(u_{n+1}))}{1 + r_\theta(u_{n-1},T(u_{n+1})) + r_\theta(u_{n+1},T(u_{n-1}))} \Bigg\} \\ &= \max\Bigg\{ r_\theta(u_{n-1},u_{n+1}), \frac{r_\theta(u_{n-1},u_n)r_\theta(u_{n+1},u_{n+2})}{1 + r_\theta(u_n,u_{n+2})}, \\ &\quad \frac{r_\theta(u_{n-1},u_n)r_\theta(u_{n+1},u_{n+2})}{1 + r_\theta(u_{n-1},u_{n+1})}, \frac{r_\theta(u_{n-1},u_n)r_\theta(u_{n-1},u_{n+2})}{1 + r_\theta(u_{n-1},u_{n+2}) + r_\theta(u_{n+1},u_n)} \Bigg\} \\ &\leq \max\Big\{ r_\theta(u_{n-1},u_{n+1}), [r_\theta(u_{n-1},u_n)r_\theta(u_{n+1},u_{n+2})] \\ &\quad [r_\theta(u_{n-1},u_n)r_\theta(u_{n+1},u_{n+2})], r_\theta(u_{n-1},u_n) \Big\}. \end{aligned}$$

Using (3.3) we get,

$$M(r_\theta(u_{n-1},u_{n+1})) \leq \max\{r_\theta(u_{n-1},u_{n+1}), r_\theta(u_{n-1},u_n), [r_\theta(u_{n-1},u_n)]^2\}.$$

First of all, let us suppose that

$$\max\{r_\theta(u_{n-1},u_{n+1}), r_\theta(u_{n-1},u_n), [r_\theta(u_{n-1},u_n)]^2\} = r_\theta(u_{n-1},u_n) or [r_\theta(u_{n-1},u_n)]^2.$$

As $\lim_{n\to\infty} r_\theta(u_{n-1},u_n) = 0$, by using (3.5), we get

$$\lim_{n\to\infty} r_\theta(u_n,u_{n+2}) = 0.$$

If the equation $\max\{r_\theta(u_{n-1},u_{n+1}), r_\theta(u_{n-1},u_n), [r_\theta(u_{n-1},u_n)]^2\} = r_\theta(u_{n-1},u_{n+1})$ is true, by using (3.5), we get

$$r_\theta(u_n,u_{n+2}) < r_\theta(u_{n-1},u_{n+1}).$$

Thus $\{r_\theta(u_n,u_{n+2})\}$ is a decreasing sequence of non-negative real numbers. Hence, there must exists $b \geq 0$ such that

$$\lim_{n\to\infty} r_\theta(u_n,u_{n+2}) = b.$$

Let us assume that $b > 0$. Then from (3.5), we get

$$\lim_{n\to\infty} r_\theta(u_n, u_{n+2}) \leq \lim_{n\to\infty} \lambda(r_\theta(u_{n-1}, u_{n+1}))M(r_\theta(u_{n-1}, u_{n+1})).$$

By using the definition of λ we get $b < \frac{1}{\theta}b$, a contradiction. Thus, we obtain

$$\lim_{n\to\infty} r_\theta(u_n, u_{n+2}) = 0. \tag{3.6}$$

Now, we have to show that $u_n \neq u_m$ for each $n = m$. On contrary we suppose that, $u_n = u_m$ for some $n > m$, then we get $u_{n+1} = T(u_n) = T(u_m) = x_{m+1}$. Then, from (3.2) we have

$$\begin{aligned} r_\theta(u_m, u_{m+1}) &= r_\theta(u_n, u_{n+1}) = r_\theta(T(u_{n-1}, T(u_n))) \\ &\leq \lambda(r_\theta(u_{n-1}, u_n))M(r_\theta(u_{n-1}, u_n)) \\ &< \frac{1}{\theta}M(r_\theta(u_{n-1}, u_n)) \leq M(r_\theta(u_{n-1}, u_n)) \\ &\leq \max\{r_\theta(u_{n-1}, u_n)r_\theta(u_n, u_{n+1})\}. \end{aligned}$$

Therefore, we get

$$\max\{r_\theta(u_{n-1}, u_n), r_\theta(u_n, u_{n+1})\} = r_\theta(u_n, u_{n+1}),$$

so that

$$r_\theta(u_m, u_{m+1}) < r_\theta(u_n, u_{n+1}),$$

which is a contradiction. Suppose

$$\max\{r_\theta(u_{n-1}, u_n), r_\theta(u_n, u_{n+1})\} = r_\theta(u_{n-1}, u_n),$$

we have

$$r_\theta(u_m, u_{m+1}) = r_\theta(u_n, u_{n+1}) < r_\theta(u_{n-1}, u_n) < r_\theta(u_{n-2}, u_{n-1}) < \cdots < r_\theta(u_m, u_{m+1})),$$

which is a contradiction. So, we can take $u_n \neq u_m \ \ \forall\, n \neq m$. Now its turn to prove that $\{u_n\}$ is $\overline{O} - r_\theta$-Cauchy sequence in $(U, r_\theta, \preceq)$. On contrary suppose that, $\{u_n\}$ is not $\overline{O} - r_\theta$-Cauchy sequence. So there must exist $\epsilon > 0$ and also two subsequences $\{n_k\}$ and $\{m_k\}$ such that $\{n_k\}$ is the index which is smallest for that

$$\{n_k\} > \{m_k\} > k \text{ and } r_\theta(u_{m_k}, u_{n_k}) \geq \frac{\epsilon}{2}, \tag{3.7}$$

which implies that

$$r_\theta(u_{m_k}, u_{n_{k-1}}) < \frac{\epsilon}{2}. \tag{3.8}$$

Now, on using rectangular inequality, we have

$$\frac{\epsilon}{2} \leq r_\theta(u_{m_k}, u_{n_k}) \leq \theta(r_\theta(u_{m_k}, u_{n_{k-1}}) + \theta(r_\theta(u_{n_{k-1}}, u_{n_{k+1}})) + \theta(r_\theta(u_{n_{k+1}}, u_{n_k}). \tag{3.9}$$

Now, using (3.4) (3.6) (3.8) and also taking limit as $k \to \infty$, we have

$$\frac{\epsilon}{2} \leq \lim_{k\to\infty} \sup r_\theta(u_{m_k}, u_{n_k}) \leq \theta\left(\frac{\epsilon}{2}\right). \tag{3.10}$$

On using (3.1) and definition of r_θ, we have

$$\begin{aligned} \lim_{k\to\infty} r_\theta(u_{m_k}, u_{n_k}) &\leq \lim_{k\to\infty} \theta(r_\theta(x_{m_{k+1}}, u_{m_k})) + \lim_{k\to\infty} \theta(r_\theta(x_{m_{k+1}}, u_{n_{k+1}}) \\ &+ \lim_{k\to\infty} \theta(r_\theta(u_{n_{k+1}}, u_{n_k}), \\ &\leq \theta \lim_{k\to\infty} \lambda(r_\theta(u_{m_k}, u_{n_k}))M(r_\theta(u_{m_k}, u_{n_k})), \end{aligned} \tag{3.11}$$

where,

$$\begin{aligned} M(r_\theta(u_{m_k}, u_{n_k})) = \max\Bigg\{ & r_\theta(u_{m_k}, u_{n_k}), \frac{r_\theta(u_{m_k}, T(u_{m_k}))r_\theta(u_{n_k}, T(u_{n_k}))}{1 + r_\theta(T(u_{m_k}), T(u_{n_k}))}, \\ & \frac{r_\theta(u_{m_k}, T(u_{m_k}))r_\theta(u_{n_k}, T(u_{n_k}))}{1 + r_\theta(u_{m_k}, u_{n_k})}, \\ & \frac{r_\theta(u_{m_k}, T(u_{m_k}))r_\theta(u_{m_k}, T(u_{n_k}))}{1 + r_\theta(u_{m_k}, T(u_{n_k})) + r_\theta(u_{n_k}, T(u_{m_k}))} \Bigg\} \end{aligned} \tag{3.12}$$

$$= \max\left\{ r_\theta(u_{m_k}, u_{n_k}), \frac{r_\theta(u_{m_k}, u_{m_{k+1}}) r_\theta(u_{n_k}, u_{n_{k+1}})}{1 + r_\theta(x_{m_{k+1}}, u_{n_{k+1}})}, \right.$$
$$\frac{r_\theta(u_{m_k}, u_{m_{k+1}}) r_\theta(u_{n_k}, u_{n_{k+1}})}{1 + r_\theta(u_{m_k}, u_{n_k})},$$
$$\left. \frac{r_\theta(u_{m_k}, u_{m_{k+1}}) r_\theta(u_{m_k}, u_{n_{k+1}})}{1 + r_\theta(u_{m_k}, u_{n_{k+1}}) + r_\theta(u_{n_k}, u_{m_{k+1}})} \right\}.$$

Taking the limit $k \to \infty$ and using (3.12), we have

$$\lim_{n\to\infty} \sup M(r_\theta(u_{m_k}, u_{n_k})) = \lim_{n,m\to\infty} \sup(r_\theta(u_{m_k}, u_{n_k})).$$

By using (3.11), we get

$$\lim_{n\to\infty} \sup r_\theta(u_{m_k}, u_{n_k}) \leq \theta \lim_{n,m\to\infty} \sup \lambda(r_\theta(u_{m_k}, u_{n_k})) \lim_{n,m\to\infty} \sup(r_\theta(u_{m_k}, u_{n_k})).$$

As we have supposed that $\lim_{k\to\infty} \sup r_\theta(u_{m_k}, u_{n_k}) \neq 0$, then from above inequality, we have

$$\frac{1}{\theta} \leq \lim_{k\to\infty} \sup \lambda r_\theta(u_{m_k}, u_{n_k}).$$

Since $\lambda \in \Lambda$, so that $\lim_{n,m\to\infty} r_\theta(u_{m_k}, u_{n_k}) = 0$, which is in general a contradiction. Hence, we assure that $\{u_n\}$ is $\overline{O} - r_\theta$-Cauchy sequence in $(U, r_\theta, \preceq)$. As $(U, r_\theta, \preceq)$ is $\overline{O} - r_\theta$-complete so there must exist $u \in U$ such that $u_n \uparrow u$ and also,

$$\lim_{n,m\to\infty} r_\theta(u_n, u_m) = 0.$$

Now coming to last condition, first of all suppose that T is $\overline{O} - r_\theta$- continuous then we will show that x is a fixed point of T.

$$x = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} T(u_n) = T(\lim_{n\to\infty} u_n) = T(u).$$

Now, we take second condition, i.e., $(U, r_\theta, \preceq)$ follows ICC-property. So there must exist a subsequence of $\{u_n\}$ which is $\{u_{n_k}\}$ such that $\{u_{n_k}\} \prec\succ\ u\ \forall\ k \in \mathbb{N}$. First we take $\{u_{n_k}\ \preceq\ x\ \forall\ k \in \mathbb{N}$ (proof for both case are alike). So by using (3.1), we get

$$\begin{aligned} \lim_{k\to\infty} r_\theta(u_{n_{k+1}}, T(u)) &= \lim_{k\to\infty} r_\theta(T(u_{n_k}), T(u)) \\ &\leq \lim_{k\to\infty} \lambda(r_\theta((u_{n_k}), u)) \lim_{k\to\infty} M(r_\theta((u_{n_k}), u) \end{aligned}$$

where

$$\begin{aligned} \lim_{k\to\infty} M(r_\theta((u_{n_k}), u) &= \lim_{k\to\infty} \left(\max\left\{ r_\theta((u_{n_k}), u), \frac{r_\theta(u_{n_k}, T(u_{n_k})), r_\theta(u, T(u))}{1 + r_\theta(T(u_{n_k}), T(u))}, \right.\right. \\ &\quad \frac{r_\theta(u_{n_k}, T(u_{n_k})), r_\theta(u, T(u))}{1 + r_\theta(u_{n_k}, u)}, \\ &\quad \left.\left. \frac{r_\theta(u_{n_k}, T(u_{n_k})), r_\theta(u_{n_k}, T(u))}{1 + r_\theta(u_{n_k}, T(u)) + r_\theta(u, T(u_{n_k}))} \right\} \right) \\ &= \lim_{k\to\infty} \left(\max\left\{ r_\theta((u_{n_k}), u), \frac{r_\theta(u_{n_k}, u_{n_{k+1}}), r_\theta(u, T(u))}{1 + r_\theta(u_{n_{k+1}}, T(u))}, \right.\right. \\ &\quad \frac{r_\theta(u_{n_k}, u_{n_{k+1}}), r_\theta(u, T(u))}{1 + r_\theta(u_{n_k}, u)}, \\ &\quad \left.\left. \frac{r_\theta(u_{n_k}, u_{n_{k+1}}), r_\theta(u_{n_k}, T(u))}{1 + r_\theta(u_{n_k}, T(u)) + r_\theta(u, u_{n_{k+1}})} \right\} \right) \\ &= \lim_{k\to\infty} r_\theta((u_{n_k}), u). \end{aligned}$$

Therefore

$$\lim_{k\to\infty} r_\theta((u_{n_k}), u.) = \lim_{k\to\infty} r_\theta(u_{n_{k+1}}, T(u)) = 0.$$

We assert that the terms u_{n_k} and $u_{n_{k+1}}$ $\forall\ k \in \mathbb{N}$ are distinct from u and $T(u)$ both. By definition ,we have

$$r_\theta(u, T(u)) \leq \theta[r_\theta(u, u_{n_k}) + r_\theta(u_{n_k}, u_{n_{k+1}}) + r_\theta(u_{n_{k+1}}, T(u))]. \tag{3.13}$$

By taking $k \to \infty$ and using (3.4) and (3.13), we have $r_\theta(u, T(u)) = 0$. Hence we can say that $T(u) = u$. So, u is a fixed point of T. □

Example 3.1. Consider $U = (-1, 0]$. Define $r_\theta : U \times U \to \mathbb{R}_+$ by (for all $u, v \in U$):

$$r_\theta(u, v) = |u - v|^2.$$

Notice that, every increasing Cauchy sequence is convergent in U. Therefore, $(U, r_\theta, \preceq)$ is an $\overline{\text{O}}$-complete $r_\theta-$metric space with coefficient $\theta(u, v) = 2$ for all $u, v \in U$.

Now, we define an ordered relation on U as under:

$$u, v \in U,\ u \preceq v \Leftrightarrow\ u = v \text{ or } \left(u, v \in \{0\} \cup \left\{\frac{-1}{n} : n = 2, 3, \cdots\right\} \text{ and } u \leq v\right),$$

where $\leq$ is the usual order. Define the mappings $T : U \to U$ as follows:

$$Tu = \begin{cases} 0, & \text{if } u = 0 \\ \frac{-1}{2n}, & \text{if } u = -1/n, n = 2, 3, \cdots. \\ -0.5, & \text{otherwise} \end{cases}$$

Observe that, T is increasing and U has the ICC-property. We distinguish two cases:

Case 1. Taking $u = -1/n$, (wherein $n = 3, 4, \cdots$) and $v = 0$. Then, from (3.1), we have

$$r_\theta(Tu, Tv) = \left|\frac{-1}{2n} - 0\right|^2 = \frac{1}{4}\left|\frac{-1}{n} - 0\right|^2 = \frac{1}{4} r_\theta(u, v). \tag{3.14}$$

Case 2. Taking $u = -1/n, v = -1/m\ m > n \geq 3$. Then, we have

$$r_\theta(Tu, Tv) = \left|\frac{-1}{2n} - \frac{-1}{2m}\right|^2 = \frac{1}{4}\left|\frac{-1}{n} - \frac{-1}{m}\right|^2 = \frac{1}{4} r_\theta(u, v). \tag{3.15}$$

If $u = v$, then condition (3.1) holds trivially. Thus, all the conditions of Theorems 3.1 are satisfied and the mapping T has a unique fixed point (namely $u = 0$).

Example 3.2. Let $U = \{1, 2, 3, 4, 5\}$ be equipped with the order relation $\preceq$ given by

$$\preceq = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (4, 1), (4, 2), (4, 3), (4, 5), (1, 3), (2, 3), (5, 3)\}$$

and let $r_\theta : U \times U \to \mathbb{R}^+$ is defined by:

$$\begin{aligned} r_\theta(u, u) &= 0, \quad \text{for all } u \in U; \\ r_\theta(u, v) &= r_\theta(v, u), \quad \text{for all } u, v \in U; \\ r_\theta(1, 3) &= r_\theta(1, 5) = r_\theta(2, 3) = r_\theta(3, 5) = 3t; \\ r_\theta(1, 4) &= r_\theta(2, 4) = r_\theta(2, 5) = r_\theta(3, 4) = r_\theta(4, 5) = 4t; \\ r_\theta(1, 2) &= 5t; \end{aligned}$$

where $0 < t < -\ln(3/4)$, that is, $e^{-t} > 3/4$. Therefore, $(U, r_\theta, \preceq)$ is an $\overline{\text{O}}$-complete $r_\theta-$metric space with coefficient $\theta(u, v) = 3$ for all $u, v \in U$. Consider a mapping $T : U \to U$ defined by:

$$T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 1 & 3 \end{pmatrix}.$$

It is easy to check that all the conditions of Theorem 3.1 are fulfilled with $\lambda(u) = e^{-u}$ for each $u > 0$ and $\lambda(0) \in [0, 1/3)$. In particular, by choosing $u, v \in \{1, 2, 3, 5\}$ such that $u \preceq v$, then $Tu = Tv = 3$ implies the condition (3.1) is trivially holds. Now, if we take $u = 4$ and $v \in \{1, 2, 3, 5\}$, such that $u \preceq v$, we obtain $Tu = 1$ and $Tv = 3$. Then by (3.1), we have

$$\begin{aligned} r_\theta(Tu, Tv) &= r_\theta(1, 3) = 3t = \frac{3}{4} 4t < e^{-t}.4t \\ &= \lambda(t) d(x, y) \leq \lambda\big(r_\theta(x, y)\big) M(x, y). \end{aligned}$$

It follows that T has a unique fixed point (which is $x = 3$).

If we replace $\overline{O}$-completeness of U and $\overline{O}$-continuity of T in Theorem 3.1, then it remains a new version as given follows:

Corollary 3.1. *Let $(U, r_\theta, \preceq)$ be an ordered extended rectangular $b-$metric space and $T : U \to U$ be an increasing mapping. Suppose these conditions holds:*
1. *T follow Geraghty-weak contraction,*
2. *$(U, r_\theta, \preceq)$ is complete,*
3. *T is continuous.*

Then we assure that T has a fixed point.

If we replace Geraghty-weak-contraction by contraction condition in Theorem 3.1, then it remains a new version of the Theorem 3.1 due to Asim et al. [4].

Corollary 3.2. *Let $(U, r_\theta, \preceq)$ be an ordered extended rectangular $b-$metric space and $T : U \to U$ be an increasing mapping. Suppose these conditions holds:*
1. *there exists an $u_0 \in U$ such that $u_0 \preceq T(u_0)$,*
2. *If $u \preceq v \ \ \forall u, v \in U$, then we get*
$$r_\theta(T(u), T(v)) \leq LM(r_\theta(u, v)),$$
where, $L \in [0, \infty)$.
3. *$(U, r_\theta, \preceq)$ is $\overline{O} - r_\theta$-complete,*
4. *either*
   (a) *T is $\overline{O} - r_\theta$-continuous or follow*
   (b) *$(U, r_\theta, \preceq)$ have the ICC-property.*

Then we assure that T has a fixed point.

Corollary 3.3. *Let (U, r_θ) be a complete extended rectangular $b-$metric space and T be a continuous and self-mapping. Also suppose T follows the property of Geraghty-weak contraction. Then we assure that T has a fixed point.*

Proposition 3.1. ([5]). *Let $(U, r_\theta, \preceq)$ be an ordered extended rectangular $b-$metric space and $T : U \to U$ is a Geraghty-weak contraction. If $u \prec\succ v$ then $u = v \ \ \forall\ u, v \in Fix(f)$.*

Definition 3.3. ([20]). *Suppose $(U, \preceq)$ is an ordered set and let T be an self-mapping. Then we define*
$$U_T = \{u \in U : u \prec\succ T(u)\}.$$
Then $(U, \preceq)$ is said to be $T-$ directed if there exists $a \in U_T$ such that $u \prec\succ a \prec\succ v \ \ \forall u, v \in U$.

Theorem 3.2. *If with all the conditions of Theorem 3.1 we add that $(U, \preceq)$ is $T-$directed. Then we assure that T has a unique fixed point.*

Proof. Let us suppose that u and v be two different points of T. Also as $(U, \preceq)$ is $T-$directed then there must exists $a \in U_T$ such that $u \prec\succ a \prec\succ v$. If we take $a = u$ or $a = v$ then by above preposition we have $u = v$, which is a contradiction. Hence, we have to suppose that $u \neq a, v \neq a$. As we know $a \in U_T$ then we get $a \prec\succ T(a)$. By putting $a = a_0$ where $a_0 \preceq T(a_0)$ we define a sequence $\{a_n\}$ as follow
$$a_{n+1} = T(a_n), n \in \mathbb{N}_0.$$
As we know that T is an increasing mapping and $u \prec\succ a \prec\succ v$, we get
$$u \prec\succ a_n \prec\succ v, \quad n \in \mathbb{N}_0.$$
If we put $a_n = a_{n+1}$ for any $n \in \mathbb{N}_0$, then we have, a_n is the fixed point of T and by relation and above preposition we get $u = a_n = v$, which is a contradiction. So, we can't say $a_n = a_{n+1}$ for all $n \in \mathbb{N}_0$. Now, by proceeding the proof of Theorem 3.1 we can prove that
$$\lim_{n,m\to\infty} r_\theta(a_n, a_m) = 0. \tag{3.16}$$
Using (3.1), we get
$$\begin{aligned} r_\theta(u, a_n) &= r_\theta(T(u), T(u_{n+1})) \leq \lambda(r_\theta(u, a_{n-1}))M(r_\theta(u, a_{n-1})) \\ &< \frac{1}{\theta}M(r_\theta(u, a_{n-1})) \leq (r_\theta(u, a_{n-1}), \end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
M(r_\theta(u,a_{n-1})) &= \max\Bigg\{(r_\theta(u,a_{n-1})), \frac{r_\theta(u,T(u))r_\theta(a_{n-1},T(a_{n-1}))}{1+r_\theta(T(u),T(a_{n-1}))}, \\
&\quad \frac{r_\theta(u,T(u))r_\theta(a_{n-1},T(a_{n-1}))}{1+r_\theta(u,a_{n-1})}, \\
&\quad \frac{r_\theta(u,T(u))r_\theta(u,T(a_{n-1}))}{1+r_\theta(u,T(a_{n-1}))+r_\theta(a_{n-1},T(u))}\Bigg\} \\
&= \max\Bigg\{r_\theta(u,a_{n-1}), \frac{r_\theta(a_{n-1},(a_n))}{1+r_\theta(u,(a_n))}, \\
&\quad \frac{r_\theta(a_{n-1},a_n)}{1+r_\theta(u,a_{n-1})}, \\
&\quad \frac{r_\theta(u,a_n)}{1+r_\theta(u,a_n)+r_\theta(a_{n-1},u)}\Bigg\} \\
&= r_\theta(u,a_{n-1}).
\end{aligned}$$

As we know that $\{r_\theta(u,a_n)\}$ is the decreasing sequence of positive real numbers. Then we choose $b \geq 0$ such that

$$\lim_{n\to\infty} r_\theta(u,a_n) = b.$$

Then suppose that $b > 0$. Then from (3.17), we get

$$\lim_{n\to\infty} r_\theta(u,a_n) \leq \lim_{n\to\infty} \lambda(r_\theta(u,a_{n-1})r_\theta(u,u_{n-1}).$$

By the definition of λ we get $r < \frac{1}{\theta}r$, which is a contradiction. Hence

$$\lim_{n\to\infty} r_\theta(u,a_n) = 0. \tag{3.18}$$

Similarly, we can prove that

$$\lim_{n\to\infty} r_\theta(v,a_n) = 0. \tag{3.19}$$

Now, using rectangle inequality, we have

$$r_\theta(u,v) \leq \theta[r_\theta(u,a_n) + r_\theta(a_n,a_{n+1}) + r_\theta(a_{n+1},v)].$$

At $n \to \infty$ and using (3.16) (3.18) (3.19), we get $r_\theta(u,v) = 0$ and we can say that $u = v$, which is a contradiction. Hence, proof is complete. □

The following example shows the importance of a T-directed condition in the Theorem 3.2 for the uniqueness of a fixed point.

Example 3.3. In Example 3.2, we take $\preceq = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$ and a mapping $T : U \to U$ defined by:

$$T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 3 & 1 & 3 \end{pmatrix}.$$

By choosing $u, v \in U$ such that $u \preceq v$ and $Tu = Tv = 1$ or 3. Thus, the contraction condition (3.1) is trivially hold. Therefore, all the conditions of Theorem 3.1 are satisfied except that $(U, \preceq)$ is not $T-$directed. Observe that the mapping T has two fixed points namely $u = 1$ and $u = 3$.

Theorem 3.3. *In Theorems* 3.1 *and* 3.2, *if we replace some conditions namely: increasing mapping T to decreasing(or monotone) mapping, $\overline{O}-$complete to $\underline{O}-$complete(or O-complete), $\overline{O}-$continuos to $\underline{O}-$continuous(or O-continuous) and ICC-property to DCC-property(or MCC-property) also replace $u_0 \preceq T(u_0)$ by $u_0 \succeq T(u_0)$(or $u_0 \prec\succ T(u_0)$). Then the result of both the remains true.*

4 Conclusion

We use geraghty-weak contraction in ordered extended rectangular b-metric space to get fixed point results, with that we have given examples to exhibit the utility of the result.

Authors Contributions

Both author contributed equally to this work. The final draft was approved by both authors. **Acknowledgement.**

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

ON A CLASS OF SLANT WEIGHTED TOEPLITZ OPERATORS

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Abstract

If $\beta = \langle\beta_n\rangle_{n\in\mathbb{Z}}$ is a sequence of positive numbers with $\beta_0 = 1$, then a slant weighted Toeplitz operator A_ϕ is an operator on $L^2(\beta)$ defined as $A_\phi = WM_\phi$ where M_ϕ is the multiplication operator on $L^2(\beta)$ given by $M_\phi e_k(z) = \dfrac{1}{\beta_k}\sum\limits_{n=-\infty}^{\infty} a_n\beta_{n+k}e_{n+k}(z)$. In this paper we investigate the closure of the set of these operators. We also discuss the C^*-algebra generated by a particular class of slant weighted Toeplitz operators and obtain the spectral radius for this class.

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1 Introduction and preliminaries

Toeplitz operators were introduced by Toeplitz [22] in the year 1911. Subsequently many mathematicians came up with different generalizations of the Toeplitz operators. In 1995, Ho [9] introduced the class of slant Toeplitz operators having the property that the matrices with respect to the standard orthonormal basis could be obtained by eliminating every alternate row of the matrices of the corresponding Toeplitz operators. These operators arise in plenty of applications like prediction theory [3], wavelet analysis [4], signal processing [17, 18, 19], and solution of differential equations [5]. However, these studies were made in the context of the usual Hardy spaces H^2 and H^p and the Lorentz spaces L^2 and L^p. Meanwhile the notion of the weighted sequence spaces $H^2(\beta)$ and $L^2(\beta)$ came up. A systematic study of the shift operator and the multiplication operator on $L^2(\beta)$ was made by Shields [20]. Lauric [13] studied particular cases of Toeplitz operators on $H^2(\beta)$.

Motivated by the increasing popularity of the spaces $L^2(\beta)$ and $H^2(\beta)$ and the diverse applications of the slant Toeplitz operators, we introduced and studied the notion of a weighted Toeplitz operator [1] and a slant weighted Toeplitz operator [2]. We also explored the properties of the k-th order slant weighted Toeplitz operator [3] and those of its compression on $H^2(\beta)$ [4]. Subsequently, others have studied the commutativity [5] and hyponormality [10] of these operators. Several approximations of related signals functions have also been explored [15] and [16] in Banach spaces and fuzzy normed spaces [14]. The essentially slant weighted Toeplitz operators and their generalisations have been studied by Gupta and Singh [7]. Amongst the recent advances in this direction is the study of a slant weighted Toeplitz operator in Calkin Algebra by Datt and Ohri [6]. The minimal reducing subspaces of the compression of a slant weighted Toeplitz operator have been explored by Hazarika [11]. The study of weighted Toeplitz operators and that of slant weighted Toeplitz operators is of interest to physicists, probalists and computer scientists. In this paper we study a particular class of the slant weighted Toeplitz operator and determine the spectral radius for it. We begin with the following preliminaries. Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$ and $0 < \dfrac{\beta_n}{\beta_{n+1}} \leq 1$ for every $n \geq 0$, $0 < \dfrac{\beta_n}{\beta_{n-1}} \leq 1$ for every $n \leq 0$. We also assume that $\dfrac{\beta_{2n}}{\beta_n} \leq M < \infty$. Consider the spaces [20]

$$L^2(\beta) = \left\{ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \,\middle|\, a_n \in \mathbb{C}, \|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2\beta_n^2 < \infty \right\},$$

and [13]

$$H^2(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \,\middle|\, a_n \in \mathbb{C}, \|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}.$$

Then $(L^2(\beta), \|\cdot\|_\beta)$ is a Hilbert space [13] with an orthonormal basis given by $\left\{ e_k(z) = \dfrac{z^k}{\beta_k} \right\}_{k\in\mathbb{Z}}$ and with an inner product defined by

$$\left\langle \sum_{n=-\infty}^{\infty} a_n z^n, \sum_{n=-\infty}^{\infty} b_n z^n \right\rangle = \sum_{n=-\infty}^{\infty} a_n \bar{b}_n \beta_n^2 .$$

Further, $H^2(\beta)$ is a subspace of $L^2(\beta)$. Now, let

$L^\infty(\beta) = \{\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \,\big|\, \phi L^2(\beta) \subseteq L^2(\beta)$ and $\exists\, c \in \mathbb{R}$ such that $\|\phi f\|_\beta \le c\|f\|_\beta$ for all $f \in L^2(\beta)\}$.

Then, $L^2(\beta)$ is a Banach space with respect to the norm defined by

$$\|\phi\|_\infty = \inf \left\{ c \,\middle|\, \|\phi f\|_\beta \le c\|f\|_\beta \text{ for all } f \in L^2(\beta) \right\}.$$

Let $P : L^2(\beta) \to H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$. Let $\phi \in L^\infty(\beta)$, then the weighted multiplication operator [20] with symbol ϕ, that is $M_\phi : L^2(\beta) \to L^2(\beta)$ is given by $M_\phi e_k(z) = \dfrac{1}{\beta_k} \sum\limits_{n=-\infty}^{\infty} a_n \beta_{n+k} e_{n+k}(z)$.

If we put $\phi_1(z) = z$, then $M_{\phi_1} = M_z$ is the operator defined as $M_z e_k(z) = w_k e_{k+1}(z)$, where $w_k = \dfrac{\beta_{k+1}}{\beta_k}$ for all $k \in \mathbb{Z}$, and is known as a weighted shift [20].

Further, the weighted Toeplitz operator T_ϕ [13] on $H^2(\beta)$ is defined as $T_\phi(f) = P(\phi f)$.

This mapping is well defined, for, if $f \in H^2(\beta) \subset L^2(\beta)$, then by definition, $\phi f \in L^2(\beta)$ and hence $P(\phi f) \in H^2(\beta)$.

The matrix of T_ϕ is:

$$\begin{bmatrix} a_0 \dfrac{\beta_0}{\beta_0} & a_{-1} \dfrac{\beta_0}{\beta_1} & a_{-2} \dfrac{\beta_0}{\beta_2} & \dots \\ a_1 \dfrac{\beta_1}{\beta_0} & a_0 \dfrac{\beta_1}{\beta_1} & a_{-1} \dfrac{\beta_1}{\beta_2} & \dots \\ a_2 \dfrac{\beta_2}{\beta_0} & a_1 \dfrac{\beta_2}{\beta_1} & a_0 \dfrac{\beta_2}{\beta_2} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}.$$

Hence the effect of T_ϕ on the orthonormal basis can be described by

$$T_\phi e_k(z) = \frac{1}{\beta_k} \sum_{n=0}^{\infty} a_{n-k} \beta_n e_n(z).$$

2 Slant Weighted Toeplitz Operator

Let $\phi \in L^\infty(\beta)$.

Definition 2.1 ([2]). *The slant weighted operator A_ϕ is an operator on $L^2(\beta)$ defined as $A_\phi : L^2(\beta) \to L^2(\beta)$ such that*

$$A_\phi e_k(z) = \frac{1}{\beta_k} \sum_{n=-\infty}^{\infty} a_{2n-k} \beta_n e_n(z).$$

If $W : L^2(\beta) \to L^2(\beta)$ such that

$$W e_{2n}(z) = \frac{\beta_n}{\beta_{2n}} e_n(z),$$

and

$$W e_{2n-1}(z) = 0 \quad \textit{for all} \;\; n \in \mathbb{Z},$$

then an alternate definition of A_ϕ is given by

$$A_\phi(f) = WM_\phi(f) = W(\phi f) \quad \text{for all} \;\; f \in L^2(\beta).$$

Clearly, $W = A_1$. In [2] we have shown that

$$M_z W = W M_{z^2} \tag{2.1}$$

$$M_{\phi(z)} W = A_{\phi(z^2)} = W M_{\phi(z^2)} \tag{2.2}$$

$$\langle A_\phi e_{j+2}, e_{i+1} \rangle = \frac{w_i}{w_j w_{j+1}} \langle A_\phi e_j, e_i \rangle. \tag{2.3}$$

Now, let S denote the shift operator on $L^2(\beta)$ given by $Se_j = \dfrac{1}{w_j}\, e_{j+1}$.

Then $S^* e_j = \dfrac{1}{w_{j-1}}\, e_{j-1}$. Also, S is bounded as $\langle w_n \rangle$ is positive and bounded.

Lemma 2.1. $S^* = M_z^{-1}$.

Proof.

$$\begin{aligned} S^* M_z e_j &= S^* w_j e_{j+1} \\ &= \frac{w_j}{w_j}\, e_j = e_j, \quad j = 0, \pm 1, \pm 2 \ldots \;. \end{aligned}$$

□

We now use Lemma 2.1 and equation (2.3) to prove the following:

Theorem 2.1. *A bounded operator A on $L^2(\beta)$ is a slant weighted Toeplitz operator on $L^2(\beta)$ if and only if $A = M_z^{-1} A M_{z^2}$ where M_z and M_{z^2} are the weighted multiplication operators an $L^2(\beta)$ induced by z and z^2 respectively.*

Proof. Let A be a slant weighted Toeplilz operator on $L^2(\beta)$. Then from equation (2.3) we get that

$$\begin{aligned} \langle Ae_j, e_i \rangle &= \frac{w_j w_{j+1}}{w_i} \langle Ae_{j+2}, e_{i+1} \rangle \\ &= \langle AM_{z^2} e_j, S_{e_i} \rangle \\ &= \langle S^* AM_{z^2} e_j, e_i \rangle \\ &= \langle M_z^{-1} AM_{z^2} e_j, e_i \rangle \qquad i, j = 0, \pm 1, \pm 2, \ldots \;. \end{aligned}$$

Hence $A = M_z^{-1} AM_{z^2}$.

Conversely, let A be a bounded operator on $L^2(\beta)$ such that $A = M_z^{-1} AM_{z^2}$. Then, for all $i, j = 0, \pm 1, \pm 2, \ldots$ we have

$$\begin{aligned} \langle Ae_j, e_i \rangle &= \langle M_z^{-1} AM_{z^2} e_j, e_i \rangle \\ &= \langle S^* AM_{z^2} e_j e_i \rangle \\ &= \langle AM_{z^2} e_j, Se_i \rangle \\ &= \frac{w_j w_{j+1}}{w_i} \langle Ae_{j+2}, e_i \rangle. \end{aligned}$$

In [2] we have proved that the necessary and sufficient condition for a bounded operator A on $L^2(\beta)$ to be a slant weighted Toeplitz operator is that its matrix entries satisfy equation (2.3). Hence we may conclude that A is a slant weighted Toeplitz operator. □

Corollary 2.1. *A bounded operator A on $L^2(\beta)$is a slant weighted Toeplitz operator on $L^2(\beta)$ if and only if $A = S^* AM_{z^2}$*

3 C^*-Algebra of Slant Weighted Toeplitz Operators

Let $L^2(\beta)$ be a given space. Let $\mathcal{A}$ denote the set of all slant weighted Toeplitz operators on $L^2(\beta)$.

Theorem 3.1. *$\mathcal{A}$ is weakly closed and hence strongly closed.*

Proof. Let A_n be a sequence of slant weighted Toeplitz operators such that $\langle A_n f, g\rangle \to \langle Af, g\rangle$ for all $f, g \in L^2(\beta)$. Then $A_n = M_z^{-1} A_n M_{z^2}$ for all n.

Therefore, as $n \to \infty$, we have

$$\begin{aligned}\langle A_n f, g\rangle &= \langle M_z^{-1} A_n M_{z^2} f, g\rangle\\ &= \langle S^* A_n M_{z^2} f, g\rangle\\ &= \langle A_n M_{z^2} f, S_g\rangle\\ &\to \langle A M_{z^2} f, S_g\rangle\\ &= \langle S^* A M_{z^2} f, g\rangle\\ &= \langle M_z^{-1} A M_{z^2} f, g\rangle.\end{aligned}$$

Thus $M_z^{-1} A_n M_{z^2} \to M_z^{-1} A M_{z^2}$ weakly. Hence $A = M_z^{-1} A M_{z^2}$. Hence from Theorem 2.1, A is a slant weighted Toeplitz operator. □

Next, to study the C^*-algebra generated by slant weighted Toeplitz operators and to obtain the spectral radius of A_ϕ, we impose a restriction on the sequence $\langle\beta_n\rangle$. Hence forth we consider only those sequences $\langle\beta_n\rangle_{n\in\mathbb{Z}}$ such that

$$\left.\begin{aligned}\beta_n &= \alpha^n \qquad \text{when } n \ge 0,\\ \beta_n &= \alpha^{-n} \qquad \text{when } n < 0.\end{aligned}\right\} \quad \text{for } 1 < \alpha < \infty.$$

Then the weight sequence $\left\langle w_n = \dfrac{\beta_{n+1}}{\beta_n}\right\rangle$ is of the form

$$\begin{aligned} w_n &= \alpha \quad \text{for } n > 0,\\ w_n &= \frac{1}{\alpha} \quad \text{for } n \le 0.\end{aligned}$$

In that case, the matrix of M_ϕ becomes

$$\left[\begin{array}{c|cccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \cdots & a_0 & a_{-1}\alpha & a_{-2}\alpha^2 & a_{-3}\alpha & a_{-4} & \cdots \\ \cdots & \dfrac{a_1}{\alpha} & a_0 & a_{-1}\alpha & a_{-2} & \dfrac{a_{-3}}{\alpha} & \cdots \\ \cdots & \dfrac{a_2}{\alpha^2} & \dfrac{a_1}{\alpha} & a_0 & \dfrac{a_{-1}}{\alpha} & \dfrac{a_{-2}}{\alpha^2} & \cdots \\ \cdots & \dfrac{a_3}{\alpha} & a_2 & a_1\alpha & a_0 & \dfrac{a_{-1}}{\alpha} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right].$$

Hence the matrix of M_ϕ^* is given by

$$\left[\begin{array}{c|cccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \cdots & \bar{a}_0 & \dfrac{\bar{a}_1}{\alpha} & \dfrac{\bar{a}_2}{\alpha^2} & \dfrac{\bar{a}_3}{\alpha} & \cdots \\ \cdots & \bar{a}_{-1}\alpha & \bar{a}_0 & \dfrac{\bar{a}_1}{\alpha} & \bar{a}_2 & \cdots \\ \cdots & \bar{a}_{-2}\alpha^2 & \bar{a}_{-1}\alpha & \bar{a}_0 & \bar{a}_1\alpha & \cdots \\ \cdots & \bar{a}_{-3}\alpha & \bar{a}_{-2} & \dfrac{\bar{a}_{-1}}{\alpha} & \bar{a}_0 & \cdots \\ \cdots & \bar{a}_{-4} & \dfrac{\bar{a}_{-3}}{\alpha} & \dfrac{\bar{a}_{-2}}{\alpha^2} & \dfrac{\bar{a}_{-1}}{\alpha} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right].$$

It is observed that the matrix entries $\langle\lambda_{ij}\rangle$ of M_ϕ^* satisfy the relation

$$\lambda_{i+1,j+1} = \frac{w_i}{w_j}\lambda_{i,j}\,. \tag{3.1}$$

We have proved in [1] that equation (3.1) is the necessary and sufficient condition for the corresponding operator to be a weighted multiplication operator.

Hence M_ϕ^* is also a weighted multiplication operator. Further, the product of two weighted multiplication operators is also a weighted multiplication operator [1]. Hence we get that

$$M_\phi M_\phi^* = M_\psi \quad \text{for some} \quad \psi \in L^\infty(\beta). \tag{3.2}$$

We suppose that $\psi = \sum\limits_{n=-\infty}^{\infty} b_n z^n$.

Theorem 3.2. *$A_\psi W^*$ is a weighted multiplication operator.*

Proof. For each $k \in \mathbb{Z}$, consider

$$\begin{aligned}
A_\psi W^* e_k(z) &= \frac{\beta_k}{\beta_{2k}} A_\psi e_{2k}(z) \\
&= \frac{\beta_k}{\beta_{2k}} \frac{1}{\beta_{2k}} \sum_{n=-\infty}^{\infty} b_{2n-2k} \beta_n e_n(z) \\
&= \frac{1}{\beta_k} \sum_{n=-\infty}^{\infty} b_{2(n-k)} \frac{\beta_k^2}{\beta_{2k}^2} \beta_n e_n(z) \\
&= M_{\theta_k} e_k(z),
\end{aligned}$$

where

$$\theta_k(z) = \sum_{n=-\infty}^{\infty} \left(b_{2n} \frac{\beta_k^2}{\beta_{2k}^2} \right) z^n \quad \text{is in} \quad L^\infty(\beta).$$

We therefore conclude that

$$A_\psi W^* = M_{\theta_k}. \tag{3.3}$$

Hence the theorem. ☐

Corollary 3.1. $A_\phi A_\phi^* = M_{\theta_k}$.

Proof.

$$\begin{aligned}
A_\phi A_\phi^* &= W M_\phi M_\phi^* W^* \\
&= W M_\psi W^* && \text{using (3.2)} \\
&= A_\psi W^* \\
&= M_{\theta_k} && \text{using (3.3)}
\end{aligned}$$

Finally $A_\phi A_\phi^* = M_{\theta_k}$. ☐

We now prove the main result of this paper:

Let $\mathcal{A}$ denote the C^*-algebra generated by all slant weighted Toeplitz operators A_ϕ on $L^2(\beta)$ with the sequence $\langle \beta_n \rangle$ discussed in this section.

Also, let $\mathcal{M}$ denote the C^*-algebra generated by all weighted multiplication operators on $L^2(\beta)$. We have proved in [2] that W does not commute with M_z. We now prove the following:

Lemma 3.1. *W commutes with the multiplication operator M_ψ if and only if $\psi =$ constant.*

Proof. Let $\psi \in L^\infty(\beta)$ be a constant. Then $M_\psi W = \alpha$ W for some constant α. Therefore

$$\begin{aligned}
M_\psi W e_{2n}(z) &= \alpha W e_{2n}(z) \\
&= \alpha \frac{\beta_n}{\beta_{2n}} e_n(z) \\
&= W \alpha e_{2n}(z) \\
&= W M_\psi e_{2n}(z).
\end{aligned} \tag{3.4}$$

Further

$$\begin{aligned}
M_\psi W e_{2n-1}(z) &= M_\psi 0 \\
&= 0 = W M_\psi e_{2n-1}(z).
\end{aligned}$$

Thus

$$M_\psi W e_n(z) = W M_\psi e_n(z), \quad n = 0, \pm 1, \pm 2 \dots .$$

Conversely, suppose that $M_\psi W = W M_\psi$ for some

$$\psi = \sum_{i=-\infty}^{\infty} b_i z^i \in L^\infty(\beta).$$

From equations (2.1) and (2.2) we infer that $b_i = 0$ for all $i \neq 0$. So, $\psi = b_0 =$ constant. Hence the result. $\square$

Theorem 3.3. $\mathcal{A}' = (I)$.

Proof. Consider the equation $A_\phi A_\phi^* = M_{\theta_k}$. This suggests that every weighted multiplication operator M_{θ_k} can be written as the product of some slant weighted Toeplitz operator A_ϕ and its adjoint A_ϕ^*. Hence $\mathcal{M} \subseteq \mathcal{A}$. We know that $\mathcal{M}$ is maximal abelian [20]. Hence $\mathcal{A}' \subseteq \mathcal{M}' = \mathcal{M}$, where $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$. Hence for a given $B \in \mathcal{A}'$ we get $B \in \mathcal{M}$. That is $B = M_\psi$ for some $\psi \in L^\infty(\beta)$. Also, $W = A_1$. Hence $(W) \subseteq \mathcal{A}$. Therefore $\mathcal{A}' \subseteq (W')$.

This implies that $B = M_\psi$ commutes with W. From the above lemma we get that $\psi =$ constant, and this is true for an arbitrary operator $B \in \mathcal{A}'$, Hence we get that $\mathcal{A}' = (I)$. $\square$

As another consequence of Corollary 3.3, we now derive the spectral radius for a slant weighted Toeplitz operator belonging to this class. For this, we use the spectral radius formula $r(T) = \lim_{n\to\infty} (\|T^n\|)^{1/n}$ and proceed as follows.

Theorem 3.4. $r(A_\phi) = \lim_{n\to\infty} (\|\theta_n\|_\infty)^{1/2}$.

Proof. We know that $A_\phi {A_\phi}^* = M_{\theta_k}$. Taking norm on both sides we get

$$\|A_\phi A_\phi^*\| = \|M_{\theta_k}\| = \|\theta_k\|_\infty.$$

So,

$$\begin{aligned} \|A_\phi\|^2 &= \|\theta_k\|_\infty \\ \|A_\phi\| &= \sqrt{\|\theta_k\|_\infty} = (\|\theta_k\|_\infty)^{1/2}. \end{aligned}$$

Now

$$\begin{aligned} A_\phi^2 A_\phi^{*2} &= W M_\phi W M_\phi M_\phi^* W^* M_\phi^* W^* \\ &= W M_\phi W M_\psi W^* M_\phi^* W^* \\ &= W M_\phi A_\psi W^* M_\phi^* W^* \\ &= W M_\phi M_{\theta_k} M_\phi^* W^* \\ &= W M_{\phi_2} W^* \qquad \text{where } \ M_{\phi_2} = M_\phi M_{\theta_k} M_\phi^* \\ &= A_{\phi_2} W^* \\ &= M_{\theta_2} \quad \text{(say)}. \end{aligned}$$

Proceeding in this manner, we can show that for each n, $A_\phi^n A_\phi^{*n}$ is a multiplication operator M_{θ_n}. Hence

$$\|A_\phi^n\|^2 = \|A_\phi^n A_\phi^{*n}\| = \|M_{\theta_n}\| = \|\theta_n\|_\infty.$$

Finally,

$$\begin{aligned} r(A_\phi) &= \lim_{n\to\infty} (\|A_\phi^n\|)^{1/n} \\ &= \lim_{n\to\infty} (\|\theta_n\|_\infty)^{1/2n}. \end{aligned}$$

$\square$

4 Conclusion

In this paper we have proved that the set of all slant weighted Toeplitz operators on $L^2(\beta)$ is weakly closed and hence strongly closed. By considering a sequence of the type $\langle \beta_n \rangle_{n\in\mathbb{Z}}$ such that $\beta_n = \alpha^n$ when $n \geq 0$ and $\beta_n = \alpha^{-n}$ when $n < 0$ we have shown that $M_\phi M_\phi^*$ is also a weighted multiplication operator. Further, for such a sequence, every weighted multiplication operator can be written as the product of some slant weighted Toeplitz operator and its adjoint.

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

A MULTIPLE REGRESSION MODEL FOR IDENTIFYING SOME RISK FACTORS AFFECTING THE CARDIOVASCULAR HEALTH ISSUES IN ADULTS

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Abstract

Multiple Regression analysis is one of the most critical and widely used statistical techniques in medical and applied research. It is defined as a multivariate technique for determining the correlation between a response variable and some combination of two or more predictor variables. Moreover, it is well-known in medical sciences that the obesity, high blood pressure and high cholesterol are major risk factors for cardiovascular health issues. The body mass index is a measure of body size, and combines a person's weight with their height, and therefore can affect their obesity, high blood pressure, high cholesterol and type 2 diabetes mellitus significantly, which are major risk factors for cardiovascular health issues in adults. Motivated by these facts, in this paper, a multiple linear regression model is developed to analyze the obesity in adults, based on a sample data of adult's age, height, weight, waist, diastolic blood pressure, systolic blood pressure, pulse, cholesterol, and the body mass index measurements. The use of multiple linear regression is illustrated in the prediction study of adult's obesity based on their body mass index. It is observed that in the presence of adult's age, weight, waist, diastolic blood pressure, systolic blood pressure, pulse, and cholesterol levels, height is a good predictor of the body mass index. Moreover, in the presence of age, height, waist, diastolic blood pressure, systolic blood pressure, pulse, and cholesterol levels, weight is a good predictor of the body mass index. Some concluding remarks are given in the end.
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Keywords and Phrases: Cardiovascular, high cholesterol levels, high blood pressure, multiple regression, obesity.

1 Introduction

Multiple linear regression is one of the most widely used statistical techniques in medical and other applied research. It is defined as a multivariate technique for determining the correlation between a response variable Y and some combination of two or more predictor variables, X. For example, it can be used to analyze data from causal-comparative, correlational, or experimental research. It can handle interval, ordinal, or categorical data. In addition, multiple regression provides estimates both of the magnitude and statistical significance of relationships between variables. For details on regression analysis and its applications, the interested readers are referred to Neter et al. [19], Draper and Smith [5], Tamhane and Dunlop [25], Mendenhall and Sincich [16], Chatterjee and Hadi [2], Montgomery [17], Surez et al. [23], Cleophas and Zwinderman [3], Guzman and Kibria [7], Johnson and Wichern [9], among others. For recent developments on linear and non-linear regression models, we refer to Kibria [12].

The purpose of the present study is to contribute to the body of knowledge pertaining to the use of multiple linear regression in medical and applied research, and, in particular, in identifying some risk factors affecting the cardiovascular health issues in adults. It appears from the literature that not much attention has been paid to this kind of studies in the multiple regression analysis of the cardiovascular health issues and problems in adults. Motivated by these facts, in this paper, a multiple linear regression model is developed to analyze the obesity in adults, based on their body mass index *(BMI)* by taking a sample data of adult's age, height, weight, waist, diastolic blood pressure, systolic blood pressure, pulse, cholesterol, and BMI measurements. The use of multiple linear regression is illustrated in the prediction study of adult's obesity based on their body mass index, along with these risk indicators.

1.1 Body Mass Index *(BMI)*

In what follows, we first present some basic ideas about the body mass index *(BMI)*, and the review of the literature relevant to the cardiovascular health issues.

Definition 1.1. *The body mass index (BMI) is defined as a measure of body size and for weight-related health risk. It combines a person's weight with their height. It can be calculated using the following formulas:*

$$BMI = Weight(kg)/[height(m)]2, \tag{1.1}$$

$$BMI = Weight(lb)/[height(in)]2 \times 703. \tag{1.2}$$

Thus, the results of a BMI measurement can give an idea about whether a person's weight is correct with respect to their height. Moreover, the BMI of a person can indicate whether they are underweight or if they have a healthy weight, or excess weight, or obesity. If a person's BMI is outside of the healthy range, their health risks may increase significantly. According to the US Centers for Disease Control and Prevention and the World Health Organization, "BMI represents the relationship between weight and height to estimate the amount of fat in the body" (Global Health Observatory. from http://www.who.int/gho/ncd/risk_factors/bmi_text/en/). Moreover, as observed by Young et al. [29], Nguyen et al. [20], and Keum et al. [13], "A higher percentage of body fat is proven to be associated with increased risk for developing certain diseases such as heart disease, high blood pressure, type 2 diabetes, breathing problems, certain cancers, and death". Furthermore, as reported by https://www.weightwatchers.com/us/science-center/bmi-calculator, there appears to be an exponential relationship between BMI and mortality rate which is illustrated in the following Figure 1.1.

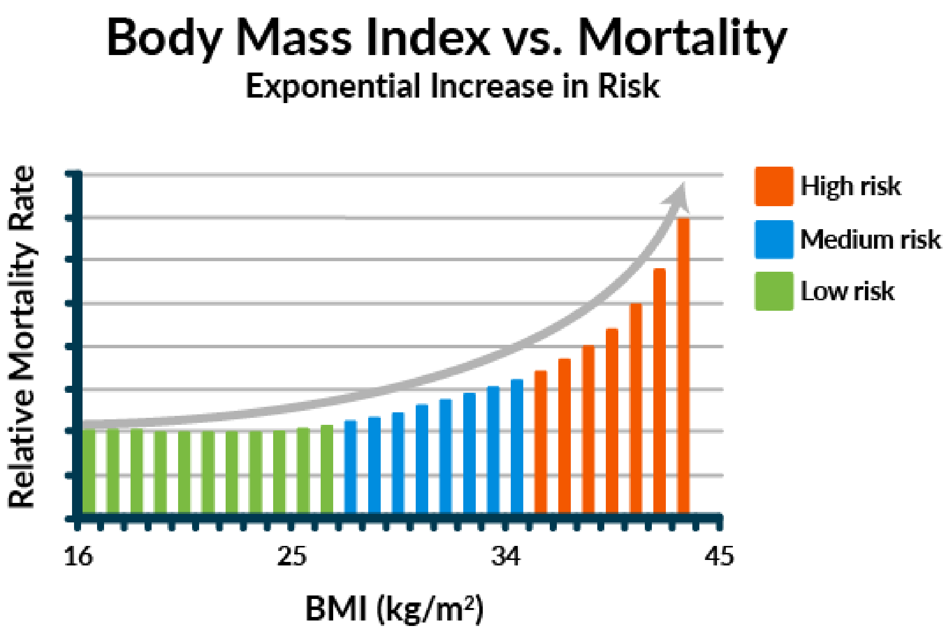

***Figure* 1.1**
(Source: https://www.weightwatchers.com/us/science-center/bmi-calculator)

According to Narkiewicz [22], "Obesity and in particular central obesity have been consistently associated with hypertension and increased cardiovascular risk. Based on population studies, risk estimates indicate that at least two-thirds of the prevalence of hypertension can be directly attributed to obesity". Further, a*s* pointed out by Hall et al. [18], "Major consequences of being overweight or obese include higher prevalence of hypertension and a cascade of associated cardiorenal and metabolic disorders. Studies in diverse populations throughout the world have shown that the relationship between BMI and systolic and diastolic blood pressure (BP) is nearly linear. Risk estimates from the Framingham Heart Study, for example, suggest that 78% of primary (essential) hypertension in men and 65% in women can be ascribed to excess weight gain. Clinical studies indicate that maintenance of a BMI <25 kg/m^2 is effective in primary prevention of hypertension and that weight loss reduces BP in most hypertensive subjects". Also, according to Jiang *et al.* [10], "Obesity can result in serious health issues that are potentially life-threatening, including hypertension, type II diabetes mellitus, increased risk for coronary disease, increased unexplained heart failure, hyperlipidemia, infertility, higher prevalence of colon, prostate, endometrial, and breast cancer. Although the relationship between obesity and hypertension is well established in children and adults, the mechanism by which obesity directly causes hypertension is under investigation".

"Having obesity puts a strain on our heart and can lead to serious health cardiovascular problems, namely, arthritis in our knees and hips, heart disease, high blood pressure, sleep apnea, type 2 diabetes, and varicose veins" (https://medlineplus.gov/ency/article/007196.htm). Moreover, a person's BMI can be categorized (Table 1.1), along with the three classes of obesity (Table 1.2), as given below:

***Table* 1.1**

(https://medlineplus.gov/ency/article/007196.htm)

BMI	**CATEGORY**
Below 18.5	Underweight
18.5 to 24.9	Healthy
25.0 to 29.9	Overweight
30.0 to 39.9	Obese
Over 40	Extreme of high-risk obesity

***Table* 1.2**

(https://medlineplus.gov/ency/article/007196.htm)

CLASS	**OBESITY**
1	BMI of 30 to less thank 35
1	BMI of 35 to less than 40
3	BMI of 40 or higher. Class 3 is considered "severe obesity".

Thus, it is obvious from the Tables 1.1 and 1.2 that a person's obesity can be significantly affected by their body mass index (BMI), high blood pressure and high cholesterol, which are all major risk factors for cardiovascular health issues. For further details on cardiovascular diseases and related issues, the interested readers are referred to Mertens and Van Gaal [18], Akil and Ahmad [1], Klop et al. [14], Vach [27], Leggio et al. [15], Seravalle and Grassi [24], Feng et al. [6], Jabeen et al. [11], Rajeshwari and Laishram [22], and references therein.

The organization of this paper is as follows. In Section 2, the proposed multiple linear regression model, and the problem and objective of this study are presented. Section 3 provides the data analysis, justification and adequacy of the multiple regression model developed. Some concluding remarks are given in Section 4.

2 Multiple Linear Regression Model

2.1 A Multiple Linear Regression Model based on a Number of Predictors

Consider following multiple linear regression model

$$Y = X\beta + \epsilon, \tag{2.1}$$

where Y is an $n \times 1$ vector of response variable (observations), β is a $k \times 1$ vector of unknown regression coefficients, X is an $n \times k(n > k)$ observed matrix of the regression, and ϵ is an $n \times 1$ vector of random errors, which is distributed as multivariate normal with mean 0 and covariance matrix $\sigma^2 I_n$, and I_n is an identity matrix of order n. The OLS estimator of β is obtained as $\widehat{\beta} = (X'X)^{-1}X'y$, and covariance matrix of $\widehat{\beta}$ is obtained as Cov $(\widehat{\beta}) = \sigma^2(X'X)^{-1}$.

2.2 Problem and Objective of Study

It is well-known in medical sciences that the obesity, high blood pressure and high cholesterol are major risk factors for cardiovascular health issues. For example, high cholesterol can affect anyone, regardless of their weight. Moreover, high blood pressure, also called hypertension, is a major risk factor for heart disease, kidney disease, stroke, and heart failure. Having excess body weight can lead to increased high blood pressure and cholesterol levels. The body mass index is a measure of body size, and combines a person's weight with their height, the results of a body mass index measurement can indicate whether a person has excess weight, and thus can affect their obesity, high blood pressure and high cholesterol significantly, which are all risk factors for cardiovascular health issues.

Thus, in view of the above facts, the objective of our present investigation would be to develop an appropriate multiple linear regression model to relate the adult's obesity, based on their body mass index (*BMI*) (considered as the dependent or response variable Y) to the adult's age, height, weight, waist, diastolic blood pressure, systolic blood pressure, pulse, cholesterol, *BMI* measurements (considered as the independent or predictor variables X). It will be examined how well the adult's age, height, weight, waist, pulse, diastolic blood pressure, systolic blood pressure, cholesterol, and *BMI* measurements could be used to predict the adult's body mass index (*BMI*), as it affects a person's obesity, high blood pressure and high cholesterol significantly, which are all risk factors for cardiovascular health issues in adults.

To pursue our studies, the data were collected from Triola [26] on the adult's age, height, weight, waist, pulse, diastolic blood pressure, systolic blood pressure, cholesterol, and *BMI* measurements, for a sample of 40 adults, (which we have provided in Appendix 1 for the sake of completeness). Using these variables and the Equation (2.1), the following eight-predictor multiple linear regression model (or the least squares prediction equation) was developed:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \beta_5 X_5 + \beta_6 X_6 + \beta_7 X_7 + \beta_8 X_8 + \varepsilon, \tag{2.2}$$

where β' s denote the population regression coefficients, ε is a random error, the response variable is the adult's *BMI* (Y), and the respective eight predictors are the adult's age (X_1), height (X_2), weight (X_3), waist ($\boldsymbol{X_4}$), pulse ($\boldsymbol{X}_5$), diastolic blood pressure ($\boldsymbol{X_6}$), systolic blood pressure ($\boldsymbol{X}_7$), and cholesterol ($\boldsymbol{X_8}$).

3 Data Analysis

The Minitab Version 17.0 regression computer programs were used to determine the regression coefficients and analyze the data. The adequacy of the multiple linear regression model for predicting the adult's body mass index (*BMI*) was conducted using the F-test for the significance of regression.

The Minitab regression computer program outputs are given below. The paragraphs that follow explain the computer program outputs.

3.1 Minitab Regression Computer Program Output: Analysis of Variance

3.1.1 Regression Analysis: *BMI* versus Age, $Ht, \ldots$

The regression equation is:

$$BMI = 52.1 + 0.00134 \text{ Age } - 0.772 Ht + 0.147 \text{ Wt } + 0.0125 \text{ Waist } + 0.00710 \text{ Pulse}$$
$$- 0.00229 \text{ Systolic } - 0.00195 \text{ Diastolic } + 0.000211 \text{ Cholesterol }.$$

***Table* 3.1**

Predictor	**Coef**	***SE* Coef**	***T***	***P***	***VIF***
Constant	52.1200000	1.8800000	27.72	0.000	
Age	0.0013420	0.0049270	0.27	0.787	2.0
Ht	-0.7721100	0.0248400	-31.08	0.000	2.4
Wt	0.1465580	0.0063350	23.13	0.000	11.7
Waist	0.0125100	0.0167500	0.75	0.461	11.5
Pulse	0.0070950	0.0047400	1.50	0.145	1.2
Systolic	-0.0022870	0.0059550	-0.38	0.704	1.6
Diastolic	-0.0019480	0.0075320	-0.26	0.798	2.0
Cholesterol	0.0002106	0.0001749	1.20	0.238	1.1

Table **3.2**

$\boldsymbol{S}=0.304262$	$R-Sq=99.4\%$	$R-Sq(\text{ adj })=99.2\%$
PRESS = 5.60841	R-Sq(pred) = 98.78%	
Durbin-Watson statistic = 2.80903		

Table **3.3**

Analysis of Variance					
Source	*DF*	*SS*	*MS*	*F*	*P*
Regression	8	456.160	57.020	615.93	0.000
Residual Error		2.870	0.093		
Total	39	459.030			

Table **3.4**

Unusual Observations						
Obs	Age	*BMI*	Fit	*SE* Fit	Residual	*St* Resid
17	41.0	33.2000	32.3881	0.1767	0.8119	3.28R
36	34.0	20.7000	21.4631	0.1542	-0.7631	−2.91R

Note: Here, in Table 4.4, R denotes an observation with a large standardized residual.

3.1.2 Interpreting the Results

I. From the Analysis of Variance Table 3.3, we observe that the p-value is (0.000). This implies that that the model estimated by the regression procedure is significant at an α-level of 0.05 . Thus at least one of the regression coefficients is different from zero.

II. From the Table 3.1, we observe that the p-values for the estimated coefficients of height (X_2) and weight (X_3) are respectively 0.000 and 0.000 , indicating that they are significantly related to the response variable is *BMI* (Y) at an α-level of 0.05. From the Table 3.1, we also observe that the p-values for the adult's age (X_1), waist (X_4), pulse (X_5), diastolic blood pressure (X_6), systolic blood pressure (X_7), and cholesterol (X_8), are relatively high, indicating that these are probably not related to the response variable *BMI* (Y) at an α-level of 0.05 .

III. **The R^2 and Adjusted R^2 Statistic:** There are several useful criteria for measuring the goodness of fit of the multiple regression model. One such criterion is to determine the square of the multiple correlation coefficient R^2 (also called the coefficient of multiple determination), (see, for example, Draper and Smith [5], and Mendenhall and Sincich [16], among others). The R^2 value in the regression output (Table 3.2) indicates that 99.4% of the total variation of the response variable $BMI(Y)$ values about their mean can be explained by the predictor variables used in the model. The adjusted R^2 value (or ${R_a}^2$) indicates that 99.2% of the total variation of the response variable $BMI(Y)$ values about their mean can be explained by the predictor variables used in the model. As the values of R^2 and R_a^2 are not very different, it appears that at least one of the predictor variables contributes information for the prediction of Y. Thus, both values indicate that the model fits the data well.

IV. **Predicted $\mathbf{R}^2$ Statistic:** Further from Table 3.2, we observe that the predicted R^2 value is 98.78%. Because the predicted R^2 value is close to the R^2 and adjusted R^2 values, the model does not appear to be overfit and has adequate predictive ability.

V. **Estimate of Variance:** The variance about the regression σ^2 of the Y values for any given set of the independent variables $X_1, X_2, \ldots, X_k$ is estimated by the residual mean square s^2, which is equal to SS (residual) divided by an appropriate number of degrees of freedom, and the standard error s is given by

$$s=\sqrt{\text{residual meansquare } s^2}.$$

For our problem, we have

$$\boldsymbol{s^2=0.093} \text{ and } \boldsymbol{s=0.30496}$$

Examination of this statistic indicates that the smaller it is the better, that is, the more precise will be the predictions. A useful way of looking at the decrease in S is to consider it in relation to response, (see, for example, Draper and Smith (1998), among others, for details). In our example, s as a percentage of mean $\bar{Y}$ (of the response variable BMI, $\boldsymbol{Y}$), that is, the coefficient of variation (CV), is given by

$$CV = \frac{0.30496}{25.9975} \times 100\% = 1.17303\%.$$

This means that the standard deviation of the adult's BMI (Y), is only **1.17303**% of their mean, which means considerably less variation.

VI. **Unusual Observations:** We also note from the Table 3.4 that the observations 17 and 36 (see Appendix 1) are identified as unusual because the absolute value of the standardized residuals is greater than 2 . This may indicate they are outliers.

VII. **Multicollinearity:** By multicollinearity, we mean that some predictor variables are correlated with other predictors. Various techniques have been developed to identify predictor variables that are highly collinear, and for possible solutions to the problem of multicollinearity, (see, for example, Draper and Smith [5], Tamhane and Dunlop [25], Mendenhall and Sincich [16], Chatterjee and Hadi [2], Montgomery et al. [17], Chatterjee and Simonoff [4], and Vittinghoff et al. [28], among others, for details). For example, we can examine the variance inflation factors (VIF), which measure how much the variance of an estimated regression coefficient increases if the predictor variables are correlated. Following Montgomery et al. [17], if the VIF is 5 - 10, the regression coefficients are poorly estimated. However, it has been observed by many researchers that for a large sample size, multicollinearity is not a big problem when compared to a small sample size. Since the variance inflation factors (VIF) for each of the estimated regression coefficient in our calculations are less than 5 for the adult's age (X_1), height (X_2), pulse (X_5), diastolic blood pressure (X_6), systolic blood pressure (X_7), and cholesterol (X_8), there does not seem to be multicollinearity for these predictors in our model. However, we observe that the VIF are fairly large for the predictor weight (X_3) and waist (X_4), implying that these are highly correlated with at least one of the other predictors in the model. In order to deal with the said multicollinearity is to remove some of the violating predictors from the model, that is, for assessing the predictive ability of a multiple linear regression model, is to examine the associated C_p-statistic. The best subsets regression method is used to choose a subset of predictor variables so that the corresponding fitted regression model optimizes the C_p-statistic, which is described in Sub-Section 3.2 below.

VIII. **Predicted Values for New Observations:** Using the model developed, some values are provided in Table 3.5 .

3.2 Best Subsets Regression:

Another important criterion function for assessing the predictive ability of a multiple linear regression model is to examine the associated Mallows' C_p-statistic, including R-Sq $\left(R^2\right)$, the percentage of variation in the response that is explained by the model, Adjusted R^2 (that is, $R\ Sq(adj)$, the percentage of the variation in the response that is explained by t for the number of predictors in the model relative to the number of observations), and s, the standard error of the estimate. The best subsets regression method is used to choose a subset of predictor variables so that the corresponding fitted regression model optimizes the Mallows' C_p-statistic, which may be interpreted as follows:

(1) A Mallows' Cp value that is close to the number of predictors plus the constant model produces relatively precise and unbiased estimates.

(2) A Mallows' Cp value that is greater than the number of predictors plus the constant model is biased and does not fit the data well.

The model with all the predictor variables should have the highest adjusted R^2, a low Mallows' Cp value, and the lowest s value. Based on these criteria, the following (Table 3.6) are the possible predictor models (X_2, X_3) or (X_1, X_2) with respective highest adjusted R^2, a low Mallows Cp value, and the lowest S value.

Note that three other predictor models, namely, [Height (X_2), Weight (X_3), Waist (X_4), Cholesterol (X_8)], or [Age (X_1), Height (X_2), Weight (X_3), Pulse (X_5)], or [Height (X_2), Weight (X_3), Cholesterol (X_8)] also exist here with respective highest adjusted R^2, a low Mallows Cp value, and the lowest S value (see the output above).

***Table* 3.5:** Predicted Values for New Observations

New				
Obs	**Fit**	***SE* Fit**	**95% *CI***	**95% *PI***
1	23.6038	0.1107	(23.3781,23.8296)	(22.9435,24.2641)
2	23.2779	0.1253	(23.0224,23.5333)	(22.6068,23.9490)
3	24.6224	0.1587	(24.2988,24.9460)	(23.9225,25.3223)
4	26.1172	0.1024	(25.9083, 26.3261)	(25.4624, 26.7720)
5	23.5401	0.1086	(23.3186,23.7616)	(22.8812,24.1990)
6	24.5249	0.1388	(24.2418,24.8081)	(23.8428,25.2070)
7	21.7545	0.1078	(21.5346,21.9744)	(21.0961,22.4128)
8	31.4276	0.1646	(31.0918,31.7634)	(30.7220,32.1331)
9	26.2895	0.1641	(25.9548,26.6243)	(25.5845,26.9946)
10	23.103	70.1407	(22.8168,23.3906)	(22.4200,23.7873)
11	27.813	60.1749	(27.4568,28.1703)	(27.0978,28.5294)
12	28.170	50.1981	(27.7665,28.5745)	(27.4301,28.9110)
13	24.948	40.1353	(24.6724,25.2244)	(24.2693,25.6276)
14	23.159	30.1732	(22.8060,23.5126)	(22.4452,23.8733)
15	31.729	90.1432	(31.4378,32.0220)	(31.0440,32.4157)
16	33.509	50.1753	(33.1521,33.8670)	(32.7934,34.2257)
17	32.388	10.1767	(32.0278,32.7485)	(31.6705,33.1057)
18	27.1068	80.1573	(26.7860,27.4276)	(26.4083,27.8054)
19	26.623	30.1234	(26.3715,26.8750)	(25.9536,27.2930)
20	19.7208	80.2088	(19.2950,20.1467)	(18.9682,20.4734)
21	27.055	10.1043	(26.8422,27.2679)	(26.3990,27.7111)
22	23.012	40.1609	(22.6842,23.3406)	(22.3104,23.7144)
23	27.202	40.1591	(26.8780,27.5268)	(26.5022,27.9026)
24	21.510	60.0911	(21.3248,21.6963)	(20.8628,22.1583)
25	30.904	70.1416	(30.6159,31.1936)	(30.2202,31.5892)
26	28.344	60.1159	(28.1083,28.5809)	(27.6806,29.0086)
27	25.344	10.1196	(25.1002,25.5881)	(24.6774,26.0109)
28	24.662	60.1623	(24.3315,24.9937)	(23.9593,25.3659)
29	23.4573	30.1171	(23.2184, 23.6961)	(22.7923, 24.1222)
30	27.437	40.1302	(27.1718,27.7030)	(26.7624,28.1124)
31	28.9268	80.1154	(28.6916,29.1621)	(28.2632,29.5905)
32	26.281	60.1592	(25.9570,26.6063)	(25.5813,26.9820)
33	26.752	50.1992	(26.3463,27.1587)	(26.0108,27.4942)
34	31.937 !	50.1318	(31.6688,32.2063)	(31.2613,32.6138)
35	19.088	30.1539	(18.7745,19.4022)	(18.3930,19.7837)
36	21.463	10.1542	(21.1486,21.7776)	(20.7674,22.1588)
37	26.280	20.1130	(26.0498,26.5106)	(25.6183,26.9421)
38	26.819	10.1417	(26.5300,27.1081)	(26.1345,27.5036)
39	25.744	20.0920	(25.5566,25.9318)	(25.0959,26.3925)
40	24.243	60.0960	(24.0478,24.4395)	(23.5929,24.8943)

Table **3.6**

Vars	$R - Sq$	$R - Sq(adj)$	$C - p$	S	Possible Predictor Models
(i) 4	99.4	99.3	2.2	0.29191	Height (X_2), Weight (X_3), Pulse (X_5), Cholesterol (X_8)
(ii) 5	99.4	99.3	3.4	0.29222	Height (X_2), Weight (X_3), Waist (X_4), Pulse (X_5), Cholesterol (X_8)
(iii) 5	99.4	99.3	3.8	0.29440	Age (X_1), Height (X_2), Weight (X_3), Pulse (X_5), Cholesterol (X_8)
(iv) 4	99.3	99.3	2.8	0.29458	Height (X_2), Weight (X_3), Waist (X_4), Pulse (X_5)
(iv) 3	99.3	99.3	2.2	0.29677	Height (X_2), Weight (X_3), Pulse (X_5)

3.3 Residual Plots for BMI

The Minitab Version 17.0 regression computer program outputs for residual plots of are given in Figure 3.1 below. The paragraphs that follow examine the goodness of fit model based on residual plots.

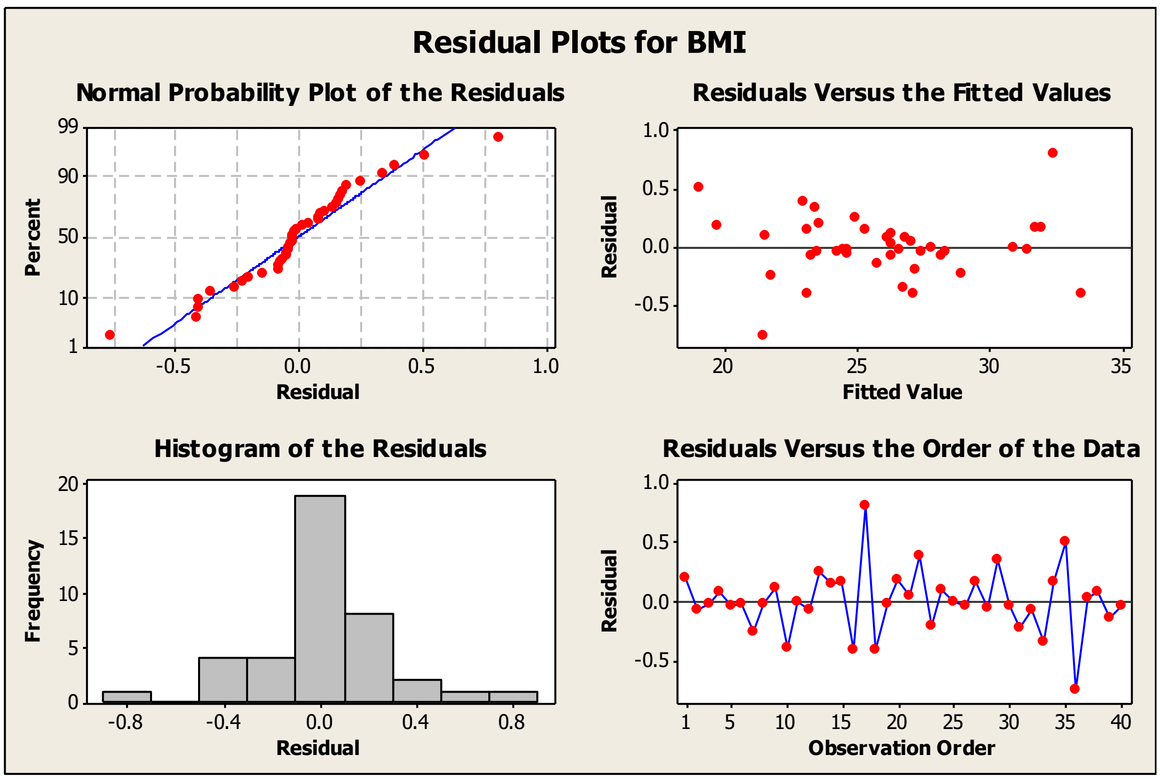

Figure **3.1**

3.3.1 Interpreting the Graphs (Figure 3.1)

A. From the normal probability plot, we observe that there exists an approximately linear pattern. This indicates the consistency of the data with a normal distribution. The outliers are indicated by the points in the upper-right and left-bottom corners of the plot.

B. From the plot of residuals versus the fitted values, it is evident that the residuals get smaller, that is, closer to the reference line, as the fitted values increase. This may indicate that the residuals have non-constant variance, (see, for example, Draper and Smith [2], among others, for details).

C. The histogram of the residuals indicates that no outliers exist in the data.

D. The plot for residuals versus order is also provided in Figure 3.1. It is defined as a plot of all residuals in the order that the data was collected. It is used to find non-random errors, especially of time-related effects. A clustering of residuals with the same sign indicates a positive correlation, whereas a negative correlation is indicated by rapid changes in the signs of consecutive residuals.

3.4 Testing the Adequacy of Multiple Regression Model for Predicting the Adults Body Mass Index *(BMI)*

This section discusses the usefulness and adequacy of the above-developed multiple regression model developed for predicting the adults body mass index (BMI), Y.

3.4.1 Confidence Interval for the Parameters β_i

If we assume that the variation of observations about the line is normal, that is, the error terms ϵ are all from the same normal distribution, $N(0, \sigma^2)$, it can be shown that we can assign $(1-\alpha)100\%$ confidence limits for β_i by calculating

$$\hat{\beta}_i \pm t(n-2, 1-\frac{\alpha}{2}), se(\hat{\beta}_i),$$

where $t(n-2, 1-\frac{\alpha}{2}))$ is the $(1-\alpha)100\%$ percentage point of a t- distribution, with $(n-2)$ degrees of freedom (the number of degrees of freedom on which the estimate s^2 is based). Suppose $\alpha = 0.05$. For $t(38, 0.975)$, we can use $t(40, 0975) = 2.021$, or interpolate in the t table. Thus, we have confidence limits for :

1. 95%; confidence limits for β_1: (-0.00862, 0.011299)
2. 95%; confidence limits for β_2: (-0.82231, -0.72191);
3. 95%; confidence limits for β_3: (0.133755, 0.159361);
4. 95%; confidence limits for β_4: (-0.02134, 0.046362);
5. 95%; confidence limits for β_5: (-0.00248, 0.016675);
6. 95%; confidence limits for β_6: (-0.01432, 0.009748);
7. 95%; confidence limits for β_7: (-0.01717, 0.013274);
8. 95%; confidence limits for β_8: (-0.00014, 0.000564).

3.4.2 Tests of Significance for Individual Parameters

$$H_0 : \beta_i = 0 \text{ versus } H_\alpha : \beta_i \neq 0$$

A test of hypothesis that a particular parameter, say, β_i equals zero, can be conducted by using a t - statistic given by $t = \frac{\hat{\beta}_i - 0}{\mathrm{se}(\hat{\beta}_i)}$. The test can also be conducted by using the F-statistic since the square of a t-statistic (with v degrees of freedom) is equal to an F-statistic with 1 degree of freedom in the numerator and v degrees of freedom in the denominator. That is, $t^2 = F$. Decision Rule: Reject H_0 if $|t| > t\left(n-2, 1-\frac{\alpha}{2}\right)$. Using the Minitab Version 17.0 multiple linear regression computer outputs, the analysis of t statistic values for different β_i 's is given in Table 3.7 below

***Table* 3.7**

Null Hypothesis	$\boldsymbol{t(38, 0.975)^*}$	$\|t\|$	Inference	Conclusion
$H_0 : \beta_1 = 0$	2.021	0.27	Fail to reject H_0	In the presence of $\boldsymbol{X}_2$, $\boldsymbol{X_3}$, $\boldsymbol{X_4}$, $\boldsymbol{X_5}$, $\boldsymbol{X_6}$, $\boldsymbol{X_7}$, and $\boldsymbol{X_8}$, X_1 is a poor predictor of Y .
$H_0 : \beta_2 = 0$	2.021	31.08	Reject H_0	In the presence of $\boldsymbol{X_1}$, $\boldsymbol{X_3}$, $\boldsymbol{X_4}$, $\boldsymbol{X_5}$, $\boldsymbol{X_6}$, $\boldsymbol{X_7}$, and $\boldsymbol{X_8}, X_2$ is a good predictor of Y .
$H_0 : \beta_3 = 0$	2.021	23.13	Reject H_0	In the presence of $\boldsymbol{X_1}$, $\boldsymbol{X_2}$, $\boldsymbol{X_4}$, $\boldsymbol{X_5}$, $\boldsymbol{X_6}$, $\boldsymbol{X_7}$, $\boldsymbol{X_8}$, X_3 is a good predictor of Y.
$\boldsymbol{H_0 : \beta_4 = 0}$	2.021	0.75	Fail to reject H_0	In the presence of $\boldsymbol{X_1}$, $\boldsymbol{X_2}$, $\boldsymbol{X_3}$, $\boldsymbol{X_5}$, $\boldsymbol{X_6}$, $\boldsymbol{X_7}$, $\boldsymbol{X_8}$, $\boldsymbol{X_4}$ is a poor predictor of Y.
$\boldsymbol{H_0 : \beta_5 = 0}$	2.021	1.50	Fail to reject H_0	In the presence of $\boldsymbol{X_1}$, $\boldsymbol{X_2}$, $\boldsymbol{X_3}$, $\boldsymbol{X_4}$, $\boldsymbol{X_6}$, $\boldsymbol{X_7}$, $\boldsymbol{X_8}$, $\boldsymbol{X_5}$ is a poor predictor of Y.
$\boldsymbol{H_0 : \beta_6 = 0}$	2.021	0.38	Fail to reject H_0	In the presence of $\boldsymbol{X_1}, \boldsymbol{X_2}, \boldsymbol{X_3}, \boldsymbol{X_4}$, $\boldsymbol{X_5}$, $\boldsymbol{X_7}$, $\boldsymbol{X_8}$, $\boldsymbol{X_6}$ is a poor predictor of Y .
$\boldsymbol{H_0 : \beta_7 = 0}$	2.021	0.26	Fail to reject H_0	In the presence of $\boldsymbol{X_1}$, $\boldsymbol{X_2}$, $\boldsymbol{X_3}$, $\boldsymbol{X_4}$, $\boldsymbol{X_5}$, $\boldsymbol{X_6}$, $\boldsymbol{X_8}$, $\boldsymbol{X_7}$ is a poor predictor of Y .
$\boldsymbol{H_0 : \beta_8 = 0}$	2.021	1.20	Fail to reject H_0	In the presence of $\boldsymbol{X_1}, \boldsymbol{X_2}$, $\boldsymbol{X_3}$, $\boldsymbol{X_4}$, $\boldsymbol{X_5}$, $\boldsymbol{X_6}$, $\boldsymbol{X_7}$, $\boldsymbol{X_8}$ is a poor predictor of Y .

*For $\boldsymbol{t}(\mathbf{38}, \mathbf{0.975})$, we can use $t(40, 0.975) = 2.021$ or interpolate in the $t-$ table.

3.4.3 F-Test for Significance of Regression

For details on it, see, for example, Draper and Smith [5], Tamhane and Dunlop [25], and Mendenhall and Sincich [16], Chatterjee and Hadi [2], Montgomery et al. [17], among others. For our proposed multiple regression model, we have

Null Hypothesis: $\boldsymbol{H_0} : \boldsymbol{\beta_1} = \boldsymbol{\beta_2} = \boldsymbol{\beta_3} = \boldsymbol{\beta_4} = \boldsymbol{\beta_5} = \boldsymbol{\beta_6} = \boldsymbol{\beta_7} = \boldsymbol{\beta_8} = \mathbf{0}$ (The regression is not significant) versus

Alternate Hypothesis: H_a : at least one of $\beta_i's \neq 0$ (The regression is significant).

Test Statistic: $F = \frac{MS_{\text{reg}}}{s^2}$.

Decision Rule: Reject H_0 if $\boldsymbol{F} > \boldsymbol{F_\alpha}\,(\boldsymbol{v_1} = \boldsymbol{k}, \boldsymbol{v_2} = \boldsymbol{n} - (\boldsymbol{k} + \mathbf{1}), \mathbf{1} - \boldsymbol{\alpha})$,

where n = number of values in the sample data = 40,

$\boldsymbol{k}$ = number of estimated β regression coefficients = 8,

$\boldsymbol{k} + \mathbf{1} = 8 + 1 = 9$ = number of estimated $\boldsymbol{\beta}$ parameter,

$\boldsymbol{v_1} = \boldsymbol{k} = \boldsymbol{df}$ in the numerator = 8,

and $\boldsymbol{v}_2 = \boldsymbol{n} - (\boldsymbol{k} + \mathbf{1}) = \boldsymbol{df}$ in the denominator = 31

In the decision rule, we compare the calculated $\boldsymbol{F}$ test statistic to a tabulated $\boldsymbol{F_\alpha}$ value based on $\boldsymbol{v_1} = \boldsymbol{kdf}$ in the numerator and $\boldsymbol{v_2} = \boldsymbol{n} - (\boldsymbol{k} + \mathbf{1})\boldsymbol{df}$ in the denominator for the considered value of $\boldsymbol{\alpha}$, using $\boldsymbol{F}$ distribution.

Thus, for our proposed multiple regression model, the decision rule is given by

Decision Rule: Reject H_0 if $\boldsymbol{F} > \boldsymbol{F_{0.05}}\,(\boldsymbol{v_1} = \mathbf{8}, \boldsymbol{v_2} = \mathbf{31}, \mathbf{0.95})$, for $\boldsymbol{\alpha} = \mathbf{0.05}$.

The value of F - statistic for testing the hypothesis is that at least one of the predictor variables contributes significant information for the prediction of the adult's body mass index (BMI), Y. In the computer output 17 (Table 4.3), it is calculated as $\boldsymbol{F} = \mathbf{615.93}$. Comparing this with the critical value of $\boldsymbol{F_{0.05}}\,(\boldsymbol{v_1} = \mathbf{8}, \boldsymbol{v_2} = \mathbf{31}, \mathbf{0.95}) = \mathbf{2.18}$ at $\alpha = 0.05$, we reject the null hypothesis: $\boldsymbol{H_0} : \boldsymbol{\beta_1} = \boldsymbol{\beta_2} = \boldsymbol{\beta}_3 = \boldsymbol{\beta}_4 = \boldsymbol{\beta}_5 = \boldsymbol{\beta}_6 = \boldsymbol{\beta_7} = \boldsymbol{\beta_8} = \mathbf{0}$, that is, the regression is not significant. Thus, the overall regression is statistically significant. In fact, $\boldsymbol{F} = \mathbf{615}.93$ exceeds $\boldsymbol{F_{0.05}}\,(\boldsymbol{v_1} = \mathbf{8}, \boldsymbol{v_2} = \mathbf{31}, \mathbf{0.95}) = 2.18$, and is significant at a p-value $(= 0.000) < 0.005$. It appears that at least one of the predictor variables contributes information for the prediction of Y.

4 Concluding Remarks

From the above analysis, it appears that our multiple regression model for predicting the adult's body mass index $(BMI), Y$, is useful and adequate. In the presence of $\boldsymbol{X_1}, \boldsymbol{X_3}, \boldsymbol{X_4}, \boldsymbol{X_5}, \boldsymbol{X_6}, \boldsymbol{X_7}$, and $\boldsymbol{X_8}$, X_2 is a good predictor of Y. In the presence of $\boldsymbol{X_1}, \boldsymbol{X_2}, \boldsymbol{X_4}, \boldsymbol{X_5}, \boldsymbol{X_6}, \boldsymbol{X_7}, \boldsymbol{X_8}, X_3$ is a good predictor of Y. As the values of R^2 and R_a^2 are not very different, it appears that at least one of the predictor variables contributes information for the prediction of Y. The coefficient of variation $\boldsymbol{CV} = \mathbf{1.17303}\%$ also tells us that the standard deviation of the adult's body mass index (BMI), Y, is only $\mathbf{1.17303}\%$ of their mean. Also, since the test statistic value of F calculated from the data, $\boldsymbol{F} = \mathbf{615.93}$, exceeds the critical value of $\boldsymbol{F_{0.05}}\,(\boldsymbol{v_1} = \mathbf{8}, \boldsymbol{v_2} = \mathbf{31}, \mathbf{0.95}) = \mathbf{2.18}$, at $\alpha = 0.05$, we reject the null hypothesis: $\boldsymbol{H_0} : \boldsymbol{\beta_1} = \boldsymbol{\beta_2} = \boldsymbol{\beta}_3 = \boldsymbol{\beta}_4 = \boldsymbol{\beta}_5 = \boldsymbol{\beta_6} = \boldsymbol{\beta_7} = \boldsymbol{\beta_8} = \mathbf{0}$, that is, the regression is not significant. Hence, our multiple regression model for predicting the adult's body mass index (BMI), Y, seems to be useful and adequate, and the overall regression is statistically significant. The C_p-statistic criterion and residual plots of Y (Figure 3.1) as discussed above also confirm the adequacy of our model. For future work, one can consider to develop and study similar models for other issues and problems associated with the fields of medical, biological, behavioral, and other applied sciences. One can also develop similar models by adding other variables, for example, the gender, marital status, employment status, race and ethnicity of the adults, as well as the squares, cubes, and, cross products of $\boldsymbol{X_1}, \boldsymbol{X}_2, X_3, \boldsymbol{X_4}, \boldsymbol{X_5}, \boldsymbol{X_6}, \boldsymbol{X_7}$, and $\boldsymbol{X_8}$. In addition, one could also study the effect of some data transformations. We believe that the present study would be useful for researchers in the fields of medical and other applied sciences.

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APPENDIX 1

(Adult's Body Mass Index (BMI) Data, $n = 40$)

(Source: Triola [26])

Age	Ht	Wt	Waist	Pulse	Systolic	Diastolic	Cholesterol	BMI
58	70.8	169.1	90.6	68	125	78	522	23.8
22	66.2	144.2	78.1	64	107	54	127	23.2
32	71.7	179.3	96.5	88	126	81	740	24.6
31	68.7	175.8	87.7	72	110	68	49	26.2
28	67.6	152.6	87.1	64	110	66	230	23.5
46	69.2	166.8	92.4	72	107	83	316	24.5
41	66.5	135	78.8	60	113	71	590	21.5
56	67.2	201.5	103.3	88	126	72	466	31.4
20	68.3	175.2	89.1	76	137	85	121	26.4
54	65.6	139	82.5	60	110	71	578	22.7
17	63	156.3	86.7	96	109	65	78	27.8
73	68.3	186.6	103.3	72	153	87	265	28.1
52	73.1	191.1	91.8	56	112	77	250	25.2
25	67.6	151.3	75.6	64	119	81	265	23.3
29	68	209.4	105.5	60	113	82	273	31.9
17	71	237.1	108.7	64	125	76	272	33.1
41	61.3	176.7	104	84	131	80	972	33.2
52	76.2	220.6	103	76	121	75	75	26.7
32	66.3	166.1	91.3	84	132	81	138	26.6
20	69.7	137.4	75.2	88	112	44	139	19.9
20	65.4	164.2	87.7	72	121	65	638	27.1
29	70	162.4	77	56	116	64	613	23.4
18	62.9	151.8	85	68	95	58	762	27
26	68.5	144.1	79.6	64	110	70	303	21.6
33	68.3	204.6	103.8	60	110	66	690	30.9
55	69.4	193.8	103	68	125	82	31	28.3
53	69.2	172.9	97.1	60	124	79	189	25.5
28	68	161.9	86.9	60	131	69	957	24.6
28	71.9	174.8	88	56	109	64	339	23.8
37	66.1	169.8	91.5	84	112	79	416	27.4
40	72.4	213.3	102.9	72	127	72	120	28.7
33	73	198	93.1	84	132	74	702	26.2
26	68	173.3	98.9	88	116	81	1252	26.4
53	68.7	214.5	107.5	56	125	84	288	32.1
36	70.3	137.1	81.6	64	112	77	176	19.6
34	63.7	119.5	75.7	56	125	77	277	20.7
42	71.1	189.1	95	56	120	83	649	26.3
18	65.6	164.7	91.1	60	118	68	113	26.9
44	68.3	170.1	94.9	64	115	75	656	25.6
20	66.3	151	79.9	72	115	65	172	24.2

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ON A DUAL CHARACTERIZATION OF THE ASYMPTOTIC CONE FOR THE SOLUTION SET OF A LINEAR OPTIMIZATION PROBLEM

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Abstract

The concept of the asymptotic cone is very useful in various branches of pure and applied mathematics, especially in optimization and variational inequalities. In recent years, many authors and researchers have studied asymptotic directions and asymptotically convergent algorithms for unbounded solution sets. In this paper, we consider the asymptotic cone of the solution set Ω of a linear optimization problem and investigate various results on its asymptotic cone, asymptotic regularity, the dual and polar cones of the asymptotic cone, the support function of the solution set, etc. Finally, we present a dual characterization of the asymptotic cone Ω_∞ for the solution set of a linear optimization problem.

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1 Introduction

The concept of an asymptotic cone appeared in the literature first time in 1913 in Steintiz [35] to deal with the unboundedness of sets, particularly unbounded convex sets. For further details on asymptotic cones for convex sets we refer to Auslender and Teboulle [8], Luc and Penot [26], and Petrovai [29] and various relevant references cited in each of them. For the notion of asymptotic cone for nonconvex sets we refer to Luc [23,24,25,26], Penot [28], and Stoker[36].

The purpose of this paper is to investigate various asymptotic properties of the solution set for a linear optimization problem and utilize them to provide a dual characterization of the asymptotic cone of the solution set.

Throughout the paper, an n-dimensional Euclidean space will be denoted by R^n. For a point or vector $x = (x_1, x_2, \ldots, x_n) \in R^n$, the Euclidean norm of x is given as $\|x\| = \left(\sum_{i=1}^{n} {x_i}^2\right)^{\frac{1}{2}}$. A sequence in R^n is written as $\{x_k\}$ or sometimes $\{x_k\}_{k\in N}$, where N is the set of natural numbers. A subsequence of this sequence is denoted by $\{x_k\}_{k\in K}$, and $K \subset N$. A sequence $\{x_k\}_{k\in N}$ is said to converge to $x \in R^n$, if $\|x_k - x\| \to 0$, as $k \to \infty$.

It is indicated by the notation $\lim_{k\to\infty} x_k = x$ or $x_k \to x$.

This is called a strong form of convergence. A sequence $\{x_k\}_{k\in N}$ in R^n may converge to $x \in R^n$, linearly, quadratically, or super linearly. For further details on the order of convergence, we refer to Petrovai [29].

The Bolzano-Weierstrass theorem, which is a fundamental result of the convergence in a finitedimensional Euclidean space R^n, states that each bounded sequence in R^n has a convergent subsequence. A point $x \in R^n$ is called a cluster point of the sequence $\{x_k\}_{k\in N}$, in R^n, if there exists a subsequence $\{x_k\}_{k\in K}$ that converges to x. Also, the sequence $\{x_k\}_{k\in N}$, in R^n converges to a point $x \in R^n$ if and only if it is bounded and x is its unique cluster point. We will make use of the Bolzano-Weierstrass theorem to prove some results associated with the asymptotic cone, asymptotic regularity, etc. of the solution set of the linear optimization problem.

Further details for dealing with the asymptotic behavior of sets and functions can be referred to $[1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 20, 21, 22, 26, 27, 31, 35, 36]$ and the relevant references cited in these papers.

1.1 Linear Optimization Problem

A linear optimization problem in standard form can be stated as

$$\text{Maximize } f(x) = c^T x = c_1x_1 + c_2x_2 + \cdots + c_nx_n, \tag{1.1}$$

such that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \\ x_i \geq 0, i = 1, 2, \ldots, n \end{aligned}$$

where $A = [a_{ij}] \in R^{m\times n}, b = (b_1, b_2, \ldots, b_m)^T \in R^m, c = (c_1, c_2, \ldots, c_n)^T \in R^n$, and $x = (x_1, x_2, \ldots, x_n)^T \in R^n$.

Here $f : R^n \to R$ is a linear map defined by $f(x) = c^T x$ and, $A : R^n \to R^m$, is also a linear map defined by $AX = b$.

If we define m hyperplanes

$$H_i = \{x \in R^n : a_{11}x_1 + a_{11}x_1 + \cdots + a_{11}x_1 = b_i\}, i = 1, 2, \cdots, m. \tag{1.2}$$

Then

$$x \in \{x \in R^n : Ax = b\} \text{ if and only if } x \in \bigcap_{i=1}^{m} H_i. \tag{1.3}$$

Let $P_+ = \{x \in R^n : x \geq 0\}$ denotes the positive orthant of R^n and

$$\Omega = [\cap_{i=1}^m H_i] \cap P_+. \tag{1.4}$$

Then the above linear optimization problem (1.1) can be stated as

$$\text{Maximize } f(x) = c^T x, \text{ such that } x \in \Omega. \tag{1.5}$$

Further, we assume that

a) $A \in {R_m}^{m\times n}$, that is A is an $m \times n$ matrix of rank m.

b) $[A, b] \in {R_m}^{m\times(n+1)}$, that is, the augmented matrix $[A, b]$ is of order $m \times (n + 1)$, and rank m.

Thus, we have, $\text{rank}[A, b] = \text{rank}(A) = m$.

The solution set Ω is a nonempty closed subset of R^n. It is easy to see that it is also a convex subset of R^n.

2 Definitions and Notations

In this section, we explicate some definitions and related notations that will be used throughout this paper.

Let $x_k \in \Omega \subset R^n$, and $\|x_k\| \to \infty$, as $k \to \infty$. Then there exists a real sequence $\{\alpha_k\}_{k\in K}$, defined as $\alpha_k := \|x_k\|$, k $\in K, K \subset N$ such that $\lim_{k\in K} \alpha_k = +\infty$, and $\lim_{k\in K} \frac{x_k}{\alpha_k} = \beta$.

Definition 2.1. *(Nonnegative orthant). The nonnegative orthant of an n-dimensional Euclidean space is denoted by R^n_+ and is given by*

$R^n_+ = \{x \in R^n \mid x_i \geq 0, i = 1, 2, 3, \ldots, n\}$.

Definition 2.2. *(Cone or nonnegative homogeneous). A set K is called a cone if $\forall x \in K$, and $\mu \geq 0, \mu x \in K$.*

Definition 2.3. *(Convex hull of a set). The convex hull of a set K is denoted by conv K, is the set of all convex combinations of the points in K :*

$$\text{conv}\, K = \left\{\sum_{i=1}^{k} \mu_i x_i : x_i \in K, \mu_i \geq 0, \forall i, \sum_{i=1}^{k} \mu_i = 1\right\}$$

Definition 2.4. *The sequence $\{x_k\}_{k\in N} \subset \Omega \subset R^n$ is said to converge to a direction $\beta_k \in R^n$, If there exists a real sequence $\{\alpha_k\}$, with $\alpha_k \to +\infty$ such that $\lim_{k\in K} \frac{x_k}{\alpha_k} = \beta$. The vector $\beta \in R^n$ is called the direction of convergence.*

Definition 2.5. *(Asymptotic Cone of the Solution set Ω). The asymptotic cone of the solution set Ω, denoted by Ω_∞ is the collection of the vector $\beta \in R^n$ that are limits in the direction of the sequence $\{x_k\}_{k\in N}$ contained in the solution set S. i.e.,*

$$\Omega_\infty = \left\{\beta \in R^n : \exists\alpha_k \to +\infty, \exists x_k \in \Omega, \text{ with } \lim_{k\to\infty}\frac{x_k}{\alpha_k} = \beta\right\} \tag{2.1}$$

Definition 2.6. *Let the solution set Ω of the linear optimization problem (1.1) be nonempty and define a set denoted by Ω^1_∞ as follows:*

$$\Omega^1_\infty = \left\{\beta \in R^n : \forall\alpha_k \to +\infty, \exists x_k \in \Omega, \text{ with } \lim_{k\to\infty}\frac{x_k}{\alpha_k} = \beta\right\} \tag{2.2}$$

Definition 2.7. *The solution set Ω of the linear optimization problem (1.1) is called asymptotically regular, if*

$$\Omega_\infty = \Omega^1_\infty \tag{2.3}$$

Definition 2.8. *The normalized set of Ω). Let the Solution set Ω of the linear optimization problem (1.1) be nonempty, then the normalized set of Ω is denoted as Ω_N, and is defined as*

$$\Omega_N = \left\{\beta \in R^n : \exists\{x_k\} \in \Omega, \|x_k\| \to +\infty, \text{ with } \beta = \lim_{k\to\infty}\frac{x_k}{\|x_k\|}\right\}. \tag{2.4}$$

Definition 2.9. *(Support Function of Ω). Let the solution set Ω of the linear optimization problem (1.1) be a nonempty, closed convex set in R^n then the support function of Ω is a map $\sigma_\Omega(x) : R^n \to \mathrm{R}$ defined by*

$$\sigma_\Omega(x) = \sup\left\{x^T y : y \in \Omega\right\}. \tag{2.5}$$

If A *and B are two convex sets in R^n. Then $\sigma_A(x) = \sigma_B(x) \Leftrightarrow A = B$.*

Definition 2.10. *(The Housdorff distance). The Houdorff distance between two nonempty compact convex sets A and B can be expressed in terms of support functions as follows:*

$$d_H(A, B) = \|\sigma_A - \sigma_B\|_\infty, \text{ where } \|.\| \text{ denotes the uniform norm.} \tag{2.6}$$

Definition 2.11. *(The Domain of the support function of Ω). The domain of the support function of the solution set Ω is given as*

$$\mathrm{Dom}\,\sigma_\Omega = \left\{x : \sup_{y\in\Omega} x^T y < \infty\right\} \tag{2.7}$$

Definition 2.12. *(The Dual cone of Ω_∞). The Dual cone of Ω_∞ is the set*

$$\Omega^*_\infty = \left\{y : y^T x \geq 0, \forall x \in \Omega_\infty\right\} \tag{2.8}$$

Definition 2.13. *(The Polar cone of Ω_∞). The polar cone of Ω_∞ is the set*

$$\Omega^p_\infty == \left\{y : y^T x \leq 0, \forall x \in \Omega_\infty\right\} \tag{2.9}$$

Remark 2.1. *The polar cone Ω^p_∞ is just the negative of the polar cone Ω^*_∞.*

3 Main Results

In this section, we will prove some theorems related to the asymptotic cone, asymptotic regularity, and the normalized set of the solution set Ω. Finally, we present a dual characterization of the asymptotic cone Ω_∞ of the solution set for the linear optimization problem (1.1) in terms of the polar cone and the support function.

Theorem 3.1. *The necessary and sufficient condition for the solution set Ω of the linear optimization problem (1.1) is bounded is that the asymptotic cone of Ω does not contain any nonzero vector. i. e., if $\Omega_\infty = \{0\}$.*

Proof. It is obvious that if the solution set Ω of the linear optimization problem (1.1) is bounded then there does not exist a direction $\beta \in \Omega_\infty$ with $\beta \neq 0$.

Conversely, suppose, if possible, Ω is unbounded, and $\Omega_\infty = \{0\}$. As Ω is unbounded so, ∃ a sequence $\{x_k\}$ contained in the solution set Ω, such that $x_k \neq 0$, and $\forall k \in N$ $\alpha_k := \|x_k\| \to \infty$.

Now we have, $\beta_k := {\alpha_k}^{-1} x_k$.

So, $\|\beta_k\| = \left\|\alpha_k^{-1} x_k\right\| = \left\|\alpha_k^{-1}\right\| \|x_k\| = \left\|\frac{\|x_k\|}{\|x_k\|}\right\| = 1$, so the sequence $\{\beta_k\}_{k\in N}$ is bounded. Now using the Bolzano-Weierstrass theorem we can pull a subsequence $\{\beta_k\}_{k\in K}, K \subset N$, out of this sequence such that $\lim_{k\in K}\beta_k = \beta, K \subset N$, and $\|\beta\| = 1$. Thus ∃ a nonzero direction $\beta \in \Omega_\infty$, which contradicts the fact that $\Omega_\infty = \{0\}$. □

Theorem 3.2. *If the solution set $\Omega \neq \varnothing$, and convex then Ω is asymptotically regular.*

Proof. It is easy to verify that Ω is a convex set. If $x_1, x_2 \in \Omega$ then both vectors will satisfy the equation $AX = b$ and therefore $Ax_1 = b$, and $Ax_2 = b$, $x_1, x_2 \geq 0$. Now we consider a convex combination of x_1 and x_2, as $\mu x_1, +(1-\mu)x_2$, with $0 \leq \mu \leq 1$. Clearly $\mu x_1, +(1-\mu)x_2 \geq 0$. and $A[\mu x_1, +(1-\mu)x_2] = \mu Ax_1 + (1-\mu)Ax_2 = \mu b + (1-\mu)b = b$. So Ω is a convex set.

Now it follows from the definitions of Ω_∞ and Ω_∞^1 that

$$\Omega_\infty^1 \subseteq \Omega_\infty. \tag{3.1}$$

Our next goal is to show that $\Omega_\infty \subseteq \Omega_\infty^1$.

Let $\beta \in \Omega$. Then it follows from the definition of Ω_∞ that there exists a sequence $\{x_k\}_{k\in N} \in \Omega$, and $\exists$ a sequence of real numbers $\{p_k\}_{k\in N}$ such that $p_k \to \infty$, and

$$\beta = \lim_{k\to\infty} p_k^{-1} x_k. \tag{3.2}$$

For, $x \in \Omega$, we define a sequence of directions $\{\beta_k\}_{k\in N} \in R^n$ as

$$\beta_k = p_k^{-1}(x_k - x). \tag{3.3}$$

Now $\beta_k = p_k^{-1}(x_k - x) \Rightarrow p_k\beta_k = x_k - x \Rightarrow x_k = x + p_k\beta_k$, As $x_k \in \Omega$, so, $x + p_k\beta_k \in \Omega$, and $\beta = \lim_{k\to\infty} \beta_k$.

Let $\{\delta_k\}_{k\in N}$ be a sequence of real numbers such that $\lim_{k\to\infty} \delta_k = +\infty$.

Now for a fixed natural number m, there exists $k(m)$ with

$$\lim_{m\to\infty} k(m) = +\infty, \text{ such that } \delta_m \leq p_{k(m)}. \tag{3.4}$$

As Ω is convex, we have $x_m^* = x + \delta_m p_{k(m)} \in \Omega$, therefore

$$\beta = \lim_{m\to\infty} \delta_m \beta_{k(m)}. \tag{3.5}$$

This implies that $\beta \in \Omega$, so we have

$$\Omega_\infty \subseteq \Omega_\infty^1. \tag{3.6}$$

Hence, it follows from (3.1) and (3.6) that

$\Omega_\infty = \Omega_\infty^1$.

Thus, the solution set Ω, of the linear optimization problem (1.1) is asymptotically regular. □

Theorem 3.3. *Let the solution set $\Omega \neq \emptyset$ and define a normalized set of Ω, as*

$\Omega_N := \{\beta \in R^n : \exists \{x_k\} \in \Omega, \|x_k\| \to +\infty$, *with* $\beta = \lim_{k\to\infty} \frac{x_k}{\|x_k\|}\}$.

Then, $\Omega_\infty = pos\ \Omega_N$, *where* $pos\ \Omega_N = \{\lambda x : x \in \Omega, \lambda \geq 0\}$ *is the positive hull of* Ω.

Proof. From the definitions of Ω_∞ and Ω_N it follows that

$$\Omega_N \subseteq \Omega_\infty. \tag{3.7}$$

To prove that $\Omega_\infty \subseteq \Omega_N$, let $\beta \in \Omega_\infty$ and $\beta \neq 0$. Then from the definition of Ω_∞ there exists a real sequence

$\{\alpha_k\}_{k\in N}$ with $\lim_{k\to\infty} \alpha_k = +\infty$. Now for $x_k \in \Omega$, we have

$$\beta = \lim_{m\to\infty} \left[\alpha_k^{-1} x_k\right] = \lim_{m\to\infty} \left[\alpha_k^{-1} \|x_k\| \frac{x_k}{\|x_k\|}\right]. \tag{3.8}$$

Thus, the sequence $\{\alpha_k^{-1}\|x_k\|\}_{k\in N}$ is a nonnegative bounded sequence, so by Bolzano-Weierstrass theorem $\exists$ a convergent subsequence $\{\alpha_k^{-1}\|x_k\|\}_{k\in K}, K \subset N$ such that

$$\lim \left[\left[\alpha_{\underset{k\to\infty}{k}}^{-1} \|x_k\|\right] = \lambda \geq 0 \right. \tag{3.9}$$

So, from (3.8) we have

$$\beta = \lim_{m\to\infty} \left[\alpha_k^{-1} \|x_k\| \frac{x_k}{\|x_k\|}\right] = \lim \left[\alpha_{\underset{k\to\infty}{k}}^{-1} \|x_k\|\right] \lim_{k\to\infty} \frac{x_k}{\|x_k\|} = \lambda\beta_N\ . \tag{3.10}$$

With normalized direction β_N and $x \in \Omega$, so $\beta \in \operatorname{pos}\Omega$

$$\Omega_\infty \subseteq \Omega_N \tag{3.11}$$

Therefore, it follows from (3.7) and (3.11) that

$$\Omega_\infty = \operatorname{pos}\Omega_N.$$

□

Theorem 3.4. *(Dual Characterization Theorem). If the solution set Ω of the linear optimization problem is nonempty and Ω_∞ and Ω_∞^P denote the asymptotic cone of Ω and the polar cone of Ω_∞ respectively. Then the following relations hold:*

a) If σ_Ω denotes the support function for the solution set Ω of the linear optimization problem, dom $\sigma_\Omega \subset \Omega_\infty^P$.

b) If the interior of the polar cone Ω_∞^P is nonempty, $\Omega_\infty^P \subset$ dom σ_Ω.

c) For the solution set Ω of the linear optimization problem, $(\text{ dom } \sigma_\Omega)^P = \Omega_\infty$.

Proof.

a) From the definitions 2.11 and 2.13 of $\operatorname{dom}\sigma_\Omega$ and Ω_∞^P, it follows that $\operatorname{dom}\sigma_\Omega \cap \Omega_\infty^P \neq \phi$.
Let $y \notin \Omega_\infty^P$. Then from the definition 2.13, $\exists \in \Omega_\infty$ such that $y^T\beta > 0$. As $\beta \in \Omega_\infty$
It follows from the definition of Ω_∞ that $\exists$ a sequence sequence $\{x_k\}_{k\in N} \subset \Omega$ in R^n and sequence $\{\alpha_k\}_{k\in N}$ in R such that $\alpha_k \to +\infty$, with $\alpha_k^{-1}x_k \to \beta$, and satisfying, the inequality $y^T\beta > 0$. Hence it follows that $y^T x_k \to +\infty$.
This implies that $y \notin \operatorname{dom}\sigma_\Omega$, so $\operatorname{dom}\sigma_\Omega \subset \Omega_\infty^P$.

b) Let $y \notin$ dom σ_Ω. Then $\exists\beta \in \Omega_\infty$ such that $y^T\beta > 0$ and $\beta \neq 0$. As $\beta \in \Omega_\infty$
$\exists$ a sequence sequence $\{x_k\}_{k\in N} \subset$ in Ω with

$$\left\{x_k^T y\right\}_{k\in N} \to +\infty$$

Considering subsequences, if necessary, without any loss of generality, we can assume that $\frac{x_k}{\|x_k\|} \longrightarrow \beta$, and $\beta \neq 0$, and $\beta \in \Omega_\infty$. Hence it follows that

$$\left(\frac{x_k}{\|x_k\|}\right)^T y \geq 0. \tag{3.12}$$

Hence for $\epsilon > 0$, we have $\beta^T(y + \epsilon\beta) \geq \epsilon\|\beta\|^2$
This implies that $y + \epsilon\beta \notin \Omega_\infty^P$.
That is, $y \notin$ int Ω_∞^P. Hence it follows that

$$\Omega_\infty^P \subset \operatorname{dom}\sigma_\Omega.$$

c) The set Ω is a closed convex set in R^n, so Ω_∞ is a closed convex cone then it follows from the definition of the polar cone that

$$\left(\Omega_\infty^P\right)^P = \Omega_\infty. \tag{3.13}$$

Now from (a) we have

$$\operatorname{dom}\sigma_\Omega \subset \Omega_\infty^P. \tag{3.14}$$

This implies that $\left(\Omega_\infty^P\right)^P \subset (\operatorname{dom}\sigma_\Omega)^P$. Using equation (3.13) we have

$$\Omega_\infty \subset (\operatorname{dom}\sigma_\Omega)^P. \tag{3.15}$$

Now in order to prove that $(\text{ dom } \sigma_\Omega)^P \subset \Omega_\infty$, suppose that $\beta \in (\operatorname{dom}\sigma_\Omega)^P$, for a real number $\alpha > 0$ and an arbitrary point $\bar{x}$ in Ω, $\alpha\beta \in (\operatorname{dom}\sigma_\Omega)^P$, so for an arbitrary $y \in \operatorname{dom}\sigma_\Omega$, we have

$$\begin{aligned}(\bar{x} + \alpha\beta)^T y &= \bar{x}^T y + (\alpha\beta)^T y \\ &\leq \bar{x}^T y \\ &\leq \sup\left\{x^T y : x \in \Omega\right\} \\ &= \sigma_\Omega(y)\end{aligned} \tag{3.16}$$

Thus, for an arbitrary $y \notin dom\sigma_\Omega$, we have

$$\sigma_\Omega(y) = +\infty.$$

The inequality (3.15) remains valid $\forall y$ in R^n.
Therefore, $\forall\alpha > 0, \bar{x} + \alpha\beta \in \Omega$, where $\operatorname{cl}\Omega$ denotes the closure of the solution set Ω.
We know that for any convex set Ω in R^n, Ω_∞ is a closed convex cone and

$$\Omega_\infty = D = \{\beta \in R^n : \mathrm{x} + \alpha\beta \in \operatorname{cl}\Omega, \forall\alpha > 0, \text{ and } \forall x \in \Omega\}. \tag{3.17}$$

Thus, $\beta \in \Omega_\infty$, and

$$(\operatorname{dom} \sigma_\Omega)^P \subset \Omega_\infty. \tag{3.18}$$

Now it follows from (3.15) and (3.17) that

$$(\operatorname{dom} \sigma_\Omega)^P = \Omega_\infty$$

Further details of the asymptotic properties of the sets and the functions can be referred to [1,2,4,5,9,10,11,12,14,16,20,21,22,23,25,26,27,31,35,36].

□

4 Concluding Remarks

The concept of the asymptotic cone is enormously useful in the study of the behavior of both convex and nonconvex sets. For example, in [26] Luc and Penot have investigated various properties of the asymptotic directions of unbounded sets to examine the perturbation of the data. Petrovai [29] has investigated the notion of asymptotic convergence which is extremely useful for the algorithms dealing with nonlinear mathematical programming Problems. The fundamental problem of linear optimization is to arrive at the best possible decision in any given set of circumstances when the functions to be optimized and the constraints are both linear. These days linear optimization is one of the most frequently used decisionmaking tools in the industry, administration, banking, finance, marketing, and various other spheres of life. A desirable property of an algorithm for solving a linear optimization problem is that it generates a well-defined solution at each iteration of the algorithm and its solution set remains bounded all the time. However, in several situations, the sequence of iterates may not remain bounded, and consequently, we get an unbounded solution set. The results obtained in this paper, together with the BolzanoWeierstrass theorem, and the notion of asymptotic convergence will be useful to deal with the unbounded solution sets of mathematical optimization problems. The results of this paper can be extended for the solution sets of the other conic optimization problems like semidefinite programming (SDP) and second-order cone programming (SOCP). These results can help to obtain some characterization results for the asymptotic cones of the solution sets for SDP and SOCP. The notion of asymptotic, polar cones and asymptotic regularity plays a considerable role in various disciplines of Mathematical Sciences.

The various applications of the asymptotic cones, polar cones, dual cones, and associated asymptotic functions in various areas of mathematical sciences can be referred to [4,5,9,10,11, 12,14,16,20,21,22,23,27,31,35,36], and the references cited in these papers.

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ON DIRECT SUM OF TOPOLOGICALLY TRANSITIVE OPERATORS

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Abstract

The purpose of this paper is to answer the question posed by Feldman [9] on topological transitivity which states that "*If E is transitive, does it follows that direct sum $E \oplus E$ is topologically transitive?*" We will show that this question has a positive answer under certain conditions. In particular, we define topologically transitive operators and use them to show that the direct sum $E \oplus E$ of two operators is topologically transitive whenever E is topologically transitive. Then, we give some examples of a topologically transitive operator which does not satisfy topologically transitive criterion and so not topologically transitive.
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Keywords and Phrases: Hypercyclic operator, topologically transitive, direct sum, transitivity criterion.

1 Introduction and Preliminaries

A bounded linear operator E on a separable Banach space X is topologically transitive if for each pair of non-empty open subsets $P \subset X$ and $Q \subset X$, one can find positive integer $k \geqslant 0$ such that $E^k(P) \cap Q \neq \varnothing$. If $E^k(P) \cap Q \neq \varnothing$ is from some $k \geqslant N$, then E is said to be topologically mixing. Birkhoff [5] developed a topological transitive operator and provided an example of how it may be used to approximate any holomorphic function in $\mathbb{H}(\mathbb{C})$. On separable Banach spaces, topological transitivity and hypercyclicity are similar concepts in linear dynamics, according to Grosse-Erdmann and Manguillot [11]. A linear operator E on a vector space X is said to be hypercyclic if there exists a vector $x \in X$ such that the set of all vectors obtained by iterating E on x, denoted by $orb(E, x) = \{x, Ex, E^2x, \dots\}$, is dense in X. For E, such a vector x is referred to as a hypercyclic vector.

Rolewicz [16] introduced the idea of hypercyclic operators and gave the first illustration of a hypercyclic operator on a Banach space. He demonstrated that if B is the backward shift on $\ell(N)$ then λB is hypercyclic for every scalar $|\lambda| > 1$. The Hypercyclicity criterion, a useful necessary condition for an operator to be hypercyclic, was later established by Kitai [13]. Gethner and Shapiro [10] also contributed to the development of this criterion. Many authors have further refined this criterion (see Grosse-Erdmann [11] and the references therein).

Recently, Madore and Martinez [14] studied hypercyclicity on subspaces. They investigated subspace-topologically transitive operators and demonstrated that any subspace-topologically transitive operator is subspace-hypercyclic. This result extends the theory of hypercyclic operators to the case of operators acting on subspaces. Further details on hypercyclicity and related topics can be found in the monographs by Grosse-Erdmann [11] and Bayart and Matheron [4].

In the study of linear dynamics, hypercyclicity and topological transitivity are important concepts that describe the behavior of bounded linear operators on Banach spaces. One question of interest is whether the hypercyclicity property is preserved under direct sums of operators. Kitai [13] showed that if a direct sum $E \oplus E$ is hypercyclic, then both E_1 and E_2 must also be hypercyclic.

However, Salas [18] constructed an operator E and its adjoint E^* such that both E and E^* are hypercyclic, but their direct sum $E \oplus E^*$ is not hypercyclic. This example raises the question of whether $E \oplus E$ is hypercyclic whenever E is hypercyclic. Herrero questioned this, and De la Rosa and Read [15] provided a hypercyclic operator E such that $E \oplus E$ is not hypercyclic, showing that the answer to Herrero's question is negative.

On the other hand, Bès and Peris [2] showed that if $E \oplus E$ is hypercyclic, then E fulfills the hypercyclic condition as well. In other words, hypercyclicity is preserved under direct sums in one direction. Further details on hypercyclicity and topological transitivity can be found in the monographs by Bayart and Matheron [4] and Grosse-Erdmann and Manguillot [11].

Definition 1.1 ([11])**.** *A bounded linear operator E acting on a Banach space X is said to be topologically transitive if for any two non-empty open subsets $P, Q \subseteq X$, there exists a positive integer k such that $E^k(P) \cap Q$ is non-empty.*

Definition 1.2. *A pair of bounded linear operators (E_1, E_2) on a Banach space X is said to be topologically mixing if for any pair of non-empty open sets $P, Q \subseteq X$, there exist positive integers M and N such that $E_1^m E_2^n(P) \cap Q \neq \varnothing$ for all $m \geq M$ and $n \geq N$.*

Intuitively, this means that after some finite number of iterations of each operator, the images of P and Q intersect.

Note that the order of the operators in the product $E_1^m E_2^n$ matters in general, and that the definition of topological mixing requires that both operators are involved in the mixing property.

Also, note that the definition of topological mixing is stronger than that of topological transitivity, as it requires the existence of two parameters M and N, whereas topological transitivity only requires the existence of one parameter n.

Definition 1.3 ([11])**.** *An operator E on a separable Hilbert space $\mathcal{H}$ is said to be chaotic if it satisfies the following conditions:*

(i) E is topologically transitive.

(ii) E has a dense set of periodic points, that is, there exists a dense subset D of $\mathcal{H}$ such that for any $a \in D$, there exists a positive integer k such that $E^k(a) = a$.

Definition 1.4 ([6])**.** *$E \in \mathcal{B}(\mathcal{H})$ is said to be weakly mixing if $E \oplus E$ is topologically transitive on $X \oplus X$, and E is mixing if for every pair of non-empty open sets $,Q \subseteq X$ there exists some $k \in N$ such that $E^k(P) \cap Q \neq \varnothing$, $\forall\ k \geqslant k_0$.*

The notions of weakly mixing and mixing are closely linked to the idea of hereditarily hypercyclic operators.

Topological mixing $\implies$ topological transitivity by definition **??**, but not vice versa.

Definition 1.5. *A dynamical system $E : X \to X$ is said to be minimal if for every $x \in X$, the orbit of x under E is dense in X.*

Example 1.1 ([11])**.** An irrational circle rotation is minimal and therefore topologically transitive, but not topologically mixing.

Proof. Let $E_\alpha : S^1 \to S^1$ be the map defined by $E_\alpha(z) = z + \alpha \pmod 1$, where α is an irrational number. This is an example of an irrational circle rotation. Where S^1 is defined as

$$S^1 = \{a \in \mathbb{C} : |a| = 1\}.$$

To show that E_α is minimal, we need to show that every point is dense in its orbit.

Let $a \in S^1$ and let $k \in \mathbb{Z}$ be arbitrary. Then, there exists a sequence of integers $(x_n)_{n=1}^\infty$ for which

$$\sum_{n=1}^{\infty} x_n \alpha = k$$

and

$$|k - \sum_{n=1}^{m} x_n \alpha| \leq |\alpha|,$$

for all $m \in \mathbb{N}$.

Using this sequence, we can construct a sequence $(a_m)_{m=0}^{\infty}$ in S^1 by setting $a_0 = a$ and $a_{m+1} = T_{\alpha}^{k_{m+1}}(a_m)$ for $m \geq 0$. Then, we have:

$$\begin{aligned} |a_{m+1} - a| &= |E_{\alpha}^{k_{m+1}}(a_m) - E_{\alpha}^{k_{m+1}}(a) + E_{\alpha}^{k_{m+1}}(a) - a|, \\ &= |\alpha k_{m+1}| + |E_{\alpha}^{k_{m+1}}(a) - a|, \\ &\leq |\alpha| + |E_{\alpha}^{k_{m+1}}(a) - E_{\alpha}^{k_m}(a)|, \\ &\leq 2|\alpha|, \end{aligned}$$

for all $m \geq 0$.

This proves that the sequence (a_m) is a Cauchy sequence, hence it converges to a limit y in S^1. Since E_α is continuous, we have

$$E_\alpha(y) = \lim_{m\to\infty} E_{\alpha}^{k_m}(a) = a + k\alpha \pmod 1 = a.$$

Therefore, y belongs to the orbit of a, and since a was arbitrary, we conclude that every point is dense in its orbit. This shows that E_α is minimal.

To show that E_α is not topologically mixing, we will construct a pair of disjoint open subsets $P \subseteq S^1$ and $Q \subseteq S^1$ for which $E_{\alpha}^{k}(P) \cap Q = \varnothing$, for all $k \in \mathbb{N}$. Let $\epsilon > 0$ be small enough so that $\epsilon < |\alpha|$.

Define $P = (-\epsilon, \epsilon)$ and $Q = (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)$. Then, for any $k \in \mathbb{N}$, we have:

$$\begin{aligned} E_{\alpha}^{k}(P) &= (k\alpha - \epsilon, k\alpha + \epsilon) \pmod 1, \\ E_{\alpha}^{k}(P) &= (\frac{1}{2} + k\alpha - \epsilon, \frac{1}{2} + k\alpha + \epsilon) \pmod 1. \end{aligned}$$

These sets are disjoint if and only if $k\alpha - \frac{1}{2} > \epsilon$ or $k\alpha - \frac{1}{2} < -\epsilon$. Since α is irrational. Then, E_α is not topologically mixing. □

Definition 1.6 ([6]). *An operator $E \in \mathcal{B}(\mathcal{H})$ is said to be hereditarily hypercyclic with respect to a strictly increasing sequence (m_k) of natural numbers if, for any subsequence (m_{k_j}) of (m_k), there is $x \in X$ such that $\{E^{m_{k_j}}x, j \in \mathbb{N}\}$ is dense in X.*

Theorem 1.1. *(Hypercyclicity Criterion) [4] Let X be a Fréchet space, and let E be a continuous linear operator on X. Assume there exist two dense subsets $\mathcal{D}_1, \mathcal{D}_2$ of X, an increasing sequence of integers $(n_k)_{k\geq 1}$, and a family of maps $(S_k)k \geq 1$ from $\mathcal{D}_2$ to X such that:*

i. For each $k \geq 1$, $E^{n_k}(x) \to 0$ for all $x \in \mathcal{D}_1$.
ii. For each $k \geq 1$, $S_k(y) \to 0$ for all $y \in \mathcal{D}_2$.
iii. For each $k \geq 1$ and each $y \in \mathcal{D}_2$, $E^{n_k} \circ S_k(y) \to y$.

Then, E is hypercyclic.

Theorem 1.2 ([2]). *Let $E \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(i) E satisfies the Hypercyclicity Criterion.
(ii) E is hereditarily hypercyclic.
(iii) $E \oplus E$ is hypercyclic.

Example 1.2. Let $(E_1, E_2, E_3) = (2I_1, \dfrac{1}{3}I_1, e^{i\theta}I_1)$ where I_1 is the identity operator on $\mathbb{C}$ and θ is an irrational multiple of π. Then E is hypercyclic on $\mathbb{C}$, but E does not satisfy the topologically transitivity criterion.

Example 1.3. If C and D be topologically transitive operators and let $E_1 = C \oplus I$ and $E_2 = I \oplus D$ then (E_1, E_2) is a topologically transitive, but neither (E_1 nor E_2) is cyclic.

Proof. First, we need to show that (E_1, E_2) is topologically transitive.

Now, consider $(x, y) \in X \oplus Y$, where X and Y are Banach spaces. We need to show that for every non-empty open subsets $P_1 \subset X$ and $P_2 \subset Y$, $\exists$ $(k, s) \in \mathbb{N} \times \mathbb{N}$ such that $E_1^k(x) \in P_1$ and $E_2^s(y) \in P_2$.

Since C and D are topologically transitive, $\exists$ $(k_1, k_2) \in \mathbb{N}$ such that $C^{k_1}(x) \in P_1$ and $D^{k_2}(y) \in P_2$.

Let $k = \max\{k_1, k_2\}$. Then, we have

$$E_1^k(x) = (C \oplus I)^k(x, y) = (C^k(x), y)$$

$$E_2^k(y) = (I \oplus D)^k(x, y) = (x, D^k(y)).$$

For all $k \geq k_1$, we have $C^k(x) \in P_1$, and for $k \geq k_2$, we have $D^k(y) \in P_2$.

Therefore, (E_1, E_2) is topologically transitive.

Next, we need to show that neither E_1 nor E_2 is cyclic.

Suppose by contradiction that E_1 is cyclic. Then, there is $x \in X$ such that the set $\left\{E_1^k(x) : k \in \mathbb{N}\right\}$ is dense in X.

Let $y \in Y$ be arbitrary. Then the set $\left\{(E_1^K(x), y) : K \in \mathbb{N}\right\}$ is dense in $X \oplus Y$.

However, we have

$$(E_1^k(x), y) = (A^k(x), y) \to (0, y)$$

as $k \to \infty$, which contradicts the density of $\left\{(E_1^k(x), y) : k \in \mathbb{N}\right\}$.

Similarly, suppose by contradiction that E_2 is cyclic. Then, there is $y \in Y$ such that the set $\left\{E_2^k(y) : k \in \mathbb{N}\right\}$ is dense in Y.

Let $x \in X$ be arbitrary. Then the set $\left\{(x, E_2^k(y)) : k \in \mathbb{N}\right\}$ is dense in $X \oplus Y$.

Nevertheless, we have that

$$(x, E_2^k(y)) = (x, D^k(y)) \to (x, 0)$$

as $k \to \infty$, which contradicts the density of $\left\{(x, E_2^k(y)) : k \in \mathbb{N}\right\}$.

Therefore, neither E_1 nor E_2 is cyclic. $\square$

Example 1.4. Let A and B be topologically transitive operators and let C be an operator with dense range that commutes with B. If we define $T_1 = A \oplus C$ and $T_2 = I \oplus B$ then (T_1, T_2) is a topologically transitive.

Proof. To show that (T_1, T_2) is topologically transitive.

We need to show that for any non-empty open subsets U_1, U_2 in $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ respectively, $\exists\, n \in \mathbb{N}$ such that

$$T_1^n(U_1) \cap T_2^n(U_2) \neq \varnothing.$$

Let U_1, U_2 be nonempty open sets in $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ respectively.

Since A and B are topologically transitive, there exist natural numbers m and n such that

$$A^m(U_1) \cap C \neq \varnothing$$

and

$$B^n(U_2) \neq \varnothing.$$

Since the range of C is dense in $\mathcal{H}_2$, $\exists\, x \in \mathcal{H}_1$ for which Cx is arbitrarily close to any given vector in $\mathcal{H}_2$.

Let $y \in B^n(U_2)$, then there exists $z \in \mathcal{H}_2$ such that $B^n z = y$.

Since C commutes with B, we have $CB^n z = BC^n z$, and since C has dense range, we can find $w \in \mathcal{H}_1$ such that $C^n w$ is arbitrarily close to $BC^n z$. Then,

$$T_1^n(A^m(U_1) \cap C) \cap T_2^n(U_2) \supseteq (A^m \oplus C)(U_1) \cap (I \oplus B)(U_2) = U_1 \oplus B^n(U_2) \neq \varnothing,$$

where we used the fact that A^m commutes with I and B^n commutes with C.

Therefore, (T_1, T_2) is topologically transitive. $\square$

Theorem 1.3 ([23])**.** *Let E be a bounded linear operator on a complex Banach space X (not necessarily separable). Suppose that there exists a strictly increasing sequence (k_i) of positive integers for which there is*

(i) a dense subset $A \subset X$ such that $E^{k_i}(x) \to 0$, for every $a \in A$ as $i \to \infty$.

(ii) a dense subset $B \subset X$ and a sequence of mappings $G_i : B \to X$ such that $G_i(b) \to 0$, for every $b \in B$ and $E^{k_i} G_i(b) \to b$, for every $b \in B$ as $i \to \infty$.

Then, E is topologically transitive.

In the next section, we investigate the properties of topologically transitive linear operators on a Banach space. Specifically, we focus on the class of operators E that are topologically transitive, and demonstrate that their direct sum $E \oplus E$ is also topologically transitive.

2 Main results

In this section, we investigate topologically transitive operator E whose direct sum $E \oplus E$ is topologically transitive. Thereby responding to the question posed by Feldman [9] which states that: *"If E is transitive, does it follows that direct sum $E \oplus E$ is topologically transitive?"* in the affirmative. Thus, we will modify Theorem 1.3 of Zagorodnyuk [23] to prove our main results of this study on topologically transitive operators.

Theorem 2.1. *Let $E = (E_1, E_2) \in L(Z \oplus Z)$ be a bounded linear operator on a topological vector space. Suppose there exists a strictly increasing sequence (k_i) of positive integers for which there is*

(i) a dense subset $A \subset Z$ such that $(E_1 \oplus E_2)^{k_i}(a_1, a_2) \to (0,0)$ for every $(a_1, a_2) \in A$ as $i \to \infty$.

(ii) a dense subset $B \subset Z$ and a sequence of mappings $G_{k_i} : B \to Z$ such that $(G_1 \oplus G_2)_{k_i}(b_1, b_2) \to (0,0)$ for every $(b_1, b_2) \in B$ and $(E_1 \oplus E_2)^{k_i}(G_1 \oplus G_2)_{k_i}(b_1, b_2) \to (b_1, b_2)$ for every $(b_1, b_2) \in B$ as $i \to \infty$.

Then $E_1 \oplus E_2$ is topologically transitive.

Proof. Let P_1, P_2, Q_1 and Q_2 be non empty open sets of Z.

Then, $(P_1 \oplus P_2)$ and $(Q_1 \oplus Q_2)$ are open in $Z \oplus Z$.

Since $(A_1 \oplus A_2)$ and $(B_1 \oplus B_2)$ are dense in $Z \oplus Z$ then there exist (a_1, a_2) and (b_1, b_2) in $(A_1 \oplus A_2)$ and $(B_1 \oplus B_2)$ respectively such that

$$(a_1, a_2) \in (P_1 \oplus P_2) \cap (A_1 \oplus A_2)$$

and

$$(b_1, b_2) \in (Q_1 \oplus Q_2) \cap (B_1 \oplus B_2).$$

For all $i \geqslant 1$, let $z_i = (a_1, a_2) + (G_1 \oplus G_2)_{k_i}(b_1, b_2)$.

By Theorem 2.1 condition (ii), we have that $(G_1 \oplus G_2)_{k_i}(b_1, b_2) \to (0,0)$ as $i \to \infty$.

$\implies z_i \to (a_1, a_2)$.

Since $(a_1, a_2) \in (P_1 \oplus P_2)$ and (P_1, P_2) is open, there exists $N_1 \in \mathbb{N}$ such that $z_i \in (P_1 \oplus P_2), \forall i \geqslant N_1$.

On the other hand,

$(E_1 \oplus E_2)^{k_i} z_i = (E_1 \oplus E_2)^{k_i}(a_1, a_2) + (E_1 \oplus E_2)^{k_i}(G_i(b_1, b_2)) \to (b_1, b_2)$. Since $(b_1, b_2) \in (Q_1 \oplus Q_2)$ and (Q_1, Q_2) is open, there exists $N_2 \in \mathbb{N}$ such that $(E_1 \oplus E_2)^{k_i} z_i \in (Q_1 \oplus Q_2), \forall i \geqslant N_2$.

Let $N = \max\{N_1, N_2\}$ then $z_i \in (P_1 \oplus P_2)$ and $(E_1 \oplus E_2)^{k_i} z_i \in (Q_1 \oplus Q_2 \forall i \geqslant N$.

It follows that,

$(E_1 \oplus E_2)^{k_i}(P_1 \oplus P_2) \cap (Q_1 \oplus Q_2) \neq \varnothing, \forall i \geqslant N$.

Hence, $E_1 \oplus E_2$ is topologically transitive. □

Remark 2.1. *If E_2 is the identity, then the conditions in Theorem 2.1 reduce to the well-known "topologically transitivity criterion" for a single operator.*

Proposition 2.1. *An operator $E = (E_1, E_2) \in \mathcal{B}(\mathcal{H})$ is topologically transitive if and only if $G = \{(E_1 \oplus E_2)^s : s \in \mathbb{N}\}$ is topologically transitive.*

Proof. We will prove the "if" part and the "only if" part separately.

If part: Suppose by contradiction that, E is not topologically transitive, that is, there exist non-empty open sets $P, Q \subseteq \mathcal{H}$ such that for all positive integers k, we have $E^k(P) \cap Q = \varnothing$. Let $G = \{(E_1 \oplus E_2)^s : s \in \mathbb{N}\}$. Then for any $p, q \in \mathcal{H}$ and any positive integer k, we have

$$(E^k \oplus E^K)(p \oplus q) = E^k p \oplus E^k q,$$

and so $E^k p \in P$ and $E^k q \in Q$ imply that $(E^k \oplus E^k)(p \oplus q) \notin P \oplus Q$.

This means that for any non-empty open sets $P', Q' \subseteq \mathcal{H} \oplus \mathcal{H}$, there exists a positive integer k such that

$$(E^k \oplus E^k)(P' \cap (P' \oplus Q')) = \varnothing$$

which is contradiction.

Therefore, E is topologically transitive.

Only if: Suppose E is topologically transitive and let $G = \{(E_1 \oplus E_2)^s : s \in \mathbb{N}\}$.

Let $P, Q \subseteq \mathcal{H} \oplus \mathcal{H}$ be non-empty open sets. Then there exist non-empty open sets $P_1, P_2, Q_1, Q_2 \subseteq \mathcal{H}$ such that $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$.

Since E is topologically transitive, there exists a positive integer K such that

$$E^k(P_1) \cap Q_1 \neq \varnothing.$$

Then

$$(E^k \oplus E^k)(P \cap (P \oplus Q)) = (E^k P_1 \oplus E^k P_2) \cap (Q_1 \oplus Q_2),$$

and since $E^k P_1 \subseteq \mathcal{H}$ and $E^k P_2 \subseteq \mathcal{H}$ are non-empty.

It follows that $(E^k \oplus E^k)(P \cap (P \oplus Q)) \neq \varnothing$.

Therefore, G is topologically transitive. $\square$

Proposition 2.2. *Every chaotic operator $E = (E_1, E_2) \in \mathcal{B}(\mathcal{H})$ on a topological vector space X satisfies the topologically transitivity criterion.*

Proof. It is enough to show that $E_1 \oplus E_2$ is topologically transitive whenever E is topologically transitive.

Now, let $E \in \mathcal{B}(\mathcal{H})$ be chaotic and also let P_1, P_2, Q_1, Q_2 be open, non-empty subsets of X. We show that there exists arbitrary large integer k satisfying

$$\begin{cases} (E_1 \oplus E_2)^k(P_1) \cap Q_1 \neq \varnothing \\ (E_1 \oplus E_2)^k(P_2) \cap Q_2 \neq \varnothing. \end{cases} \tag{2.1}$$

Now, since E is topologically transitive, there exists m arbitrarily large with

$$(E_1 \oplus E_2)^m(P_1) \cap Q_1 \neq \varnothing.$$

Furthermore, since E is chaotic there exists some $p_1 \in P_1$ and $s > 0$ with

$$(E_1 \oplus E_2)^m(p_1) \in Q_1$$
$$(E_1 \oplus E_2)^s(p_1) = p_1.$$

By Proposition 2.1, the operator $G = (E_1 \oplus E_2)^s \in L(X)$ is also topologically transitive, and so there exists a positive integer d satisfying

$$(E_1 \oplus E_2)^{ds}(P_2) \cap (E_1 \oplus E_2)^{-m}(Q_2) \neq \varnothing.$$

Let $k = ds + m$. Then we have that,

$$(E_1 \oplus E_2)^k(P_2) \cap Q_2 \neq \varnothing.$$
$$(E_1 \oplus E_2)^k(p_1) = (E_1 \oplus E_2)^m((E_1 \oplus E_2)^{ds} p_1) = (E_1 \oplus E_2)^m(p_1) \in Q_1.$$

Therefore (2.1) holds. $\square$

Proposition 2.3. *A bounded linear operator $E : X \to X$ is called topologically transitive if $E \oplus E$ is topologically transitive.*

Proof. To show that E is topologically transitive if and only if $E \oplus E$ is topologically transitive, we need to prove two implications.

($\Rightarrow$) Suppose E is topologically transitive.

Let P, Q be non-empty open subsets of $X \oplus X$. Then $P = \bigcup_{i=1}^k P_i \oplus Q_i$ and $Q = \bigcup_{j=1}^m P'_j \oplus Q'_j$ for some $k, m \in \mathbb{N}$ and non-empty open subsets P_i, Q_i, P'_j, Q'_j of X.

Since E is topologically transitive, there exists $k \in \mathbb{N}$ such that $E^n(P_i) \cap Q_j \neq \varnothing$, $\forall\ i, j)$. Then, $E^k(P) \cap Q = \bigcup_{i=1}^k \bigcup_{j=1}^m E^k(P_i) \cap Q'_j \neq \varnothing$.

Thus, $E \oplus E$ is topologically transitive.

($\Leftarrow$) Conversely, suppose $E \oplus E$ is topologically transitive.

Let P and Q be non-empty open subsets of X. Then $P \oplus Q$ is a non-empty open subset of $X \oplus X$.

Since $E \oplus E$ is topologically transitive, there exists $k \in \mathbb{N}$ such that

$$(E \oplus E)^k(P \oplus Q) \cap (X \oplus X) \neq \varnothing.$$

Let $(a, b) \in (E \oplus E)^k(P \oplus Q) \cap (X \oplus X)$. Then $(a, b) = (E^k(p), E^k(q))$ for some $p \in P$ and $q \in Q$.

Thus, $E^k(p) = a$ and $E^k(q) = b$, so $E^k(P) \cap Q \neq \varnothing$.

Therefore, E is topologically transitive. $\square$

Proposition 2.4. *If two operators T_1 and T_2 are topologically transitive, their direct sum $T_1 \oplus T_2$ is also topologically transitive.*

Proof. Suppose X is a Banach space and T_1 and T_2 are bounded linear operators on X that are topologically transitive. We want to show that $T_1 \oplus T_2$ is also topologically transitive on $X \oplus X$.

Let U and V be non-empty open subsets of $X \oplus X$.

Assuming that T_1 is topologically transitive, there exist $m \in \mathbb{N}$ and $(x_n) \in U$ such that

$$T_1^m(x_n) \in V.$$

Similarly, since T_2 is topologically transitive, $\exists\ n \in \mathbb{N}$ and $(y_k) \in U$ for which

$$T_2^n(y_k) \in V.$$

Now, consider the element $(x_n, y_k) \in U$ and compute its image under $T_1 \oplus T_2$:

$$(T_1 \oplus T_2)(x_n, y_k) = (T_1(x_n), T_2(y_k)).$$

By our choice of m and (x_n), there exists $0 \leq j < m$ such that

$$(T_1^j(x_n), 0) \in U.$$

Similarly, there exists $0 \leq l < n$ such that $(0, T_2^l(y_k)) \in U$.

Consider the element $(T_1^j(x_n), T_2^l(y_k)) \in U$. Then,

$$(T_1 \oplus T_2)^{j+l}(T_1^j(x_n), T_2^l(y_k)) = (T_1^{j+l}(x_n), T_2^{j+l}(y_k)).$$

Since T_1 and T_2 are topologically transitive, there exist $p, q \in \mathbb{N}$ such that

$$T_1^p(x_n) \in U$$

and

$$T_2^q(y_k) \in U.$$

Then, we can choose $r = j + l + p + q$ and see that

$$(T_1 \oplus T_2)^r(x_n, y_k) = (T_1^r(x_n), T_2^r(y_k)) \in V.$$

Thus, $T_1 \oplus T_2$ is topologically transitive on $X \oplus X$. □

The following corollary is due to Feldman [8] on the hypercyclicity criterion.

Corollary 2.1 ([8])**.** *If (E_1, E_2) satisfies the hypercyclicity criterion, then $(E_1 \oplus E_1, E_2 \oplus E_2)$ also satisfies the hypercyclicity criterion, hence is a hypercyclic pair.*

We extend Corollary 2.1 to the direct sum of the same operators, especially when they satisfy the topologically transitive criterion.

Corollary 2.2. *If (E_1, E_2) satisfies topologically transitive criterion, then $(E_1 \oplus E_1, E_2 \oplus E_2)$ also satisfies the topologically transitive criterion, hence is a topologically transitive pair.*

Proof. Suppose (E_1, E_2) satisfies the topologically transitive criterion on a topological vector space X.

That is, for any open sets $P, Q \subseteq X$, there exist $n, m \in \mathbb{N}$ such that $E_1^n(P) \cap E_2^m(Q) \neq \varnothing$.

We need to show that $(E_1 \oplus E_1, E_2 \oplus E_2)$ satisfies the topologically transitive criterion on $X \oplus X$.

Let $P\oplus$ P and $Q \oplus Q$ be open sets in $X \oplus X$.

Then P, Q are open sets in X, then there exist $n, m \in \mathbb{N}$ such that $E_1^n(P) \cap E_2^m(Q) \neq \varnothing$. Let $(a, b) \in E_1^n(P) \cap E_2^m(Q)$, then

$$(E_1 \oplus E_1)^n(a, b) = (E_1^n(a), E_1^n(b)) \in P \oplus P$$

and

$$(E_2 \oplus E_2)^m(a, b) = (E_2^m(a), E_2^m(b)) \in Q \oplus Q.$$

Therefore, $(E_1 \oplus E_1)^n(P \oplus P) \cap (E_2 \oplus E_2)^m(Q \oplus Q) \neq \varnothing$,

Hence $(E_1 \oplus E_1, E_2 \oplus E_2)$ satisfies the topologically transitive criterion.

Thus, $(E_1 \oplus E_1, E_2 \oplus E_2)$ is a topologically transitive pair on $X \oplus X$. □

Here are some examples of direct sums of topologically transitive operators that are topologically transitive pairs:

Example 2.1. Let T_1 and T_2 be the left and right shift operators on $\ell^2(\mathbb{N})$, respectively. Then $T_1 \oplus T_2$ is a topologically transitive pair, since it is known that both T_1 and T_2 are topologically transitive.

Proof. Required to show that $T_1 \oplus T_2$ is topologically transitive, that is, for any non-empty open subsets U_1 and U_2 of $\ell^2(\mathbb{N})$, there exist $m, n \in \mathbb{N}$ such that

$(T_1 \oplus T_2)^k(U_1 \times U_2) \cap (U_1 \times U_2) \neq \varnothing$, $\forall\ k \geq m + n$.

Consider two non-empty open sets U_1 and U_2 in the separable Hilbert space $\ell^2(\mathbb{N})$. Then $U_1 \times U_2$ is a non-empty open subset of $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, which is the Hilbert space direct sum of two copies of $\ell^2(\mathbb{N})$.

Since T_1 and T_2 are both topologically transitive, there exist $m_1, m_2, n_1, n_2 \in \mathbb{N}$ such that

$$T_1^{m_1}(U_1) \cap U_1 \neq \varnothing, \quad T_2^{m_2}(U_2) \cap U_2 \neq \varnothing$$

and

$$T_1^{n_1}(U_1) \cap U_1 \neq \varnothing, \quad T_2^{n_2}(U_2) \cap U_2 \neq \varnothing.$$

Now, consider $(T_1 \oplus T_2)^{m+n}(u_1, u_2)$, where $u_1 \in U_1$ and $u_2 \in U_2$.

We have that,

$$(T_1 \oplus T_2)^{m+n}(u_1, u_2) = (T_1^m(u_1), T_2^n(u_2))$$

Thus, $(T_1 \oplus T_2)^{m+n}(U_1 \times U_2)$ contains the non-empty open set

$(T_1^{m_1}(U_1) \cap U_1) \times (T_2^{n_2}(U_2) \cap U_2)$.

Therefore, we have $(T_1 \oplus T_2)^{m+n}(U_1 \times U_2) \cap (U_1 \times U_2) \neq \varnothing$ which proves that $T_1 \oplus T_2$ is topologically transitive. $\square$

Example 2.2. Consider the unilateral shift operator E on the separable Hilbert space $\ell^2(\mathbb{N})$, defined by

$$E(a_1, a_2, a_3 \ldots) = (a_2, a_3, a_4 \ldots).$$

Also, let S be the operator on $\ell^2(\mathbb{N})$ given by $S(a_k) = 2^k a_k$ for $k \geq 1$.

Then $E \oplus S$ is a topologically transitive pair, since both E and S are topologically transitive.

Proof. In order to establish that $E \oplus S$ is a topologically transitive pair of operators, it is necessary to demonstrate that for any pair of nonempty open sets P_1 and P_2 in $\ell^2(\mathbb{N})$, there exists an integer $k \in \mathbb{N}$ such that,

$(E \oplus S)^k(P_1 \times P_2) \neq \varnothing$, where $(E \oplus S)^k$ denotes the k-th power of the operator $E \oplus S$.

Let P_1, P_2 be non-empty open subsets in $\ell^2(\mathbb{N})$. Then, there exist $\epsilon_1, \epsilon_2 > 0$ and sequences $(a^{(1)}k)$ and $(a^{(2)}k)$ in $\ell^2(\mathbb{N})$ such that $B\epsilon_1(a^{(1)}) \subseteq P_1$ and $B\epsilon_2(a^{(2)}) \subseteq P_2$, where $B_\epsilon(a)$ denotes the open ball of radius ϵ centered at a.

We claim that there exists $k \in \mathbb{N}$ such that $(E \oplus S)^k(a^{(1)} \times a^{(2)}) \in P_1 \times P_2$.

Notice that,

$(E \oplus S)^k(a^{(1)} \times a^{(2)}) = (E^k a^{(1)}) \times (2^k a^{(2)})\ \forall\ k \geq 1$.

Since E is topologically transitive, there exists $k_1 \geq 1$ such that

$E^{k_1} a^{(1)} \in B_{\epsilon_1}(a^{(1)}) \subseteq P_1$.

Similarly, since S is topologically transitive, there exists $k_2 \geq 1$ such that

$2^{k_2} a^{(2)} \in B_{\epsilon_2}(a^{(2)}) \subseteq P_2$.

Let $k = \max\{k_1, k_2\}$ then,

$(E \oplus S)^k(a^{(1)} \times a^{(2)}) = (a^k a^{(1)}) \times (2^k a^{(2)}) \in P_1 \times P_2$.

As we have, $E^k a^{(1)} \in P_1$ and $2^k a^{(2)} \in P_2$.

Therefore, $(E \oplus S)^k(P_1 \times P_2) \neq \varnothing$ for some $k \in \mathbb{N}$.

Thus, $E \oplus S$ is a topologically transitive pair. $\square$

2.1 Subspace mixing operators and their direct sum

In this section, we focus on the direct sum of a topologically transitive operator in the context of a separable Hilbert space $\mathcal{H}$, where $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded linear operators on $\mathcal{H}$. Throughout our discussion, we assume that M is a closed topologically transitive subspace of $\mathcal{H}$.

Several researchers have studied the direct sum of operators in linear dynamics, as illustrated in works such as [2, 18, 15, 22, 21, 12, 3]. In particular, the idea of topological transitivity on the direct sum of operators is related to other concepts, such as topological weak mixing and the hypercyclicity criterion.

Definition 2.1 ([7]). *Let M_1 and M_2 be subspaces of a Banach space X, then the direct sum of M_1 and M_2 is defined as:*

$$M_1 \oplus M_2 = \{(a, b) : a \in M_1, b \in M_2\}$$

and the norm $||(a,b)||^2 = ||a||^2 + ||b||^2$ on $M_1 \oplus M_2$ defines the space $M_1 \oplus M_2$ to be Banach space. For more information and details on the direct sum of Banach spaces, the reader may refer [7].

Definition 2.2 ([20]). *Let $E \in \mathcal{L}(\mathcal{B})$ and let M be a closed non-zero subspace of X. We say E is subspace mixing or (M-mixing), if for all non-empty sets $P, Q \subseteq M$ both relatively open, there exists a positive integer N such that $E^k(P) \cap Q \neq \varnothing \ \forall\ k > N$.*

Theorem 2.2 ([1]). *If F_1 is M_1-hypercyclic and F_2 is M_2-hypercyclic, and at least one of them is subspace mixing, then $F_1 \oplus F_2$ is $(M_1 \oplus M_2)$-hypercyclic.*

The following results is obtained by extending the Theorem 2.2 to topologically transitive operators.

Theorem 2.3. *If F_1 is M_1-topologically transitive and F_2 is M_2-topologically transitive and at least one of them is subspace mixing, then $F_1 \oplus F_2$ is $(M_1 \oplus M_2)$-topologically transitive.*

Proof. By Theorem 2.2 we have $F_1 \oplus F_2$ is $(M_1 \oplus M_2)$-hypercyclic. Now we need to show that $F_1 \oplus F_2$ is $(M_1 \oplus M_2)$-topologically transitive.

Suppose that F_1 is M_1-mixing. Let $P_1 \oplus Q_1$ and $P_2 \oplus Q_2$ be open sets in $M_1 \oplus M_2$, then P_1, P_2 and Q_1, Q_2 are open in M_1 and M_2 respectively.

By hypothesis, there exist two numbers $N_1, N_2 \in \mathbb{N}$ such that

$$F_1^{-N_1}(P_1) \cap P_2 \neq \varnothing \quad \text{and} \quad F_1^{N_1}(M_1) \subseteq M_1$$

and

$$F_2^{-n}(Q_1) \cap Q_2 \neq \varnothing \quad \text{and} \quad F_2^{n}(M_2) \subseteq M_2 \quad \forall n \geqslant N_2.$$

As F_2 is M_2-topologically transitive, we have

$\left\{F_2^{-n}(Q_1) \cap Q_2 : n \in \mathbb{N}\right\}$ and $F_2^n(M_2) \subseteq M_2$ is infinite.

Then, there exists $k \in \mathbb{N}$ such that $F_1^{-k}(P_1) \cap P_2 \neq \varnothing$, $F_2^{-k}(Q_1) \cap Q_2 \neq \varnothing$, $F_1^k(M_1) \subseteq M_1$ and $F_2^k(M_2) \subseteq M_2$.

Notice that

$(F_1 \oplus F_2)^{-k}(P_1 \oplus Q_1) \cap (P_2 \oplus Q_2) \neq \varnothing$ and $(F_1 \oplus F_2)^k(M_1 \oplus M_2) \subseteq (M_1 \oplus M_2)$

Hence, $F_1 \oplus F_2$ is $(M_1 \oplus M_2)$-topologically transitive. □

The implication of Theorem 2.3 is that the following result holds.

Corollary 2.3. *Let M_1 and M_2 be closed subspaces on Hilbert space X, then F_1 and F_2 are M_1-topologically mixing and M_2-topologically mixing; respectively, if and only if $(F_1 \oplus F_2)$ is $(M_1 \oplus M_2)$-topologically mixing.*

Proof. For the "If" part.

Let P_1, P_2 be open sets in M_1 and Q_1, Q_2 be open sets in M_2, then

$P_1 \oplus Q_1$ and $P_2 \oplus Q_2$ are open in $M_1 \oplus M_2$. Thus, there is an $N \in \mathbb{N}$ such that

$$(F_1 \oplus F_2)^{-n}(P_1 \oplus Q_1) \cap (P_2 \oplus Q_2) \neq \varnothing$$

and

$$(F_1 \oplus F_2)^k(M_1 \oplus M_2) \subseteq (M_1 \oplus M_2)$$

$\forall\ n \geq N$.

Then,

$F^{-n}(P_1) \cap P_2 \neq \varnothing$, $F^{-n}(Q_1) \cap Q_2 \neq \varnothing$, $F^n(M_1) \subseteq M_1$ and $F^n(M_2) \subseteq M_2$.

Therefore, F_1 is M_1-topologically mixing and F_2 is M_2-topologically mixing.

We skip the proof of "only if" part since it is similar to the proof of Theorem 2.3. □

Corollary 2.4. *If E satisfies subspace-topologically transitive criterion, then $E \oplus E$ is subspace-topologically transitive.*

Proof. Let X be a topological space and $E : X \to X$ be a subspace-topologically transitive operator. We show that the operator $E \oplus E : X \oplus X \to X \oplus X$ defined by $(E \oplus E)(a, b) = (Ta, Tb)\ \forall\ (a, b) \in X \oplus X$ is also subspace-topologically transitive.

Let $Y \subset X \oplus X$ be a non-empty open subset.

We need to show that there exists $n \in \mathbb{N}$ such that $(E \oplus E)^n(Y) = X \oplus X$.

Since Y is non-empty and open, it contains some basic open set of the form $P \oplus Q$ for some non-empty open subsets $P, Q \subset X$.

Since E is subspace-topologically transitive, there exists $m \in \mathbb{N}$ such that $E^m(P) = X$.

Similarly, there exists $k \in \mathbb{N}$ such that $E^k(Q) = X$.

Then, for any $(a, b) \in X \oplus X$, we have $(E \oplus E)^{m+k}(a, b) = (E^m(E^k(A)), E^k(E^m(b)))$.

Since $E^m(P) = X$ and $E^k(Q) = X$.

It follows that

$(E \oplus E)^{m+k}(a, b) \in P \oplus Q \subset Y$, which implies that $(E \oplus E)^{m+k}(X \oplus X) \subset Y$.

Therefore, $(E \oplus E)^{m+k}(X \oplus X) = X \oplus X$.

Thus, $E \oplus E$ is subspace-topologically transitive. □

The famous tent map shown below is an example of subspace-topologically transitive, which will support the results obtained in the corollary 2.4.

Example 2.3. Let $X = [0, 1]$ with the usual topology, and let $T : Y \to Y$ be defined by

$$Tx = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Proof. We need to show that $E \oplus E : Y \oplus Y \to Y \oplus Y$ is also subspace-topologically transitive.

Suppose that $X = (a, b) \times (c, d) \subset Y \oplus Y$ be a non-empty open subset. Then $P = (a, b)$ and $Q = (c, d)$ are non-empty open subsets of Y.

Since E is subspace-topologically transitive, there exists $m \in \mathbb{N}$ such that

$E^m(P) = Y$ and $E^m(Q) = Y$.

Let $n = 2m$. Then for any $(x, y) \in Y \oplus Y$, we have that

$(E \oplus E)^n(x, y) = (E^m(E^m(x)), E^m(E^m(y)))$.

As we have that $E^m(P) = Y$ and $E^m(Q) = Y$.

It follows that $(E \oplus E)^n(x, y) \in P \oplus Q \subset X$.

Therefore, $(E \oplus E)^n(Y \oplus Y) \subset X$.

Hence, $E \oplus E$ is subspace-topologically transitive. □

In his paper [17], Salas presented the first example of a bounded linear operator E on a separable complex Hilbert space X that is topologically transitive whose adjoint T^* is also topologically transitive. Later, in [19], Salas showed that such an operator exists in any separable complex Hilbert space X with a separable dual space. This prompts the following question.

Question 2.1. Let X be a separable complex Hilbert space. Is there a bounded linear operator $E \in \mathcal{B}(X)$ that is not topologically transitive and such that both E^* and E are $\mathbb{J}$-class operators in a subspace of X?

3 Conclusion

In this paper, we investigated the topologically transitive operators and topologically mixing features of dynamical systems. In particular, we established that the transitivity property does not necessarily carry over to direct sums of operators. We establish this result through a rigorous mathematical proof, which builds on prior research in this area. Our findings contribute to a deeper understanding of the behavior of topologically transitive operators, and have potential implications for a wide range of applications in mathematics and related fields. Overall, this study contributes to the advancement of mathematical knowledge and lays the groundwork for further research in this area.

Authors Contributions.
Each author contributed in the writing of this work. All authors read and approved the final draft.

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ON THE CHARACTERISTIC POLYNOMIAL OF CHEBYSHEV MATRICES

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Abstract

We exhibit that the coefficients of the characteristic polynomial of any matrix $\mathbf{A}_{nxn}$ can be written in terms of the complete Bell polynomials, and this result is applied to Chebyshev matrices which generates the concept of Associated Polynomials of Chebyshev.

Keywords and Phrases: Bell and Chebyshev polynomials, Characteristic polynomial, Chebyshev matrices, Gauss hypergeometric function.

1 Introduction

For an arbitrary matrix $\mathbf{A}_{\mathrm{nxn}} = (A^i{}_j)$ its characteristic polynomial [9, 10, 19]

$$P_n(\lambda) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n, \tag{1.1}$$

can be obtained, through several procedures [11, 19, 25, 33, 34], directly from the condition

$$P_n(\lambda) = det(\lambda\delta^i_j - A^i{}_j).$$

The approach of Leverrier-Takeno [2, 8, 15, 20, 21, 32, 33, 35] is a simple and interesting technique to construct (1.1) based in the traces of the powers $A^r, r = 1, \ldots, n$. In fact, if we define the quantities

$$a_0 = 1, s_k = tr\mathbf{A}^k, k = 1, 2, \ldots, n, \tag{1.2}$$

then by (1.2) the process of Leverrier-Takeno implies (1.1) wherein the a_i are determined with the recurrence relation

$$\mathrm{r}\ a_r + s_1a_{r-1} + s_2a_{r-2} + \ldots + s_{r-1}\ \mathrm{a}_1 + s_r = 0, r = 1, 2, \ldots, n, \tag{1.3}$$

therefore

$$\begin{gathered} a_1 = -s_1, 2!a_2 = (s_1)^2 - s_2, 3!a_3 = -(s_1)^3 + 3s_1\ \mathrm{s}_2 - 2s_3, \\ 4!a_4 = (s_1)^4 - 6(s_1)^2 s_2 + 8s_1\ \mathrm{s}_3 + 3(s_2)^2 - 6s_4, \\ 5!a_5 = -(s_1)^5 + 10(s_1)^3\ \mathrm{s}_2 - 20(s_1)^2\ \mathrm{s}_3 - 15s_1(s_2)^2 + 30\ \mathrm{s}_1\ \mathrm{s}_4 + 20s_2s_3 - 24s_5, \ldots, \end{gathered} \tag{1.4}$$

in particular, det $A = (-1)^n a_n$, that is, the determinant of any matrix only depends on the traces s_r, which means that A and its transpose have the same determinant.

2 Complete Bell polynomials in terms of the determinant

In this section we make an appeal to recurrence relations (1.3) and (1.4) and thus due to [1, 5, 22] find the general expression

$$a_m = \frac{(-1)^m}{m!}\begin{vmatrix} s_1 & s_2 & s_3 & \cdots & s_{m-1} & s_m \\ m-1 & s_1 & s_2 & \cdots & s_{m-2} & s_{m-1} \\ 0 & m-2 & s_1 & \cdots & s_{m-3} & s_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & s_1 \end{vmatrix}, m = 1, \ldots, n. \tag{2.1}$$

which allows reproduce the expressions (1.4). The formula (2.1) permits relate the coefficients of the characteristic polynomial (1.1) with the complete Bell polynomials [3, 4, 29, 30, 36]. In [12, 23] we find the following expression for the Bell polynomials

$$
(2.2)\qquad B_m\left(x_1,x_2,\ldots,x_m\right)=\begin{vmatrix}
\binom{m-1}{0}x_1 & \binom{m-1}{1}x_2 & \cdots & \binom{m-1}{m-2}x_{m-1} & \binom{m-1}{m-1}x_m \\
-1 & \binom{m-2}{0}x_1 & \cdots & \binom{m-2}{m-3}x_{m-2} & \binom{m-2}{m-2}x_{m-1} \\
0 & -1 & \cdots & \binom{m-3}{m-4}x_{m-3} & \binom{m-3}{m-3}x_{m-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \binom{1}{0}x_1 & \binom{1}{1}x_2 \\
0 & 0 & \cdots & -1 & \binom{0}{0}x_1
\end{vmatrix}.
$$

Therefore

$$
(2.3)\qquad B_0=1, B_1=x_1, B_2=x_1^2+x_2, B_3=x_1^3+3x_1x_2+x_3, B_4=x_1^4+6\,x_1^2\,x_2+4\,x_1x_3+3\,x_2^2+x_4,
$$
$$
B_5=x_1^5+10\,x_1^3\,x_2+10\,x_1^2\,x_3+15x_1\,x_2^2+5\,x_1x_4+10x_2x_3+x_5,\ldots
$$

We see that with (2.3) we can deduce (1.4) if we employ $x_1=-s_1, x_2=-s_2, x_3=-2s_3, x_4=-6s_4, x_5=-24s_5, \ldots$, that is

$$
(2.4)\qquad a_m=\frac{1}{m!}B_m(-0!s_1,-1!s_2,-2!s_3,-3!s_4,\ldots,-(m-2)!s_{m-1},-(m-1)!s_m).
$$

In fact, it is simple to prove that (2.2) with $x_k=-(k-1)!s_k$ implies (2.1), thus the coefficients of the characteristic polynomial (1.1) are generated by the complete Bell polynomials [3, 4, 12, 23, 29, 30, 36].

3 Chebyshev matrices

The first-kind Chebyshev polynomials $T_n(x), |x|\leq 1$, verify the differential equation [6, 17, 18, 19, 26, 28]

$$
(3.1)\qquad \left(1-x^2\right)\frac{d^2}{dx^2}T_n-x\frac{d}{dx}T_n+n^2T_n=0, n=0,1,2,\ldots
$$

which is equivalent to the following expression in terms of the Gauss hypergeometric function [7, 24, 31]

$$
(3.2)\qquad T_n(x)={}_2F_1\left(-n,n;\frac{1}{2};\frac{1-x}{2}\right),
$$

thus

$$
(3.3)\qquad T_0=1, T_1=x, T_2=2x^2-1, T_3=4x^3-3x, T_4=8x^4-8x^2+1,\ldots
$$

Alternatively, we can employ the Chebyshev matrices [27]

$$
(3.4)\qquad A_{nxn}(x)=\begin{pmatrix}
x & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 2x & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 2x & 1 & \cdots & 0 & 0 \\
0 & 0 & 1 & 2x & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \ddots & 1 & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\
0 & 0 & 0 & 0 & \cdots & 1 & 2x
\end{pmatrix},
$$

whose determinant generates the Chebyshev polynomials

$$
(3.5)\qquad T_n(x)=\det A_{nxn}(x).
$$

That is

$$
(3.6)\qquad T_1=det(x), T_2=det\begin{pmatrix} x & 1 \\ 1 & 2x \end{pmatrix}, T_3=det\begin{pmatrix} x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{pmatrix}, T_4=det\begin{pmatrix} x & 1 & 0 & 0 \\ 1 & 2x & 1 & 0 \\ 0 & 1 & 2x & 1 \\ 0 & 0 & 1 & 2x \end{pmatrix},\ldots.
$$

Therefore from (1.4), (2.4) and (3.5)

$$T_n(x) = \frac{(-1)^n}{n!} B_n\left(-0!\,\mathrm{s}_1, -1!s_2, -2!s_3, -3!s_4, \dots, -(n-2)!s_{n-1}, -(n-1)!\,\mathrm{s}_n\right), \tag{3.7}$$

where s_j are the traces of the powers of the matrix (3.4); hence the complete Bell polynomials allow construct the Chebyshev polynomials of the first kind.

It is natural to investigate the characteristic polynomial of (3.4) for several values of n, thus

$$\begin{aligned}
&P_1 = \lambda - T_1, P_2 = \lambda^2 - 3x\lambda + T_2, P_3 = \lambda^3 - 5x\lambda^2 + \left(8x^2 - 2\right)\lambda - T_3,\\
&P_4 = \lambda^4 - 7x\lambda^3 + \left(18x^2 - 3\right)\lambda^2 - \left(20x^3 - 10x\right)\lambda + T_4,\\
&P_5 = \lambda^5 - 9x\lambda^4 + \left(32x^2 - 4\right)\lambda^3 - \left(56x^3 - 21x\right)\lambda^2 + \left(48x^4 - 36x^2 + 3\right)\lambda - T_5,\\
&P_6 = \lambda^6 - 11x\lambda^5 + \left(50x^2 - 5\right)\lambda^4 - \left(120x^3 - 36x\right)\lambda^3 + \left(160x^4 - 96x^2 + 6\right)\lambda^2 -\\
&\qquad - \left(112x^5 - 112x^3 + 21x\right)\lambda + T_6,\\
&P_7 = \lambda^7 - 13x\lambda^6 + \left(72x^2 - 6\right)\lambda^5 - \left(220x^3 - 55x\right)\lambda^4 + \left(400x^4 - 200x^2 + 10\right)\lambda^3 -\\
&\qquad - \left(432x^5 - 360x^3 + 54x\right)\lambda^2 + \left(256x^6 - 320x^4 + 96x^2 - 4\right)\lambda - T_7,\\
&P_8 = \lambda^8 - 15x\lambda^7 + \left(98x^2 - 7\right)\lambda^6 - \left(364x^3 - 78x\right)\lambda^5 + \left(840x^4 - 360x^2 + 15\right)\lambda^4 -\\
&\qquad - \left(1232x^5 - 880x^3 + 110x\right)\lambda^3 + \left(1120x^6 - 1200x^4 + 300x^2 - 10\right)\lambda^2 -\\
&\qquad - \left(576x^7 - 864x^5 + 360x^3 - 36\right)\lambda + T_8, \dots\ .
\end{aligned} \tag{3.8}$$

That is

$$P_n(\lambda) = \sum_{m=0}^{n} T_n^m(x)\lambda^{n-m},\ T_n^0 = 1, T_n^n = (-1)^n T_n. \tag{3.9}$$

Then it is clear that $T_n^m(x), m = 0, 1, \dots, n$ is a polynomial in x of degree m, and they may be named as Associated Polynomials of Chebyshev.

We know that if the operator $\frac{d^N}{\mathrm{dx}^N}$ is applied to the Legendre polynomials we obtain their associated polynomials, then now we shall show that this process can be employed for the first-kind Chebyshev polynomials $T_n(x)$ to construct the new polynomials $T_n^m(x)$ in terms of the Gauss hypergeometric function. In fact, we know the property

$$\frac{d^N}{dx^N}\ {}_2F_1(a, b; c; z) \propto {}_2F_1(a + N, b + N; c + N; z), \tag{3.10}$$

then we apply the operator $\frac{d^{n-m}}{\mathrm{dx}^{n-m}}$ to (3.2) and we use (3.10) with an adequate factor of proportionality to obtain the expression

$$\begin{aligned}
T_n^m(x) &= (-1)^m \binom{2n-m}{m}\, {}_2F_1\left(-m, 2n-m; n-m+\frac{1}{2}; \frac{1-x}{2}\right),\\
&= 2^{m-1}\frac{(n-1)!(2n-m)}{m!(n-m)!}\sum_{k=0}^{m}(-1)^{k-m}\binom{m}{k}\, {}_2F_1(k-m, -1-2m; -2m; 1)x^k
\end{aligned} \tag{3.11}$$

$m = 0, 1, \dots, n$, verifying the differential equation

$$\left(1 - x^2\right)\frac{d^2}{dx^2}T_n^m - (2n - 2m + 1)x\frac{d}{dx}T_n^m + m(2n - m)T_n^m = 0; \tag{3.12}$$

with (3.11) it is simple to calculate these associated polynomials of Chebyshev, for example

$$T_3^1 = -5x, T_3^2 = 8x^2 - 2, T_5^2 = 32x^2 - 4, T_5^3 = -56x^3 + 21x, T_5^4 = 48x^4 - 36x^2 + 3,\ \text{etc.}$$

in accordance with (3.8). The relations (3.11) and (3.12) reproduce (3.1) and (3.2) for the case $m = n$.

Finally, it is easy to show that the associated polynomials (3.11) can generate the other types of Chebyshev polynomials [13, 14, 16, 26, 27]

$$U_n(x) = \frac{2(-1)^n}{2+n}T_{n+1}^n(x), V_n(x) = \frac{(-1)^n}{n+1}T_{2n+1}^{2n}\left(\sqrt{\frac{1-x}{2}}\right), W_n(x) = \frac{1}{n+1}T_{2n+1}^{2n}\left(\sqrt{\frac{1+x}{2}}\right). \tag{3.13}$$

4 An Abel type integral equation representation involving Chebyshev determinants (3.5)

In this section, on using orthogonal property of the Chebyshev polynomials [26, 27], we obtain orthogonal property of product of two Chebyshev determinants (3.5). Then we derive an Abel type integral equation representation involving these Chebyshev determinants.

Making an use of the Eqn. (3.5) and the orthogonal property of the Chebyshev polynomials [26, 27] given by

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x) T_m(x) \mathrm{dx} = \begin{cases} 0, m \neq n; \\ \frac{\pi}{2}, m = n \neq 0; \\ \pi, m = n = 0, \end{cases} \tag{4.1}$$

due to (4.1), we get an interesting orthogonal property in terms of product of two Chebyshev determinants as

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \{\det A_{nxn}(x) \det A_{mxm}(x)\} \mathrm{dx} = \begin{cases} 0, m \neq n; \\ \frac{\pi}{2}, m = n \neq 0; \\ \pi, m = n = 0. \end{cases} \tag{4.2}$$

Theorem 4.1. *For $x > 0$, if $|xt| \leq 1$, and any function $f : (x,\ t) \to \mathbb{R}$, is defined by*

$$f(x,t) = \sum_{n=1}^{\infty} C_n \{\det A_{nxn}(xt)\}, \ C_n \text{ an arbitrary constant,} \tag{4.3}$$

may be represented as an integral equation

$$f(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \{\det A_{nxn}(xt)\} \int_{-1}^{1} \frac{f(x, ux^{-1})}{\sqrt{1-u^2}} \det A_{nxn}(u) du. \tag{4.4}$$

Proof. Consider a function in terms of the series of Chebyshev determinants (3.5) as

$$f(x,t) = \sum_{n=1}^{\infty} C_n \{\det A_{nxn}(\mathrm{xt})\}. \tag{4.5}$$

Then in both sides of Eqn. (4.5) multiply by $\frac{\det A_{mxm}(\mathrm{xt})}{\sqrt{1-(\mathrm{xt})^2}}$ and thus integrate that sides with respect to t from $t = -\frac{1}{x}$ to $t = \frac{1}{x}$, $\forall x > 0$, we obtain

$$\int_{-\frac{1}{x}}^{\frac{1}{x}} \frac{f(x,\ t)}{\sqrt{1-(xt)^2}} \det A_{mxm}(xt) dt = \sum_{n=1}^{\infty} C_n \int_{-\frac{1}{x}}^{\frac{1}{x}} \frac{1}{\sqrt{1-(xt)^2}} \det A_{nxn}(xt) \det A_{mxm}(xt) dt. \tag{4.6}$$

After some manipulations in (4.6), we find that

$$\int_{-1}^{1} \frac{f(x,\ ux^{-1})}{\sqrt{1-u^2}} \det A_{mxm}(u) du = \sum_{n=1}^{\infty} C_n \int_{-1}^{1} \frac{1}{\sqrt{1-u^2}} \det A_{nxn}(u) \det A_{mxm}(u) du. \tag{4.7}$$

Now in the Eqn. (4.8) use the orthogonality formula (4.2) we derive the coefficients

$$C_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x,\ ux^{-1})}{\sqrt{1-u^2}} \det A_{nxn}(u) du \ \forall n = 1, 2, 3, \ldots. \tag{4.8}$$

Finally, with the aid of the formulae (4.5) and (4.9), we get an integral equation (4.4).

Specially, by Eqn. (4.4) for $n = 1$ we find an Abel type integral equation

$$f(x,t) = \frac{2xt}{\pi} \int_{-1}^{1} \frac{f(x,\ ux^{-1})}{\sqrt{1-u^2}} u du, \ \forall x > 0. \tag{4.9}$$

□

5 Conclusions

In the Section 2, complete Bell polynomials are expressed in terms of determinant. The Section 3 consists of Chebyshev matrices. In the Section 4, on using orthogonal property of the Chebyshev polynomials [26, 27], an orthogonal property of product of two Chebyshev determinants (3.5) is derived. Again an integral equation representation involving these Chebyshev determinants is also obtained. The results obtained in the Eqns. (3.13) and (4.9) are very applicable in computational work of various scientific problems consisting of Abel's type integrals and Chebyshev polynomials.

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A NEW α-LAPLACE TRANSFORM ON TIME SCALES

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Abstract

In this paper we introduce a new $\alpha-$ Laplace transform which is a generalization of nabla version of Laplace transform on time scales. In particular for $0 < \alpha < 1$ this transform will serve as fractional Laplace transform on time scales. Existence theorem and some important properties such as linearity, initial and final value theorem, transform of integral, shifting theorem, transform of derivative are proved. Additionally convolution theorem and formulae for fractional integral, Riemann-Liouville fractional derivative, Liouville-Caputo fractional derivative, Mittag Leffler function are given. At last for a suitable value of α a fractional dynamic equation with given initial condition is solved.

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1 Introduction and Motivation

Integral transforms are mathematical techniques that play a crucial role in various fields of Science and Engineering. The important significance of integral transform is their ability to simplify mathematical equations often involving derivatives and integrals, into algebraic equations or simpler differential equations. The theory of the Laplace transform has a great theoretical interest as it is one of the former integral transforms invented by Pierre-Simon Laplace. Furthermore many researchers have introduced several integral transforms such as Laplace-Carson, Sumudu, Elzaki, Natural and Shehu transforms [7, 10, 11, 12, 27], all of which represent a family of Laplace transforms. Subsequently an ample development regarding classical integral transforms in form of generalization in distribution spaces, formulation of multidimensional and fractional transform has been done due to [5, 16, 18, 23, 26]. Recently H.M.Srivastava [24] has explored recent developments in the Laplace and Hankel transforms and their extensions and variations. Using Srivastava's generalized Whittaker transform [17], Hardy's generalized Hankel transform [9] and Srivastava's $\epsilon-$ generalized Hankel transform [16], the properties, characteristics and relationships among integral transforms representing the family of Laplace transform are studied. Further eminent results regarding integral transforms and fractional calculus have been studied in [19, 20, 21, 22].

Time scale calculus is an unification tool that encompasses both continuous (*e.g.* $\mathbb{R}$) and discrete (*e.g.* $\mathbb{Z}$) domains. Integral transforms on time scales a mathematical framework that extends the concepts of classical integral transforms into functions defined on time scale domains. Thus far Laplace, Fourier, Sumudu and Shehu transforms have been introduced on time scales and have served as a powerful tool for modeling and solving problems that bridge the gap between continuous and discrete dynamic systems [1, 2, 3, 4, 6, 8, 15, 25, 28]. In 2016 Medina Gustavo et al.[13] introduced a new $\alpha-$integral Laplace transform which is a generalization of Laplace transform when $\alpha \to 1$. Subsequently, in 2007, a the fractional Laplace transform was formulated and is applied to solve fractional differential equations [14]. In our work we develop a new $\alpha-$ Laplace transform on time scales and discuss its fundamental properties. Using this transform, we solve the fractional dynamic equation on time scales with given initial conditions.

The next section is concerned with precursory concepts needed for comprehension of our work.

2 Preliminaries

Note that the discussion in this section follows from [2, 3, 4, 6, 15, 25]. Here we will assume that a time scale $\mathbb{T}$ is unbounded above and $t_0 \in \mathbb{T}$ is fixed. For $t \in \mathbb{T}$, the *forward jump operator* $\sigma(t) : \mathbb{T} \to \mathbb{T}$ is given as $\sigma(t) := inf\{s \in \mathbb{T} : s > t\}$. And the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is given as $\rho(t) := sup\{s \in \mathbb{T} : s < t\}$.

If $\sigma(t) > t$, t is said to be *right-scattered*, while if $\rho(t) < t$ then t is *left-scattered*. Also, if $t < sup\mathbb{T}$ and $\sigma(t) = t$,then t is called *right-dense,* and if $t > inf\mathbb{T}$ and $\rho(t) = t$, then t is *left-dense.*

For $t \in \mathbb{T}$ the *forward graininess* function $\mu : \mathbb{T} \to [0, \infty)$ is $\mu(t) = \sigma(t) - t$. And the *backward graininess* function $\nu : \mathbb{T} \to [0, \infty)$ is $\nu(t) = t - \rho(t)$.

Definition 2.1. *A function $h : \mathbb{T} \to \mathbb{C}$ is said to be* ld-continuous *if it is continuous at every* left-dense point*, and the right sided limit exists at every* right-dense point. *It is expressed as $h(t) \in \mathcal{C}_{ld}(\mathbb{T}, \mathbb{C})$.*

Note that if time scale $\mathbb{T}$ has a right scattered miminum m, the $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$

Definition 2.2. *An* ld-continuous *function $h : \mathbb{T} \to \mathbb{C}$ is called complex $\nu-$regressive if $1 - \nu h \neq 0$ for all $t \in \mathbb{T}_k$. It is denoted as $\mathcal{R}^{\nu}_c(\mathbb{T}, \mathbb{C})$.*

For $h > 0$, we have $\mathbb{C}_h = \{z \in \mathbb{C} : z \neq \frac{1}{h}\}$ and $\mathbb{Z}_h = \{z \in \mathbb{C} : \frac{-\pi}{h} < Im(z) \leq \frac{\pi}{h}\}$ with $\mathbb{C}_0 = \mathbb{Z}_0 = \mathbb{C}$. Further the Hilger real part and imaginary part of a complex numbers are given by $\mathscr{R}e_h(z) = \frac{1}{h}(1 - |1 - hz|)$ and $\mathscr{I}m_h(z) = \frac{1}{h}Arg(1 - hz)$ respectively, where Arg denotes principal argument of a complex number. In particular we have, $\mathscr{R}e_0(z) = \mathscr{R}e(z)$ and $\mathscr{I}m_0(z) = \mathscr{I}m(z)$.

Definition 2.3. *If $f \in \mathcal{R}^{\nu}_c(\mathbb{T}, \mathbb{C})$, then the nabla exponential function is given by, $\hat{e}_f(t, t_0) := exp\big[\int_{t_0}^{t} \hat{\xi}_{\nu(s)}(f(s))\,\nabla s\big]$ for $t, t_0 \in \mathbb{T}$ where, the $\nu-$cylinder transformation $\hat{\xi}_h : \mathbb{C}_h \to \mathbb{Z}_h$ is $\hat{\xi}_h(z) = \frac{-1}{h}Log(1 - zh)$.*

Theorem 2.1. *Let the first-order linear dynamic equation $x^{\nabla} = f(t)x$ is $\nu-$regressive and $t_0 \in \mathbb{T}$ is fixed. Then $\hat{e}_f(\cdot, t_0)$ is the solution of the initial value problem $x^{\nabla} = f(t)x, \quad x(t_0) = 1$ on $\mathbb{T}$.*

Lemma 2.1. *If $f \in \mathcal{R}^{\nu}_c(\mathbb{T}, \mathbb{C})$ then, $\hat{e}^{\rho}_{\ominus f}(t, t_0) = \frac{\hat{e}_{\ominus f}(t,t_0)}{1-\nu(t)f}$.*

Definition 2.4. *A function h belongs to the space of functions $\mathcal{A}(\mathbb{T})$ if*

(1) h is *piecewise ld-continuous in every interval* $[t_0, \tau] \cap \mathbb{T}$.
(2) h is of *exponential order k ($k \in \mathcal{R}^{+\nu}_c([t_0, \infty))$* on $[t_0, \infty)$, *that is there exists constant $M > 0$ such that $|f(t)| \leq Me_k(t, t_0)$ for all $t \in [t_0, \infty)$.*

The *minimal-graininess* function $\nu_* : \mathbb{T} \to \mathbb{R}^+_0$ is given as $\nu_*(t_0) = inf\ \nu(t)$ for $t \in [t_0, \infty)_{\mathbb{T}}$.

Theorem 2.2 (**Decay of the nabla-exponential function**). *For $sup\ \mathbb{T} = \infty$, let $t_0 \in \mathbb{T}$ and $\lambda \in \mathcal{R}^{+\nu}_c([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$. Then for any $z \in \mathbb{C}_{\nu_*(t_0)}(\lambda)$, we have the following properties,*

(1) $|\hat{e}_{\lambda \ominus z}(t, t_0)| \leq \hat{e}_{\lambda \ominus \mathscr{R}e_{\nu_*(t_0)}(z)}(t, t_0)$ *for all $t \in [t_0, \infty)_{\mathbb{T}}$.*
(2) $\lim\limits_{t\to\infty} \hat{e}_{\lambda \ominus \mathscr{R}e_{\nu_*(t_0)}(z)}(t, t_0) = 0$.
(3) $\lim\limits_{t\to\infty} \hat{e}_{\lambda \ominus z}(t, t_0) = 0$.

Definition 2.5. *Let $t_0, t \in \mathbb{T}$ and $\lambda_1, \lambda_2 > -1$. The time scale power functions $\hat{h}_{\lambda_1}(t, t_0)$ are defined as a family of non-negative functions satisfying,*

(1) $\int_{t_0}^{t} \hat{h}_{\lambda_1}(t, \rho(s))\hat{h}_{\lambda_2}(s, t_0)\nabla s = \hat{h}_{\lambda_1+\lambda_2+1}(t, t_0)$ *for $t \geq t_0$.*
(2) $\hat{h}_0(t, t_0) = 1$ *for $t \geq t_0$.*
(3) $\hat{h}_{\lambda_1}(t, t) = 0$ *for $\lambda_1 \in (0, 1)$.*

Definition 2.6. *Let $t_0, t' \in \mathbb{T}$ and $a \geq 0, \beta, \lambda > 0$, then for $h \in \mathcal{C}_{ld}([t_0, t']_{\mathbb{T}}, \mathbb{C})$ one defines,*

(1) *The Riemann-Liouville fractional integral of order $a > 0$ with the lower limit t_0 as*
$${}_{t_0}(\nabla^{-a}h)(t) := \int_{t_0}^{t} \hat{h}_{a-1}(t, \rho(\tau))h(\tau)\nabla\ \tau$$
and for $a = 0$ one have $({}_{t_0}\nabla^0 h)(t) = h(t)$.
(2) *The Riemann-Liouville fractional derivative of order $\beta > 0$ with lower limit t_0 as*
$$({}_{t_0}\nabla^{\beta}h)(t) := [{}_{t_0}\nabla^{-(n-\beta)}h]^{\nabla^n} \quad t \in [\sigma(t_0), t']_{\mathbb{T}},$$
where $n = [\beta] + 1$.
(3) *The Caputo fractional derivative ${}^{\mathcal{C}}_{t_0}\nabla^{\lambda}h(t)$ on $[\sigma(t), t']_{\mathbb{T}}$ is defined via the Riemann-Liouville fractional derivative by,*
$${}^{\mathcal{C}}_{t_0}\nabla^{\lambda}h(t) := ({}_{t_0}\nabla^{-(n-\lambda)}h^{\nabla^n})(t),$$
where $n = [\lambda] + 1$.

3 Main Results

In this section we define $\alpha-$Laplace transform and give some of its salient properties.

Definition 3.1. *Let $h : \mathbb{T} \to \mathbb{C}$ is an ld-continuous function and α is a real number, then we define the $\alpha-$Laplace transform $\mathscr{L}^{\alpha}\{h(t)\} = \mathscr{H}_{\mathbb{T}}(z)$ of $h(t)$ of order α as*

$$\mathscr{H}_{\mathbb{T}}(z) = \mathscr{L}^{\alpha}\{h(t)\} = \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t, t_0)h(t)\ \nabla t \quad \text{for all } \ t \in \mathcal{D}_{\nu},$$

where $\mathcal{D}_{\nu} \subset \mathbb{C}$ consists of all complex numbers $z \in \mathcal{R}_c(\mathbb{T}, \mathbb{C})$ for which the improper integral converges.

Next we define fractional Laplace transform on time scales using above definition as.

Definition 3.2. *Let $h : \mathbb{T} \to \mathbb{C}$ is an ld-continuous function then for a real number $0 < \alpha < 1$, $\mathscr{L}^{\alpha}\{h(t)\} = \mathscr{H}_{\mathbb{T}}(z)$ will be the fractional Laplace transform of $h(t)$.*

Theorem 3.1 (**Existence Theorem**). *Let $h : \mathbb{T} \to \mathbb{C}$ is a function of class $\mathcal{A}(\mathbb{T})$ of exponential order k, then the α-Laplace transform $\mathscr{L}^{\alpha}\{h(t)\}$ of $h(t)$ exists for all $z \in \mathbb{C}_{(\nu_*(t_0))^{1/\alpha}}$ with $\mathscr{R}e_{(\nu_*(t_0))^{1/\alpha}}(z) > k$ and converges absolutely.*

Proof. We have

$$\begin{aligned}
|\mathscr{L}^{\alpha}\{h(t)\}| &= \left| \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t_0, t)h(t)\ \nabla t \right| \\
&\leq \int_{t_0}^{\infty} |\hat{e}^{\rho}_{\ominus z^{\alpha}}(t_0, t)h(t)\ \nabla t| \\
&\leq M \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t_0, t)\hat{e}_k(t, t_0)\ \nabla t \\
&= M \int_{t_0}^{\infty} \frac{\hat{e}_{\ominus z^{\alpha}}(t_0, t)\hat{e}_k(t, t_0)}{1 - \nu(t)z^{\alpha}}\ \nabla t \\
&= M \int_{t_0}^{\infty} \frac{\hat{e}_{k\ominus z^{\alpha}}(t, t_0)}{1 - \nu(t)z^{\alpha}}\ \nabla t \\
&= \frac{M}{k - z^{\alpha}} \int_{t_0}^{\infty} \frac{k - z^{\alpha}}{(1 - \nu(t)z^{\alpha})}\hat{e}_{k\ominus z^{\alpha}}(t, t_0)\ \nabla t \\
&= \frac{M}{k - z^{\alpha}} \int_{t_0}^{\infty} (k \ominus z^{\alpha})\hat{e}_{k\ominus z^{\alpha}}(t, t_0)\ \nabla t \\
&= \frac{M}{k - z^{\alpha}} \int_{t_0}^{\infty} \hat{e}^{\nabla}_{k\ominus z^{\alpha}}(t, t_0)\nabla t \\
&= \frac{M}{z^{\alpha} - k}.
\end{aligned}$$

Last step follows from Theorem 2.2. □

Theorem 3.2 (**Linearity**). *Let $ah_1(t)$ and $bh_2(t)$ are functions of class $\mathcal{A}(\mathbb{T})$ for constants $a, b \in \mathbb{R}$ then,*

$$\mathscr{L}^{\alpha}\{ah_1(t) + bh_2(t)\} = a\mathscr{L}^{\alpha}\{h_1(t)\} + b\mathscr{L}^{\alpha}\{h_2(t)\}.$$

Proof. Let $ah_1(t)$ and $bh_2(t)$ are functions of class $\mathcal{A}(\mathbb{T})$ with exponential order k_1 and k_2 respectively, then $\mathscr{L}^{\alpha}\{ah_1(t)\}$ exists for all $z \in \mathbb{C}_{(\nu_*(t_0))^{1/\alpha}}(k_1)$ with $\mathscr{R}e_{(\nu_*(t_0))^{1/\alpha})}(z) > k_1$ and $\mathscr{L}^{\alpha}\{bh_2(t)\}$ exists for all $z \in \mathbb{C}_{(\nu_*(t_0)^{1/\alpha})}(k_2)$ with $\mathscr{R}e_{(\nu_*(t_0))^{1/\alpha}}(z) > k_2$. Then $\mathscr{L}^{\alpha}\{ah_1(t) + bh_2(t)\}$ exists for all $z \in \mathbb{C}_{(\nu_*(t_0))^{1/\alpha}}$ with $\mathscr{R}e_{(\nu_*(t_0))^{1/\alpha}}(z) > max\{k_1, k_2\}$. Thus,

$$\begin{aligned}
&\mathscr{L}^{\alpha}\{ah_1(t) + bh_2(t)\} \\
&= \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t, t_0)[ah_1(t) + bh_2(t)]\ \nabla t \\
&= a \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t, t_0)h_1(t)\ \nabla t + b \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t, t_0)h_2(t)\ \nabla t \\
&= a\mathscr{L}^{\alpha}\{h_1(t)\} + b\mathscr{L}^{\alpha}\{h_2(t)\}.
\end{aligned}$$

□

We apply definition 3.1 to find transform of some elementary functions which are given in the table below

$h(t)$	1	$\hat{e}_a(t,t_0)$	$sin_a(t,t_0)$	$cos_a(t,t_0)$
$\mathscr{L}^\alpha\{h(t)\}$	$\frac{1}{z^\alpha}$	$\frac{a}{z^\alpha - a}$	$\frac{a}{z^{2\alpha}+a^2}$	$\frac{z^\alpha}{z^{2\alpha}+a^2}$

$h(t)$	$sinh_a(t,t_0)$	$cosh_a(t,t_0)$	$h_k(t,t_0)$	$\hat{h}_\lambda(t,t_0)$
$\mathscr{L}^\alpha\{h(t)\}$	$\frac{a}{z^{2\alpha}-a^2}$	$\frac{z^\alpha}{z^{2\alpha}-a^2}$	$\frac{1}{z^{(k+1)\alpha}}$	$\frac{1}{z^{(\lambda+1)\alpha}}$

.

Here $\hat{h}_\lambda(t,t_0)$ is the time scale power function defined in definition 2.5 and it's transform was found using Convolution theorem which we are going to prove further.

Theorem 3.3. *Assume that $h(t)$ is regulated function such that $H(t) = \int_{t_0}^t h(s)\nabla s$ for $t, t_0 \in \mathbb{T}$ is of class $\mathcal{A}(\mathbb{T})$ then $\mathscr{L}^\alpha\{H(t)\} = \frac{1}{z^\alpha}\mathscr{L}^\alpha\{h(t)\}$.*

Proof.

$$\begin{aligned}
\mathscr{L}^\alpha\{H(t)\} &= \int_{t_0}^{\infty} H(t)\hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)\ \nabla t\\
&= -\int_{t_0}^{\infty} H(t)\frac{\ominus z^\alpha}{z^\alpha}\hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t\\
&= \frac{-1}{z^\alpha}\int_{t_0}^{\infty} H(t)\ominus z^\alpha\hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t\\
&= \frac{-1}{z^\alpha}\int_{t_0}^{\infty} H(t)\hat{e}^{\nabla}_{\ominus z^\alpha}(t,t_0)\ \nabla t
\end{aligned}$$

Using integration by parts

$$\begin{aligned}
\mathscr{L}^\alpha\{H(t)\} &= \frac{-1}{z^\alpha}\Big[\big[H(t)\hat{e}_{\ominus z^\alpha}(t,t_0)\big]_{t=t_0}^{t\to\infty} - \int_{t_0}^{\infty} H^{\nabla}(t)\hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t\Big]\\
&= \frac{-1}{z^\alpha}\Big[-H(t_0) - \int_{t_0}^{\infty} h(t)\hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t\Big]\\
&= \frac{1}{z^\alpha}\int_{t_0}^{\infty} h(t)\hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t\\
&= \frac{1}{z^\alpha}\mathscr{L}^\alpha\{h(t)\},
\end{aligned}$$

provided $\lim_{t\to\infty} H(t)\hat{e}_{\ominus z^\alpha}(t,t_0) = 0$.

□

Theorem 3.4 (Second Shifting Theorem). *If $h(t) \in \mathcal{A}(\mathbb{T})$ and $u_a(t) = \begin{cases} 1 & if\ \ t \in \mathbb{T}\cap(-\infty,a] \\ 0 & if\ \ t\in\mathbb{T}\cup(a,\infty)\end{cases}$ where $a \in \mathbb{T}$ with $a > 0$, then $\mathscr{L}^\alpha\{u_a(t)h(t)\} = \hat{e}_{\ominus z^\alpha}(a,t_0)\mathscr{L}_\alpha\{h(t)\}$.*

Proof.

$$\begin{aligned}
&\mathscr{L}^\alpha\{u_a(t)h(t)\}\\
&= \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)u_a(t)h(t)\ \nabla t\\
&= \int_{a}^{\infty} \hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)h(t)\ \nabla t\\
&= \int_{a}^{\infty} \frac{\hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)}{1-\nu(t)z^\alpha}h(t)\ \nabla t\\
&= \int_{a}^{\infty} \frac{\hat{e}_{\ominus z^\alpha}(t,a)\hat{e}_{\ominus z^\alpha}(a,t_0)}{1-\nu(t)z^\alpha}h(t)\ \nabla t
\end{aligned}$$

$$
\begin{aligned}
&= \hat{e}_{\ominus z^\alpha}(a,t_0)\int_a^\infty \frac{\hat{e}_{\ominus z^\alpha}(t,a)}{1-\nu(t)z^\alpha} h(t)\ \nabla t \\
&= \hat{e}_{\ominus z^\alpha}(a,t_0)\int_a^\infty \hat{e}^{\rho}_{\ominus z^\alpha}(t,a) h(t)\ \nabla t \\
&= \hat{e}_{\ominus z^\alpha}(a,t_0)\mathscr{L}^\alpha\{h(t)\}.
\end{aligned}
$$

□

Theorem 3.5 (**Transform of derivative**). *Let* $h, h^\nabla \in \mathcal{A}(\mathbb{T})$, *then* $\mathscr{L}^\alpha\{h^\nabla(t)\} = z^\alpha \mathscr{L}^\alpha\{h(t)\} - h(t_0)$ *for those regressive* $z \in \mathbb{C}$ *satisfying* $\lim_{t\to\infty}\{h(t)\hat{e}_{\ominus z^\alpha}(t,t_0)\} = 0$

Proof.

$$
\begin{aligned}
\mathscr{L}^\alpha\{h^\nabla(t)\} &= \int_{t_0}^\infty \hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0) h^\nabla(t)\ \nabla t \\
&= \int_{t_0}^\infty \Big[[h(t)\hat{e}_{\ominus z^\alpha}(t,t_0)]^\nabla - h(t)\hat{e}^\nabla_{\ominus z^\alpha}(t,t_0)\Big]\ \nabla t \\
&= [h(t)\hat{e}_{\ominus z^\alpha}(t,t_0)]_{t=t_0}^{t\to\infty} - \int_{t_0}^\infty h(t)\hat{e}^\nabla_{\ominus z^\alpha}(t,t_0)\ \nabla t \\
&= -h(t_0) - \int_{t_0}^\infty h(t) \ominus z^{1/\alpha}\hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t \\
&= -h(t_0) + z^\alpha \int_{t_0}^\infty h(t)\frac{-\ominus z^\alpha \hat{e}_{\ominus z^\alpha}(t,t_0)}{z^\alpha}\ \nabla t \\
&= -h(t_0) + z^\alpha \int_{t_0}^\infty h(t)\hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)\ \nabla t \\
&= z^\alpha \mathscr{L}^\alpha\{h(t)\} - h(t_0),
\end{aligned}
$$

provided, $\lim_{t\to\infty} h(t)\hat{e}_{\ominus z^\alpha}(t,t_0) = 0$. □

In a similar way for $h, h^\nabla, h^{\nabla\nabla} \in \mathcal{A}(\mathbb{T})$ then

$$
\begin{aligned}
\mathscr{L}^\alpha\{h^{\nabla\nabla}(t)\} &= \int_{t_0}^\infty \hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0) h^{\nabla\nabla}(t)\ \nabla t \\
&= \int_{t_0}^\infty \Big[[h^\nabla(t)\hat{e}_{\ominus z^\alpha}(t,t_0)]^\nabla - h^\nabla(t)\hat{e}^\nabla_{\ominus z^\alpha}(t,t_0)\Big]\ \nabla t \\
&= [h^\nabla(t)\hat{e}_{\ominus z^\alpha}(t,t_0)]_{t=t_0}^{t\to\infty} - \int_{t_0}^\infty h^\nabla(t)\hat{e}^\nabla_{\ominus z^\alpha}(t,t_0)\ \nabla t \\
&= -h^\nabla(t_0) + z^\alpha \int_{t_0}^\infty h^\nabla(t)\frac{\ominus z^\alpha \hat{e}_{\ominus z^\alpha}(t,t_0)}{-z^\alpha}\ \nabla t \\
&= -h^\nabla(t_0) + z^\alpha \int_{t_0}^\infty h^\nabla \hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)\ \nabla t \\
&= -h^\nabla(t_0) + z^\alpha \mathscr{L}^\alpha\{h^\nabla(t)\} \\
&= -h^\nabla(t_0) + z^\alpha[z^\alpha \mathscr{L}^\alpha\{h(t)\} - h(t_0)] \\
&= z^{2\alpha}\mathscr{L}^\alpha\{h(t)\} - z^\alpha h(t_0) - h^\nabla(t_0).
\end{aligned}
$$

More generally we get, $\mathscr{L}^\alpha\{h^{\nabla^n}(t)\} = z^{n\alpha}\mathscr{L}^\alpha\{h(t)\} - \sum_{k=0}^{n-1} z^{(n-(k+1))\alpha} h^{\nabla^k}(t_0)$.

Theorem 3.6 (**Initial and Final Value Theorem**). $h, h' \in \mathcal{A}(\mathbb{T})$ *with* $H_{\mathbb{T}}(z) = \mathscr{L}^\alpha\{h(t)\}$ *then* $h(t_0) = \lim_{z\to\infty} z^\alpha H_{\mathbb{T}}(z)$ *and* $\lim_{t\to\infty} h(t) = \lim_{z\to 0} z^\alpha H_{\mathbb{T}}(z)$.

Proof. We have,

$$
\mathscr{L}^\alpha\{h^\nabla(t)\}
$$

$$
\begin{aligned}
&= \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t,t_0) h^{\nabla}(t) \ \nabla t \\
&= \int_{t_0}^{\infty} \left[(h(t)\hat{e}_{\ominus z^{\alpha}}(t,t_0))^{\nabla} - h(t)\hat{e}^{\nabla}_{\ominus z^{\alpha}}(t,t_0) \right] \ \nabla t \\
&= \left[h(t) e_{\ominus z^{\alpha}}(t,t_0) \right]_{t=t_0}^{\infty} - \int_{t_0}^{\infty} h(t)\hat{e}^{\nabla}_{\ominus z^{\alpha}}(t,t_0) \ \nabla t \\
&= -h(t_0) - \int_{t_0}^{\infty} h(t) \ominus z^{\alpha} \hat{e}_{\ominus z^{\alpha}}(t,t_0) \ \nabla t \\
&= -h(t_0) + z^{\alpha} \int_{t_0}^{\infty} h(t) \frac{\ominus z^{\alpha}}{-z^{\alpha}} \hat{e}_{\ominus z^{\alpha}}(t,t_0) \ \nabla t \\
&= -h(t_0) + z^{\alpha} \int_{t_0}^{\infty} h(t) \hat{e}^{\rho}_{\ominus z^{\alpha}}(t,t_0) \ \nabla t \\
&= z^{\alpha} \mathscr{L}^{\alpha}\{h(t)\} - h(t_0).
\end{aligned}
$$

Provided $\lim\limits_{t\to\infty} \hat{e}_{\ominus z^{\alpha}}(t,t_0) = 0$.
Taking $\lim\limits_{z\to\infty}$ on both sides,

$$
\lim_{z\to\infty} \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t,t_0) h^{\nabla}(t) = 0 = \lim_{z\to\infty} z^{\alpha} \mathscr{L}^{\alpha} h(t) - \lim_{z\to\infty} h(t_0)
$$

$$
\lim_{z\to\infty} z^{\alpha} H_{\mathbb{T}}(z) = h(t_0).
$$

Now taking $\lim\limits_{z\to 0}$ on both sides we get,

$$
\begin{aligned}
\lim_{z\to 0} \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t,t_0) h^{\nabla}(t) &= \lim_{z\to 0} z^{\alpha} \mathscr{L}^{\alpha} h(t) - \lim_{z\to 0} h(t_0) \\
\int_{t_0}^{\infty} h^{\nabla}(t) \nabla t &= \lim_{z\to 0} z^{\alpha} \mathscr{L}^{\alpha}\{h(t)\} - h(t_0) \\
\lim_{t\to\infty} h(t) - h(t_0) &= \lim_{z\to 0} z^{\alpha} \mathscr{L}^{\alpha}\{h(t)\} - h(t_0) \\
\lim_{t\to\infty} h(t) &= \lim_{z\to 0} z^{\alpha} H_{\mathbb{T}}(z).
\end{aligned}
$$

□

Definition 3.3 ([28]). *For given functions $h_1, h_2 : \mathbb{T} \to \mathbb{C}$ their convolution $h_1 * h_2$ is defined by,*

$$
(h_1 * h_2)(t) = \int_{t_0}^{t} \tilde{h_1}(t, \rho(\tau)) h_2(\tau) \nabla \tau \qquad t \in \mathbb{T}
$$

where $\tilde{h}$ is the shift of $h : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$ is the solution of the initial value problem

$$
\begin{aligned}
g^{\nabla_t}(t, \rho(s)) &= -g^{\nabla_s}(t,s), \quad t, s \in \mathbb{T},\ t \le s \le t_0 \\
g(t, t_0) &= h(t) \quad t \in \mathbb{T},\ t \ge t_0.
\end{aligned}
$$

Theorem 3.7 (**Convolution theorem**). *If $h_1(t), h_2(t) \in \mathcal{A}(\mathbb{T})$ having $\alpha-$Laplace transforms $\mathscr{L}^{\alpha}\{h_1(t)\}$ and $\mathscr{L}^{\alpha}\{h_2(t)\}$ respectively, then*

$$
\mathscr{L}^{\alpha}\{h_1(t) * h_2(t)\} = \mathscr{L}^{\alpha}\{h_1(t)\} \cdot \mathscr{L}^{\alpha}\{h_2(t)\}.
$$

Proof.

$$
\begin{aligned}
&\mathscr{L}^{\alpha}\{h_1(t) * h_2(t)\} \\
&= \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t,t_0) [h_1(t) * h_2(t)] \ \nabla t \\
&= \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t,t_0) \Big[\int_{t_0}^{t} h_1(t, \rho(\tau)) h_2(\tau) \nabla \tau \Big] \nabla t \\
&= \int_{t_0}^{\infty} h_2(\tau) \Big[\int_{\rho(\tau)}^{\infty} h_1(t, \rho(\tau)) \hat{e}^{\rho}_{\ominus z^{\alpha}}(t,t_0) \ \nabla t \Big] \nabla \tau
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{t_0}^{\infty} \Big[\int_{t_0}^{\infty} h_1(t,\rho(\tau)) u_{\rho(\tau)}(t) \hat{e}^{\rho}_{\ominus z^{\alpha}}(t,t_0) \nabla t \Big] \nabla \tau \\
&= \int_{t_0}^{\infty} h_2(\tau) \mathscr{L}^{\alpha}\{u_{\rho(\tau)}(t) h_1(t)\} \nabla \tau \\
&= \int_{t_0}^{\infty} h_2(\tau) \hat{e}_{\ominus z^{\alpha}}(\rho(t),t_0)\ \mathscr{L}^{\alpha}\{h_1(t)\} \nabla \tau \\
&= \mathscr{L}^{\alpha}\{h_1(t)\} \int_{t_0}^{\infty} h_2(\tau) \hat{e}^{\rho}_{\ominus z^{\alpha}}(\tau,t_0) \nabla \tau \\
&= \mathscr{L}^{\alpha}\{h_1(t)\} \cdot \mathscr{L}^{\alpha}\{h_2(t)\}.
\end{aligned}
$$

□

Theorem 3.8 (**$\alpha-$Laplace transform of Riemann-Liouville Fractional Integral**). *For $h \in \mathcal{C}_{ld}([t_0,t']_{\mathbb{T}},\mathbb{C})$ and $a > 0$, the $\alpha-$ Laplace transform of Riemann-Liouville fractional integral $({}_{t_0}\nabla^{\alpha} f)(t)$ is $\mathscr{L}^{\alpha}\{{}_{t_0}\nabla^{-a} f\}(t)$ and is given by $\mathscr{L}^{\alpha}\{{}_{t_0}\nabla^{-a} f\}(t) = z^{-a\alpha}\mathscr{L}^{\alpha}\{f(t)\}$.*

Proof. From Definition 2.6, Riemann-Liouville fractional integral can be written in form of convolution as

$$({}_{t_0}\nabla^{a} f)(t) = \hat{h}_{a-1} * h(t)$$

$$
\begin{aligned}
\text{Thus,}\quad \mathscr{L}^{\alpha}\{({}_{t_0}\nabla^{-a} h)(t)\} &= \mathscr{L}^{\alpha}\{\hat{h}_{a-1}(t,t_0) * h(t)\} \\
&= \mathscr{L}^{\alpha}\{\hat{h}_{a-1}(t,t_0)\}\mathscr{L}^{\alpha}\{h(t)\} \\
&= \frac{1}{z^{a\alpha}}\mathscr{L}^{\alpha}\{h(t)\} \\
&= z^{-a\alpha}\mathscr{L}^{\alpha}\{h(t)\}.
\end{aligned}
$$

□

Theorem 3.9 (**$\alpha-$Laplace transform of Riemann-Liouville Fractional derivative**). *For $h \in \mathcal{C}_{ld}([t_0,t']_{\mathbb{T}},\mathbb{C})$ and $\beta > 0$, the $\alpha-$ Laplace transform of Riemann-Liouville fractional derivative $({}_{t_0}\nabla^{\beta} h)(t)$ is $\mathscr{L}^{\alpha}\{({}_{t_0}\nabla^{b} h)(t)\}$ and is given by*

$$\mathscr{L}^{\alpha}\{({}_{t_0}\nabla^{\beta} f)(t)\} = z^{\beta\alpha}\mathscr{L}^{\alpha}\{h(t)\} - \sum_{k=0}^{m-1} z^{(m-k-1)\alpha}\big[{}_{t_0}\nabla^{-(m-\beta)} h\big]^{\nabla^k}(t_0).$$

Proof. From Definition 2.6 the Riemann-Liouville fractional derivative can be written as,

$$({}_{t_0}\nabla^{\beta} h)(t) = (\chi^{\nabla^m})(t) \quad \text{where} \quad \chi(t) = ({}_{t_0}\nabla^{-(m-\beta)} h)(t),$$

$$
\begin{aligned}
\mathscr{L}^{\alpha}\{({}_{t_0}\nabla^{\beta} h)(t)\} &= \mathscr{L}^{\alpha}\{\chi^{\nabla^m}(t)\} \\
&= z^{m\alpha}\mathscr{L}^{\alpha}\{\chi(t)\} - \sum_{k=0}^{m-1} z^{(m-(k+1))\alpha}\chi^{\nabla^k}(t_0) \\
&= z^{m\alpha} z^{(\beta-m)\alpha}\mathscr{L}^{\alpha}\{h(t)\} - \sum_{k=0}^{m-1} z^{(m-k-1)\alpha}\chi^{\nabla^k}(t_0) \\
&= z^{\beta\alpha}\mathscr{L}^{\alpha}\{h(t)\} - \sum_{k=0}^{m-1} z^{(m-k-1)\alpha}\big[{}_{t_0}\nabla^{-(m-\beta)h}\big]^{\nabla^k}(t_0).
\end{aligned}
$$

This is equivalent to

$$\mathscr{L}^{\alpha}\{({}_{t_0}\nabla^{\beta} h)(t)\} = z^{\beta\alpha}\mathscr{L}^{\alpha}\{h(t)\} - \sum_{j=1}^{l} z^{(j-1)\alpha}({}_{t_0}\nabla^{\beta-j} h)(t_0) \qquad l-1 < \beta < l.$$

□

Theorem 3.10 (**$\alpha-$Laplace transform of Liouville-Caputo fractional derivative**). *For $h \in \mathcal{C}_{ld}([t_0,t']_{\mathbb{T}},\mathbb{C})$ and $\lambda > 0$, the $\alpha-$ Laplace transform of Liouville-Caputo fractional derivative ${}^{\mathcal{C}}_{t_0}\nabla^{\lambda} h(t)$ is $\mathscr{L}^{\alpha}\{{}^{\mathcal{C}}_{t_0}\nabla^{\lambda} h(t)\}$ and is given by*

$$\mathscr{L}^{\alpha}\{{}^{\mathcal{C}}_{t_0}\nabla^{\lambda} h(t)\} = z^{\lambda\alpha}\mathscr{L}^{\alpha}\{h(t)\} - \sum_{k=0}^{m-1} z^{(\lambda-k-1)\alpha}\ h^{\nabla^k}(t_0).$$

Proof. From Definition 2.6 the Caputo fractional derivative can be written as,

$$ {}^{\mathcal{C}}_{t_0}\nabla^{\lambda}h(t) = ({}_{t_0}\nabla^{-(m-\lambda)}\chi)(t) \quad \text{where} \;\; \chi(t) = h^{\nabla^m}(t), $$

$$
\begin{aligned}
\mathscr{L}^{\alpha}\{{}^{\mathcal{C}}_{t_0}\nabla^{\lambda}h(t)\} &= \mathscr{L}^{\alpha}\{{}_{t_0}\nabla^{-(m-\lambda)}\chi(t)\} \\
&= z^{-(m-\lambda)\alpha}\mathscr{L}^{\alpha}\{h^{\nabla^m}(t)\} \\
&= z^{-(m-\lambda)\alpha}\big[z^{m\alpha}\mathscr{L}^{\alpha}\{h(t)\sum_{k=0}^{m-1} z^{(m-k-1)\alpha}h^{\nabla^k}(t_0)\}\big] \\
&= z^{\lambda\alpha}\mathscr{L}^{\alpha}\{h(t)\} - \sum_{k=0}^{m-1} z^{(m-k-1)\alpha}h^{\nabla^k}(t_0).
\end{aligned}
$$

□

Definition 3.4 ([15]). *For $n > 0, m, \lambda \in \mathbb{R}$ and $t, t_0 \in \mathbb{T}$. The time scale Mittag-Leffler function is defined as*

$$ E^{\lambda}_{n,m}(t,t_0) = \sum_{k=0}^{\infty} \lambda^k \hat{h}_{nk+m-1}(t,t_0) $$

provided the right hand side series is convergent.

Theorem 3.11 (**$\alpha-$ Laplace transform of Mittag-Leffler function**). *For $n, m, \lambda \in \mathbb{T}$ and $t_0, t \in \mathbb{T}$*

$$ \mathscr{L}^{\alpha}\{E^{\lambda}_{n,m}(t,t_0)\} = \frac{z^{(m-n)\alpha}}{z^{m\alpha}-\lambda}. $$

Proof.

$$
\begin{aligned}
&\mathscr{L}^{\alpha}\big\{E^{\lambda}_{n,m}(t,t_0)\big\} \\
&= \mathscr{L}^{\alpha}\big\{\sum_{k=0}^{\infty}\lambda^k\hat{h}_{nk+m-1}(t,t_0)\big\} \\
&= \sum_{k=0}^{\infty}\lambda^k\mathscr{L}^{\alpha}\{\hat{h}_{nk+m-1}(t,t_0)\} \\
&= \sum_{k=0}^{\infty}\lambda^k\frac{1}{z^{(mk+n)\alpha}} \\
&= \frac{1}{z^{n\alpha}}\sum_{k=0}^{\infty}\frac{\lambda^k}{z^{mk\alpha}} \\
&= \frac{1}{z^{n\alpha}}\Big[1+\frac{\lambda}{z^{n\alpha}}+\frac{\lambda^2}{z^{2m\alpha}}+\ldots\Big] \\
&= \frac{1}{z^{n\alpha}}\Big[\frac{1}{1-\frac{\lambda}{m\alpha}}\Big] \qquad\qquad \text{provided} \quad \Big|\frac{\lambda}{z^{m\alpha}}\Big| < 1 \\
&= \frac{1}{z^{n\alpha}}\Big[\frac{z^{m\alpha}}{z^{m\alpha}-\lambda}\Big] \\
&= \frac{z^{(m-n)\alpha}}{z^{m\alpha}-\lambda}.
\end{aligned}
$$

□

In this last section we will solve a fractional dynamic equation using our defined transform.

4 Application

Consider the following fractional dynamic equation with initial condition,

$$ {}_0\nabla^{1/2}g(t) + ag(t) = 0, \qquad ({}_0\nabla^{-1/2}g)(0) = k. $$

Applying the $\alpha-$ Laplace transform with $\alpha = \frac{1}{2}$, we obtain

$$ \mathscr{L}^{\frac{1}{2}}\big\{{}_0\nabla^{\frac{1}{2}}g(t) + ag(t)\big\} = 0, $$

$$z\mathscr{L}^{\frac{1}{2}}\{g(t)\} - ({}_0\nabla^{-\frac{1}{2}}g)(0) + a\mathscr{L}^{\frac{1}{2}}\{g(t)\} = 0,$$
$$z\mathscr{L}^{\frac{1}{2}}\{g(t)\} - k + a\mathscr{L}^{\frac{1}{2}}\{g(t)\} = 0,$$
$$(z+a)\mathscr{L}^{\frac{1}{2}}\{g(t)\} = k,$$
$$\mathscr{L}^{\frac{1}{2}}\{g(t)\} = \frac{k}{(z+a)}.$$

Taking inverse required solution is,

$$g(t) = kE^{-a}_{\frac{1}{2},\frac{1}{2}}(t,0).$$

5 Conclusion

In this paper, we introduce a new α-Laplace transform on time scales. This transform for $\alpha = 1$ coinsides with a nabla Laplace transform on time scales and for $0 < \alpha < 1$ will serve as a fractional Laplace transform on time scales. Accompained by the existence theorem we have proved some of its important properties, including the convolution theorem and found, transform of the Riemann-Liouville fractional integral, Riemamm-Liouville fractional derivative, Liouville-Caputo derivative and Mittag Leffter function on time scales. A fractional dynamic equation with a given initial condition is solved for a suitable value of α showing efficiency of this integral transform.

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CERTAIN PROPERTIES OF A NEW SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract

The purpose of this paper is to study a new subclass of close-to-convex functions associated with generalized Janowski's function. Various properties such as coefficient estimates, inclusion relationship, distortion property, argument property and radius of convexity, are established for this class. The results mentioned here, generalize some earlier known results.
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Keywords and Phrases: Analytic functions, Univalent functions, Subordination, Close-to-convex functions, Distortion theorem, Argument theorem.

1 Introduction

By $\mathcal{A}$, we denote the class of functions f of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which are analytic in the open unit disc $E = \{z :| z |< 1\}$. Further, the class of functions $f \in \mathcal{A}$ and which are univalent in E, is denoted by $\mathcal{S}$. A function w is said to be a Schwarz function if it has expansion of the form $w(z) = \sum_{n=1}^{\infty} c_n z^n$ and satisfy the conditions $w(0) = 0$ and $|w(z)| \leq 1$. The class of Schwarz functions is denoted by $\mathcal{U}$.

For two analytic functions f and g in E, f is said to be subordinate to g, if there exists a Schwarz function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$. If f is subordinate to g, then it is denoted by $f \prec g$. Further, if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

By $\mathcal{S}^*$ and $\mathcal{K}$, we denote the classes of starlike functions and of convex functions respectively, which are defined as follows:

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}$$

and

$$\mathcal{K} = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}.$$

A function $f \in \mathcal{A}$ is said to be close-to-convex function if there exists a function $g \in \mathcal{S}^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0 (z \in E).$$

The class of close-to-convex functions is denoted by $\mathcal{C}$ and was given by Kaplan [6]. Several subclasses of close-to-convex functions were studied by various authors and recently by Singh and Singh [14], but here we mention those which are relevant to our study.

Gao and Zhou [3] studied the class $\mathcal{K}_S$ defined as

$$\mathcal{K}_s = \left\{ f : f \in \mathcal{A}, Re\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > 0, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E \right\}.$$

Further, Kowalczyk and Les-Bomba [7] extended the class $\mathcal{K}_S$ by introducing the class $\mathcal{K}_S(\gamma), (0 \leq \gamma < 1)$, which is mentioned below:

$$\mathcal{K}_s(\gamma) = \left\{ f : f \in \mathcal{A}, Re\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > \gamma, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E \right\}.$$

For $\gamma = 0$, the class $\mathcal{K}_S(\gamma)$ reduces to the class $\mathcal{K}_S$.

Later on, Seker [12] established the class $\mathcal{K}_s^{(k)}(\gamma)$ $(0 \le \gamma < 1)$ of close-to-convex analytic functions $f \in \mathcal{A}$ which satisfy the condition

$$Re\left(\frac{z^k f'(z)}{g_k(z)}\right) > \gamma,$$

where

$$g_k(z) = \Pi_{\nu=0}^{k-1} \epsilon^{-\nu} g(\epsilon^\nu z)(\epsilon^k = 1; k \ge 1), \tag{1.1}$$

and $g \in \mathcal{S}^*\left(\frac{k-1}{k}\right)$.

As a generalization, Seker and Cho [13] introduced the class $\mathcal{K}_s^{(k)}(\gamma;\delta;\eta)$ of the functions $f \in \mathcal{A}$ which satisfy the condition

$$\frac{z^k f'(z)}{g_k(z)} \prec \frac{1+\eta[1-(1+\delta)\gamma]z}{1-\eta\delta z}$$

where g_k is defined in (1.1) and $0 \le \gamma < 1, 0 \le \delta \le 1$ and $0 < \eta \le 1$.

Raina et al. [10] established the class of strongly close-to-convex functions of order β, as below:

$$\mathcal{C}'_\beta = \left\{ f : f \in \mathcal{A}, \left| arg\left\{\frac{zf'(z)}{g(z)}\right\}\right| < \frac{\beta\pi}{2}, g \in \mathcal{K}, 0 < \beta \le 1, z \in E \right\},$$

which can also be expressed as

$$\mathcal{C}'_\beta = \left\{ f : f \in \mathcal{A}, \frac{zf'(z)}{g(z)} \prec \left(\frac{1+z}{1-z}\right)^\beta, g \in \mathcal{K}, 0 < \beta \le 1, z \in E \right\}.$$

For $-1 \le B < A \le 1$, Janowski [5] introduced the class of functions in $\mathcal{A}$ which are of the form $p(z) = 1 + \sum_{n=1}^\infty p_n z^n$ and satisfying the condition $p(z) \prec \frac{1+Az}{1+Bz}$. This class plays an important role in the study of various subclasses of analytic-univalent functions. As a generalization of Janowski's class, Polatoglu et al. [9] established the class $\mathcal{P}(A, B; \alpha)$ $(0 \le \alpha < 1)$, the subclass of $\mathcal{A}$ which consists of functions of the form $p(z) = 1 + \sum_{n=1}^\infty p_n z^n$ such that $p(z) \prec \frac{1+[B+(A-B)(1-\alpha)]z}{1+Bz}$. Also for $\alpha = 0$, the class $\mathcal{P}(A, B; \alpha)$ agrees with the class defined by Janowski [5].

Inspired by the above mentioned classes, now we define the following generalized class which is to study in this paper.

Definition 1.1. *Let $\mathcal{K}_s^{(k)}(A, B; \alpha; \beta)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the conditions,*

$$\frac{z^k f'(z)}{g_k(z)} \prec \left(\frac{1+[B+(A-B)(1-\alpha)]z}{1+Bz}\right)^\beta, -1 \le B < A \le 1, z \in E,$$

where $g(z) = z + \sum_{n=2}^\infty b_n z^n \in \mathcal{S}^*\left(\frac{k-1}{k}\right)$, $0 \le \alpha < 1, 0 < \beta \le 1, -1 \le B < A \le 1$ and $g_k(z)$ is defined in (1.1).

The following observations are obvious:

(i) $\mathcal{K}_s^{(k)}(\eta[1-(1+\delta)\gamma], -\eta\delta; 0; 1) \equiv \mathcal{K}_s(\gamma, \delta, \eta)$, *the class established by Seker and Cho [13].*

(ii) $\mathcal{K}_s^{(k)}(1-2\gamma, -1; 0; 1) \equiv \mathcal{K}_s^{(k)}(\gamma)$, *the class studied by Seker [12].*

(iii) $\mathcal{K}_s^{(2)}(1, -1; 0; 1) \equiv \mathcal{K}_s$, *the class introduced by Gao and Zhou [3].*

(iv) $\mathcal{K}_s^{(2)}(1-2\gamma, -1; 0; 1) \equiv \mathcal{K}_s(\gamma)$, *the class established by Kowalczyk and Les Bomba [7].*

As $f \in \mathcal{K}s^{(k)}(A, B; \alpha; \beta)$, by definition of subordination, it follows that

$$\frac{z^k f'(z)}{g_k(z)} = \left(\frac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)}\right)^\beta, w \in \mathcal{U}. \tag{1.2}$$

We study various properties such as coefficient estimates, inclusion relationship, distortion theorem, argument theorem and radius of convexity for the functions in the class $\mathcal{K}_s^{(k)}(A, B; \alpha; \beta)$. The results proved by various authors follow as special cases.

Throughout this paper, we assume that $-1 \le B < A \le 1, 0 \le \alpha < 1, 0 < \beta \le 1, 0 \le \gamma < 1, 0 < \eta \le 1, 0 \le \delta \le 1, k \ge 1, z \in E$.

2 Preliminary Results

For the derivation of our main results, we must require the following lemmas:

Lemma 2.1 ([2, 11])**.** *Let,*

$$\left(\frac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)}\right)^{\beta}=(P(z))^{\beta}=1+\sum_{n=1}^{\infty}p_n z^n, \tag{2.1}$$

then

$$|p_n|\le\beta(1-\alpha)(A-B), n\ge 1.$$

Lemma 2.2 ([10])**.** *Let $-1\le B_2\le B_1<A_1\le A_2\le 1$, then*

$$\left(\frac{1+A_1z}{1+B_1z}\right)^{\beta}\prec\left(\frac{1+A_2z}{1+B_2z}\right)^{\beta}.$$

Lemma 2.3 ([8])**.** *If $g\in\mathcal{S}^*$, then for $|z|=r, 0<r<1$, we have*

$$\frac{r}{(1+r)^2}\le|g(z)|\le\frac{r}{(1-r)^2}.$$

Lemma 2.4 ([15])**.** *For $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{S}^*\left(\frac{k-1}{k}\right)$, then*

$$G_k(z)=\frac{g_k(z)}{z^{k-1}}=z+\sum_{n=2}^{\infty}d_nz^n\in\mathcal{S}^*.$$

Lemma 2.5 ([1, 2])**.** *If $P(z)=\frac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)}, -1\le B<A\le 1, w\in\mathcal{U}$, then for $|z|=r<1$, we have*

$$Re\frac{zP'(z)}{P(z)}\ge\begin{cases}-\frac{(A-B)(1-\alpha)r}{(1-[B+(A-B)(1-\alpha)]r)(1-Br)}, & \text{if } R_1\le R_2,\\ 2\frac{\sqrt{(1-B)(1-[B+(A-B)(1-\alpha)])(1+[B+(A-B)(1-\alpha)]r^2)(1+Br^2)}}{(A-B)(1-\alpha)(1-r^2)} & \\ -\frac{(1-[B+(A-B)(1-\alpha)]Br^2)}{(A-B)(1-\alpha)(1-r^2)}+\frac{(A+B)-\alpha(A-B)}{(A-B)(1-\alpha)}, & \text{if } R_1\ge R_2,\end{cases}$$

where $R_1=\sqrt{\frac{(1-[B+(A-B)(1-\alpha)])(1+[B+(A-B)(1-\alpha)]r^2)}{(1-B)(1+Br^2)}}$ and $R_2=\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}$.

3 Main Results

Theorem 3.1. *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{K}_s^{(k)}(A,B;\alpha;\beta)$, then*

$$|a_n|\le\ 1+\frac{\beta(1-\alpha)(n-1)(A-B)}{2}. \tag{3.1}$$

Proof. As $f\in\mathcal{K}_s^{(k)}(A,B;\alpha;\beta)$, therefore (1.2) can be written as

$$\frac{z^kf'(z)}{g_k(z)}=(P(z))^{\beta},$$

which can be further expressed as

$$\frac{zf'(z)}{G_k(z)}=(P(z))^{\beta}, \tag{3.2}$$

where

$$G_k(z)=\frac{g_k(z)}{z^{k-1}}=z+\sum_{n=2}^{\infty}d_nz^n. \tag{3.3}$$

By Lemma 2.4, we have $G_k\in\mathcal{S}^*$.

Using (2.1) and (3.3) in (3.2), it yields

$$1+\sum_{n=2}^{\infty}na_nz^{n-1}=\left(1+\sum_{n=2}^{\infty}nd_nz^{n-1}\right)\left(1+\sum_{n=1}^{\infty}p_nz^n\right). \tag{3.4}$$

As $G_k(z)=z+\sum_{n=2}^{\infty}d_nz^n\in\mathcal{S}^*$, it is well known that $|d_n|\le n$.

Comparing the coefficients of z^{n-1} in (3.4), we have

$$na_n=d_n+d_{n-1}p_1+d_{n-2}p_2+...+d_2p_{n-2}+p_{n-1}. \tag{3.5}$$

Applying triangle inequality, using Lemma 2.1 and the inequality $|d_n|\le n$ in (3.5), it gives

$$n|a_n|\le n+\beta(1-\alpha)(A-B)[(n-1)+(n-2)+...+2+1], \tag{3.6}$$

which proves Theorem 3.1. □

For $A=\eta[1-(1+\delta)\gamma], B=-\eta\delta, \alpha=0, \beta=1$, Theorem 3.1 gives the following result:

Corollary 3.1. *If $f\in\mathcal{K}_s^{(k)}(\gamma;\delta;\eta)$, then*

$$|a_n|\le 1+\frac{\eta(n-1)(1+\delta)(1-\gamma)}{2}.$$

Putting $A=1-2\gamma$, $B=-1$, $\alpha=0$ and $\beta=1$ in Theorem 3.1, the following result is obvious:

Corollary 3.2. *If $f\in\mathcal{K}_s^{(k)}(\gamma)$, then*

$$|a_n|\le n-(n-1)\gamma.$$

Substituting for $k=2, A=1-2\gamma$, $B=-1$, $\alpha=0$ and $\beta=1$ in Theorem 3.1, we can easily obtain the following result:

Corollary 3.3. *If $f\in\mathcal{K}_s(\gamma)$, then*

$$|a_n|\le n-(n-1)\gamma.$$

Taking $k=2, A=1$, $B=-1$, $\alpha=0$ and $\beta=1$, Theorem 3.1 yields the following result:

Corollary 3.4. *If $f\in\mathcal{K}_s$, then*

$$|a_n|\le n.$$

Theorem 3.2. *If $-1\le B_2=B_1<A_1\le A_2\le 1$ and $0\le\alpha_2\le\alpha_1<1$, then*

$$\mathcal{K}_s^{(k)}(A_1,B_1;\alpha_1;\beta)\subset\mathcal{K}_s^{(k)}(A_2,B_2;\alpha_2;\beta).$$

Proof. As $f\in\mathcal{K}_s^{(k)}(A_1,B_1;\alpha_1;\beta)$, so

$$\frac{z^kf'(z)}{g_k(z)}\prec\left(\frac{1+[B_1+(A_1-B_1)(1-\alpha_1)]z}{1+B_1z}\right)^\beta.$$

As $-1\le B_2=B_1<A_1\le A_2\le 1$ and $0\le\alpha_2\le\alpha_1<1$, we have

$$-1\le B_1+(1-\alpha_1)(A_1-B_1)\le B_2+(1-\alpha_2)(A_2-B_2)\le 1.$$

Thus by Lemma 2.2, it yields

$$\frac{z^kf'(z)}{g_k(z)}\prec\left(\frac{1+[B_2+(A_2-B_2)(1-\alpha_2)]z}{1+B_2z}\right)^\beta,$$

which implies $f\in\mathcal{K}_s^{(k)}(A_2,B_2;\alpha_2;\beta)$. □

Theorem 3.3. *If $f\in\mathcal{K}_s^{(k)}(A,B;\alpha;\beta)$, then for $|z|=r, 0<r<1$, we have*

$$\left(\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}\right)^\beta.\frac{1}{(1+r)^2}\le|f'(z)|\le\left(\frac{1+[B+(A-B)(1-\alpha)]r}{1+Br}\right)^\beta.\frac{1}{(1-r)^2}\tag{3.7}$$

and

$$\int_0^r\left(\frac{1-[B+(A-B)(1-\alpha)]t}{1-Bt}\right)^\beta.\frac{1}{(1+t)^2}dt\le|f(z)|\tag{3.8}$$

$$\le\int_0^r\left(\frac{1+[B+(A-B)(1-\alpha)]t}{1+Bt}\right)^\beta.\frac{1}{(1-t)^2}dt.$$

Proof. From (3.2), we have

$$|f'(z)|=\frac{|G_k(z)|}{|z|}(P(z))^\beta.\tag{3.9}$$

Aouf [2] proved that

$$\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}\le|P(z)|\le\frac{1+[B+(A-B)(1-\alpha)]r}{1+Br},$$

which implies

$$\left(\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}\right)^\beta\le|P(z)|^\beta\le\left(\frac{1+[B+(A-B)(1-\alpha)]r}{1+Br}\right)^\beta.\tag{3.10}$$

Since $G_k\in\mathcal{S}^*$, so by Lemma 2.3, we have

$$\frac{r}{(1+r)^2}\le|G_k(z)|\le\frac{r}{(1-r)^2}.\tag{3.11}$$

Relation (3.9) together with (3.10) and (3.11) yields (3.7). On integrating (3.7) from 0 to r, (3.8) follows.

For $A=\eta[1-(1+\delta)\gamma], B=-\eta\delta, \alpha=0, \beta=1$, Theorem 3.3 gives the following result: □

Corollary 3.5. *If* $f \in \mathcal{K}_s^{(k)}(\gamma;\delta;\eta)$, *then*

$$\left(\frac{1-\eta[1-(1+\delta)\gamma]r}{1+\eta\delta r}\right)\cdot\frac{1}{(1+r)^2} \le |f'(z)| \le \left(\frac{1+\eta[1-(1+\delta)\gamma]r}{1-\eta\delta r}\right)^{\beta}\cdot\frac{1}{(1-r)^2}$$

and

$$\int_0^r \left(\frac{1-\eta[1-(1+\delta)\gamma]t}{1+\eta\delta t}\right)\cdot\frac{1}{(1+t)^2}dt \le |f(z)| \le \int_0^r \left(\frac{1+\eta[1-(1+\delta)\gamma]t}{1-\eta\delta t}\right)\cdot\frac{1}{(1-t)^2}dt.$$

Putting $A = 1-2\gamma$, $B=-1$, $\alpha = 0$ *and* $\beta = 1$ *in Theorem 3.3, the following result is obvious:*

Corollary 3.6. *If* $f \in \mathcal{K}_s^{(k)}(\gamma)$, *then*

$$\frac{2\gamma r}{(1+r)^3} \le |f'(z)| \le \frac{2(1-\gamma)r}{(1-r)^3}.$$

and

$$\int_0^r \left(\frac{2\gamma t}{(1+t)^3}\right)dt \le |f(z)| \le \int_0^r \left(\frac{2(1-\gamma)t}{(1-t)^3}\right)dt.$$

Substituting for $k=2, A=1-2\gamma$, $B=-1$, $\alpha=0$ *and* $\beta=1$ *in Theorem 3.3, we can easily obtain the following result:*

Corollary 3.7. *If* $f \in \mathcal{K}_s(\gamma)$, *then*

$$\frac{1-(1-2\gamma)r}{(1+r)^3} \le |f'(z)| \le \frac{1+(1-2\gamma)r}{(1-r)^3}.$$

and

$$\int_0^r \left(\frac{1-(1-2\gamma)t}{(1+t)^3}\right)dt \le |f(z)| \le \int_0^r \left(\frac{1+(1-2\gamma)t}{(1-t)^3}\right)dt.$$

Taking $k=2, A=1$, $B=-1$, $\alpha=0$ *and* $\beta=1$, *Theorem 3.3 yields the following result:*

Corollary 3.8. *If* $f \in \mathcal{K}_s$, *then*

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}.$$

and

$$\int_0^r \left(\frac{1-t}{(1+t)^3}\right)dt \le |f(z)| \le \int_0^r \left(\frac{1+t}{(1-t)^3}\right)dt.$$

Theorem 3.4. *Let* $f \in \mathcal{K}_s^{(k)}(A,B;\alpha;\beta)$, *then*

$$Re\frac{(zf'(z))'}{f'(z)} \ge \begin{cases} \frac{1-r}{1+r} - \beta\frac{(A-B)(1-\alpha)r}{(1-[B+(A-B)(1-\alpha)]r)(1-Br)}, & \text{if } R_1 \le R_2, \\ \frac{1-r}{1+r} + \frac{(A+B)-\alpha(A-B)}{(A-B)(1-\alpha)} & \\ +2\frac{\sqrt{(1-B)(1-[B+(A-B)(1-\alpha)])(1+[B+(A-B)(1-\alpha)]r^2)(1+Br^2)}}{(A-B)(1-\alpha)(1-r^2)} & \\ -2\frac{(1-[B+(A-B)(1-\alpha)]Br^2)}{(A-B)(1-\alpha)(1-r^2)}, & \text{if } R_1 \ge R_2, \end{cases}$$

where R_1 *and* R_2 *are defined in Lemma 2.5.*

Proof. Proof. As $f \in \mathcal{K}_s^{(k)}(A,B;\alpha;\beta)$, we have

$$zf'(z) = G_k(z)(P(z))^{\beta}.$$

Differentiating logarithmically, we get

$$\frac{(zf'(z))'}{f'(z)} = \frac{zG_k'(z)}{G_k(z)} + \beta\frac{zP'(z)}{P(z)}. \tag{3.12}$$

As $G_k \in \mathcal{S}^*$, so by the result due to Mehrok [8], we have

$$Re\left(\frac{zG_k'(z)}{G_k(z)}\right) \ge \frac{1-r}{1+r}. \tag{3.13}$$

Hence, using (3.13) and Lemma 2.5 in (3.12), the proof of Theorem 3.4 is obvious. □

Theorem 3.5. *If $f \in \mathcal{K}_s^{(k)}(A,B;\alpha;\beta)$ and let $F(z) = zf'(z)$, then for $|z| = r, 0 < r < 1$, we have*
$$\left|arg\frac{F(z)}{z}\right| \leq \beta sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right) + 2sin^{-1}r.$$

Proof. *Proof*. From (3.2), we have
$$\frac{zf'(z)}{G_k(z)} = (P(z))^\beta,$$
which can be expressed as
$$F(z) = G_k(z)(P(z))^\beta.$$
Therefore, we have
$$\left|arg\frac{F(z)}{z}\right| \leq \beta|argP(z)| + \left|arg\frac{G_k(z)}{z}\right|. \tag{3.14}$$
It is well known that
$$|argP(z)| \leq sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right). \tag{3.15}$$
It was proved by Goel and Mehrok [4] that, for $G_k(z) \in S^*$,
$$\left|arg\frac{G_k(z)}{z}\right| \leq 2sin^{-1}r. \tag{3.16}$$
Using (3.15) and (3.16) in (3.14), Theorem 3.5 is obvious. □

4 Conclusion and Open Problems

Close-to-convex functions are of great importance in the study of univalent functions. In the present paper, we introduce a new and generalized subclass of close-to-convex functions using subordination and established various properties for this class. Many earlier known results follow as particular cases of our results. This study will motivate the other researchers to investigate other such classes and to discuss their properties.

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FUZZY CONTINUOUS AND FUZZY BOUNDED LINEAR OPERATORS OVER ANTI-FUZZY FIELDS

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Abstract

In this paper, we have studied the concept of anti-norm and anti-inner product function on anti-fuzzy linear space over anti-fuzzy field, we have also given fuzzy continuous linear operator from an anti-normed anti-fuzzy linear space to another anti-normed anti-fuzzy linear space and also introduced three types (strong, weak and sequential) of fuzzy bounded linear operators.

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Keywords and Phrases: Fuzzy field, fuzzy linear space, anti-fuzzy field, anti-fuzzy linear space, anti-norm, anti- inner product, weak fuzzy continuity, strong fuzzy continuity and sequential fuzzy continuity.

1 Introduction

During the last few years there is a growing interest in the extension of fuzzy set theory which is a useful tool to describe the situation in which data are imprecise or vague or uncertain. Fuzzy set theory handles the situation, by attributing a degree of membership to which a certain object belongs to a set. The fundamental concept of fuzzy set theory was introduced by Zadeh [23] in 1965 and thereafter, the concept of fuzzy set theory applied on different branches of pure and applied mathematics in different ways. The fuzzy topology was introduced by Chang [4] in 1968, while the concept of fuzzy norm was introduced by Katsaras [9] in 1984. Thereafter Wu and Fang [20] introduced a fuzzy normed space. In 1991, Biswas [1] defined fuzzy norm and fuzzy inner product function on a linear space. In 1992, Felbin [8] introduced fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space. Another important approach of fuzzy norm on a linear space was introduced in 1994 by Cheng and Morderson [5], on a parallel line as the corresponding fuzzy metric due to Kramosil and Michelek [11] type. Krishna and Sarma [10], Xiao and Zhu [22] discussed fuzzy norms on linear spaces at different points of view. In 2005, Bag and Samanta [2], introduced an idea of fuzzy norm of a linear operator from a fuzzy normed linear space to another fuzzy normed linear space and defined various notions of continuities and boundedness of linear operators over fuzzy normed linear spaces such as fuzzy continuity, sequential fuzzy continuity, weakly fuzzy continuity, strongly fuzzy continuity, weakly and strongly fuzzy boundedness. All these Researchers have done their work in the area of crisp linear space. Wenxiang and Tu [21] were the first to introduce the concept of fuzzy fields and fuzzy linear spaces over fuzzy fields. In 2011, Santosh and Ramakrishnan [18] introduced norm and inner product on fuzzy linear spaces over fuzzy field. In 2012, Srinivas, Swamy and Nagaiah [19] introduced anti-fuzzy near-algebras over anti-fuzzy fields. In 2022, Barge and Yadav [3] defined (λ, μ)-anti-fuzzy linear spaces. In 2022, Chandra, Srivastava, and Sinha [6]; Srivastava, Sinha and Chandra [17] introduced 2-norm and 2-inner product on fuzzy linear spaces over fuzzy field. For more recent work of the area under study, we refer to [7,12,13,14,15,16]. In the present paper we introduce the idea of anti-norm and anti-inner product function on anti-fuzzy linear space over anti-fuzzy field and also given fuzzy continuous and fuzzy bounded linear operators on anti-fuzzy linear space over anti-fuzzy field.

2 Preliminaries

This section contains some definitions and preliminary results which are used in the paper.

Definition 2.1 ([21])**.** *Let X be a field and F a fuzzy set in X with the following conditions:*

(i) $F(x+y) \geq \min\{F(x), F(y)\}, x, y \in X$,
(ii) $F(-x) \geq F(x), x \in X$,

(iii) $F(xy) \geq \min\{F(x), F(y)\}, x, y \in X$,
(iv) $F\left(x^{-1}\right) \geq F(x), x(\neq 0) \in X$.
Then we call F a fuzzy field in X and denoted by (F, X) and it is also called a fuzzy field of X.

Theorem 2.1 ([21]). *If (F, X) is a fuzzy field of X, then*
(i) $F(0) \geq F(x), x \in X$.
(ii) $F(1) \geq F(x), x(\neq 0) \in X$.
(iii) $F(0) \geq F(1)$.

Theorem 2.2 ([21]). *Let X and Y be field and f a homomorphism of X into Y suppose that (F, X) is a fuzzy field of X and (G, Y) is a fuzzy field of Y. Then*
(i) $(f(F), Y)$ *is a fuzzy field of* Y.
(ii) $\left(f^{-1}(G), X\right)$ *is a fuzzy field of* X.

Definition 2.2 ([21]). *Let X be a field and (F, X) be a fuzzy field of X. Let Y be a linear space over X and V a fuzzy set of Y. Suppose the following condition hold:*
(i) $V(x+y) \geq \min\{V(x), V(y)\}, x, y \in Y$,
(ii) $V(\lambda x) \geq \min\{F(\lambda), V(x)\}, \lambda \in X, x \in Y$,
(iii) $V(-x) \geq V(x), x \in Y$,
(iv) $F(1) \geq V(0)$.
Then (V, Y) is called a fuzzy linear space over (F, X).

Theorem 2.3 ([21]). *If (V, Y) is a fuzzy linear space over fuzzy field (F, X), then*
(i) $F(0) \geq V(0)$.
(ii) $V(0) \geq V(x), x \in Y$.
(iii) $F(0) \geq V(x), x \in Y$.

Theorem 2.4 ([21]). *Let (F, X) be a fuzzy field of X and Y a linear space over X. Let V be a fuzzy set of Y. Then (V, Y) is a fuzzy linear space over (F, X) if and only if*
(i) $V(\lambda x+\mu y) \geq \min\{F(\lambda), F(\mu), V(x), V(y)\}, \lambda, \mu \in X$ *and* $x, y \in Y$.
(ii) $F(1) \geq V(x), x \in Y$.

Definition 2.3 ([18]). *Let (F, K) be a fuzzy field of K (K denotes either R or C), X be a linear space over K and (V, X) be a fuzzy linear space over (F, K). A norm on (V, X) is a function $\|\,\|: X \rightarrow[0, \infty)$ such that*
(i) $F(\|x\|) \geq V(x)$ *for all* $x \in X$,
(ii) $\|x\| \geq 0 \;\forall\; x \in X$ *and* $\|x\|=0$ *if and only if* $x=0$,
(iii) $\|x+y\| \leq\|x\|+\|y\|$ *for all* $x, y \in X$,
(iv) $\|k x\|=|k|\|x\|$ *for all* $k \in K$ *and for all* $x \in X$.
Then $(V, X,\|\|)$ is called a normed anti-fuzzy linear space (NFLS) over fuzzy field.

Definition 2.4 ([18]). *An inner product on a fuzzy linear space (V, X) over a fuzzy field (F, K) is a function $\langle,\rangle :, X \times X \rightarrow K$ such that for all $x, y, z \in X$ and $k \in K$,*
(i) $F(\langle x, y\rangle) \geq V \times V(x, y)$,
(ii) $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle=0$ *if and only if* $x=0$,
(iii) $\langle x+y, z\rangle=\langle x, z\rangle+\langle y, z\rangle$ *and* $\langle k x, y\rangle=k\langle x, y\rangle$,
(iv) $\langle y, x\rangle=\overline{\langle x, y\rangle}$.
Thus, $(V, X,\langle,\rangle)$ is called an inner product on fuzzy linear space over fuzzy field.

Definition 2.5 ([19]). *Let X be a field and F a fuzzy set in X with the following conditions:*
(i) $F(x+y) \leq \max\{F(x), F(y)\}, x, y \in X$,
(ii) $F(-x) \leq F(x), x \in X$,
(iii) $F(xy) \leq \max\{F(x), F(y)\}, x, y \in X$,
(iv) $F\left(x^{-1}\right) \leq F(x), x(\neq 0) \in X$.
An anti-fuzzy field F of X is denoted by (F, X).

Theorem 2.5 ([19]). *If (F, X) is an anti-fuzzy field of X, then*

(i) $F(0) \leq F(x)$, *for any* $x \in X$.
(ii) $F(1) \leq F(x)$, *for any* $x(\neq 0) \in X$.
(iii) $F(0) \leq F(1)$.

Definition 2.6 ([19]). *Let X be a field and (F, X) be an anti-fuzzy field of X. Let Y be a linear space over X and V a fuzzy set of Y. Suppose the following condition hold:*
(i) $V(x+y) \leq \max\{V(x), V(y)\}, x, y \in Y$
(ii) $V(\lambda x) \leq \max\{F(\lambda), V(x)\}, \lambda \in X, x \in Y$,
(iii) $V(-x) \leq V(x), x \in Y$,
(iv) $F(1) \leq V(0)$.
Then (V, Y) is called an anti-fuzzy linear space over (F, X).

3 Anti-norm and anti-inner product function on anti-fuzzy linear space over anti-fuzzy field

In this section, we define anti-norm and anti-inner product function on anti-fuzzy linear space over anti-fuzzy field and also establish relationship between them.

Here, K denotes either R (set of real numbers) or C (set of complex numbers).

Definition 3.1. *Let (F, K) be an anti-fuzzy field of K, X be a linear space over K and (V, X) be an anti-fuzzy linear space over (F, K). An anti-norm on (V, X) is function $||.|| : X \to [0, \infty)$ such that:*
(i) $F(\|x\|) \leq \mathrm{V(x)}$ *for all* $x \in X$,
(ii) $\|x\| \geq 0\ \forall\ x \in X$ *and* $\|x\| = 0$ *if and only if* $x = 0$,
(iii) $\|x+y\| \leq \|x\| + \|y\|$,
(iv) $\|kx\| \leq |k| \|x$, *for all* $k \in K$ *and for all* $x \in X$.
Then $(V, X, \|.\|)$ is called anti-normed anti-fuzzy linear space (ANAFLS).

Theorem 3.1. *Let (V, X) be an anti-fuzzy linear space over an anti-fuzzy field $(F, K), Y$ be a linear space over K and T be an isomorphism of X onto Y. (V, X) is an anti-normed anti-fuzzy linear space over (F, K) if and only if $(T(V), Y)$ is an anti-normed anti-fuzzy linear space over (F, K).*

Proof. Let $\|\cdot\|_{\mathrm{X}}$ be an anti-norm on (V, X). Let x $\in$ X so, T(x) $\in$ Y. Take T(x) = y. Now consider the anti-norm $\|\cdot\|_Y$ on Y defined $\|y\|_Y = \|x\|_X$. Then $\mathrm{F}\left(\|y\|_Y\right) = F\left(\|x\|_X\right) \leq V(x) = T(V)T(x) = T(V)(y)$. Therefore $\|\cdot\|_{\mathrm{Y}}$ is an anti-norm on $(T(V), Y)$.

Conversely, assume that $\|\cdot\|_Y$ is an anti-norm on $(T(V), Y)$. Consider the anti-norm $\|\cdot\|_X$ on X as $\|x\|_X = \|Tx\|_Y$

Then $F\left(\|x\|_X\right) = F\left(\|Tx\|_Y\right) \leq T(V)(T(x)) = V(x)$.

Therefore, $\|\cdot\|_X$ is an anti-norm on (V, X). □

Theorem 3.2. *Let X be a linear space over $K, (W, Y)$ an anti-fuzzy linear space over an anti-fuzzy field (F, K) and $T : X \to Y$ be an injective linear transformation. If (W, Y) is an anti-normed anti-fuzzy linear space over (F, K). Then $\left(T^{-1}(W), X\right)$ is an anti-normed anti-fuzzy linear space over (F, K).*

Proof. Let $\|.\|_Y$ be an anti-norm on (W, Y). Consider the anti-norm $\|.\|_X$ on X as

$$\|x\|_X = \|Tx\|_Y \quad \text{Then}$$
$$F\left(\|x\|_X\right) = F\left(\|Tx\|_Y\right) \leq W\left(T(x) = T^{-1}W(x).\right.$$

Hence $\|\cdot\|_X$ is an anti-norm on $\left(T^{-1}(W), X\right)$. □

Theorem 3.3. *Let (V, X) be an anti-normed anti-fuzzy linear space over an anti-fuzzy field (F, K) and $T : X \to X$ be an injective linear transformation. Then $\left(T^{-1}(V), X\right)$ is an anti-normed anti-fuzzy linear space over (F, K). Proof. Obvious by Theorem 3.2.*

Definition 3.2. *An anti-inner product on an anti-fuzzy linear space (V, X) over an anti-fuzzy field (F, K) is a function $\langle , \rangle : X \times X \to K$ such that for all $x, y, z \in X$ and $k \in K$,*
(i) $F(\langle x, y\rangle) \leq V \times V(x, y)$
(ii) $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$
(iii) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ *and* $\langle kx, y\rangle = k\langle x, y\rangle$
(iv) $\langle y, x\rangle = \overline{\langle x, y\rangle}$.

Thus, $(V, X, \langle,\rangle)$ is called an anti-inner product on anti-fuzzy linear space over anti-fuzzy field.

Example 3.1. Let F be an anti-fuzzy field of R. The anti-inner product $\langle,\rangle$ on R^n defined by $\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i$ is an anti-inner product on an anti-fuzzy linear space $(\underbrace{F \times F \times \ldots \times F}_{n\ times}, R^n)$.

Proof. Let $V = (\underbrace{F \times F \times \ldots \times F}_{n\ times}, R^n)$

$$\begin{aligned}
F(\langle x,y\rangle) &= F(x_1y_1 + x_2y_2 + \cdots + x_ny_n) \\
&\leq \max\left\{F\left(x_1y_1\right), F\left(x_2y_2\right), \ldots, F\left(x_ny_n\right)\right\} \\
&\leq \max\left\{\max\left\{F\left(x_1\right), F\left(y_1\right)\right\}, \ldots, \max\left\{F\left(x_n\right), F\left(y_n\right)\right\}\right\} \\
&= \max\left\{\max\left\{F\left(x_1\right), \ldots, F\left(x_n\right)\right\}, \max\left\{F\left(y_1\right), \ldots, F\left(y_n\right)\right\}\right. \\
&= \max\{V(x), V(y)\} \\
&= V \times V(x,y).
\end{aligned}$$

So, $\langle,\rangle$ is an anti-inner product on $(\underbrace{F \times F \times \ldots \times F}_{n\ times}, R^n)$. □

Theorem 3.4. *If $\langle,\rangle$ is an anti-inner product on the anti-fuzzy linear space (V, X) over the anti-fuzzy field (F, K), then for all $x, y, z \in X$ and $k \in K$.*
(i) $F\langle \underline{x+y}, z\rangle \leq V \times V(x+y, z)$
(ii) $F(\overline{\langle x,y\rangle}) \leq V \times V(y,x)$
(iii) $F(\lambda\langle x,y\rangle) \leq V \times V(\lambda x, y)$.

Proof.

$$\begin{aligned}
\text{(i) } F\langle x+y, z\rangle &= F\{\langle x,z\rangle + \langle y,z\rangle\} \\
&= F\langle x,z\rangle + F\langle y,z\rangle \\
&\leq V \times V(x,z) + V \times V(y,z) \\
&\leq V \times V(x+y, z). \\
\text{(ii) } \quad F(\overline{\langle x,y\rangle}) &= F(\langle y,x\rangle) \\
&\leq V \times V(y,x). \\
\text{(iii) } F(\lambda\langle x,y\rangle) &= F(\langle \lambda x, y\rangle). \\
&\leq V \times V(\lambda x, y).
\end{aligned}$$

□

Theorem 3.5. *If $\langle,\rangle$ is an anti-inner product on the anti-fuzzy linear space (V, X) over the anti-fuzzy field (F, K), then*
(i) $F(\langle x+y, z\rangle) \leq \max\{V(x), V(y), V(z)\}$,
(ii) $F(\langle kx, y\rangle) \leq \max\{F(k), V(x), V(y)\}$.

Proof.

$$\begin{aligned}
\text{(i) } F(\langle x+y, z\rangle) &\leq V \times V(x+y, z) \\
&= \max\{V(x+y), V(z)\} \\
&\leq \max\{\max\{V(x), V(y), V(z)\} \\
&= \max\{V(x), V(y), V(z)\}. \\
\text{(ii) } \quad F(\langle kx, y\rangle) &\leq V \times V(kx, y) \\
&= \max\{V(kx), V(y)\} \\
&\geq \max\{\max\{F(k), V(x), V(y)\} \\
&= \max\{F(k), V(x), V(y)\}.
\end{aligned}$$

□

Theorem 3.6. *Let (V, X) be an anti-fuzzy linear space over an anti-fuzzy field $(F, K), Y$ a linear space over K and T is an isomorphism of X onto Y. Then there exists an anti-inner product on $(V,\ X)$ if and only if there exists an anti-inner product on $(T(V), Y)$.*

Proof. ($\Rightarrow$) Let $\langle , \rangle_X$ be an anti-inner product on (V, X). Consider the anti-inner product $\langle , \rangle_Y$ on Y defined by $\langle y_1, y_2\rangle_Y = \langle x_1, x_2\rangle_X$ where $y_1 = Tx_1$ and $y_2 = Tx_2$.
$F(\langle y_1, y_2\rangle_Y) = F(\langle x_1, x_2\rangle_X) \leq V \times V(x_1, x_2) = T(V) \times T(V)(Tx_1, Tx_2) = T(V) \times T(V)(y_1, y_2)$.

So $\langle , \rangle_Y$ is an anti-inner product on $(T(V), Y)$.

($\Leftarrow$) Assume that $\langle , \rangle_Y$ is an anti-inner product on $(T(V), Y)$. Consider, the anti-inner product $\langle , \rangle_X$ on X defined by $\langle x_1, x_2\rangle_X = \langle Tx_1, Tx_2\rangle_Y$.

$$F\left(\langle x_1, x_2\rangle_X\right) = F\left(\langle Tx_1, Tx_2\rangle_Y \leq T(V) \times T(V)\left(Tx_1, Tx_2\right) = V \times V\left(x_1, x_2\right).$$

So, $\langle , \rangle_X$ is an anti-inner product on (V, X). □

Theorem 3.7. *Let X be a linear space over $K, (W, Y)$ be an anti-fuzzy linear space over an anti-fuzzy field (F, X) and $T : X \to Y$ be an injective linear transformation. If there exists an anti-inner product on (W, Y), then there exists an anti-inner product on $\big(T^{-1}(W), X\big)$.*

Proof. Let $\langle , \rangle_Y$ be an anti-inner product on (W, Y). Consider the anti-inner product $\langle , \rangle_X$ on X defined by $\langle x_1, x_2\rangle_X = \langle Tx_1, Tx_2\rangle_Y$.

$$\begin{aligned} F\left(\langle x_1, x_2\rangle_X\right) = F\left(\langle Tx_1, Tx_2\rangle_Y\right) &\leq W \times W\left(Tx_1, Tx_2\right) = \max\left\{W\left(Tx_1\right), W\left(Tx_2\right)\right\} \\ &= \max\left\{T^{-1}(w)\left(x_1\right), T^{-1}(w)\left(x_2\right)\right\} = T^{-1}(w) \times T^{-1}(w)\left(x_1, x_2\right). \end{aligned}$$

Therefore $\langle , \rangle_X$ is an anti-inner product on $\big(T^{-1}(W), X\big)$. □

Theorem 3.8. *Let (V, X) be an anti-fuzzy linear space over (F, K) and $T : X \to X$ be an injective linear transformation. If there exists an anti-inner product on (V, X) then there exists an anti-inner product on $\big(T^{-1}(V), X\big)$.*

Proof. Let $\langle , \rangle_X$ be an anti-inner product on (V, X). Consider the anti-inner product $\langle , \rangle_X$ on X defined by $\langle x_1, x_2\rangle_X = \langle Tx_1, Tx_2\rangle_X$.

$$\begin{aligned} F\left(\langle x_1, x_2\rangle_X\right) = F\left(\langle Tx_1, Tx_2\rangle_X\right) &\leq V \times V\left(Tx_1, Tx_2\right) = \max\left\{V\left(Tx_1\right), V\left(Tx_2\right)\right\} \\ &= \max\left\{T^{-1}(v)\left(x_1\right), T^{-1}(v)\left(x_2\right)\right\} = T^{-1}(v) \times T^{-1}(v)\left(x_1, x_2\right). \end{aligned}$$

Therefore $\langle , \rangle_X$ is an anti-inner product on $\big(T^{-1}(V), X\big)$. □

Theorem 3.9. *Let (V, X) be an anti-fuzzy linear space over (F, K). An anti-norm on (V, X) satisfying the parallelogram law induces an anti-inner product on (V, X) if $F(4), F(i) \leq V(x)$ for all $x \in X$.*

Proof. If $\|.\|$ is an anti-norm on (V, X) satisfying the parallelogram law, then $\mathrm{F}(\|x\|) \leq V(x)$ for all $x \in X$ and $\|.\|$ induces an anti-inner product $\langle , \rangle$ on X given by

$$\begin{aligned} \langle x, y\rangle &= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right) \\ F(\langle x, y\rangle) &= F\left(\frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)\right) \\ &\leq \max\left\{F\left(\frac{1}{4}\right), F\left(\|x+y\|^2\right), \mathrm{F}\left(-\|x-y\|^2\right), \mathrm{F(i)}, \mathrm{F}\left(\|x+iy\|^2\right), \mathrm{F}\left(-\|x-iy\|^2\right)\right\} \\ &= \max\left\{F(4), \mathrm{F(i)}, F\left(\|x+y\|^2\right), \mathrm{F}\left(\|x-y\|^2\right), \mathrm{F}\left(\|x+iy\|^2\right), \mathrm{F}\left(\|x-iy\|^2\right)\right\} \\ &\leq \max\{F(4), \mathrm{F(i)}, F(\|x+y\|), \mathrm{F}(\|x-y\|), \mathrm{F}(\|x+iy\|), \mathrm{F}(\|x-iy\|)\} \\ &\leq \max\{F(4), F(i), V(x+y), V(x-y), V(x+iy), V(x-iy)\} \\ &\leq \max\{F(4), F(i), V(x), V(y)\} \\ &= \max\{V(x), V(y)\} \text{ if } \mathrm{F(4), F(i)} \leq \mathrm{V(x)} \text{ for all } x \in X \\ &= V \times V(x, y). \end{aligned}$$

Hence anti-norm induces an anti-inner product on (V, X) if $F(4), F(i) \leq V(x)$ for all $x \in X$. □

4 Fuzzy continuous mapping and fuzzy bounded linear operators

In this section we define different types of continuity such as weak fuzzy continuity, strong fuzzy continuity and sequential fuzzy continuity of an operator over anti-normed anti-fuzzy linear spaces. The notion of weakly fuzzy boundedness and strongly fuzzy boundedness are defined for linear operators over anti-normed anti-fuzzy linear spaces.

Definition 4.1. *A mapping T from $(V_1, X, \|.\|_1)$ to $(V_2, Y, \|.\|_2)$ is said to be weakly fuzzy continuous at $x_0 \in X$ if for each $\epsilon > 0, \exists\ \delta > 0$ such that $\forall\ x \in X$*

$\|T(x) - T(x_0)\|_2 < \epsilon$ *whenever* $\|x - x_0\|_1 < \delta$,

and $F\|x_0\|_1 \leq V_1(x_0)$ *and* $F\|Tx_0\|_2 \leq V_2 T(x_0)$.

If T is weakly fuzzy continuous at each point of X then we say that T is weakly fuzzy continuous on X.

Example 4.1. Let $T : (V_1, X, \|.\|_1) \to (V_2, Y, \|.\|_2)$ be a mapping where $(V_1, X, \|.\|_1)$ and $(V_2, Y, \|.\|_2)$ are anti-normed anti-fuzzy linear spaces where $\|x\|_1 = |x|$ and $\|x\|_2 = \frac{|x|}{2}$, and consider $T(x) = x$, here T is weakly fuzzy continuous.

Definition 4.2. *A mapping T from $(V_1, X, \|.\|_1)$ to $(V_2, Y, \|.\|_2)$ is said to be strongly fuzzy continuous at $x_0 \in X$ if for each $\epsilon > 0, \exists\ \delta > 0$ such that $\forall\ x \in X$*

$\|T(x) - T(x_0)\|_2 < \epsilon$ *whenever* $\|x - x_0\|_1 < \delta$,

and $\max\{F\|x_0\|_1, F\|Tx_0\|_2\} \leq \max\{V_1(x_0), V_2(Tx_0)\}$.

If T is strongly fuzzy continuous at each point of X then T is said to be strongly fuzzy continuous on X.

Example 4.2. Let $(V, X, \|.\|)$ be an anti-normed anti-fuzzy linear space where X = R and $\|x\| = |x|\ \forall\ x \in R$. Define two functions $\|.\|_1 \& \|.\|_2 : X \times R \to [0,1]$ by $\|x\|_1 = |x|, \|x\|_2 = 2|x|$.

Then it can be easily verified that $\|x\|_1$ and $\|x\|_2$ are anti-norms on X and thus $(V, X\|x\|_1)$ and $(V, X\|x\|_2)$ are anti-normed anti-fuzzy linear spaces.

Now we consider a function $T(x) = 4x$. Therefore

$$\begin{aligned}
\|Tx - Tx_0\|_2 &= \|4x - 4x_0\|_2 \\
&= 2\,|4x - 4x_0| \\
&= 8\,|x - x_0| < \epsilon \\
&= |x - x_0| < \frac{\epsilon}{8} \\
&\|x - x_0\|_1 = |x - x_0| < \delta, \quad \text{Take } \delta = \frac{\epsilon}{8}. \\
&F(\|x_0\|_1) = F(|x_0|), \\
&F(\|Tx_0\|_2 = F(\|4x_0\|_2) = F(8\,|x_0|), \\
&V_1(x_0) \geq F(\|x_0\|_1) \quad \text{(by Def. 3.1(i))}. \\
&V_2(Tx_0) = V_2(4x_0) \geq F\|Tx_0\|_2 = F(\|4x_0\|_2) = F(8x_0), \\
&V_2(4x_0) \geq F(8x_0). \\
\max\{F(\|x_0\|_1), F(\|Tx_0\|_2)\} &\leq \max\{V_1(x_0), V_2T(x_0)\} \\
&\leq \max\{V_1(x_0), V_2(Tx_0)\}.
\end{aligned}$$

Hence it is strongly fuzzy Continuous.

Definition 4.3. *A mapping T from anti-normed anti-fuzzy linear space $(V_1, \|.\|_1, X)$ to anti-normed anti-fuzzy linear space $(V_2, \|.\|_2, Y)$ over (F, K) is said to be sequentially fuzzy continuous at x_0 if for any sequence $\{x_n\}$ with $x_n \to x_0 \Rightarrow T(x_n) \to T(x_0)$*

$$\begin{aligned}
&\|T(x_n) - T(x_0)\|_2 \to 0 \ \textit{Whenever} \ \|x_n - x_0\|_1 \to 0 \ \textit{and} \\
&F\|x_n - x_0\|_1 \leq V_1(x_n - x_0), \\
&F\|T(x_n) - T(x_0)\|_2 \leq V_2(T(x_n) - T(x_0)).
\end{aligned}$$

If T is sequentially fuzzy continuous at each point of X then T is said to be sequentially fuzzy Continuous on X.

Example 4.3. Let $T : (V_1, X, \|.\|_1) \to (V_2, Y, \|.\|_2)$ be a mapping where $(V_1, X, \|.\|_1)$ and $(V_2, Y\|.\|_2)$ are anti-normed anti-fuzzy linear spaces and $\|x\|_1 = |x|$ and $\|x\|_2 = \frac{|x|}{2}$ and consider the function $T(x) = x$

So whenever $\|x_n - x\|_1 \to 0, \Rightarrow |x_n - x| \to 0$

Then $\|T(x_n) - T(x)\|_2 = \|x_n - x\|_2 = \frac{1}{2}|x_n - x_0| \to 0.$

Also, $F(\|x_n - x_0\|_1) \leq V_1(x_n - x_0)$ (by Def. 3.1. (i)).

$$F(\|Tx_n - Tx_0\|_2) = F(\|x_n - x_0\|_2) = F\left(\frac{|x_n - x_0|}{2}\right) \leq \max\left\{F\left(2^{-1}\right), F(x_n - x_0)\right\}$$
$$= \max\{F(2), F(x_n - x_0)\}.$$

As $n \to \infty$ then $F(x_n - x_0) \to F(0)$

$= \max\{F(2), F(0)\} = F(2)$.

$V_2(Tx_n - Tx_0) = V_2(x_n - x_0)$.

But as $n \to \infty$ then $V_2(x_n - x_0) \to V_2(0)$,

While as $V_2(0) \geq F(2)$

$V_2(Tx_n - Tx_0) \geq F(\|Tx_n - Tx_0\|_2)$.

Hence sequentially fuzzy continuous.

Definition 4.4. *Let us denote the set of all fuzzy bounded linear operators from anti- normed anti-fuzzy linear space* $(V_1, X, \|x\|_1)$ *to* $(V_2, Y, \|x\|_2)$ *by* B(X, Y).

$\|Tx\|_2 \leq k\|x\|_1, \quad V_1(x) \geq F\|x\|_1$

and $\quad V_2(T(x)) \geq F\|x\|_2$.

Example 4.4. Let us take, $\|x\|_1 = |x|, \quad \|x\|_2 = 4|x|$

Define, a linear map $T(x) = \frac{x}{2}$,

Now,

$$\|Tx\|_2 = \left\|\frac{x}{2}\right\|_2 = 2|x|,$$
$$\|Tx\|_2 \leq k|x|, \quad \text{For } k \geq 2.$$
$$V_1(x) \geq F(\|x\|_1), \quad \text{(by Definition 4.4)}$$
$$V_2(x) \geq F(\|x\|_2), \quad \text{(by Definition 4.4).}$$

From above it is clear that, the set $B(X, Y)$ is bounded linear operator on anti-normed anti-fuzzy linear space over anti-fuzzy field.

Theorem 4.1. *Let* $(V_1, X, \|.\|_1)$ *and* $(V_2, Y, \|.\|_2)$ *be two anti-normed anti-fuzzy linear spaces and* T *is a linear operator from* X *to* Y *then*

T *is weakly fuzzy continuous iff it is fuzzy bounded.*

Proof. Let T be fuzzy bounded.

$$\|T(x - x_0)\|_2 \leq k\|x - x_0\|_1$$
$$\|Tx - Tx_0\|_2 \leq \frac{\epsilon}{\delta}\|x - x_0\|_1 \cdot \quad \text{Take } k = \frac{\epsilon}{\delta}.$$
$$\|Tx - Tx_0\|_2 \leq \epsilon, \text{ whenever } \|x - x_0\|_1 \leq \delta.$$

So, T is weakly fuzzy continuous.

Now, T take weakly fuzzy continuous

$\|Tx - Tx_0\|_2 < \epsilon$, whenever $\|x - x_0\|_1 < \delta$.

Let $y \in X$, $x_1 = x_0 + \frac{\delta}{2}\frac{y}{\|y\|_1}$,

$$x_1 - x_0 = \frac{\delta}{2}\frac{y}{\|y\|_1}$$
$$\Rightarrow \|x_1 - x_0\|_1 = \left\|\frac{\delta}{2}\frac{y}{\|y\|_1}\right\|_1 = \frac{\delta}{2} < \delta$$
$$\Rightarrow \|Tx_1 - Tx_0\|_2 < \varepsilon \quad \text{(given)}$$

$$\Rightarrow \|T(x_1 - x_0)\|_2 = \left\| T\frac{\delta}{2}\frac{y}{\|y\|_1} \right\|_2 = \frac{\delta}{2\|y\|_1}\|Ty\|_2 < \varepsilon$$
$$\Rightarrow \|Ty\|_2 \le \frac{2\varepsilon}{\delta}\|y_1\|$$
$$\Rightarrow \|Ty\|_2 \le k\|y_1\| \quad (\text{Taking } k = \frac{2\varepsilon}{\delta}).$$

Therefore T is fuzzy bounded. □

5 Conclusion

In this paper, we developed a theory of anti-norm, anti-inner product on anti-fuzzy linear space over anti-fuzzy field and relation between them. We proved fuzzy continuity theory and their related examples. In the future we will work on open mapping theorem and uniform boundedness principle over anti-fuzzy field.

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CERTAIN INTEGRALS OF PRODUCT OF MITTAG-LEFFLER FUNCTION, M-SERIES AND I-FUNCTION OF TWO VARIABLES

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Abstract

The object of this paper is to establish certain unified integrals associated with I-function of two variables. First, we have evaluated integrals whose integrand is the product of generalized Mittag-Leffler function, generalized M-series and I-function of two variables. Moreover, the integrand of the last integral is the product of generalized Mittag-Leffler function, generalized M-series, H-function of one variables and I-function of two variables. We have evaluated this integral by means of Mellin transform of H-function of one variables. In consequence of general nature of I-function of two variables, some special cases also have been considered.
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1 Introduction

Using various special functions, numerous integrals have been established. For example, in 2003, Garg and Mittal [13] obtained new unified integrals whose integrands contain product of general class of polynomial and H-function having general arguments. Saha *et al.*[19] presented certain new type of integrals having the product of I-function with exponential function, hypergeometric function and H-function in 2011. In 2011, Agarwal *et al.*[1] established some new finite integrals containing Jacobi polynomials and I-function of one variable. In 2019, Agarwal *et al.*[2] established some new integral formulas with the involvement of $\aleph$-function associated with Laguerre-type polynomials. Abeye and Suthar[3] evaluated three definite integrals involving the $\overline{H}$-function together with the Srivastava's general class of polynomial in 2019. Ayant *et al.*[4] established two finite integrals containing the product of Legendre function, generalized hypergeometric function and the modified generalized multivariable I-function in 2020. For similar work, we may also refer to Kumar *et al.*[15], Suthar *et al.* [20], Bohara and Jain [6], Singh and Chandel [32], Goyal and Agrawal [12].

Motivated by these results, in this paper, we have established certain unified integrals associated with two variable's I-function defined by Goyal and Agrawal[11]. In first to sixth integrals, we have evaluated integrals whose integrand is the product of generalized Mittag-Leffler function, generalized M-series and I-function of two variables. The integrand of the seventh integral is the product of generalized Mittag-Leffler function, generalized M-series, H-function of one variables and I-function of two variables. We have evaluated this integral by means of Mellin transform of H-function of one variables. The results of all the integrals are expressed in terms of I-function of two variables.

The results evaluated here are quite general and a large number of known and new integrals can be evaluated as special cases by specializing the parameters in I-function of two variables. For the sake of illustrations, we have recorded some special cases of our main findings at the end of the paper.

The I-function of two variables defined by Goyal and Agrawal[11] in 1995 due to double Mellin-Barnes type contour integral and they discussed the asymptotic behavior and convergence conditions also. The I-function of two variables is very general in nature and specializing the parameters we obtain I-function of one variable, H-function of one variable, H-function of two variables and many more as its special case. For current research of I-function of two variables, see [26, 28].

The I-function of two variables is expressed in the following manner:

(1.1)

$$I^{m_1,n_1:m_2,n_2;m_3,n_3}_{p,q:p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r}\left[\begin{array}{c|c} z_1 & [(e_p:E_p,E_p^{'})]:[(a_\tau,\alpha_\tau)_{1,n_2}],[(a_{\tau i},\alpha_{\tau i})_{n_2+1,p_i^{(1)}}];[(c_\tau,\gamma_\tau)_{1,n_3}],[(c_{\tau i},\gamma_{\tau i})_{n_3+1,p_i^{(2)}}] \\ z_2 & [(f_q:F_q,F_q^{'})]:[(b_\tau,\beta_\tau)_{1,m_2}],[(b_{\tau i},\beta_{\tau i})_{m_2+1,q_i^{(1)}}];[(d_\tau,\delta_\tau)_{1,m_3}],[(d_{\tau i},\delta_{\tau i})_{m_3+1,q_i^{(2)}}] \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}\,d\xi\,d\eta,$$

where $\omega=\sqrt{-1}$ and $\phi_1(\xi)$, $\phi_2(\eta)$, $\psi(\xi,\eta)$ are given by

$$\phi_1(\xi)=\frac{\prod_{\tau=1}^{m_2}\Gamma(b_\tau-\beta_\tau\xi)\prod_{\tau=1}^{n_2}\Gamma(1-a_\tau+\alpha_\tau\xi)}{\sum_{i=1}^{r}\left[\prod_{\tau=m_2+1}^{q_i^{(1)}}\Gamma(1-b_{\tau i}+\beta_{\tau i}\xi)\prod_{\tau=n_2+1}^{p_i^{(1)}}\Gamma(a_{\tau i}-\alpha_{\tau i}\xi)\right]}, \tag{1.2}$$

$$\phi_2(\eta)=\frac{\prod_{\tau=1}^{m_3}\Gamma(d_\tau-\delta_\tau\eta)\prod_{\tau=1}^{n_3}\Gamma(1-c_\tau+\gamma_\tau\eta)}{\sum_{i=1}^{r}\left[\prod_{\tau=m_3+1}^{q_i^{(2)}}\Gamma(1-d_{\tau i}+\delta_{\tau i}\eta)\prod_{\tau=n_3+1}^{p_i^{(2)}}\Gamma(c_{\tau i}-\gamma_{\tau i}\eta)\right]}, \tag{1.3}$$

$$\psi(\xi,\eta)=\frac{\prod_{\tau=1}^{m_1}\Gamma(f_\tau-F_\tau\xi-F_\tau^{'}\eta)\prod_{\tau=1}^{n_1}\Gamma(1-e_\tau+E_\tau\xi+E_\tau^{'}\eta)}{\prod_{\tau=m_1+1}^{q}\Gamma(1-f_\tau+F_\tau\xi+F_\tau^{'}\eta)\prod_{\tau=n_1+1}^{p}\Gamma(e_\tau-E_\tau\xi-E_\tau^{'}\eta)}, \tag{1.4}$$

where an empty product is termed as unity, z_1, z_2 are two non zero complex variables, and L_1, L_2 are two Mellin-Barnes type contour integrals.

(i) $m_1,n_1;m_2,n_2;m_3,n_3$ and $p,q;\ p_i^{(1)},q_i^{(1)};\ p_i^{(2)},q_i^{(2)}$ are non-negative integers satisfying the conditions $0\le n_1\le p$, $0\le n_2\le p_i^{(1)}$, $0\le n_3\le p_i^{(2)}$, $0\le m_1\le q$, $0\le m_2\le q_i^{(1)}$, $0\le m_3\le q_i^{(2)}$ for all $i=1,2,3,\dots,r$ where r is also a positive integer.

(ii) $\alpha_\tau(\tau=1,\dots,n_2),\beta_\tau(\tau=1,\dots,m_2),\gamma_\tau(\tau=1,\dots,n_3),\delta_\tau(\tau=1,\dots,m_3),\alpha_{\tau i}(\tau=n_2+1,\dots,p_i^{(1)})$, $\beta_{\tau i}(\tau=m_2+1,\dots,q_i^{(1)}),\gamma_{\tau i}(\tau=n_3+1,\dots,p_i^{(2)}),\delta_{\tau i}(\tau=m_3+1,\dots,q_i^{(2)})$ are termed to be positive quantities for standardization purposes.$E_\tau,E_\tau^{'},F_\tau,F_\tau^{'}$ are also positive quantities.

(iii) $a_\tau(\tau=1,\dots,n_2),b_\tau(\tau=1,\dots,m_2),c_\tau(\tau=1,\dots,n_3),d_\tau(\tau=1,\dots,m_3),a_{\tau i}(\tau=n_2+1,\dots,p_i^{(1)}),b_{\tau i}(\tau=m_2+1,\dots,q_i^{(1)}),c_{\tau i}(\tau=n_3+1,\dots,p_i^{(2)}),d_{\tau i}(\tau=m_3+1,\dots,q_i^{(2)})$ are complex for all $i=1,2,3,\dots,r$.

(iv) The contour L_1 lies in the complex ξ-plane which runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(b_\tau-\beta_\tau\xi)(\tau=1,\dots,m_2),\Gamma(f_\tau-F_\tau\xi-F_\tau^{'}\eta)(\tau=1,\dots,m_1)$ lies to the right and the poles of $\Gamma(1-a_\tau+\alpha_\tau\xi)(\tau=1,\dots,n_2),\Gamma(1-e_\tau+E_\tau\xi+E_\tau^{'}\eta)(\tau=1,\dots,n_1)$ to the left of the contour L_1.

(v) The contour L_2 lies in the complex η-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(d_\tau-\delta_\tau\eta)(\tau=1,\dots,m_3),\Gamma(f_\tau-F_\tau\xi-F_\tau^{'}\eta)(\tau=1,\dots,m_1)$ lies to the right and the poles of $\Gamma(1-c_\tau+\gamma_\tau\xi)(\tau=1,\dots,n_3),\Gamma(1-e_\tau+E_\tau\xi+E_\tau^{'}\eta)(\tau=1,\dots,n_1)$ to the left of the contour L_2. All the poles are simple poles.

Convergence conditions are as follows:

$$|\arg\,z_1|<\frac{A_i\pi}{2},\ |\arg\,z_2|<\frac{B_i\pi}{2}, \tag{1.5}$$

where

$$A_i = \sum_{\tau=1}^{n_1} E_\tau + \sum_{\tau=1}^{m_1} F_\tau - \sum_{\tau=n_1+1}^{p} E_\tau - \sum_{\tau=m_1+1}^{q} F_\tau + \sum_{\tau=1}^{m_2} \beta_\tau + \sum_{\tau=1}^{n_2} \alpha_\tau - \sum_{\tau=m_2+1}^{q_i^{(1)}} \beta_{\tau i} - \sum_{\tau=n_2+1}^{p_i^{(1)}} \alpha_{\tau i} > 0 \tag{1.6}$$

and

$$B_i = \sum_{\tau=1}^{n_1} E_\tau^{'} - \sum_{\tau=n_1+1}^{p} E_\tau^{'} + \sum_{\tau=1}^{m_1} F_\tau^{'} - \sum_{\tau=m_1+1}^{q} F_\tau^{'} + \sum_{\tau=1}^{m_3} \delta_\tau - \sum_{\tau=m_3+1}^{q_i^{(2)}} \delta_{\tau i} + \sum_{\tau=1}^{n_3} \gamma_\tau - \sum_{\tau=n_3+1}^{p_i^{(2)}} \gamma_{\tau i} > 0, \tag{1.7}$$

for $i = 1, \ldots, r$.

For the sake of brevity throughout the paper, following notations will be used:
$P = m_2, n_2; m_3, n_3,$
$Q = p_i^{(1)}, q_i^{(1)}; p_i^{(2)}, q_i^{(2)} : r,$
$U = [(a_\tau, \alpha_\tau)_{1,n_2}], [(a_{\tau i}, \alpha_{\tau i})_{n_2+1, p_i^{(1)}}]; [(c_\tau, \gamma_\tau)_{1,n_3}], [(c_{\tau i}, \gamma_{\tau i})_{n_3+1, p_i^{(2)}}],$
$V = [(b_\tau, \beta_\tau)_{1,m_2}], [(b_{\tau i}, \beta_{\tau i})_{m_2+1, q_i^{(1)}}]; [(d_\tau, \delta_\tau)_{1,m_3}], [(d_{\tau i}, \delta_{\tau i})_{m_3+1, q_i^{(2)}}],$

The generalized Mittag-Leffler function $E_{\theta,\varphi}(z)$ is a complex function involving two complex parameters θ and φ. It is defined by means of following series when $\Re(\theta)$ is strictly positive

$$E_{\theta,\varphi}(z) = \sum_{k\geq 0} \frac{z^k}{\Gamma(\theta k + \varphi)}. \tag{1.8}$$

If θ and φ are positive and real, the function converges for all z. By specializing the parameters, Mittag-Leffler function reduces to the exponential function, error function, hyperbolic sine function, hyperbolic cosine function.

This function was studied by Wiman [33] in 1905, Agrawal [5] in 1953, Humbert and Agrawal [14] in 1953 and Dzrbashjan [8, 9, 10]. Kilbas et al. [16] studied the several properties of the Mittag-Leffler function related to the generalized fractional calculus operators.

During the last some decades, the special importance to Mittag-Leffler function is given by the mathematicians due to its vast and vivid involvement to solve the problems of probability, engineering and statistical distribution theory. The solution of fractional order differential and integral equations occurs naturally in terms of Mittag-Leffler function.

A detailed description about the basic properties of Mittag-Leffler function has been described in the third volume of Batemann Manuscript Project which was written by Erdélyi et al in 1955. For current research of Mittag-Leffler function, see [29].

Sharma and Jain [24] introduced generalized M-series which is defined as

$$\begin{aligned} {}_pM_q^{\theta,\varphi}(z) &= {}_pM_q^{\theta,\varphi}(c_1, \ldots, c_p; d_1, \ldots, d_q; z) \\ &= \sum_{k\geq 0} \frac{(c_1)_k \ldots (c_p)_k}{(d_1)_k \ldots (d_q)_k} \frac{z^k}{\Gamma(\theta k + \varphi)}, \end{aligned} \tag{1.9}$$

where $\theta, \varphi \in \mathbb{C}$, $z \in \mathbb{C}$, $\Re(\theta) > 0$; $(c_\tau)_k$ $(\tau = 1, \ldots, p)$ and $(d_\varsigma)_k$ $(\varsigma = 1, \ldots, q)$ are Pochhammer symbols. The series (1.9) is defined when no parameters d_ς $(\varsigma = 1, \ldots, q)$ is a negative integer or zero; if any numerator parameter c_τ is a negative integer or zero, then series terminates to a polynomial in z. The series (1.9) is convergent for all z if $p \leq q$; it is convergent for $|z| < \delta = \theta^\theta$ if $p = q + 1$ and divergent if $p > q + 1$. When $p = q + 1$ and $|z| = \delta$, the series is convergent on conditions depending on the parameters. The detailed description of the M-Series can be seen in the paper [24]. The M-series has interesting relationship with various classical functions, for instance, see [25, 27, 30].

2 Required Results

We require following results for our study.

In view of Mellin inversion theorem and using the definition of H-function, The Mellin transform of H-function is given by

$$\int_0^\infty x^{s-1} H_{p,q}^{m,n}\left[ax \middle| \begin{matrix} (a_\tau, \alpha_\tau)_{1,p} \\ (b_\tau, , \beta_\tau)_{1,q} \end{matrix}\right] dx = a^{-s}\chi(-s) \tag{2.1}$$

$$= a^{-s} \frac{\prod_{\tau=1}^{m} \Gamma(b_\tau + \beta_\tau s) \prod_{\tau=1}^{n} \Gamma(1 - a_\tau - \alpha_\tau s)}{\prod_{\tau=m+1}^{q} \Gamma(1 - b_\tau - \beta_\tau s) \prod_{\tau=n+1}^{p} \Gamma(a_\tau + \alpha_\tau s)},$$

where

$$|\arg a| < \frac{\pi A}{2},\ \delta = -\sum_{\tau=1}^{p} \alpha_\tau + \sum_{\tau=1}^{q} \beta_\tau > 0,\ A > 0,$$

$$A = \sum_{\tau=1}^{n} \alpha_\tau + \sum_{\tau=1}^{m} \beta_\tau - \sum_{\tau=n+1}^{p} \alpha_\tau - \sum_{\tau=m+1}^{q} \beta_\tau > 0,$$

and

$$-\min_{1\le\tau\le m} \Re\left(\frac{b_\tau}{\beta_\tau}\right) < \Re(s) < \min_{1\le\tau\le n} \Re\left(\frac{1-a_\tau}{\alpha_\tau}\right).$$

From Rainville [18], we have

$$\sum_{f\ge0}\sum_{u\ge0} A(u,f) = \sum_{f\ge0}\sum_{u=0}^{f} A(u, f-u), \tag{2.2}$$

$$\int_{-1}^{1} (1+x)^{\varsigma-1}(1-x)^{e-1}\, dx = 2^{\varsigma+e-1} B(\varsigma, e),\ \ \varsigma > 0, e > 0. \tag{2.3}$$

3 Main Results

In this section, we evaluate certain type of new unified integrals with the involvement of the product of I-function of two variables with generalized Mittag-Leffler function, generalized M-series and Fox's H-function.

Result 3.1.

$$I_1 \equiv \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1} E_{\mu,\lambda}\{(t-x)z\}\, {}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\} \tag{3.1}$$

$$\times I_{p,q:Q}^{m_1,n_1:P}\left[\begin{matrix} z_1 x^{\mu_1}(t-x)^{\nu_1} \\ z_2 x^{\mu_2}(t-x)^{\nu_2} \end{matrix} \middle| \begin{matrix} [(e_p : E_p, E_p^{'})] : U \\ [(f_q : F_q, F_q^{'})] : V \end{matrix}\right] dx$$

$$= t^{\rho_1+\sigma_1-1} \sum_{k\ge0}\sum_{m=0}^{k} f(m) z^{k-m} t^{(\rho_2+\sigma_2-1)m+k}$$

$$\times I_{p+2,q+1:Q}^{m_1,n_1+2:P}\left[\begin{matrix} z_1 t^{\mu_1+\nu_1} \\ z_2 t^{\mu_2+\nu_2} \end{matrix} \middle| \begin{matrix} E_1, [(e_p : E_p, E_p^{'})] : U \\ [(f_q : F_q, F_q^{'})], E_2 : V \end{matrix}\right],$$

where

$$E_1 = [(1-\rho_1-\rho_2 m : \mu_1, \mu_2)], [(1-\sigma_1-(\sigma_2-1)m-k : \nu_1, \nu_2)],$$
$$E_2 = [(1-\sigma_1-\rho_1-(\rho_2+\sigma_2-1)m-k : \mu_1+\nu_1, \mu_2+\nu_1)],$$

and

$$f(m) = \frac{a^m (a_1')_m \cdots (a_u')_m}{(b_1')_m \cdots (b_u')_m \Gamma(\mu(k-m)+\lambda)\Gamma(Gm+T)},$$

provided

(i) $\Re(\mu) > 0$, $\Re(\lambda) > 0$, $\Re(G) > 0$, $\Re(T) > 0$,
(ii) $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\nu_1 \geq 0$, $\nu_2 \geq 0$. (Not all zero simultaneously),
(iii) ρ_2, σ_2 are positive integers such that $\rho_2 + \sigma_2 \geq 1$,
(iv) $A_i > 0$, $B_i > 0$, $|\arg z_1| < \frac{\pi A_i}{2}$, $|\arg z_2| < \frac{\pi B_i}{2}$,
(v) $\Re(\rho_1) + \mu_1 \min\limits_{1\leq\tau\leq m_2} \Re\left(\frac{b_\tau}{\beta_\tau}\right) + \mu_2 \min\limits_{1\leq\tau\leq m_3} \Re\left(\frac{d_\tau}{\delta_\tau}\right) > 0,$

$$\Re(\sigma_1) + \nu_1 \min_{1\leq\tau\leq m_2} \Re\left(\frac{b_\tau}{\beta_\tau}\right) + \nu_2 \min_{1\leq\tau\leq m_3} \Re\left(\frac{d_\tau}{\delta_\tau}\right) > 0.$$

Proof.

$$I_1 \equiv \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1} E_{\mu,\lambda}\{(t-x)z\}\, {}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\}$$
$$\times I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c} z_1 x^{\mu_1}(t-x)^{\nu_1} \\ z_2 x^{\mu_2}(t-x)^{\nu_2} \end{array}\middle|\begin{array}{c} [(e_p : E_p, E_p^{'})] : U \\ [(f_q : F_q, F_q^{'})] : V \end{array}\right] dx.$$

Now expressing Mittag-Leffler function and M-series in summation form and I-function in its well known Mellin-Barnes contour integral, we get

$$I_1 = \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1} \sum_{k\geq0} \frac{(t-x)^k z^k}{\Gamma(\mu k+\lambda)} \sum_{m\geq0} \frac{(a_1')_m \cdots (a_u')_m}{(b_1')_m \cdots (b_v')_m} \frac{a^m x^{\rho_2 m}(t-x)^{\sigma_2 m}}{\Gamma(Gm+T)}$$

$$\times \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^\xi\, z_2^\eta x^{\mu_1\xi} x^{\mu_2\eta}(t-x)^{\nu_1\xi}(t-x)^{\nu_2\eta}\, d\xi\, d\eta\, dx$$
$$= \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1} \sum_{k\geq0}\sum_{m\geq0} \frac{(a_1')_m \cdots (a_u')_m}{(b_1')_m \cdots (b_v')_m} \frac{z^k a^m x^{\rho_2 m}}{\Gamma(\mu k+\lambda)} \frac{(t-x)^{\sigma_2 m+k}}{\Gamma(Gm+T)}$$
$$\times \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^\xi\, z_2^\eta x^{\mu_1\xi+\mu_2\eta}(t-x)^{\nu_1\xi+\nu_2\eta}\, d\xi\, d\eta\, dx.$$

Now by an application of (2.2), the above result turns to

$$I_1 = \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1} \sum_{k\geq0}\sum_{m=0}^{k} \frac{(a_1')_m \cdots (a_u')_m}{(b_1')_m \cdots (b_v')_m} \frac{z^{k-m} a^m x^{\rho_2 m}}{\Gamma(\mu(k-m)+\lambda)} \frac{(t-x)^{\sigma_2 m+k-m}}{\Gamma(Gm+T)}$$
$$\times \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^\xi\, z_2^\eta x^{\mu_1\xi+\mu_2\eta}(t-x)^{\nu_1\xi+\nu_2\eta}\, d\xi\, d\eta\, dx.$$

Changing the order of integral and summation which is valid due to the conditions mentioned with the equation (3.1), we obtain

$$I_1 = \sum_{k\geq0}\sum_{m=0}^{k} f(m) z^{k-m} \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^\xi\, z_2^\eta$$
$$\times \left\{ \int_0^t x^{\mu_1\xi+\mu_2\eta+\rho_2 m+\rho_1-1}(t-x)^{\nu_1\xi+\nu_2\eta+\sigma_2 m+k-m+\sigma_1-1}\, dx \right\} d\xi\, d\eta,$$

where $f(m)$ is given with the integral (3.1).

On putting $x = st$ in the x-integral, the above expression becomes

$$I_1 = t^{\rho_1+\sigma_1-1} \sum_{k\geq0}\sum_{m=0}^{k} f(m) z^{k-m} t^{(\rho_2+\sigma_2-1)m+k} \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^\xi\, z_2^\eta$$

$$\times\, t^{(\mu_1+\nu_1)\xi+(\mu_2+\nu_2)\eta}\left\{\int_0^1 s^{\mu_1\xi+\mu_2\eta+\rho_2 m+\rho_1-1}(1-s)^{\nu_1\xi+\nu_2\eta+\sigma_2 m+k-m+\sigma_1-1}\,ds\right\}d\xi\,d\eta$$

$$= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m) z^{k-m} t^{(\rho_2+\sigma_2-1)m+k}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}$$

$$\times\frac{\Gamma(\mu_1\xi+\mu_2\eta+\rho_2 m+\rho_1)\Gamma(\nu_1\xi+\nu_2\eta+\sigma_2 m+k-m+\sigma_1)}{\Gamma(\mu_1\xi+\mu_2\eta+\rho_2 m+\rho_1+\nu_1\xi+\nu_2\eta+\sigma_2 m+k-m+\sigma_1)}t^{(\mu_1+\nu_1)\xi+(\mu_2+\nu_2)\eta}\,d\xi\,d\eta,$$

Finally, by re-arranging the double Mellin-Barnes contour integrals by means of I-function of two variables represented by (1.1) , we get

$$I_1 = t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m) z^{k-m} t^{(\rho_2+\sigma_2-1)m+k}$$
$$\times\, I^{m_1,n_1+2:P}_{p+2,q+1:Q}\left[\begin{array}{c|c} z_1 t^{\mu_1+\nu_1} & E_1,[(e_p:E_p,E_p^{'})]:U \\ z_2 t^{\mu_2+\nu_2} & [(f_q:F_q,F_q^{'})],E_2:V\end{array}\right],$$

where E_1 and E_2 are given with (3.1). Hence the desired result.

Result 3.2.

$$I_2 \equiv \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\} \tag{3.2}$$
$$\times\, I^{m_1,n_1:P}_{p,q:Q}\left[\begin{array}{c|c} z_1 x^{-\mu_1}(t-x)^{-\nu_1} & [(e_p:E_p,E_p^{'})]:U \\ z_2 x^{-\mu_2}(t-x)^{-\nu_2} & [(f_q:F_q,F_q^{'})]:V\end{array}\right]dx$$
$$= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m) z^{k-m} t^{(\rho_2+\sigma_2-1)m+k}$$
$$\times\, I^{m_1+2,n_1:P}_{p+1,q+2:Q}\left[\begin{array}{c|c} z_1 t^{-\mu_1-\nu_1} & [(e_p:E_p,E_p^{'})],E_3:U \\ z_2 t^{-\mu_2-\nu_2} & E_4,[(f_q:F_q,F_q^{'})]:V\end{array}\right],$$

where

$$E_3 = [((\rho_2+\sigma_2-1)m+\rho_1+\sigma_1+k:\mu_1+\nu_1,\mu_2+\nu_2)],$$
$$E_4 = [(\rho_1+\rho_2 m:\mu_1,\mu_2)],[((\sigma_2-1)m+\sigma_1+k:\nu_1,\nu_2)],$$

provided

$$\Re(\rho_1)-\left[\mu_1\max_{1\leq\tau\leq n_2}\Re\left(\frac{a_\tau-1}{\alpha_\tau}\right)+\mu_2\max_{1\leq\tau\leq n_3}\Re\left(\frac{c_\tau-1}{\gamma_\tau}\right)\right]>0,$$
$$\Re(\sigma_1)-\left[\nu_1\max_{1\leq\tau\leq n_2}\Re\left(\frac{a_\tau-1}{\alpha_\tau}\right)+\nu_2\max_{1\leq\tau\leq n_3}\Re\left(\frac{c_\tau-1}{\gamma_\tau}\right)\right]>0,$$

and also satisfies the conditions (i) to (iv) (3.1) and $f(m)$ is given with (3.1).

Result 3.3.

$$I_3 \equiv \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\} \tag{3.3}$$
$$\times\, I^{m_1,n_1:P}_{p,q:Q}\left[\begin{array}{c|c} z_1 x^{\mu_1}(t-x)^{-\nu_1} & [(e_p:E_p,E_p^{'})]:U \\ z_2 x^{\mu_2}(t-x)^{-\nu_2} & [(f_q:F_q,F_q^{'})]:V\end{array}\right]dx$$
$$= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m) z^{k-m} t^{(\rho_2+\sigma_2-1)m+k}$$
$$\times\, I^{m_1+1,n_1+1:P}_{p+1,q+2:Q}\left[\begin{array}{c|c} z_1 t^{\mu_1-\nu_1} & E_5,[(e_p:E_p,E_p^{'})]:U \\ z_2 t^{\mu_2-\nu_2} & E_6,[(f_q:F_q,F_q^{'})],E_7:V\end{array}\right],$$

where

$$E_5 = [(1-\rho_1-\rho_2 m:\mu_1,\mu_2)],$$
$$E_6 = [(\sigma_1+(\sigma_2-1)m+k:\nu_1,\nu_2)],$$
$$E_7 = [(1-\rho_1-\sigma_1-(\rho_2+\sigma_2-1)m-k:\mu_1-\nu_1,\mu_2-\nu_2)],$$

provided $\mu_1>0,\mu_2>0,\nu_1\geq 0,\nu_2\geq 0$ such that $\mu_1-\nu_1\geq 0,\mu_2-\nu_2\geq 0$,

$$\Re(\rho_1)+\mu_1\min_{1\leq\tau\leq m_2}\Re\left(\frac{b_\tau}{\beta_\tau}\right)+\mu_2\min_{1\leq\tau\leq m_3}\Re\left(\frac{d_\tau}{\delta_\tau}\right)>0,$$

$$\Re(\sigma_1)-\left[\nu_1\max_{1\leq\tau\leq n_2}\Re\left(\frac{a_\tau-1}{\alpha_\tau}\right)+\nu_2\max_{1\leq\tau\leq n_3}\Re\left(\frac{c_\tau-1}{\gamma_\tau}\right)\right]>0,$$

and also satisfies the conditions (i) to (iv) of (3.1) and $f(m)$ is given with (3.1).

Result 3.4.

$$\begin{aligned}(3.4)\qquad I_4 &\equiv \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\}\\ &\quad\times I^{m_1,n_1:P}_{p,q:Q}\left[\begin{array}{c} z_1x^{\mu_1}(t-x)^{-\nu_1}\\ z_2x^{\mu_2}(t-x)^{-\nu_2}\end{array}\middle|\begin{array}{c}[(e_p:E_p,E_p^{'})]:U\\ [(f_q:F_q,F_q^{'})]:V\end{array}\right]dx\\ &= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k}f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k}\\ &\quad\times I^{m_1+1,n_1+1:P}_{p+2,q+1:Q}\left[\begin{array}{c} z_1t^{\mu_1-\nu_1}\\ z_2t^{\mu_2-\nu_2}\end{array}\middle|\begin{array}{c}E_8,[(e_p:E_p,E_p^{'})],E_9:U\\ E_{10},[(f_q:F_q,F_q^{'})]:V\end{array}\right],\end{aligned}$$

where

$$E_8 = [(1-\rho_1-\rho_2 m:\mu_1,\mu_2)],$$
$$E_9 = [(\rho_1+\sigma_1+k+(\rho_2+\sigma_2-1)m:\nu_1-\mu_1,\nu_2-\mu_2)],$$
$$E_{10} = [(\sigma_1+(\sigma_2-1)m+k:\nu_1,\nu_2)],$$

provided $\mu_1\geq 0,\mu_2\geq 0,\nu_1>0,\nu_2>0$ such that $\nu_1-\mu_1\geq 0,\nu_2-\mu_2\geq 0$,

$$\Re(\rho_1)+\mu_1\min_{1\leq\tau\leq m_2}\Re\left(\frac{b_\tau}{\beta_\tau}\right)+\mu_2\min_{1\leq\tau\leq m_3}\Re\left(\frac{d_\tau}{\delta_\tau}\right)>0,$$

$$\Re(\sigma_1)-\nu_1\max_{1\leq\tau\leq n_2}\Re\left(\frac{a_\tau-1}{\alpha_\tau}\right)-\nu_2\max_{1\leq\tau\leq n_3}\Re\left(\frac{c_\tau-1}{\gamma_\tau}\right)>0,$$

and also satisfies the conditions (i) to (iv) of (3.1) and $f(m)$ is given with (3.1).

Result 3.5.

$$\begin{aligned}(3.5)\qquad I_5 &\equiv \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\}\\ &\quad\times I^{m_1,n_1:P}_{p,q:Q}\left[\begin{array}{c} z_1x^{-\mu_1}(t-x)^{\nu_1}\\ z_2x^{-\mu_2}(t-x)^{\nu_2}\end{array}\middle|\begin{array}{c}[(e_p:E_p,E_p^{'})]:U\\ [(f_q:F_q,F_q^{'})]:V\end{array}\right]dx\\ &= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k}f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k}\\ &\quad\times I^{m_1+1,n_1+1:P}_{p+2,q+1:Q}\left[\begin{array}{c} z_1t^{-\mu_1+\nu_1}\\ z_2t^{-\mu_2+\nu_2}\end{array}\middle|\begin{array}{c}E_{11},[(e_p:E_p,E_p^{'})],E_{12}:U\\ E_{13},[(f_q:F_q,F_q^{'})]:V\end{array}\right],\end{aligned}$$

where

$$E_{11} = [(1-\sigma_1-(\sigma_2-1)m-k:\nu_1,\nu_2)],$$

$$E_{12} = [(\rho_1 + \sigma_1 + (\rho_2 + \sigma_2 - 1)m + k : \mu_1 - \nu_1, \mu_2 - \nu_2)],$$
$$E_{13} = [(\rho_1 + \rho_2 m : \mu_1, \mu_2)],$$

provided $\mu_1 > 0, \mu_2 > 0, \nu_1 \geq 0, \nu_2 \geq 0$ such that $\mu_1 - \nu_1 \geq 0, \mu_2 - \nu_2 \geq 0$,

$$\Re(\rho_1) - \mu_1 \max_{1\leq\tau\leq n_2} \Re\left(\frac{a_\tau - 1}{\alpha_\tau}\right) - \mu_2 \max_{1\leq\tau\leq n_3} \Re\left(\frac{c_\tau - 1}{\gamma_\tau}\right) > 0,$$

$$\Re(\sigma_1) + \nu_1 \min_{1\leq\tau\leq m_2} \Re\left(\frac{b_\tau}{\beta_\tau}\right) + \nu_2 \min_{1\leq\tau\leq m_3} \Re\left(\frac{d_\tau}{\delta_\tau}\right) > 0,$$

and also satisfies the conditions (i) to (iv) of (3.1) and $f(m)$ is given with (3.1).
Result 3.6.

$$I_6 \equiv \int_0^t x^{\rho_1 - 1}(t-x)^{\sigma_1 - 1} E_{\mu,\lambda}\{(t-x)z\}\, {}_u M_v^{G,T}\{a x^{\rho_2}(t-x)^{\sigma_2}\} \tag{3.6}$$
$$\times I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c} z_1 x^{-\mu_1}(t-x)^{\nu_1} \\ z_2 x^{-\mu_2}(t-x)^{\nu_2} \end{array}\middle| \begin{array}{c} [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')] : V \end{array}\right] dx$$

$$= t^{\rho_1+\sigma_1-1} \sum_{k\geq 0} \sum_{m=0}^{k} f(m) z^{k-m} t^{(\rho_2+\sigma_2-1)m+k}$$
$$\times I_{p+1,q+2:Q}^{m_1+1,n_1+1:P}\left[\begin{array}{c} z_1 t^{-\mu_1+\nu_1} \\ z_2 t^{-\mu_2+\nu_2} \end{array}\middle| \begin{array}{c} E_{14}, [(e_p : E_p, E_p')] : U \\ E_{15}, [(f_q : F_q, F_q')], E_{16} : V \end{array}\right],$$

where

$$E_{14} = [(1 - \sigma_1 - (\sigma_2 - 1)m - k : \nu_1, \nu_2)],$$
$$E_{15} = [(\rho_1 + \rho_2 m : \mu_1, \mu_2)],$$
$$E_{16} = [(1 - \rho_1 - \sigma_1 - (\rho_2 + \sigma_2 - 1)m - k : \nu_1 - \mu_1, \nu_2 - \mu_2)],$$

provided $\mu_1 \geq 0, \mu_2 \geq 0, \nu_1 > 0, \nu_2 > 0$ such that $\nu_1 - \mu_1 \geq 0, \nu_2 - \mu_2 \geq 0$,

$$\Re(\rho_1) - \left[\mu_1 \max_{1\leq\tau\leq n_2} \Re\left(\frac{a_\tau - 1}{\alpha_\tau}\right) + \mu_2 \max_{1\leq\tau\leq n_3} \Re\left(\frac{c_\tau - 1}{\gamma_\tau}\right)\right] > 0,$$

$$\Re(\sigma_1) + \nu_1 \min_{1\leq\tau\leq m_2} \Re\left(\frac{b_\tau}{\beta_\tau}\right) + \nu_2 \min_{1\leq\tau\leq m_3} \Re\left(\frac{d_\tau}{\delta_\tau}\right)\Bigg] > 0,$$

and also satisfies the conditions (i) to (iv) of (3.1) and $f(m)$ is given with (3.1).

The integrals (3.2) to (3.6) can be established on similar lines as of integral (3.1).
Result 3.7.

$$I_7 \equiv \int_0^\infty x^{l-1} E_{\mu,\lambda}(ax)\, {}_u M_v^{G,T}(a x^{\rho}) I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c} z_1 x^{\sigma_1} \\ z_2 x^{\sigma_2} \end{array}\middle| \begin{array}{c} [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')] : V \end{array}\right] \tag{3.7}$$
$$\times H_{p',q'}^{m,n}\left[wx \middle| \begin{array}{c} (c_\tau',\, \gamma_\tau')_{1,n}, (c_\tau',\, \gamma_\tau')_{n+1,p'} \\ (d_\tau',\, \delta_\tau')_{1,m}, (d_\tau',\, \delta_\tau')_{m+1,q'} \end{array}\right] dx$$

$$= w^{-l} \sum_{k\geq 0} \sum_{m=0}^{k} f(m) w^{-(\rho-1)m-k} I_{p+q',q+p':Q}^{m_1+n,n_1+m:P}\left[\begin{array}{c} z_1 w^{-\sigma_1} \\ z_2 w^{-\sigma_2} \end{array}\middle| \begin{array}{c} E_{17}, [(e_p : E_p, E_p')], E_{18} : U \\ E_{19}[(f_q : F_q, F_q')], E_{20} : V \end{array}\right],$$

where

$$E_{17} = [(1 - d_\tau' - \delta_\tau'(k + (\rho - 1)m + l) : \sigma_1\delta_\tau', \sigma_2\delta_\tau')_{1,m}],$$
$$E_{18} = [(1 - d_\tau' - \delta_\tau'(k + (\rho - 1)m + l) : \sigma_1\delta_\tau', \sigma_2\delta_\tau')_{m+1,q'}],$$
$$E_{19} = [(1 - c_\tau' - \gamma_\tau'(k + (\rho - 1)m + l) : \sigma_1\gamma_\tau', \sigma_2\gamma_\tau')_{1,n}],$$
$$E_{20} = [(1 - c_\tau' - \gamma_\tau'(k + (\rho - 1)m + l) : \sigma_1\gamma_\tau', \sigma_2\gamma_\tau')_{n+1,p'}],$$

and

$$f(m)=\frac{a^k(a'_1)_m...(a'_u)_m}{(b'_1)_m...(b'_u)_m\Gamma(\mu(k-m)+\lambda)\Gamma(Gm+T)},$$

provided

(i) $\Re(\mu)>0$, $\Re(\lambda)>0$, $\Re(G)>0$, $\Re(T)>0$,

(ii) $A_i>0, |\arg\, z_1|<\frac{\pi A_i}{2}$,

(iii) $B_i>0, |\arg\, z_2|<\frac{\pi B_i}{2}$,

(iv) $\Delta>0, |\arg\, w|<\frac{\pi\Delta}{2}$,

(v) $\Delta\geq 0, |\arg\, w|\leq\frac{\pi\Delta}{2}, \Re(\Omega+1)<0$,

(vi) $\sigma_1>0, \sigma_2>0, -\sigma_1\min\limits_{1\leq\tau\leq m_2}\Re\left(\frac{b_\tau}{\beta_\tau}\right)-\sigma_2\min\limits_{1\leq\tau\leq m_3}\Re\left(\frac{d_\tau}{\delta_\tau}\right)-\min\limits_{1\leq\tau\leq m}\Re\left(\frac{d'_\tau}{\delta'_\tau}\right),$

$$<\Re(l)<\sigma_1\min_{1\leq\tau\leq n_2}\Re\left(\frac{1-a_\tau}{\alpha_\tau}\right)+\sigma_2\min_{1\leq\tau\leq n_3}\Re\left(\frac{1-c_\tau}{\gamma_\tau}\right)+\min_{1\leq\tau\leq n}\Re\left(\frac{1-c'_\tau}{\gamma'_\tau}\right),$$

where

$$\Delta=\sum_{\tau=1}^{m}\delta'_\tau+\sum_{\tau=1}^{n}\gamma'_\tau-\sum_{\tau=m+1}^{q'}\delta'_\tau-\sum_{\tau=n+1}^{p'}\gamma'_\tau,$$

$$\Omega=\frac{1}{2}(p'-q')+\sum_{\tau=1}^{q'}d'_\tau-\sum_{\tau=1}^{p'}c'_\tau,$$

A_i and B_i are same as given in (1.6) and (1.7).

Proof.

$$I_7\equiv\int_0^\infty x^{l-1}E_{\mu,\lambda}(ax)\,{}_uM_v^{G,T}(ax^\rho)I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1x^{\sigma_1} & [(e_p:E_p,E'_p)]:U\\ z_2x^{\sigma_2} & [(f_q:F_q,F'_q)]:V\end{array}\right]$$
$$\times H_{p',q'}^{m,n}\left[wx\left|\begin{array}{c}(c'_\tau,\,\gamma'_\tau)_{1,n},(c'_\tau,\,\gamma'_\tau)_{n+1,p'}\\ (d'_\tau,\,\delta'_\tau)_{1,m},(d'_\tau,\,\delta'_\tau)_{m+1,q'}\end{array}\right.\right]dx,$$

now expressing Mittag-Leffler function and M-series in summation form and I-function in its well known Mellin-Barnes contour integral, we get

$$I_7=\int_0^\infty x^{l-1}\sum_{k\geq 0}\frac{a^kx^k}{\Gamma(\mu k+\lambda)}\sum_{m\geq 0}\frac{(a'_1)_m\cdots(a'_u)_m}{(b'_1)_m\cdots(b'_v)_m}\frac{a^mx^{\rho m}}{\Gamma(Gm+T)}$$
$$\times\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^\xi\,z_2^\eta x^{\sigma_1\xi}x^{\sigma_2\eta}$$
$$\times H_{p',q'}^{m,n}\left[wx\left|\begin{array}{c}(c'_\tau,\,\gamma'_\tau)_{1,n},(c'_\tau,\,\gamma'_\tau)_{n+1,p'}\\ (d'_\tau,\,\delta'_\tau)_{1,m},(d'_\tau,\,\delta'_\tau)_{m+1,q'}\end{array}\right.\right]d\xi\,d\eta\,dx$$
$$=\int_0^\infty x^{l-1}\sum_{k\geq 0}\sum_{m\geq 0}\frac{(a'_1)_m\cdots(a'_u)_m}{(b'_1)_m\cdots(b'_v)_m}\frac{a^{k+m}}{\Gamma(\mu k+\lambda)}\frac{x^{k+\rho m}}{\Gamma(Gm+T)}$$
$$\times\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^\xi\,z_2^\eta x^{\sigma_1\xi}x^{\sigma_2\eta}$$
$$\times H_{p',q'}^{m,n}\left[wx\left|\begin{array}{c}(c'_\tau,\,\gamma'_\tau)_{1,n},(c'_\tau,\,\gamma'_\tau)_{n+1,p'}\\ (d'_\tau,\,\delta'_\tau)_{1,m},(d'_\tau,\,\delta'_\tau)_{m+1,q'}\end{array}\right.\right]d\xi\,d\eta\,dx.$$

Now with an appeal to (2.2), the above mentioned result reduces to

$$I_7 = \int_0^\infty \sum_{k\geq 0}\sum_{m=0}^{k} \frac{(a'_1)_m \cdots (a'_u)_m}{(b'_1)_m \cdots (b'_v)_m} \frac{a^k}{\Gamma(\mu(k-m)+\lambda)} \frac{x^{k-m+\rho m+l-1}}{\Gamma(Gm+T)}$$
$$\times \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^\xi\, z_2^\eta x^{\sigma_1\xi} x^{\sigma_2\eta}$$
$$\times H^{m,n}_{p',q'}\left[wx \,\middle|\, \begin{matrix} (c'_\tau,\,\gamma'_\tau)_{1,n}, (c'_\tau,\,\gamma'_\tau)_{n+1,p'} \\ (d'_\tau,\,\delta'_\tau)_{1,m}, (d'_\tau,\,\delta'_\tau)_{m+1,q'} \end{matrix}\right] d\xi\, d\eta\, dx.$$

Changing the order of integral and summation which is valid due to the convergence conditions given with (3.7),

$$I_7 = \sum_{k\geq 0}\sum_{m=0}^{k} f(m) \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^\xi\, z_2^\eta \left\{ \int_0^\infty x^{k+(\rho-1)m+l+\sigma_1\xi+\sigma_2\eta-1}\right.$$
$$\left.\times H^{m,n}_{p',q'}\left[wx \,\middle|\, \begin{matrix} (c'_\tau,\,\gamma'_\tau)_{1,n}, (c'_\tau,\,\gamma'_\tau)_{n+1,p'} \\ (d'_\tau,\,\delta'_\tau)_{1,m}, (d'_\tau,\,\delta'_\tau)_{m+1,q'} \end{matrix}\right] dx \right\} d\xi\, d\eta,$$

where $f(m)$ is given with (3.7).
Now using Mellin transform of H-function by means of (2.1), we obtain

$$I_7 = w^{-l} \sum_{k\geq 0}\sum_{m=0}^{k} f(m) w^{-(\rho-1)m-k} \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,(z_1 w^{-\sigma_1})^\xi\,(z_2 w^{-\sigma_2})^\eta$$
$$\times \frac{\prod_{\tau=1}^{m} \Gamma(d'_\tau + \delta'_\tau(l+(\rho-1)m+k+\sigma_1\xi+\sigma_2\eta))}{\prod_{\tau=m+1}^{q'} \Gamma(1-d'_\tau - \delta'_\tau(l+(\rho-1)m+k+\sigma_1\xi+\sigma_2\eta))}$$
$$\times \frac{\prod_{\tau=1}^{n} \Gamma(1-c'_\tau - \gamma'_\tau(l+(\rho-1)m+k+\sigma_1\xi+\sigma_2\eta))}{\prod_{\tau=n+1}^{p'} \Gamma(c'_\tau + \gamma'_\tau(l+(\rho-1)m+k+\sigma_1\xi+\sigma_2\eta))}$$

$$= w^{-l} \sum_{k\geq 0}\sum_{m=0}^{k} f(m) w^{-(\rho-1)m-k} \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,(z_1 w^{-\sigma_1})^\xi\,(z_2 w^{-\sigma_2})^\eta$$
$$\times \frac{\prod_{\tau=1}^{m} \Gamma(d'_\tau + \delta'_\tau(l+(\rho-1)m+k)+\sigma_1\delta'_\tau\xi+\sigma_2\delta'_\tau\eta)}{\prod_{\tau=m+1}^{q'} \Gamma(1-d'_\tau - \delta'_\tau(l+(\rho-1)m+k)-\sigma_1\delta'_\tau\xi-\sigma_2\delta'_\tau\eta)}$$
$$\times \frac{\prod_{\tau=1}^{n} \Gamma(1-c'_\tau - \gamma'_\tau(l+(\rho-1)m+k)-\sigma_1\gamma'_\tau\xi-\sigma_2\gamma'_\tau\eta))}{\prod_{\tau=n+1}^{p'} \Gamma(c'_\tau + \gamma'_\tau(l+(\rho-1)m+k)+\sigma_1\gamma'_\tau\xi+\sigma_2\gamma'_\tau\eta)}.$$

Finally, by re-arranginging the double MB contour integrals by means of two variables I-function defined by (1.1), we establish

$$I_7 = w^{-l} \sum_{k\geq 0}\sum_{m=0}^{k} f(m) w^{-(\rho-1)m-k} I^{m_1+n,n_1+m:P}_{p+q',q+p':Q}\left[\begin{matrix} z_1 w^{-\sigma_1} \\ z_2 w^{-\sigma_2}\end{matrix} \,\middle|\, \begin{matrix} E_{17}, [(e_p : E_p, E'_p)], E_{18} : U \\ E_{19}[(f_q : F_q, F'_q)], E_{20} : V \end{matrix}\right],$$

where E_{17}, E_{18},E_{19} and E_{20} are given with (3.7). Hence the desired result.

4 Special Cases

I-function of two variables is of very general nature, it can be reduced in a large number of special functions by suitably specializing the parameters involved in the function. Here we record some special cases of main results.

(i) If we set $m_1 = 0$ and $r = 1$ in integral (3.1), the I-function of two variables occurring in integral (3.1) reduces into two variable's H-function [23] then we have following result

$$\tag{4.1} \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\} \times H^{0,n_1:m_2,n_2;m_3,n_3}_{p,q:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1x^{\mu_1}(t-x)^{\nu_1} \\ z_2x^{\mu_2}(t-x)^{\nu_2}\end{array}\middle| \begin{array}{c}[(e_p:E_p,E_p^{'})]:T_1\\ [(f_q:F_q,F_q^{'})]:T_2\end{array}\right]dx$$

$$= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k} \times H^{0,n_1+2:m_2,n_2;m_3,n_3}_{p+2,q+1:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1t^{\mu_1+\nu_1} \\ z_2t^{\mu_2+\nu_2}\end{array}\middle| \begin{array}{c}E_1,[(e_p:E_p,E_p^{'})]:T_1\\ [(f_q:F_q,F_q^{'})],E_2:T_2\end{array}\right],$$

where

$$T_1 = [(\,a_\tau,\alpha_\tau\,)_{1,p_1^{(1)}}];[(\,c_\tau,\gamma_\tau\,)_{1,p_1^{(2)}}],\quad T_2 = [(\,b_\tau,\beta_\tau\,)_{1,q_1^{(1)}}];[(\,d_\tau,\delta_\tau\,)_{1,q_1^{(2)}}].$$

Also E_1, E_2 and $f(m)$ are similar as given with integral (3.1).

The validity conditions of above mentioned result easily followed from integral (3.1).

(ii) If we set $m_1 = n_1 = p = q = 0$ in the integral (3.1) then we have following result in terms of product of I-function of one variable introduced by Saxena [31].

$$\tag{4.2} \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\} \times I^{m_2,n_2}_{p_i^{(1)},q_i^{(1)}:r}\left[z_1x^{\mu_1}(t-x)^{\nu_1}\middle| \begin{array}{c}(a_\tau,\alpha_\tau)_{1,n_2},(a_{\tau i},\alpha_{\tau i})_{n_2+1,p_i^{(1)}}\\ (b_\tau,\beta_\tau)_{1,m_2},(b_{\tau i},\beta_{\tau i})_{m_2+1,q_i^{(1)}}\end{array}\right] \times I^{m_3,n_3}_{p_i^{(2)},q_i^{(2)}:r}\left[z_2x^{\mu_2}(t-x)^{\nu_2}\middle| \begin{array}{c}(c_\tau,\gamma_\tau)_{1,n_3},(c_{\tau i},\gamma_{\tau i})_{n_3+1,p_i^{(2)}}\\ (d_\tau,\delta_\tau)_{1,m_3},(d_{\tau i},\delta_{\tau i})_{m_3+1,q_i^{(2)}}\end{array}\right]dx$$

$$= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k} I^{0,2:m_2,n_2;m_3,n_3}_{2,1:p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r}\left[\begin{array}{c} z_1t^{\mu_1+\nu_1} \\ z_2t^{\mu_2+\nu_2}\end{array}\middle| \begin{array}{c}E_1,...,:U\\ ...,E_2:V\end{array}\right],$$

where E_1, E_2 and $f(m)$ are similar as given in integral (3.1).

The validity conditions of above mentioned result easily followed from integral (3.1).

(iii) If we set $m_1 = n_1 = p = q = 0$ and $r = 1$ in the integral (3.1) then we have following result in terms of product of H-function of one variable [23].

$$\tag{4.3} \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\} \times H^{m_2,n_2}_{p_1^{(1)},q_1^{(1)}}\left[z_1x^{\mu_1}(t-x)^{\nu_1}\middle| \begin{array}{c}(a_\tau,\alpha_\tau)_{1,p_1^{(1)}}\\ (b_\tau,\beta_\tau)_{1,q_1^{(1)}}\end{array}\right] \times H^{m_3,n_3}_{p_1^{(2)},q_1^{(2)}}\left[z_2x^{\mu_2}(t-x)^{\nu_2}\middle| \begin{array}{c}(c_\tau,\gamma_\tau)_{1,p_1^{(2)}}\\ (d_\tau,\delta_\tau)_{1,q_1^{(2)}}\end{array}\right]dx$$

$$= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k} H^{0,2:m_2,n_2;m_3,n_3}_{2,1:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1t^{\mu_1+\nu_1} \\ z_2t^{\mu_2+\nu_2}\end{array}\middle| \begin{array}{c}E_1,...,:T_1\\ ...,E_2:T_2\end{array}\right],$$

where E_1, E_2 and $f(m)$ are same as given in integral (3.1). T_1 and T_2 are also same as given in (4.1).
The validity conditions of above mentioned result easily followed from integral (3.1).
Special cases of the integral (3.2) to integrals (3.6) can be obtained on following similar procedure but we do not mention them here.
(iv) If we set $r = 1$ in integral (3.7), we obtain following result in terms of two variable H-function introduced by Prasad and Gupta[17],

$$\text{(4.4)} \qquad \int_0^\infty x^{l-1} E_{\mu,\lambda}(ax)\, {}_uM_v^{G,T}(ax^\rho) H^{m_1,n_1:m_2,n_2;m_3,n_3}_{p,q:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}} \left[\begin{array}{c} z_1 x^{\sigma_1} \\ z_2 x^{\sigma_2} \end{array} \middle| \begin{array}{c} [(e_p : E_p, E_p')] : T_1 \\ [(f_q : F_q, F_q')] : T_2 \end{array} \right]$$
$$\times H^{m,n}_{p',q'} \left[wx \middle| \begin{array}{c} (c_\tau', \gamma_\tau')_{1,n}, (c_\tau', \gamma_\tau')_{n+1,p'} \\ (d_\tau', \delta_\tau')_{1,m}, (d_\tau', \delta_\tau')_{m+1,q'} \end{array} \right] dx$$

$$= w^{-l} \sum_{k\geq 0} \sum_{m=0}^{k} f(m) w^{-(\rho-1)m-k}$$
$$\times H^{m_1+n,n_1+m:m_2,n_2;m_3,n_3}_{p+q',q+p':p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}} \left[\begin{array}{c} z_1 w^{-\sigma_1} \\ z_2 w^{-\sigma_2} \end{array} \middle| \begin{array}{c} E_{17}, [(e_p : E_p, E_p')], E_{18} : T_1 \\ E_{19}[(f_q : F_q, F_q')], E_{20} : T_2 \end{array} \right],$$

where $E_{17}, E_{18}, E_{19}, E_{20}$ and $f(m)$ are same as given in integral (3.7). T_1 and T_2 are also similar as given in (4.1).
The validity conditions of above mentioned result easily followed from integral (3.7).
(v) If we set $m_1 = n_1 = p = q = 0$ in the integral (3.7) then we have following result in terms of product of I-function and H-function of one variable.

$$\text{(4.5)} \qquad \int_0^\infty x^{l-1} E_{\mu,\lambda}\{ax\}\, {}_uM_v^{G,T}\{ax^\rho\} \times I^{m_2,n_2}_{p_i^{(1)},q_i^{(1)}:r} \left[z_1 x^{\sigma_1} \middle| \begin{array}{c} (a_\tau, \alpha_\tau)_{1,n_2}, (a_{\tau i}, \alpha_{\tau i})_{n_2+1,p_i^{(1)}} \\ (b_\tau, \beta_\tau)_{1,m_2}, (b_{\tau i}, \beta_{\tau i})_{m_2+1,q_i^{(1)}} \end{array} \right]$$
$$\times I^{m_3,n_3}_{p_i^{(2)},q_i^{(2)}:r} \left[z_2 x^{\sigma_2} \middle| \begin{array}{c} (c_\tau, \gamma_\tau)_{1,n_3}, (c_{\tau i}, \gamma_{\tau i})_{n_3+1,p_i^{(2)}} \\ (d_\tau, \delta_\tau)_{1,m_3}, (d_{\tau i}, \delta_{\tau i})_{m_3+1,q_i^{(2)}} \end{array} \right] \times H^{m,n}_{p',q'} \left[wx \middle| \begin{array}{c} (c_\tau', \gamma_\tau')_{1,n}, (c_\tau', \gamma_\tau')_{n+1,p'} \\ (d_\tau', \delta_\tau')_{1,m}, (d_\tau', \delta_\tau')_{m+1,q'} \end{array} \right] dx$$
$$= w^{-l} \sum_{k\geq 0} \sum_{m=0}^{k} f(m) w^{-(\rho-1)m-k} I^{n,m:m_2,n_2;m_3,n_3}_{q',p':p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r} \left[\begin{array}{c} z_1 w^{-\sigma_1} \\ z_2 w^{-\sigma_2} \end{array} \middle| \begin{array}{c} E_{17}, \cdots, E_{18} : U \\ E_{19}, \cdots, E_{20} : V \end{array} \right],$$

where $E_{17}, E_{18}, E_{19}, E_{20}$ and $f(m)$ are similar as given in integral 3.7.
The validity conditions of above mentioned result easily followed from integral (3.7).
(vi) If we set $m_1 = n_1 = p = q = 0$ and $r = 1$ in the integral (3.7) then we have following result in terms of product of three H-function of one variable.

$$\text{(4.6)} \qquad \int_0^\infty x^{l-1} E_{\mu,\lambda}\{ax\}\, {}_uM_v^{G,T}\{ax^\rho\} \times H^{m_2,n_2}_{p_1^{(1)},q_1^{(1)}} \left[z_1 x^{\sigma_1} \middle| \begin{array}{c} (a_\tau, \alpha_\tau)_{1,p_1^{(1)}} \\ (b_\tau, \beta_\tau)_{1,q_1^{(1)}} \end{array} \right]$$
$$\times H^{m_3,n_3}_{p_1^{(2)},q_1^{(2)}} \left[z_2 x^{\sigma_2} \middle| \begin{array}{c} (c_\tau, \gamma_\tau)_{1,p_1^{(2)}} \\ (d_\tau, \delta_\tau)_{1,q_1^{(2)}} \end{array} \right] \times H^{m,n}_{p',q'} \left[wx \middle| \begin{array}{c} (c_\tau', \gamma_\tau')_{1,n}, (c_\tau', \gamma_\tau')_{n+1,p'} \\ (d_\tau', \delta_\tau')_{1,m}, (d_\tau', \delta_\tau')_{m+1,q'} \end{array} \right] dx$$
$$= w^{-l} \sum_{k\geq 0} \sum_{m=0}^{k} f(m) w^{-(\rho-1)m-k} H^{n,m:m_2,n_2;m_3,n_3}_{q',p':p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}} \left[\begin{array}{c} z_1 w^{-\sigma_1} \\ z_2 w^{-\sigma_2} \end{array} \middle| \begin{array}{c} E_{17}, \cdots, E_{18} : T_1 \\ E_{19}, \cdots, E_{20} : T_2 \end{array} \right],$$

where $E_{17}, E_{18}, E_{19}, E_{20}$ and $f(m)$ are similar as given in integral (3.7). T_1 and T_2 are also similar as given in (4.1).
The validity conditions of above mentioned result easily followed from integral (3.7).

5 Conclusion

This paper has successfully achieved its objective of establishing unified integrals associated with the I-function of two variables. Through a systematic exploration, the authors have derived integrals encompassing a wide range of mathematical functions, including the generalized Mittag-Leffler function, generalized M-series, and H-function of one variable in addition to the I-function of two variables. The utilization of the Mellin transform technique for the evaluation of the integral (3.7) demonstrates the versatility and effectiveness of the methods employed in this study.

Moreover, the authors have highlighted the generality of the I-function of two variables, allowing for the consideration of various special cases. This not only adds depth to the understanding of these integrals but also opens the door to potential applications in diverse areas of mathematics and science.

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

MODIFIED HOMOTOPY PERTURBATION METHOD BASED SOLUTION OF LINEARLY DAMPED DUFFING OSCILLATOR AND COMPARISON WITH SIMULATED SOLUTION*

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Abstract

The Duffing oscillator provides a basis for studying nonlinear dynamics as its phase space trajectory is fairly complex and depends on the parameter of the system viz., initial amplitude, phase, frequency, linear damping coefficient and non-linearity parameter. In order to understand the complexity of the system, three variable effective expansions have been introduced in the usual homotopy perturbation framework to obtain the solution of damped Duffing system which finds application in several areas in engineering sciences such as vibration of bars, plates and electronic circuits, etc. The necessity of the extended homotopy frame work has been further discussed for non-conservative system. Simulation results for different parameters of the systems, such as, linear damping coefficient (μ), amplitude (α) and nonlinearity parameter (ϵ) are compared with the corresponding results based on perturbative homotopy analysis up to third order by changing (i) the magnitude of linear damping coefficient (μ), (ii) the magnitude of the nonlinearity of the system (ϵ). Even though the simulated result matches satisfactorily with the perturbative solution over the entire evolutionary time scale, noticeable divergence and phase shift are observed only lately for increased value μ and ϵ, respectively.
2020 Mathematical Sciences Classification: 34D10: 34A34: 37M05: 70K60.
Keywords and Phrases: Dynamical system, Duffing Oscillator, regular solution, Homotopy method, three control parameter expansion.

1 Introduction

Many engineering applications, such as large amplitude magneto-elastic system, centrifugal governor, vibration of bars and plates involve the Duffing oscillator as basic nonlinear oscillator [10,14]. The Duffing oscillator has been used to explain many observed phenomena in science, engineering, biological systems in particular nano-tubes, microtubules and hence, dynamical analysis of this oscillator attracted many workers [1,3,10,13,15]. Numerous researchers contributed to both analytical and numerical solutions of the Duffing oscillator with and without damping [4,16,17,18]. In a similar way the problem related to synchronization of chaotic Duffing system has also been taken up in recent years in [1,15]. Duffing equation without a damping term represents a conservative system. In view of the nonlinear characteristic of the basic Duffing oscillator, several authors have developed different analytical methods to obtain approximate analytical solution so as to understand the complexity of the involved dynamics [10,15,18]. Interestingly, the solution of Duffing oscillator, in case of non-conservative system, involves intricacies that led to several methods for the situation when damping coefficient is large [10,12]. Among the various perturbative methods, the homotopy perturbation method (*HPM)* has been extensively used, in general, for finding analytical solution of nonlinear oscillators [2,5,6,7,8,9,12,19]. In this work, we revisited the *HPM* to investigate in detail the complexity of

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dynamics of duffing system in the presence of large damping coefficient. For this, a revised framework of the *HPM* involving three parameter expansion method has been used to elucidate the convergence of obtained solution with the numerically simulated solution for different values of parameters defining the system.

This paper is organized as follows: in section 2, overview of the scheme involved in Homotopy perturbation method (*HPM*) given by He [6] is revisited and three parameter expansion formalism due to He and El-Dib [4,7] is explained. Analytical and numerical solutions for non-conservative Duffing oscillator are obtained. In sections 3, we provide the results of numerical simulation for various control parameters of the system and compare them with those obtained using *HPM*.

2 Homotopy Perturbation Method to Solve Non-conservative Duffing system

In the following, we describe briefly *HPM* for solving nonlinear differential equation and in particular the one that governs the dynamics of a damped Duffing oscillator. Further, the methodology used provides a basis for using three parameter expansion.

2.1 Homotopy perturbation scheme

For a general nonlinear ordinary differential equation, we may write it as [5,12],

$$A(Q) - f(r) = 0, \; r \in \Omega, \tag{2.1}$$

with boundary conditions

$$B\left(Q, \frac{\partial Q}{\partial n}\right) = 0, \; r \in \Gamma, \tag{2.2}$$

where A, B refer to general differential operator and boundary operator respectively and further $f(r)$, a known analytic function with Γ referring to boundary of the domain Ω. We may divide the operator A into linear (L) and nonlinear part (N) resulting in the following form.

$$L(Q) + N(Q) - f(r) = 0. \tag{2.3}$$

Homotopy method formulated earlier in [5] involves constructing a homotopy $q(r,p) : \Omega \times [0,1] \to R$ satisfying

$$\mathcal{H}(q,p) = (1-p)\left[L(q) - L(Q_0)\right] + p\left[A(q) - f(r)\right] = 0, \; p \in [0,1], \; r \in \Omega$$

or

$$\mathcal{H}(q,p) = L(q) - L(Q_0) + pL(Q_0) + p\left[N(q) - f(r)\right] = 0, \tag{2.4}$$

where $p \in [0,1]$ defines an embedding parameter and Q_0 refers to an initial approximate solution of equation (2.1) satisfying the boundary conditions. From equation (2.4), we observe that

$$\begin{aligned} \mathcal{H}(q,0) &= L(q) - L(Q_0) = 0, \\ \mathcal{H}(q,1) &= A(q) - f(r) = 0. \end{aligned} \tag{2.5}$$

This implies that as p changes from $0 \to 1$, the homotopy q goes from $Q_0 \to Q$. If we write the solution of equation (2.4) as a power series in p as

$$Q = Q_0 + pq_1 + p^2 q_2 + p^3 q_3 + \cdots . \tag{2.6}$$

then the solution of equation (2.1) would be

$$Q = \lim_{p \to 1} Q = Q_0 + q_1 + q_2 + q_3 + \cdots . \tag{2.7}$$

It may be noted that use of standard *HPM* results in inconsistency, as described briefly in Box: 2.1.

2.2 Three parameter expansion formalism

In view of the description in Box: 2.1, a need for modification of the *HPM* method arises. In the context of Damped Duffing equation (*DDE*), we note that the solution comprising of three variables i.e., homotopy function, oscillation amplitude (A), and frequency ω. Following He and El. Dib [6] and He [7] Homotopy Perturbation Method (*HPM*), we write the homotopy equation corresponding to *DDE* as,

$$\ddot{Q}(t) + \omega_0^2 Q(t) + p\left\{\mu \dot{Q}(t) + \epsilon Q^3(t)\right\} = 0, \quad p \in [0,1]. \tag{2.8}$$

where ω_0, μ, ϵ refers to the natural frequency, linear damping coefficient and magnitude of nonlinearity of the system.

The Homotopy Q is now expressed as a power series in p, given as

$$Q = q_0(t) + p^1 q_1(t) + p^2 q_2(t) + p^3 q_3(t) + \cdots . \tag{2.9}$$

Substitution of equation (2.9) in (2.8) and equating coefficients of p^0 to zero gives us

$$p^0 \ : \ \ddot{q}_0 + \omega_0^2 q_0 = 0, \tag{2.10}$$

whose exact solution would be

$$q_0(t) = A\cos(\omega_0 t + \phi), \tag{2.11}$$

where A and ϕ are real constants. To solve the non-conservative nonlinear equation a further expansion of linear frequency ω_0 and the time dependent amplitude A in powers of p is suggested [4,6]. Expanding ω_0 and $A(t)$ as follows

$$\omega_0^2 = \omega^2 - p\omega_1 - p^2\omega_2 - p^3 om_3 - \cdots . \tag{2.12}$$

$$A(t) = \alpha(1 + pc_1 + p^2c_2 + p^3c_3 + \cdots), \tag{2.13}$$

which implies that

$$q_0(t) = \alpha\{1 + pc_1 + p^2c_2 + p^3c_3 + \cdots\}\cos(\omega t + \phi). \tag{2.14}$$

Box: 2.1

Considering the following *DDE* where fundamental frequency is taken as 1, *i.e.*, $\omega_0 = 1$

$$\ddot{Q} + \mu\dot{Q} + Q + \epsilon Q^3 = 0. \qquad (B.1)$$

Observe that

$$A(Q) = L(Q) + N(Q), \qquad (B.2)$$

where

$$L(Q) = \ddot{Q}(t) + Q(t); \quad L(q_0) = \ddot{q}_0(t) + q_0; \quad N(Q) = \mu\dot{Q}(t) + \epsilon Q^3.$$

Following the standard *HPM* framework, we may write equation $(B.1)$ as,

$$p^0 \ : \ \ddot{q}_0(t) + q_0(t) - \ddot{Q}_0(t) - Q_0(t) = 0, \qquad (B.3)$$

$$p^1 \ : \ \ddot{q}_1(t) + q_1 + \ddot{Q}_0(t) + Q_0(t) + \mu\dot{Q}_0(t) + \epsilon Q_0^3 = 0, \qquad (B.4)$$

For $p \to 0$, $Q_0 \to q_0$, we may write equation $(B.4)$ as

$$\ddot{q}_1(t) + q_1 + \ddot{q}_0(t) + q_0(t) + \mu\dot{q}_0(t) + \epsilon q_0^3 = 0. \qquad (B.5)$$

Considering the initial approximate solution as $Q_0 = q_0 = A\cos\omega t$, where A is the amplitude and ω is the frequency of the output, which when substituted back in last equation, results in

$$\ddot{q}_1 + q_1 + = A\left\{\omega^2 - 1 - \frac{3}{4}A^2\epsilon\right\}\cos\omega t - \mu\omega A\sin\omega t - \frac{1}{4}\epsilon A^3\cos 3\omega t, \qquad (B.6)$$

where we have two secular terms. One of them gives us frequency ω of the output, as

$$\omega = \sqrt{1 + \frac{3}{4}\epsilon A^2}. \qquad (B.7)$$

which is the frequency obtained up to first order for conservative Duffing system and the other indicates that $\mu = 0$, *i.e.*, no damping. It is to be noted that even a second expansion, *i.e.*, expansion of amplitude A does not work [4,7]. Keeping these facts, to deal with damped Duffing oscillator, the modified *HPM*, explained in *section 2.2*, is considered.

Making an application of equations (2.11)-(2.14) in equation (2.8) and equating coefficients of p^i, $i = 1, 2, 3, \cdots$, we obtain following equations for $q_1,\ q_2,\ q_3,\ \cdots$ as

$$\ddot{q}_1(t) + \omega^2 q_1(t) = \alpha\omega\{2\dot{c}_1 + \mu\}\sin(\omega t + \phi) + \alpha\left\{\omega_1 - \ddot{c}_1 - \frac{3}{4}\epsilon\alpha^2\right\}\cos(\omega t + \phi) + \\ -\frac{1}{4}\epsilon\alpha^3\cos 3(\omega t + \phi), \tag{2.15}$$

$$\ddot{q}_2(t) + \omega^2 q_2(t) = \omega_1 q_1 - \mu\dot{q}_1 + \alpha\omega(2\dot{c}_2 + \mu c_1)\sin(\omega t + \phi) - \frac{3}{4}\epsilon\alpha^3 c_1\cos 3(\omega t + \phi) + \\ -\frac{3}{2}\epsilon\alpha^2 q_1(1 + \cos 2(\omega t + \phi)) + \alpha\left\{\omega_2 - \ddot{c}_2 + \omega_1 c_1 - \mu\dot{c}_1 - \frac{9}{4}\epsilon\alpha^2 c_1\right\}\cos(\omega t + \phi), \tag{2.16}$$

$$(2.17)\quad \ddot{q}_3(t)+\omega^2 q_3(t)=\omega_1 q_2+\omega_2 q_1-\mu\dot{q}_2-3\epsilon q_0 q_1(q_0+q_1)+\alpha\omega(2\dot{c}_3+\mu c_2)\sin(\omega t+\phi)$$
$$+\alpha\left\{\omega_2 c_1+\omega_3+\omega_1 c_2-\mu\dot{c}_2-\ddot{c}_3-\frac{9}{4}\epsilon\alpha^3(c_1^2+c_2)-\frac{3}{64}\frac{\epsilon^2\alpha^5}{\omega^2}\ c_1\right\}\cos(\omega t+\phi)$$
$$-\left\{\frac{3}{4}\epsilon\alpha^3(c_1^2+c_2)+\frac{3}{32}\frac{\epsilon^2\alpha^5}{\omega^2}c_1\right\}\cos 3(\omega t+\phi)-\frac{3}{64}\frac{\epsilon^2\alpha^5}{\omega^2}c_1\ \cos 5(\omega t+\phi).$$

The solutions for $q_1(t),\ q_2(t)$ and $q_3(t)$ could easily be obtained by removing the secular terms in the respective equations. Removal of secular terms from equation (2.15) results in following conditions on c_1 and ω_1,

$$(2.18)\qquad \dot{c}_1=-\frac{1}{2}\mu\quad\Longrightarrow c_1=-\frac{1}{2}\mu t,\quad \ddot{c}_1=0,\quad \text{and}\quad \omega_1=\frac{3}{4}\ \epsilon\alpha^2,$$

which further leads to

$$(2.19)\qquad q_1(t)=\frac{1}{32}\ \frac{\epsilon\alpha^3}{\omega^2}\quad \cos 3(\omega t+\phi).$$

Removing secular terms from equation (2.16), we get

$$(2.20)\qquad \dot{c}_2=\frac{1}{4}\ \mu^2 t\ \Longrightarrow \ddot{c}_2=\frac{1}{4}\mu^2,\ c_2=\frac{1}{8}\mu^2t^2\ \text{ and }\ \omega_2=-\frac{1}{4}\mu^2-\frac{3}{4}\epsilon\alpha^2\mu t+\frac{3}{128}\frac{\epsilon^2\alpha^4}{\omega^2},$$

and the solution for q_2 could be written as,

$$(2.21)\qquad q_2(t)\ =\ \frac{3}{64}\frac{\epsilon\alpha^3}{\omega^2}\left[\left\{\frac{1}{16}\frac{\epsilon\alpha^2}{\omega^2}-\mu t\right\}\cos 3(\omega t+\phi)+\frac{1}{2}\mu\sin 3(\omega t+\phi)\right.$$
$$\left.+\ \frac{1}{16}\frac{\epsilon\alpha^2}{\omega^4}\cos 5(\omega t+\phi)\right].$$

Similarly removal of secular terms in equation (2.17), results in following conditions on c_3 and ω_3 as,

$$(2.22)\qquad \dot{c}_3=-\frac{1}{2}\mu c_2+\frac{9}{1024}\frac{\epsilon^2\alpha^4}{\omega^5},\ \mu=-\frac{1}{16}\mu^3t^2+\frac{9}{1024}\frac{\epsilon^2\alpha^4}{\omega^4}\mu$$
$$\Longrightarrow\qquad \ddot{c}_3=-\frac{1}{8}\mu^3 t\qquad \&\qquad \mathrm{c}_3=-\frac{1}{48}\mu^3\mathrm{t}^3+\frac{9}{1024}\frac{\epsilon^2\alpha^4}{\omega^4}\mu,$$
$$\text{and}\qquad \omega_3=\frac{3}{8}\epsilon\alpha^2\mu t\left\{\frac{1}{16}\epsilon\alpha^2+\mu t\right\}-\frac{3}{4096}\frac{\epsilon^3\alpha^6}{\omega^4},$$

and solution for $q_3(t)$ may be written as,

$$(2.23)\qquad q_3(t)\ =\ -\frac{1}{8\omega^2}\left\{(X_1+X_2t)-\frac{7}{16\omega^2}\left(X_2+2X_3\right)+(X_4+X_5t)\right\}\ \cos 3(\omega t+\phi)$$
$$+\ \frac{1}{8\omega^2}\left\{\frac{3}{4\omega}\left(X_2+X_5+2X_3t\right)\right\}\ \sin 3(\omega t+\phi)-\frac{1}{24\omega^2}\left\{(X_6+X_7t+X_8)\right\}\ \cos 5(\omega t+\phi)$$
$$+\ \frac{5}{288\omega^3}X_7\ \sin 5(\omega t+\phi)-\frac{1}{48\omega^2}X_9\ \cos 7(\omega t+\phi).$$

Therefore, from equation (2.9), the solution of non-conservative duffing oscillator up to third order would be

$$(2.24)\quad q(t)=\lim_{p\to 1}\ Q=q_0+q_1+q_2+q_3$$
$$=\alpha\left\{1-\frac{1}{2}\mu t+\frac{1}{8}\mu^2t^2-\frac{1}{48}\mu^3t^3+X_{13}\ t\right\}\cos(\omega t+\phi)+\left[X_{14}\left\{1-\frac{3}{2}\mu t+\frac{9}{8}\mu^2t^2\right\}\right.$$
$$\left.-\frac{1}{8\omega^2}\left\{(X_1+X_2t)-\frac{7}{16\omega^2}\left(X_2+2X_3\right)+X_4+X_{10}+X_5\ t\right\}\right]\ \cos 3(\omega t+\phi)$$
$$+\frac{3}{32\omega^2}\left\{X_2+X_5+2X_3\ t+X_{11}\right\}\ \sin 3(\omega t+\phi)-\frac{1}{24\omega^2}\left\{X_6+X_7\ t+X_8+X_{13}\right\}\ \cos 5(\omega t+\phi)$$
$$+\frac{5}{288\omega^3}X_7\ \sin 5(\omega t+\phi)-\frac{1}{48\omega^2}X_9\ \cos 7(\omega t+\phi),$$

where

$$X_1=-\left\{\frac{\epsilon\alpha^3}{32\omega^2}\ \mu^2+\frac{9\epsilon^3\alpha^7}{4096\omega^4}\right\},\qquad X_2=\frac{15\epsilon^2\alpha^5}{256\omega^2}\ \mu,\quad X_3=-\frac{9}{32}\epsilon\alpha^3\ \mu^2,\quad X_4=-\frac{9\epsilon^2\alpha^5}{1024\omega^3},$$

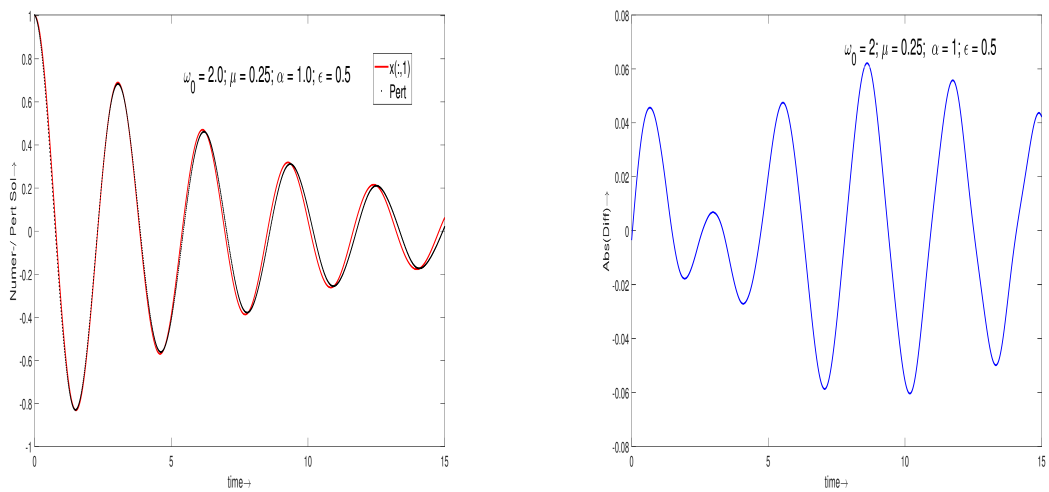
$$X_5 = -\frac{9\epsilon\alpha^3}{64\omega}\ \mu^2, \quad X_6 = -\frac{15\epsilon^3\alpha^7}{4096\omega^4}, \quad X_7 = \frac{15\epsilon^2\alpha^5}{1024\omega^2}\ \mu, \quad X_8 = -\frac{13\epsilon^2\alpha^5}{1024\omega^3}\ \mu, \quad X_9 = -\frac{3\epsilon^3\alpha^7}{4096\omega^4},$$

$$X_{10} = -\frac{3\epsilon^2\alpha^5}{128\omega^4}, \qquad X_{11} = \frac{\epsilon\alpha^3}{4}\ \mu, \qquad X_{12} = \frac{3\epsilon^2\alpha^5}{128\omega^2}, \qquad X_{13} = \frac{9\epsilon^2\alpha^4}{1024\omega^4}\ \mu, \qquad X_{14} = \frac{\epsilon\alpha^3}{32\omega^2}.$$

Following [4,6], we may rewrite equation (2.24) in a compact form as

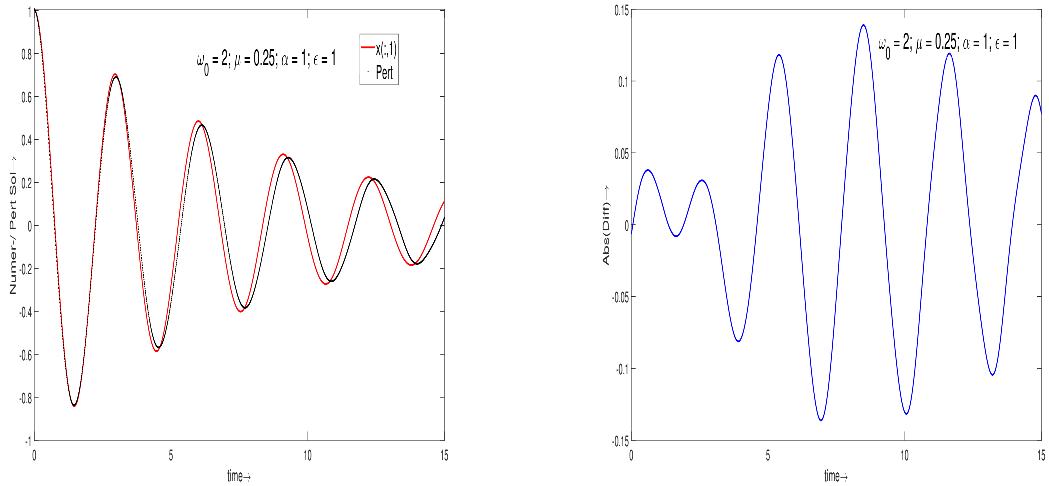
$$\begin{aligned}
q(t) =& \alpha \left[e^{-\mu t/2} + X_{13}\ t\right] \cos(\omega t + \phi) \\
&+ \left[X_{14}\ e^{-3\mu t/2} - \frac{1}{8\omega^2}\left\{(X_1 + X_2\ t) - \frac{7}{16\omega^2}\,(X_2 + 2X_3) + X_4 + X_{10} + X_5\ t\right\}\right]\ \cos 3(\omega t + \phi) \\
&+ \frac{3}{32\omega^2}\,\{X_2 + X_5 + 2X_3\ t + X_{11}\}\ \sin 3(\omega t + \phi) \\
&- \frac{1}{24\omega^2}\,\{X_6 + X_7\ t + X_8 + X_{13}\}\ \cos 5(\omega t + \phi) \\
&+ \frac{5}{288\omega^3} X_7\ \sin 5(\omega t + \phi) - \frac{1}{48\omega^2} X_9\ \cos 7(\omega t + \phi).
\end{aligned} \tag{2.25}$$

where $X's$ are defined as mentioned above.



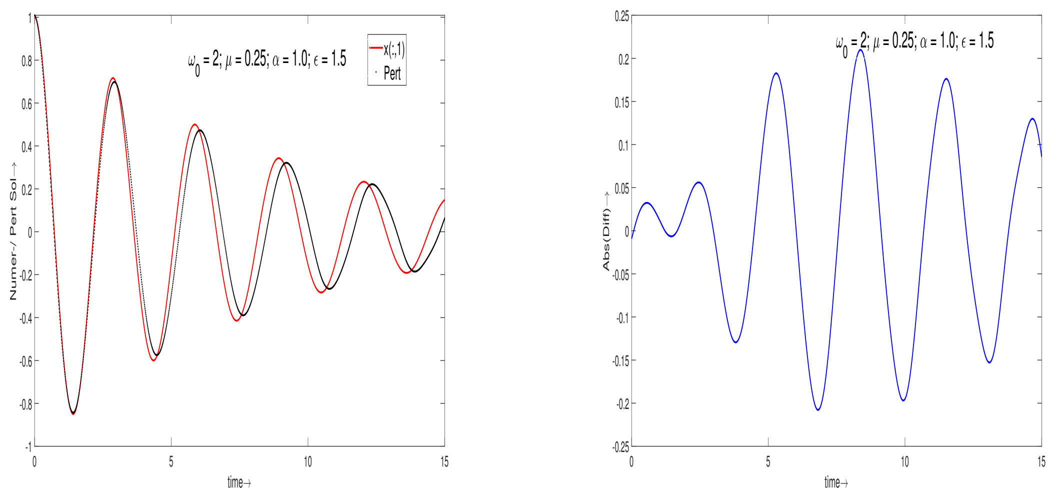
(a) **(b)**

***Figure* 2.1:** (a) Comparison of simulated result (shown in red color) and the homotopy based solution (shown in black color) of Duffing oscillator up to third order. (b) the evolution of absolute error.



(a) **(b)**

***Figure* 2.2:** (a) Comparison of simulated result (shown in red color) and the homotopy based solution (shown in black color) of Duffing oscillator up to third order. (b) the evolution of absolute error.



(a) **(b)**

***Figure* 2.3:** (a) Comparison of simulated result (shown in red color) and the homotopy based solution (shown in black color) of Duffing oscillator up to third order. (b) the evolution of absolute error.

2.3 Amplitude, Frequency and Stability condition with $\mu > 0$

Applying conditions (2.18), (2.20) and (2.22), in equations (2.12) and (2.13), the frequency ω and the amplitude A are obtained in terms of damping coefficient μ and nonlinearity parameter ϵ, respectively, as

$$(2.26)\qquad \omega^2 = \omega_0^2 + \frac{3}{4}\,\epsilon\alpha^2 - \frac{1}{4}\mu^2 - \frac{3}{4}\epsilon\alpha^2\mu t + \frac{3}{128}\frac{\epsilon^2\alpha^4}{\omega^2} + \frac{3}{8}\epsilon\alpha^2\mu t\left\{\frac{1}{16}\epsilon\alpha^2 + \mu t\right\} - \frac{3}{4096}\frac{\epsilon^3\alpha^6}{\omega^4}.$$

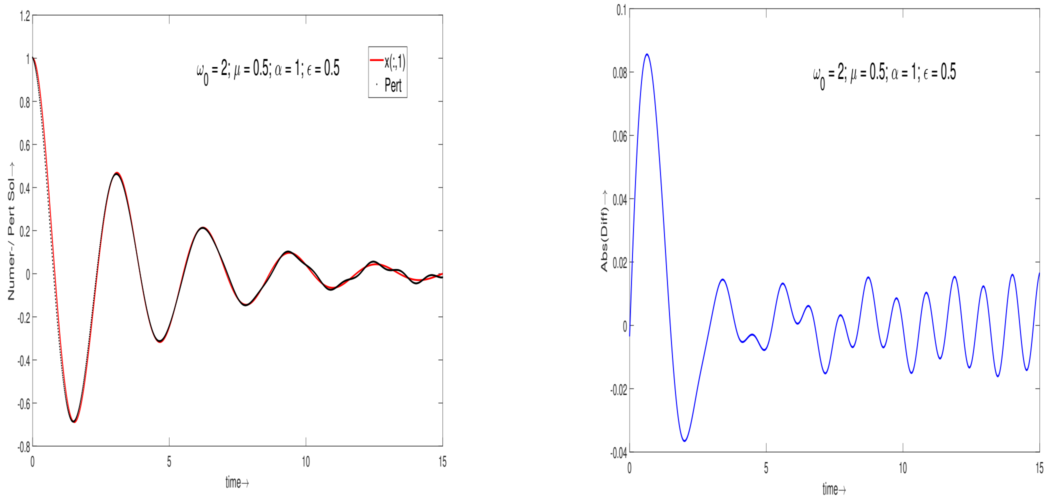
$$\implies \omega^2 = \omega_0^2 - \frac{1}{4}\mu^2 + \frac{3}{4}\,\epsilon\alpha^2\left\{1 - \mu\, t + \frac{1}{2}\mu^2 t^2\right\} + \frac{3}{128}\frac{\epsilon^2\alpha^4}{\omega^2} + \frac{3}{128}\epsilon^2\alpha^4\mu t - \frac{3}{4096}\frac{\epsilon^3\alpha^6}{\omega^4}.$$

$$A(t) = \alpha \left\{ 1 - \frac{1}{2}\mu \; t \frac{1}{8}\mu^2 \; t^2 - \frac{1}{48}\mu^3 \; t^3 + \frac{9}{1024} \; \frac{\epsilon^2\alpha^4}{\omega^4}\mu \; t \right\}. \tag{2.27}$$

Following [5], equation (2.26) and (2.27) may be written as

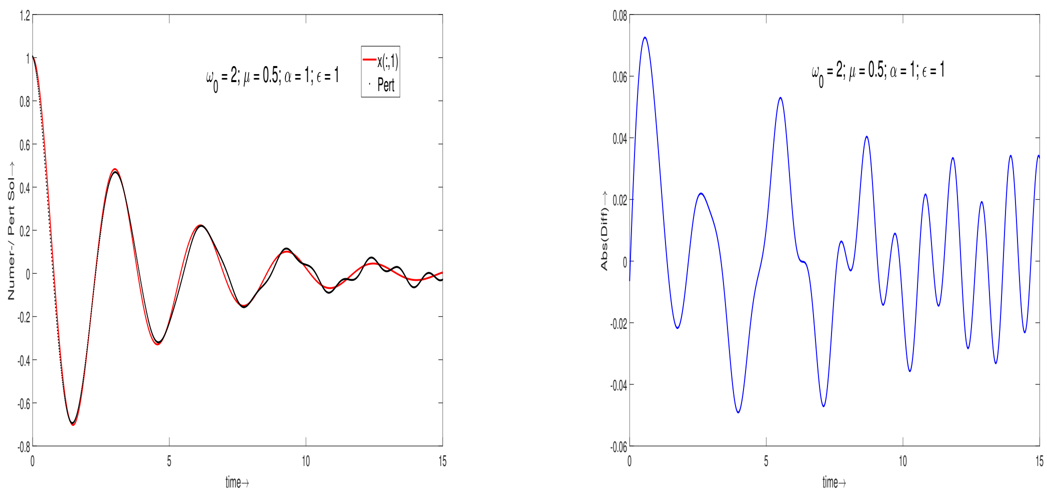
$$\omega^2 = \omega_0^2 - \frac{1}{4}\mu^2 + \frac{3}{4} \; \epsilon\alpha^2 \; e^{-\mu \; t} + \frac{3}{128}\epsilon^2\alpha^4\mu t + \frac{3}{128} \frac{\epsilon^2\alpha^4}{\omega_0^2} - \frac{3}{128} \frac{\epsilon^2\alpha^4}{\omega_0^4}\omega_1 - \frac{3}{4096} \frac{\epsilon^3\alpha^6}{\omega_0^4}, \tag{2.28}$$

$$A(t) = \alpha \left\{ e^{-\frac{1}{2}\mu \; t} + \frac{9}{1024} \; \frac{\epsilon^2\alpha^4}{\omega^4}\mu \; t \right\}. \tag{2.29}$$



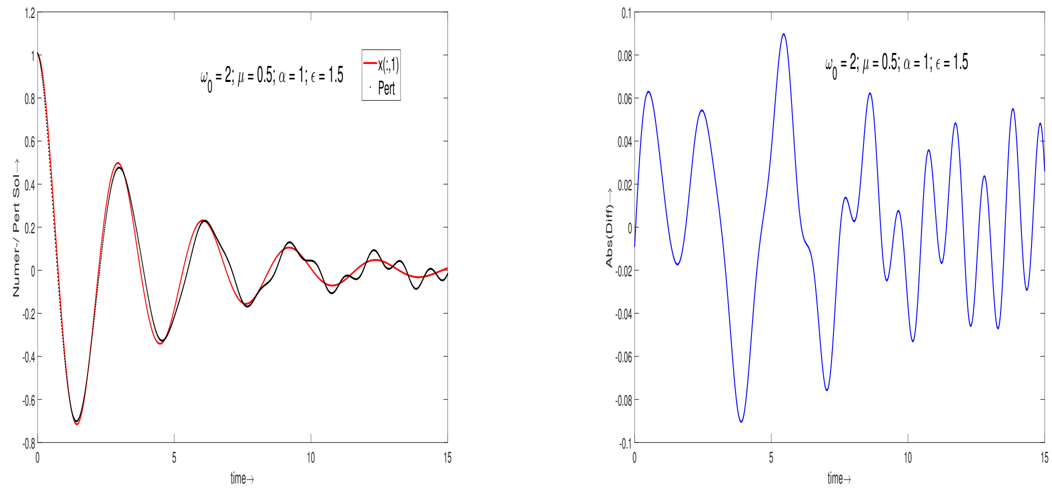
(a) **(b)**

***Figure* 2.4:** (a) Comparison of simulated result (shown in red color) and the homotopy based solution (shown in black color) of Duffing oscillator up to third order. (b) the evolution of absolute error.



(a) **(b)**

***Figure* 2.5:** (a) Comparison of simulated result (shown in red color) and the homotopy based solution (shown in black color) of Duffing oscillator up to third order. (b) the evolution of absolute error.



(a) **(b)**

***Figure* 2.6:** (a) Comparison of simulated result (shown in red color) and the homotopy based solution (shown in black color) of Duffing oscillator up to third order. (b) the evolution of absolute error.

For the values of parameters considered for numerical simulation, it is observed that the third term on the right hand side decreases at a faster rate than the corresponding rise in the fourth term and thereby resulting in a constant value of the frequency on larger time scale, say, $\omega_f > 0$, which may be considered also as a stability condition for the system.

3 Conclusion

In the present work, numerical simulation of the linearly damped Duffing system has been carried out by fixing the values of the initial amplitude, $\alpha = 1$ and the frequency, $\omega_0 = 2.0$, keeping the damping parameter (i) $\mu = 0.25$ and (ii) $\mu = 0.5$ while varying the nonlinearity parameter, ϵ. The modified version of the homotopy based perturbative solution, as obtained in equation (2.25), for various parameters are subsequently compared with the direct numerical simulation results. It is observed that the *HPM* based solutions compares well with those obtained numerically (Figs.2.1a-2.6a). The magnitude of errors between simulated and *HPM* based solution are observed to be nominal for $\epsilon < 1.0$. However, it is also observed that for lower values of damping parameter μ, noticeable changes in phase relationship between the numerical and *HPM* based solution occurs for moderately higher values of the nonlinearity parameter *i.e.*, $\epsilon \sim 1.5$. (Figs. 2.1b-2.6b) further illustrate the time variation of small deviation between the *HPM* and simulated solutions for various control parameters.

The foregoing *HPM* method allows one to obtain solution of the non-conservative Duffing system with larger damping coefficient (μ) and nonlinearity parameter (ϵ). We plan to use it subsequently to analyze the complex response of micro-nanosystems *i.e.*, , resonator used for mass detection, vibration of carbon nanotube, micro tubules, etc., which play important role in biological system [3].

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

IDENTITIES RELATED TO GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS

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Abstract

The famous Fibonacci and Lucas polynomials possess various astonishing properties and identities. The Fibonacci polynomial has been generalized in many ways by preserving the recurrence relation and others by preserving the initial condition. In this paper, we define generalized Fibonacci and Lucas polynomials and proved some famous identities in our settings.
2020 Mathematical Sciences Classification: 11B37, 11B39.
Keywords and Phrases: Fibonacci polynomial; Lucas Polynomial; Generalized Fibonacci polynomial; Generalized Lucas polynomial; Generating function.

1 Introduction

Belgian mathematician Eugene Charles Catalan and the German Mathematician E. Jacobsthal [7] were first studied Fibonacci polynomials in 1883. Fibonacci polynomials are of great importance in Mathematics. The Fibonacci and Lucas polynomials are extensively explored by many mathematicians like, Basin [2] , Horadam and Mahon [6], and Lucas [11] (for details see Koshy [7]) and connected to various branches of mathematics. Recently, many new identities of Generalized Fibonacci and Lucas polynomials are studied by Agrawal et al. [1].

A set of Fibonacci polynomials generated by the Q matrix, satisfying the following recurrence relation, was proved by Basin [2].

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x), n \geq 2 \text{ with } f_0(x) = 0, f_1(x) = 1. \tag{1.1}$$

The initial terms of the Fibonacci polynomials are

$$f_2(x) = x, f_3(x) = x^2 + 1, f_4(x) = x^3 + 2x, f_5(x) = x^4 + 3x^2 + 1 \text{ and so on }. \tag{1.2}$$

Jacobsthal polynomials are given by (for more details see Koshy [7])

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x), n \geq 3 \text{ with } J_1(x) = 1 = J_2(x). \tag{1.3}$$

Pell polynomials due to Horadam and Mahon [6] are defined by

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), n \geq 2 \text{ with } P_0(x) = 0, P_1(x) = 1. \tag{1.4}$$

The generating function of Fibonacci and Lucas polynomials due to Doman and Williams [4] is given by

$$\sum_{n=0}^{\infty} f_n(x)t^n = t(1 - xt - t^2)^{-1}, \sum_{n=0}^{\infty} L_n(x)t^n = (2 - xt)(1 - xt - t^2)^{-1}. \tag{1.5}$$

For Fibonacci and Lucas polynomials, the explicit sum formula due to Horadam and Mahon [6] and Koshy [7] is given by

$$f_n(x) = \sum_{n=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} x^{n-1-2k}, L_n(x) = \sum_{n=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}. \tag{1.6}$$

where $\binom{n-k}{k}$ is a binomial coefficient and $[x]$ is defined as the greatest integer less than or equal to x.

Many more interesting properties for Fibonacci and Lucas polynomials have been studied by Doman and Williams [4], Koshy [7], and Lucas [11].

Many famous identities which we have proved for our polynomial have been studied for generalized Fibonacci sequence in [9].

In this paper, we derive famous identities such as Catalan's, d'Ocagne's, and many other for our generalized Lucas polynomials which is derived for the generalized Fibonacci polynomials by Rathore et al. [8]. Also, we proved some identities for our generalized Fibonacci polynomials with the help of generating function and Binet's formula.

2 Preliminaries

In this section, we give some basic definitions which are useful throughout the paper.

Definition 2.1. ***Fibonacci Polynomials:*** *A polynomial sequence that can be considered as a generalization of Fibonacci numbers are Fibonacci polynomials (for more details see Lucas* [11]*). The Fibonacci polynomial due to Koshy* [7] *is defined by the following recurrence relation,*

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x), n \geq 3 \text{ with } f_1(x) = 1, f_2(x) = x.$$

Definition 2.2. ***Lucas Polynomials:*** *The Lucas Polynomials due to Bicknell* [3] *and Lucas* [8] *are defined by the recurrence relation,*

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), n \geq 2 \text{ with } L_0(x) = 2, L_1(x) = x.$$

Definition 2.3. ***Generalized Fibonacci Polynomials:*** *The generalized Fibonacci polynomials are defined by*

$$f_n(x) = \begin{cases} s, & \text{if } n = 0; \\ sx, & \text{if } n = 1; \\ xf_{n-1}(x) + f_{n-2}(x), & \text{if } n \geq 2. \end{cases} \tag{2.1}$$

Definition 2.4. ***Generalized Lucas Polynomials:*** *The generalized Lucas polynomials are defined by*

$$l_n(x) = \begin{cases} 2s, & \text{if } n = 0; \\ sx, & \text{if } n = 1; \\ xl_{n-1}(x) + l_{n-2}(x), & \text{if } n \geq 2. \end{cases} \tag{2.2}$$

Definition 2.5. ***Generating Function:*** *Let $a_0, a_1, a_2,$ be a sequence of real numbers. Then the function (2.5) $g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ is called a generating function for the sequence $\{a_n\}$. Generating functions provides a powerful tool for solving linear homogeneous recurrence relations with constant coefficients (for more details see Lucas* [11]*).*

3 Generalized Fibonacci Polynomials

The generalization of Fibonacci polynomials can be done in many ways by changing the initial condition and others by changing the recurrence relation. Rathore et al. [9] defined the generalized Fibonacci polynomials $w_n(x)$ by recurrence relation $w_n = xw_{n-1} + w_{n-2}, n \geq 2$, with $w_0(x) = 2b, w_1(x) = a + b$ where a and b are integer. Sikhwal et al. [9] defined the generalized Fibonacci polynomials $u_n(x)$ by recurrence relation with $u_n = xu_{n-1} + u_{n-2}, n \geq 2$, with $u_0(x) = a, u_1(x) = 2a + 1$ where a is an integer. In this paper, we define generalized Fibonacci polynomials $g_n(x)$ by the recurrence relation

$$g_n^{(x)} = xg_{n-1}^{(x)} + g_{n-2}^{(x)}, n \geq 2 \text{ with } g_0(x) = a + b, g_1(x) = 2a + 1 \tag{3.1}$$

where a and b are integers.

The starting few terms of a generalized Fibonacci polynomials are given by

$$g_0(x) = a + b, g_1(x) = 2a + 1, g_2(x) = x(2a + 1) + a + b, g_3(x) = x^2(2a + 1) + x(a + b) + 2a + 1.$$

For $x = 1, a = 0, b = 0$, we obtain the classical Fibonacci sequence.

Binet's Formula for generalized Fibonacci polynomial is given by $g_n(x) = (A\alpha^n + B\beta^n)$, where

$$A = \frac{(2a + 1) - (a + b)\beta}{\alpha - \beta}, B = \frac{(a + b)\alpha - (2a + 1)}{\alpha - \beta}.$$

Also, Note that $\alpha\beta = -1, \alpha + \beta = x, \alpha + \beta = \sqrt{4 + x^2}$ where α and $\beta =$ are the roots of the quadratic multline given by $\lambda^2 - x\lambda - 1 = 0$ (Koshy [7]).

Lemma 3.1. *The generating function for generalized Fibonacci polynomials defined in equation* (3.1) *is given by*

$$\sum_{n=0}^{\infty} g_n(x)t^n = \frac{(a+b)(1-xt)+(2a+1)t}{1-xt-t^2}.$$

Proof. Replace n by $n+1$ in (3.1), we have

$$g_{n+1}(x) = xg_n(x) + g_{n-1}(x); n \geq 1. \tag{3.2}$$

Let

$$F(t) = \sum_{n=0}^{\infty} g_n(x)t^n. \tag{3.3}$$

From equation (3.2), we have

$$\sum_{n\geq 1} g_{n+1}(x)t^n = x\sum_{n\geq 1} g_n(x)t^n + \sum_{n\geq 1} g_{n-1}(x)t^n. \tag{3.4}$$

Now,

$$\sum_{n\geq 1} g_n(x)t^n = \sum_{n\geq 1} g_n(x)t^n + g_0(x) - g_0(x) = F(t) - (a+b). \tag{3.5}$$

and

$$\sum_{n\geq 1} g_{n-1}(x)t^n = tF(t). \tag{3.6}$$

Therefore, R.H.S of (3.4) becomes

$$\sum_{n\geq 1} g_{n+1}(x)t^n = x[F(t)-(a+b)] + tF(t). \tag{3.7}$$

Now,

$$\sum_{n\geq 1} g_{n+1}(x)t^n = \sum_{n\geq 1} g_n(x)t^n + g_0(x) - g_0(x) + g_1(x) - g_1(x) = \frac{1}{t}[F(t)-(a+b)-t(2a+1)] \tag{3.8}$$

Therefore, (3.7) becomes

$$\frac{1}{t}[F(t)-(a+b)-t(2a+1)] = x[F(t)-(a+b)] + tF(t)$$

i.e.

$$F(t)(1-xt-t^2) = [(a+b)(1-xt)+(2a+1)t].$$

Thus,

$$\sum_{n=0}^{\infty} g_n(x)t^n = \frac{(a+b)(1-xt)+(2a+1)t}{1-xt-t^2}. \tag{3.9}$$

□

4 Generalized Lucas Polynomials

We define generalized Lucas polynomials $k_n(x)$ by the recurrence relation

$$k_n(x) = xk_{n-1}(x) + k_{n-2}(x); n \geq 2 \text{ with } k_0(x) = a, k_1(x) = x, \tag{4.1}$$

where a is an integer. The first few terms of generalized Lucas polynomials are given by

$$k_0(x) = a, k_1(x) = x, k_2(x) = ax + a = a(x+1), k_3(x) = x(a+1) + a.$$

For $x = 1, a = 2$, we obtain Lucas sequence.

Following the same idea as in proof of Lemma 3.1, we can derive a generating function for generalized Lucas polynomials (defined as above), is given by

$$\sum_{n=0}^{\infty} k_n(x)t^n = \frac{a(1-xt)+xt}{1-xt-t^2}.$$

Binet's Formula for generalized Lucas polynomials is given by

$$k_n(x) = A(\alpha^n + \beta^n), \text{ where } A = \frac{a}{2} \text{ (Koshy [7])} .$$

5 Some Identities of generalized Fibonacci polynomials

In this section, we investigate some of the identities of our generalized Fibonacci polynomials with the help of a generating function and Binet's formula.

Theorem 5.1. *If the nth term of a generalized Fibonacci polynomial is $g_n(x)$ and $g'_n(x)$ denotes the derivative of $g_n(x)$ with respect to x, then*

(5.1) $$g'_n(x) = xg'_{n-1}(x) + g'_{n-2}(x) + g_{n-1}(x), n \geq 2.$$

Proof. The generating function of generalized Fibonacci polynomials is given by

$$\sum_{n=0}^{\infty} g_n(x)t^n = [(a+b)(1-xt) + (2a+1)t](1-xt-t^2)^{-1}.$$

Differentiating both sides with respect to x, we get

$$\begin{aligned}\sum_{n=0}^{\infty} g'_n(x)t^n =&[(a+b)(1-xt) + (2a+1)t](-t)(-1)(1-xt-t^2)^{-2}\\ &+[-t(a+b)](1-xt-t^2)^{-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}(1-xt-t^2)\sum_{n=0}^{\infty} g'_n(x)t^n &= t[(a+b)(1-xt) + (2a+1)t](1-xt-t^2)^{-1} - t(a+b)\\ &= t\sum_{n=0}^{\infty} g_n(x)t^n - t(a+b).\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} g'_n(x)t^n - x\sum_{n=0}^{\infty} g'_n(x)t^{n+1} - \sum_{n=0}^{\infty} g'_n(x)t^{n+2} = \sum_{n=0}^{\infty} g_n(x)t^{n+1} - t(a+b).$$

Equating the coefficients of t^n on both sides, we have

$$g'_n(x) = xg'_{n-1}(x) + g'_{n-2}(x) + g_{n-1}(x),$$

which proves the Theorem 5.1.

Replacing n by $n+1$, we also derive,

$$g'_{n+1}(x) = xg'_n(x) + g'_{n-1}(x) + g_n(x).$$

□

Theorem 5.2. *Let $g_n(x)$ be the n^{th} term of a generalized Fibonacci polynomial, then*

(5.2) $$ng_n(x) - x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x). = xg_n(x) + (2-x^2)g_{n-1}(x) - 3xg_{n-2}(x) - 2g_{n-3}(x); n \geq 3.$$

Proof. The generating function of a generalized Fibonacci polynomials is given by

(i) $$\sum_{n=0}^{\infty} g_n(x)t^n = [(a+b)(1-xt) + (2a+1)t](1-xt-t^2)^{-1}.$$

Differentiate it both sides partially with respect to t, we get

(ii) $$\sum_{n=0}^{\infty} ng_n(x)t^{n-1} = [(a+b)(1-xt) + (2a+1)t](x+2t)(1-xt-t^2)^{-2} + [-x(a+b) + (2a+1)](1-xt-t^2)^{-1}.$$

Differentiating (i) both sides partially with respect to x, we have

(iii) $$\sum_{n=0}^{\infty} g'_n(x)t^n = [(a+b)(1-xt) + (2a+1)t](t)(1-xt-t^2)^{-2} + [-t(a+b)](1-xt-t^2)^{-1}.$$

On dividing both sides by t, we derive

$$\sum_{n=0}^{\infty} g'_n(x)t^{n-1} = [(a+b)(1-xt) + (2a+1)t](1-xt-t^2)^{-2} + [-(a+b)](1-xt-t^2)^{-1}.$$

Hence,

$$\text{(iv)} \qquad \sum_{n=0}^{\infty} g_n'(x)t^(n-1) + (a+b)(1-xt-t^2)^{-1} = [(a+b)(1-xt) + (2a+1)t](1-xt-t^2)^{-2}.$$

On substituting the value of R.H.S of equation (iv) in equation (ii), we have

$$\sum_{n=0}^{\infty} ng_n(x)t^{n-1} = (x+2t)\sum_{n=0}^{\infty} g_n'(x)t^{n-1} + (a+b)(1-xt-t^2)^{-1}$$
$$+[-x(a+b) + (2a+1)](1-xt-t^2)^{-1}$$
$$= x\sum_{n=0}^{\infty} g_n'(x)t^{n-1} + 2\sum_{n=0}^{\infty} g_n'(x)t^{n-1} + (x+2t)(a+b)(1-xt-t^2)^{-1}$$
$$+[-x(a+b) + (2a+1)](1-xt-t^2)^{-1}$$
$$= x\sum_{n=0}^{\infty} g_n'(x)t^{n-1} + 2\sum_{n=0}^{\infty} g_n'(x)t^{n} + (2t)(a+b) + (2a+1)(1-xt-t^2)^{-1}.$$

Therefore,

$$(1-xt-t^2)\sum_{n=0}^{\infty} ng_n(x)t^{n-1}$$
$$= x(1-xt-t^2)\sum_{n=0}^{\infty} g_n'(x)t^{n-1} + 2(1-xt-t^2)\sum_{n=0}^{\infty} g_n'(x)t^{n} + (2t)(a+b) + (2a+1).$$

Hence

$$\sum_{n=0}^{\infty} ng_n(x)t^{n-1} - x\sum_{n=0}^{\infty} ng_n(x)t^{n} - \sum_{n=0}^{\infty} ng_n(x)t^{n+1}$$
$$= x\sum_{n=0}^{\infty} g_n'(x)t^{n-1} - x^2\sum_{n=0}^{\infty} g_n'(x)t^{n} - x\sum_{n=0}^{\infty} g_n'(x)t^{n+1}$$
$$+2\sum_{n=0}^{\infty} g_n'(x)t^{n} - 2x\sum_{n=0}^{\infty} g_n'(x)t^{n+1} - 2\sum_{n=0}^{\infty} g_n'(x)t^{n+2} + (2t)(a+b) + (2a+1).$$

By equating the coefficients of t^{n-1} on both the sides, we finally derive (5.2). □

Theorem 5.3. *For the generalized Fibonacci polynomials $g_n(x)$, we derive the following identities*

(i) $g_{n+1}'(x) - g_{n-1}'(x) = xg_n'(x) + g_n(x)$,

(ii) $g_{n+1}'(x) - (1-x^2)g_{n-1}'(x)$
$= (x+1)g_n(x) - x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x) + 3xg_{n-2}'(x) + 2g_{n-3}'(x); n \geq 3$,

(iii) $(2-x^2)g_{n-1}'(x)$
$= xg_n(x) - x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x) - xg_n'(x) + 3xg_{n-2}'(x) + 2g_{n-3}'(x); n \geq 3$,

(iv) $(2-x^2)g_{n-1}'(x) = x(1-x^2)g_n'(x) + 3xg_{n-2}'(x) + 2g_{n-3}'(x) + (n+2-x^2)g_n(x)$
$- x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x); n \geq 3$.

Proof. Differentiating (3.2) both sides with respect to x, we obtain

$$\text{(i)} \qquad g_{n+1}'(x) - g_{n-1}'(x) = xg_n'(x) + g_n(x).$$

Using Theorem 5.2 in (i), we derive

$$\text{(ii)} \quad g_{n+1}'(x) + (1-x^2)g_{n-1}'(x) = (n+1)g_n(x) - x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x) + 3xg_{n-2}'(x) + 2g_{n-3}'(x).$$

On subtracting (i) from (ii), we prove

$$\text{(iii)} \quad (2-x^2)g_(n-1)'(x) = ng_n(x) - x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x) - xg_n'(x) + 3xg_{n-2}'(x) + 2g_{n-3}'(x).$$

On multiplying (i) by $(1-x^2)$ and adding it to (ii), we establish

$$\text{(iv)} \qquad (2-x^2)g_{n+1}'(x) = x(1-x^2)g_n'(x) + 3xg_{n-2}'(x) + 2g_{n-3}'(x) + (n+2-x^2)g_n(x).$$

□

Theorem 5.4. *Let $g_n(x)$ be the nth term of a generalized Fibonacci polynomial, then*

$$ng'_{n+1}(x) - (n+2-x^2)g'_{n-1}(x) - (n+1)xg'_n(x) + 2g'_{n-3}(x) + 3xg'_{n-2}(x) = x(n-1)g_{n-1}(x) + (n-2)g_{n-2}(x); n \geq 3. \tag{5.3}$$

Proof. From Theorem 5.3(i), we have

$$g'_{n+1}(x) - g'_{n-1}(x) - xg'_n(x) = g_n(x). \tag{I}$$

and from Theorem 5.3(ii), we have

(II) $g'_{n+1}(x) + (1-x^2)g'_{n-1}(x) = (n+1)g_n(x) - x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x) + 3xg'_{n-2}(x) + 2g'_{n-3}(x)$.

Substituting the value of $g_n(x)$ from (I) in (II), we finally derive

$g'_{n+1}(x) + (1-x^2)g_(n-1)'(x)$
$= (n+1)g'_{n+1}(x) - g'_{n-1}(x) - xg'_n(x) - x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x) + 3xg'_{n-2}(x) + 2g'_{n-3}(x)$,

which is (5.3). □

Theorem 5.5 (**Explicit Summation formula**). *For generalized Fibonacci polynomials*

$$g_n(x) = (a+b)\left\{\sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{n-2k} - \sum_{k=0}^{[n/2]} \binom{n-k-1}{k} x^{n-2k-1}\right\} + (2a+1)\sum_{k=0}^{[n/2]} \binom{n-k-1}{k} x^{n-2k-1}.$$

Proof. The generating function for generalized Fibonacci polynomials is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} g_n(x)t^n &= [(a+b)(1-xt) + (2a+1)t](1-xt-t^2)^{-1} \\
&= [(a+b)(1-xt) + (2a+1)t]\sum_{n=0}^{\infty}(x+t)^n t^n \\
&= [(a+b)(1-xt) + (2a+1)t]\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k} (nk) x^{n-k}t^k \\
&= [(a+b)(1-xt) + (2a+1)t]\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k}t^{n+k} \\
&= [(a+b)(1-xt) + (2a+1)t]\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(n+k)!}{k!(n)!} x^n t^{n+2k}.
\end{aligned}$$

On equating the coefficients of t^n on both sides, we prove

$$g_n(x) = (a+b)\left\{\sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{n-2k} - \sum_{k=0}^{[n/2]} \binom{n-k-1}{k} x^{n-2k-1}\right\} + (2a+1)\sum_{k=0}^{[n/2]} \binom{n-k-1}{k} x^{n-2k-1}.$$

□

Theorem 5.6. ***A Variant Property:*** *For generalized Fibonacci polynomials*

$$g_{n-2}(x)g_{n+1}(x) - g_{n-1}(x)g_n(x) = (-1)^{n-2}x[(2a+1)(a+b)x + (2a+1)^2 + (a+b)^2].$$

Proof. We know that the Binet's formula for generalized Fibonacci polynomials is given by

$$g_n(x) = (A\alpha^n + B\beta^n).$$

Therefore,

$$g_{n-2}(x)g_{n+1}(x) - g_{n-1}(x)g_n(x)$$

$$
\begin{aligned}
&= (A\alpha^{n-2} + B\beta^{n-2})(A\alpha^{n+1} + B\beta^{n+1}) - (A\alpha^{n-1} + B\beta^{n-1})(A\alpha^{n} + B\beta^{n}) \\
&= (A^2\alpha^{2n-1} + AB\alpha^{n-2}\beta^{n+1} + AB\alpha^{n+1}\beta^{n-2} + B^2\beta^{2n-1}) \\
&-(A^2\alpha^{2n-1} + AB\alpha^{n-1}\beta^{n} + AB\alpha^{n}\beta^{n-1} + B^2\beta^{2n-1}) \\
&= AB(\alpha^{n-2}\beta^{n+1} + \alpha^{n+1}\beta^{n-2} + \alpha^{n-1}\beta^{n} + \alpha^{n}\beta^{n-1}) \\
&= AB(\alpha\beta)[(\alpha+\beta)(\beta^2 - \alpha\beta + \alpha^2) - \alpha\beta(\alpha+\beta)] \\
&= AB(\alpha\beta)^{n-2}(\alpha+\beta)(\alpha-\beta)^2 \\
&= (-1)^{n-2}x[(2a+1)(a+b)x - (2a+1)^2 + (a+b)^2].
\end{aligned}
$$

For $a = 0, b = 0, x = 1$, the above identity reduces to the identity for classical Fibonacci sequence. □

6 Some Identities of generalized Lucas polynomial

Next, we explore the Lucas counterparts of Catalan's identity which have been stated for Fibonacci due to Sikhwal et al. [9].

Theorem 6.1. *Let $k_n(x)$ be the nth term of generalized Lucas polynomial, then*

$$k_n^2(x) - k_{n+r}(x)k_{n-r}(x) = (-1)^{n-r}\left[\frac{a^2(-1)^r}{2} - \frac{ak_{2r}(x)}{2}\right].$$

Proof. Binet's formula for Lucas polynomial is given by

$$k_n(x) = A(\alpha^n + \beta^n).$$

Therefore,

$$
\begin{aligned}
&k_n^2(x) - k_{n+r}(x)k_{n-r}(x) = [A(\alpha^n + \beta^n)]^2 - A(\alpha^{n+r} + \beta^{n+r})A(\alpha^{n-r} + \beta^{n-r}) \\
&= [A(\alpha^2 n + \beta^2 n + 2\alpha^n\beta^n)]^2 - A^2(\alpha^2 n + \alpha^(n+r)\beta^{n-r} + \alpha^{n-r}\beta^{n+}) + \beta^2 n) \\
&= 2A^2(\alpha\beta)^n - A^2(\alpha\beta)^{n-r}(\alpha^2 r + \beta^2 r) \\
&= 2A^2(-1)^n - A(-1)^{n-r}k_{2r}(x) \\
&= (-1)^n 2A^2 - A(-1)^{-r}k_{2r}(x) \\
&= (-1)^{n-r}\left\{\frac{a^2(-1)^r}{2} - \frac{ak_{2r}(x)}{2}\right\}.
\end{aligned}
$$

□

The following theorem gives the identity for Lucas polynomial which is already derived for generalized Fibonacci polynomials known as d'Ocagne's identity in Sikhwal et al. [10].

Theorem 6.2. *If the nth term of generalized Lucas polynomial is $k_n(x)$, then*

$$k_m(x)k_{n+1}(x) - k_{m+1}(x)k_n(x) = \frac{a}{2}\{(-1)^{n+1}k_{m-n-1}(x) - (-1)^{m+1}k_{n-m-1}(x)\}.$$

Proof. Binet's formula for Lucas polynomials is given by

$$k_n(x) = A(\alpha^n + \beta^n).$$

Therefore,

$$
\begin{aligned}
&k_m(x)k_{n+1}(x) - k_{m+1}(x)k_n(x) \\
&= A(\alpha^m + \beta^m)A(\alpha^{n+1} + \beta^{n+1}) - A(\alpha^{m+1} + \beta^{m+1})A(\alpha^n + \beta^n) \\
&= A^2(\alpha^m\beta^{n+1} + \alpha^{n+1}\beta^m - \alpha^{m+1}\beta^n - \alpha^n\beta^{m+1}) \\
&= A^2\{(\alpha\beta)^{n+1}(\alpha^{m-n-1} + \beta^{m-n-1}) - (\alpha\beta)^{m+1}(\alpha^{n-m-1} + \beta^{n-m-1})\} \\
&= A\{(-1)^{n+1}k_{m-n-1}(x) - (-1)^{m+1}k_{n-m-1}(x)\} \\
&= \frac{a}{2}\{(-1)^{n+1}k_{m-n-1}(x) - (-1)^{m+1}k_{n-m-1}(x)\}.
\end{aligned}
$$

□

The next theorem gives the relevant results to Theorems 6.1 and 6.2 for our Lucas polynomials.

Theorem 6.3. *Let $k_n(x)$ be the nth term of generalized Lucas polynomial, then* *(i)* $k_n^2(x) + k_{n+r}(x)k_{n-r}(x) = ak_2n(x) + \frac{a^2}{2}(-1)^n + \frac{a}{2}(-1)^{n-r}k_{2r}(x)$,

(ii) $k_m(x)k_{n+1}(x) + k_{m+1}(x)k_n(x)$
$= \frac{a}{2}\{2k_{m+n+1}(x) + (-1)^{n+1}k_{m-n-1}(x) + (-1)^{m+1}k_{n-m-1}(x)\}$.

Proof. (i). With the help of Binet's formula, we establish

$$\begin{aligned}
&k_n^2(x) + k_{n+r}(x)k_{n-r}(x) = [A(\alpha^n + \beta^n)]^2 + A(\alpha^{n+r} + \beta^{n+r})A(\alpha^{n-r} + \beta^{n-r})\\
&= [A(\alpha^{2n} + \beta^{2n} + 2\alpha^n\beta^n)]^2 + A^2(\alpha^{2n} + \alpha^{n+r}\beta^{n-r} + \alpha^{n-r}\beta^{n+r} + \beta^{2n})\\
&= 2A^2(\alpha^{2n} + \beta^{2n}) + 2A^2(\alpha\beta)^n + A^2(\alpha\beta)^{n-r}(\alpha^{2r} + \beta^{2r})\\
&= 2Ak_{2n}(x) + 2A^2(-1)^n + A(-1)^{n-r}k_{2r}(x)\\
&= ak_{2n}(x) + \frac{a^2}{2}(-1)^n + \frac{a}{2}(-1)^{n-r}k_{2r}(x).
\end{aligned}$$

Proof (ii). With the help of Binet's formula, we derive

$$\begin{aligned}
&k_m(x)k_{n+1}(x) + k_{m+1}(x)k_n(x)\\
&= A(\alpha^m + \beta^m)A(\alpha^{n+1} + \beta^{n+1}) + A(\alpha^{m+1} + \beta^{m+1})A(\alpha^n + \beta^n)\\
&= A\{2A(\alpha^{m+n+1} + \beta^{m+n+1}) + (\alpha\beta)^{n+1}A(\alpha^{m-n-1} + \beta^{m-n-1}) + (\alpha\beta)^{m+1}A(\alpha^{n-m-1} + \beta^{n-m-1})\}\\
&= A\{2k_{m+n+1}(x) + (-1)^{n+1}k_{m-n-1}(x) + (-1)^{m+1}k_{n-m-1}(x)\}\\
&= \frac{a}{2}\{2k_{m+n+1}(x) + (-1)^{n+1}k_{m-n-1}(x) + (-1)^{m+1}k_{n-m-1}(x)\}.
\end{aligned}$$

□

7 Conclusion

In this paper, we have defined generalized Fibonacci and generalized Lucas polynomials. We have stated and derived many properties of our generalized Fibonacci polynomial and generalized Lucas polynomial through generating function and Binet's formula. Many other identities like Catalan's identity and d'Ocagne's identity can be derived easily from our generalized Fibonacci polynomial. Similarly, identities proved in section 5 for our generalized Fibonacci polynomial can also be proved for the generalized Lucas polynomial.

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ON THE DEGREE OF APPROXIMATION OF FUNCTION $f \in W(L_p, \xi(t))$ CLASS BY $(C,2)(e,c)$ MEANS OF ITS FOURIER SERIES

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Abstract

We study on degree of approximation of function belonging to weighted $(L_p, \xi(t))$ class by $(C,1)(e,c)$ mean and weighted $(L_p, \xi(t))$ class by $(C,2)(E,q)$ has been discussed by Rathore and Shrivastava. Since (e,c) includes (E,q) method, so for obtaining more generalized result we replace (E,q) by (e,c) mean. Which is a regular method of summation for $c > 0$. In this paper we obtain the degree of approximation of the function belonging to weighted $(L_p, \xi(t))$ class by $(C,2)(e,c)$ product means of its Fourier series has been proved.

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1 Introduction

The (e,c) summability method was introduced by Hardy and Littlewood [6], which is a regular for $c > 0$ including the method of summability for Borel, (E,q) etc. We study on approximation of f belonging to many classes. Also $W(L_p, (\xi(t))$ by Cesăro mean, Nörlund mean, has been discussed by several researchers like Alexits [1], Khan [6], Chandra [3], Sahney and Goel [17], Quereshi [12], Shrivastava and Verma [19], Mishra et al.[10] etc. Rathore and Shrivastava [13] extended the result on degree of approximation of a function belonging to $W(L_r, \xi(t))$ class by $(C,1)(e,c)$ means of Fourier series. Further Rathore and Shrivastava [14] studied about product summability on approximation of a function belonging to $W(L_r, \xi(t))$ class by $(C,2)$ (E,q) means. In this direction several researchers like Lal and Singh [9], Lal and Kushwaha [8], Nigam [11], Albayrak, Koklu and Bayramov [2], Rathore, Shrivastava and Mishra ([15], [16]) etc. Recently Kushwaha [7] has determined on approximation of function by $(C,2)(E,1)$ product summability method of Fourier series, but till now no work done to extend the result on approximation of function $f \epsilon W(L_p, \xi(t))$ class by $(C,2)(e,c)$ mean has been seen.

2 Definition and Notations

Let $f(x)$ be periodic with period -2π and integrable in the sense of Lebesgue. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

with n^{th} partial sum $S_n(f;x)$.

A series $\sum_{n=0}^{\infty} u_n$ with the sequence of partial sum $\{S_n\}$ is said to be summable (e,c), $(c > 0)$ to sum S.

Let $\left\{t_n^{(e,c)}\right\}$ denotes the sequence of (e,c) mean of the sequence $\{S_n\}$. If the (e,c) transform of S_n defined as

$$\lim_{n\to\infty} t_n^{(e,c)} = \lim_{n\to\infty} \sqrt{\frac{c}{\pi n}} \sum_{r=-\infty}^{\infty} \exp\left(-\frac{cr^2}{k}\right) S_{k+r} \tag{2.2}$$

exists, where $S_{k+r} = 0$, when $k + r < 0$.

We write

$$\left\| t_n^{(e,c)} - f \right\| = \sup_{-\pi \le x \le \pi} \left| t_n^{(e,c)}(f : x) - f(x) \right|, \tag{2.3}$$

where $t_n^{(e,c)}(f:x)$ is n^{th} (e,c) means of the Fourier series f at x. Thus if

$$t_n^{(e,c)}(f;x) - f(x) = \frac{1}{\pi}\sqrt{\frac{c}{\pi n}} \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \left[\sum_{r=-k}^{\infty} \exp\left(-\frac{cr^2}{k}\right) \sin\left(k+r+\frac{1}{2}\right) t \right] dt. \tag{2.4}$$

The series $\sum_{n=0}^{\infty} u_n$ with the partial sum S_n is said to be summable (e,c) to the definite number S, (see Hardy [4]).

Let $\left\{t_n^{(C,2)}\right\}$ denote the sequence of $(C,2)$ mean of the sequence $\{S_n\}$. If the $(C,2)$ transform of S_n is defined as

$$t_n^{(C,2)}(f:x) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) S_k \to S \quad \text{as } n \to \infty \tag{2.5}$$

then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number S by $(C,2)$ method. Thus if

$$t_n^{(C,2)(e,c)}(f:x) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) t_k^{(e,c)} \to S \text{ as } n \to \infty \tag{2.6}$$

where $t_n^{(C,2)(e,c)}$ denotes the sequence of $(C,2)(e,c)$ product mean of the sequence S_n, the series $\sum_{n=0}^{\infty} \mathrm{u_n}$ is said to be summable to the number S by $(C,2)(e,c)$ method.

We observe that $(C,2)(e,c)$ method is regular if $c > 0$.

A function $f \in W(L_p, \xi(t))$ class, if

$$\left(\int_0^{2\pi} \left(\left| [f(x+t) - f(x)] \sin^{\beta} x \right|^p dx \right) \right)^{1/p} = O(\xi(t)), (\beta \geq 0). \tag{2.7}$$

Given a positive increasing function $\xi(t)$ and an integer $p \geq 1$, we observe that

$$W(L_p, \xi(t)) \stackrel{\beta=0}{\longrightarrow} L(\xi(t), p) \stackrel{\xi(t)=t^{\alpha}}{\longrightarrow} L(\alpha, p) \stackrel{p\to\infty}{\longrightarrow} \text{Lip } \alpha.$$

That is

$$\text{Lip}\,\alpha \subseteq \text{Lip}(\alpha, p) \subseteq \text{Lip}(\xi(t), p) \subseteq W(L_p, \xi(t)), \text{ for } 0 < \alpha \leq 1, p \geq 1.$$

Now we define norm by

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p dx \right)^{1/p}, p \geq 1. \tag{2.8}$$

The degree of approximation $E_n(f)$ be given by

$$E_n(f) = \min \|T_n - f\|_p, \tag{2.9}$$

where $T_n(x)$ is a trigonometric polynomial of degree n by (see Zygmund [21]).

We shall use following notation:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x). \tag{2.10}$$

3 Inequalities

We use the following inequalities in our further investigations

$$\sum_{r=k+1}^{\infty} r \exp\left(-\frac{cr^2}{k}\right) \leq \frac{k}{2c} \exp(-ck) \tag{3.1}$$

$$\left| \sum_{r=k+1}^{\infty} \exp\left(-\frac{cr^2}{k}\right) \sin\left(k+r+\frac{1}{2}\right) t \right| \leq \frac{kt}{2c} \exp(-ck), \tag{3.2}$$

$$\sum_{r=k+1}^{\infty} \exp\left(-\frac{cr^2}{k}\right) \cos(rt) = O\left\{ \frac{\exp(-ck)}{t} \right\}, \tag{3.3}$$

$$1 + 2\sum_{r=1}^{\infty} \exp\left(-\frac{cr^2}{k}\right) \cos(rt) = \sqrt{\frac{\pi k}{c}} \left\{ \exp\left(\frac{-kt^2)}{4c}\right) + O\left(\exp\left(\frac{-k\pi}{4c}\right)\right) \right\}. \tag{3.4}$$

The inequality (3.2) follows from (3.1). (3.3) may be obtained by using Able's Lemma and (3.4) may be obtained by classical formula for theta function by Siddiqui [18] and (3.1) is due to Shrivastava & Verma [19].

4 Main Theorem

We prove the following theorem

Theorem 4.1. *If $f : R \to R$ is 2π-periodic function, Lebesgue integrable on $[0, 2\pi]$ and belonging to the $W(L_p, \xi(t))$ class then the degree of approximation of f by the $(C, 2)(e, c)$ product summability means of Fourier series satisfies*

$$\left\| t_n^{(C,2)(e,c)} - f(x) \right\|_p = O\left[(n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right] \tag{4.1}$$

provided $\xi(t)$ satisfies the following condition :

$$\left\{ \frac{\xi(t)}{t} \right\} \text{ be a decreasing sequence,} \tag{4.2}$$

$$\left\{ \int_0^{1/n+1} \left(\frac{t|\phi(t)|}{\xi(t)} \right)^p \sin^{\beta p} t dt \right\}^{1/p} = O\left(\frac{1}{n+1}\right), \tag{4.3}$$

$$\left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}\phi(t)}{\xi(t)} \right)^p \sin^{\beta p} t dt \right\}^{1/p} = O\left((n+1)^{\delta}\right), \tag{4.4}$$

where δ is an arbitrary number such that $q(1-\delta) - 1 > 0, \frac{1}{p} + \frac{1}{q} = 1$. conditions (4.3) and (4.4) hold uniformly in x and $t_n^{(C,2)(e,c)}$ is (C, 2) (e, c) mean of the Fourier series (2.1).

5 Lemmas

We shall use the following Lemmas

Lemma 5.1. *Let* $M_n(t) = \frac{1}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (n-k+1) \left[\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin t/2} \right]$.

Then $|M_n(t)| = O(n+1)$, for $0 < t < \frac{\pi}{(n+1)}$.

Proof. Applying $\sin nt \le n \sin t$, for $0 < t < \frac{\pi}{(n+1)}$, we have

$$\begin{aligned} |M_n(t)| &\le \frac{1}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (n-k+1) \left[\frac{(2k+1)\sin t/2}{\sin t/2} \right] \\ &\le \frac{1}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (n-k+1)(2k+1) \\ &= \frac{(n+1)}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (2k+1) - \frac{1}{(n+1)(n+2)\pi} \sum_{k=0}^{n} k(2k+1) \\ &= \frac{(n+1)^2}{(n+2)\pi} - \frac{1}{(n+1)(n+2)\pi} \left[\sum_{k=0}^{n} \left(2k^2 + k\right) \right] \\ &= \frac{(n+1)^2}{(n+2)\pi} - \frac{1}{(n+1)(n+2)\pi} \cdot \frac{n(n+1)(2n+1)}{3} - \frac{1}{(n+1)(n+2)\pi} \frac{n(n+1)}{2} \\ &= O(n+1). \end{aligned} \tag{5.1}$$

□

Lemma 5.2. *Let $|M_n(t)| = O\left(\frac{1}{t}\right)$, for $\frac{\pi}{(n+1)} \le t \le \pi$.*

Proof. Applying Jordon's Lemma $\sin\left(\frac{t}{2}\right) \ge t/\pi$ and $\sin kt \le 1$ for $\frac{\pi}{(n+1)} \le t \le \pi$

$$\begin{aligned} |M_n(t)| &\le \frac{1}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (n-k+1) \left[\frac{1}{t/\pi} \right] \\ &= \frac{(n+1)\pi}{(n+1)(n+2)t\pi} - \frac{\pi}{(n+1)(n+2)t\pi} \sum_{k=0}^{n} k \end{aligned} \tag{5.2}$$

$$= \frac{1}{(n+2)t} - \frac{n(n+1)}{2(n+1)(n+2)t}$$
$$= O\left(\frac{1}{t}\right).$$

□

6 Proof of the Main Theorem

Using (Titchmarsh [20]) and Riemann - Lebesgue theorem, the partial sum $S_n(f;x)$ of the series (2.1) is given by

$$S_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}} \sin\left(n+\frac{1}{2}\right) t dt. \tag{6.1}$$

If $t_n^{(e,c)}$ denotes (e,c) transform of $S_n(f;x)$ then

$$\begin{aligned}
t_n^{(e,c)}(f;x) - f(x) &= \frac{1}{2\pi}\sqrt{\frac{c}{\pi n}}\int_0^\pi \frac{\emptyset_x(t)}{\sin t/2}\left[\sum_{r=-k}^{\infty} \exp\left(-\frac{cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t\right] dt \\
&= \frac{1}{2\pi}\sqrt{\frac{c}{\pi n}}\int_0^\pi \frac{\emptyset_x(t)}{\sin t/2}\left[\left\{1+2\sum_{r=1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos rt\right\}\sin\left(k+\frac{1}{2}\right)t\right. \\
&\qquad \left. + \sum_{r=k+1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t\right] dt \\
&= \frac{1}{\pi}\sqrt{\frac{c}{\pi n}}\int_0^\pi \frac{\phi_x(\mathrm{t})}{\sin t/2}\left[\left\{1+2\sum_{r=1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos rt\right\}\sin\left(k+\frac{1}{2}\right)t\right] dt \\
&-2\sum_{r=k+1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos rt \sin\left(k+\frac{1}{2}\right)t + \sum_{r=1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right) dt.
\end{aligned}$$

For $(C,2)(e,c)$ transform, $t_n^{(C,2)(e,c)}(f;x)$ of $S_n(f;x)$, we write

$$t_n^{(C,2)(e,c)}(f;x) - f(x) = \frac{2}{2\pi(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\sqrt{\frac{c}{\pi k}}\int_0^\pi \frac{\emptyset_x(t)}{\sin t/2}$$
$$\cdot\left[\begin{array}{c} \left\{1+2\sum_{r=1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos rt\right\}\sin\left(k+\frac{1}{2}\right)t \\ -2\sum_{r=k+1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos rt\sin\left(k+\frac{1}{2}\right)t \\ +\sum_{r=k+1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t \end{array}\right] dt,$$

$$I = I_1 + I_2 + I_3 \text{ (say)} . \tag{6.2}$$

Now,

$$\begin{aligned}
I_1 \leq &\frac{1}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\sqrt{\frac{c}{\pi k}}\int_0^\pi \frac{\emptyset_x(t)}{\sin t/2} \\
&\cdot\left[\left\{1+2\sum_{r=1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos rt\right\}\sin\left(k+\frac{1}{2}\right)t\right] dt \\
= &\frac{1}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\sqrt{\frac{c}{\pi k}}\int_0^\pi \frac{\emptyset_x(t)}{\sin t/2} \\
&\cdot\sqrt{\frac{\pi k}{c}}\left[\left\{\exp\left(\frac{-kt^2}{4c}\right) + O\left(\exp\left(\frac{-k\pi}{4c}\right)\right)\right\}\sin\left(k+\frac{1}{2}\right)t\right] dt \\
= &\frac{1}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\int_0^\pi \frac{\emptyset_x(t)}{\sin t/2}\sin\left(k+\frac{1}{2}\right)t\exp\left(\frac{-kt^2}{4c}\right) dt
\end{aligned}$$

$$+\frac{1}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\left[\left\{\int_0^{\pi}\frac{\emptyset_x(t)}{\sin t/2}\sin\left(k+\frac{1}{2}\right)t\cdot O\left(\exp\left(\frac{-k\pi}{4c}\right)\right)\right\}dt\right],$$

$$I_1 = I_{1.1} + I_{1.2} \quad \text{(say)}. \tag{6.3}$$

Now,

$$I_{1.1} = \frac{1}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\int_0^{\pi}\frac{\emptyset_x(t)}{\sin t/2}\left[\left\{\sin\left(k+\frac{1}{2}\right)t\exp\left(\frac{-kt^2)}{4c}\right)\right\}\right]dt$$

$$= O\left(\exp\left(\frac{-nt^2)}{4c}\right)\right)\int_0^{\pi}\emptyset_x(t)M_n(t)dt, \quad \text{using Lemma 5.1}$$

Then

$$|I_{1.1}| \le O(1)\left[\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}.\right]\emptyset_x(t)M_n(t)dt$$

$$I_{1.1} = I_{1.11} + I_{1.12} \tag{6.4}$$

Now

$$|I_{1.11}| \le \int_0^{\pi/n+1}|\varnothing_x(t)|\,|M_n(t)|\,dt$$

We have

$$|\emptyset_x(x+t) - \emptyset_x(x)| \le |f(v+x+t) - f(v+x)| + |f(v-x-t) - f(v-x)|.$$

Hence by Minkowiski inequality

$$\left[\int_0^{2\pi}|\,\{|\,\emptyset_x(x+t)\ \ -\emptyset_x(x)\}\,\sin^\beta x|^p\,dx\right]^{1/p}$$

$$\le\left[\int_0^{2\pi}\left|\{f(v+x+t)-f(v+x)\}\sin^\beta x\right|^p dx\right]^{1/p}$$

$$+\left[\int_0^{2\pi}\left|\{f(v-x-t)-f(v-x)\}\sin^\beta x\right|^p dx\right]^{1/p}$$

$$=O(\xi(t)).$$

Then $f\in W\left(L_p,\xi(t)\right)\Rightarrow \emptyset_x(t)\in W\left(L_p,\xi(t)\right)$,
Applying Hölder's inequality and second mean value theorem for integral

$$|I_{1.11}| \le \left[\int_0^{\pi/n+1}\left\{\frac{t\,|\emptyset_x(t)|\sin^\beta t}{\xi(t)}\right\}^p dt\right]^{1/p}\left[\int_0^{\pi/n+1}\left\{\frac{\xi(t)\,|M_n(t)|}{t\sin^\beta t}\right\}^q dt\right]^{1/q} \tag{6.5}$$

$$= O\left(\frac{\pi}{(n+1)}\right)\left[\int_0^{\pi/n+1}\left\{\frac{\xi(t)(n+1)}{t^{1+\beta}}\right\}^q dt\right]^{1/q}$$

$$= O\left\{\xi\left(\frac{\pi}{(n+1)}\right)\right\}\left[\left(t^{-(1+\beta)q++1}\right)^{1/q}\right]_0^{\pi/n+1}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\}.$$

Now

$$I_{1.12} \le \int_{\pi/n+1}^{\pi}\emptyset_x(t)M_n(t)dt$$

Using Hölder's inequality $|\sin t|<1, |\sin t|\ge\left(\frac{2t}{\pi}\right)$, Lemma 5.2 and second mean value theorem

$$|I_{1.12}| \le \left[\int_{\pi/n+1}^{\pi}\left\{\frac{t^{-\delta}\,|\emptyset_x(t)|\sin^\beta t}{\xi(t)}\right\}^p dt\right]^{1/p}\left[\int_{\pi/n+1}^{\pi}\left\{\frac{\xi(t)\,|M_n(t)|}{t^{-\delta}\sin\beta}\right\}^q dt\right]^{1/q} \tag{6.6}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{\pi/n+1}^{\pi}\left\{\frac{\xi(t)}{t^{1+\beta-\delta}}\right\}^{q}dt\right]^{1/q}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{1}^{n+1}\left\{\frac{\xi(\pi/y)}{\left(\left({}^{1}/y\right)^{(1+\beta-\delta)q}\right.}\right\}^{q}\frac{dy}{y^{2}}\right]^{1/q} \qquad \text{Lput } t=(\pi/y)\Big]$$

$$= O\left\{(n+1)^{\delta}\right\}\xi\left(\frac{\pi}{n+1}\right)\left[\int_{1}^{n+1}\frac{dy}{y^{-(1+\beta-\delta)q+2}}\right]^{1/q}$$

$$= O\left\{(n+1)^{\delta}\right\}\xi\left(\frac{\pi}{n+1}\right)\left[y^{(1+\beta-\delta)-1/q}\right]_{1}^{n+1}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\}.$$

Now,

$$I_{1.2} = \frac{1}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\left[\left\{\int_{0}^{\pi}\frac{\emptyset_x(t)}{\sin t/2}\sin\left(k+\frac{1}{2}\right)tO\left(\exp\left(\frac{-k\pi}{4c}\right)\right)\right\}dt\right]$$

$$= O\left(\exp\left(\frac{-n\pi}{4c}\right)\right)\int_{0}^{\pi}\emptyset_x(t)M_n(t)dt \qquad \text{(using Lemma 5.1).}$$

Then,

$$I_{1.2} = \left[\int_{0}^{\pi/n+1}+\int_{\pi/n+1}^{\pi}\cdot\right]\emptyset_x(t)M_n(t)dt, \tag{6.7}$$

$$I_{1.2} = I_{1.21}+I_{1.22}.$$

Now,

$$I_{1.21} = \int_{0}^{\pi/n+1}\emptyset_x(t)M_n(t)dt$$

Using Hölder's inequality and second mean value theorem

$$|I_{1.21}| \le \left[\int_{0}^{\pi/n+1}\left\{\frac{t\,|\emptyset_x(t)|\sin^{\beta}t}{\xi(t)}\right\}^{p}dt\right]^{1/p}\left[\int_{0}^{\pi/n+1}\left\{\frac{\xi(t)\,|M_n(t)|}{t\sin^{\beta}t}\right\}^{q}dt\right]^{1/q} \tag{6.8}$$

$$= O\left(\frac{\pi}{n+1}\right)\left[\int_{0}^{\pi/n+1}\left\{\frac{\xi(t)(n+1)}{t^{\beta+1}}\right\}^{q}dt\right]^{1/q}$$

$$= O\left\{(n+1)^{\beta+1/p}\xi\left(\frac{1}{n+1}\right)\right\}, \text{ since } \frac{1}{p}+\frac{1}{q}=1.$$

Now,

$$I_{1.22} = \int_{\pi/n+1}^{\pi}\emptyset_x(t)M_n(t)dt$$

Using Hölder inequality and $|\sin t|<1, |\sin t|\ge\left(\frac{2t}{\pi}\right)$

$$|I_{1.22}| \le \left[\int_{\pi/n+1}^{\pi}\left\{\frac{t^{-\delta}\,|\emptyset_x(t)|\sin^{\beta}t}{\xi(t)}\right\}^{p}dt\right]^{1/p}\left[\int_{\pi/n+1}^{\pi}\left\{\frac{\xi(t)M_n(t)}{t^{-\delta}\sin\beta}\right\}^{q}dt\right]^{1/q} \tag{6.9}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{\pi/n+1}^{\pi}\left\{\frac{\xi(t)}{t^{1+\beta-\delta}}\right\}^{q}dt\right]^{1/q}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{1}^{n+1}\left\{\frac{\xi(\pi/y)}{(\pi/y)^{(1+\beta-\delta)q}}\right\}^{q}\frac{dy}{y^{2}}\right]^{1/q} \qquad \text{put } t=(\pi/y)$$

$$= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\}.$$

Now,

$$I_2 = -\frac{1}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (n-k+1) \sqrt{\frac{c}{\pi k}} \int_0^{\pi} \frac{\emptyset_x(t)}{\sin t/2}$$
$$\cdot \left[2 \sum_{r=k+1}^{\infty} \exp\left(-\frac{cr^2}{k}\right) \cos rt \sin\left(k+\frac{1}{2}\right) t \right] dt$$

$$I_2 \leq \frac{-2}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (n-k+1) \sqrt{\frac{c}{\pi k}} \int_0^{\pi} \frac{\emptyset_x(t)}{\sin t/2} \left[O\left(\frac{\exp(-ck)}{t}\right) \sin\left(k+\frac{1}{2}\right) t \right] dt$$
$$= O\left(n^{-1/2} \exp(-cn)\right) \int_0^{\pi} \emptyset_x(t) \frac{M_n(t)}{t} dt, \text{ (using inequality (3.3))}$$

Then

$$I_2 = \left[\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi} \cdot \right] \emptyset_x(t) \frac{M_n(t)}{t} dt, \tag{6.10}$$
$$= I_{2.1} + I_{2.2}.$$

Now, using Hölder's inequality

$$I_{2.1} \leq \left[\int_0^{\pi/n+1} \left\{ \frac{t\,|\emptyset_x(t)| \sin^{\beta} t}{\xi(t)} \right\}^p dt \right]^{1/p} \left[\int_0^{\pi/n+1} \left\{ \frac{\xi(t)\,|M_n(t)|}{t^2 \sin \beta} \right\}^q dt \right]^{1/q} \tag{6.11}$$
$$= O\left(\frac{1}{n+1}\right) \left[\int_0^{\pi/n+1} \left\{ \frac{\xi(t)(n+1)}{t^{\beta+2}} \right\}^q dt \right]^{1/q}$$
$$= O\left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\} \quad \left(\text{since } \frac{1}{p} + \frac{1}{q} = 1 \right).$$

Now

$$I_{2.2} = \int_{\pi/n+1}^{\pi} \emptyset_x(t) \frac{M_n(t)}{t} dt.$$

Using Hölder's inequality and similarly

$$I_{2.2} = O\left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\} \quad (\text{ from } I_{1.22}). \tag{6.12}$$

Similarly,

$$I_3 = O\left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\}. \tag{6.13}$$

Now combining (6.2) to (6.13), we set

$$\left| t_n^{(C,2)(e,c)}(f;x) - f(x) \right| = O\left[(n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right],$$
$$\left\| t_n^{(C,2)(e,c)}(f;x) - \mathrm{f(x)} \right\|_p = \left\{ \int_0^{2\pi} \left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\}^p dx \right\}^{1/p}$$
$$= 0\left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\} \left[\left\{ \int_0^{2\pi} dx \right\}^{1/p} \right]$$
$$= 0\left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\}.$$

This completes the proof of the theorem.

7 Corollaries

Following corollaries can be derived from main theorem:

Corollary 7.1. *If $\beta = 0$ and $\xi(t) = t^\alpha$ then the degree of approximation of a function $f \in \mathrm{Lip}(\alpha, p)$, $\quad 0 < \alpha \leq 1$ is given by*

$$\left\| t_n^{(C,2)(e,c)}(f;x) - \mathrm{f(x)} \right\|_p = \mathrm{O}\left\{ \frac{1}{(n+1)^{\alpha - 1/p}} \right\}$$

Corollary 7.2. *If $p \to \infty$ in corollary (7.1), and for $0 < \alpha < 1$.*

$$\left\| t_n^{(C,2)(e,c)}(f;x) - \mathrm{f(x)} \right\|_\infty = 0\left\{ \frac{1}{(n+1)^{\alpha}} \right\}.$$

8 Conclusion

The summability method (e,c) includes method of summability like Borel, $(E,1), (E,q), F(a,q)$ and $[F, d_n]$ then by using the result of main theorem we can derive more generalizing result and also the result of Kushwaha [7] can be derived directly.

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ON ARITHMETIC FUNCTIONS, THEIR EXTENDED COEFFICIENTS: VARIOUS RESULTS AND RELATIONS

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Abstract

In this article, we represent a recurrence relation of the arithmetic function connected with an ascending factorial function, Lah and Stirling numbers. We then obtain a relation of harmonic numbers and again extend the coefficients of these arithmetic functions involving Bell polynomials through introducing the sequence of Hankel type integrals. On the other hand, making some of the extensions of these arithmetic functions, we derive some more results and the summation formulae in terms of Riemann Zeta function.

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1 Introduction

In this representation, we consider the following recurrence relation recently studied by Pathan et al. [17] as

$$f_k(n) = \frac{k}{n}\sum_{j=1}^{n} g_j f_k(n-j), \quad f_k(0) = 1, \quad n, k \geq 1, \tag{1.1}$$

which is satisfied by interesting arithmetic functions [2, 11]. From (1.1), it is clear that $f_k(n)$ is a polynomial of degree n in k

$$f_k(n) = a(n,1)k + a(n,2)k^2 + \cdots + a(n,n-1)k^{n-1} + a(n,n)k^n, \quad n \geq 1, \tag{1.2}$$

where the coefficients $a(n,m)$ are in terms of the quantities g_j, in fact due to [17], we have

$$a(n,n) = \frac{1}{n!}\left(g_1\right)^n, \quad n \geq 1. \tag{1.3}$$

Here in (1.2) the coefficients are

$$a(n,m) = \frac{1}{m!(n-m)!}\sum_{j=1}^{n-m}\left(g_1\right)^{m-j}\binom{m}{j} j! B_{n-m,j}\left(\frac{1!}{2}g_2, \frac{2!}{3}g_3, \ldots, \frac{(n-m-j+1)!}{n-m-j+2}g_{n-m-j+2}\right) \quad \forall n \geq m+1, \tag{1.4}$$

that involving the incomplete exponential Bell polynomials [9, 16, 17, 18].

The relations (1.3) and (1.4) are in harmony with the expressions of Jakimczuk [12, Eqns. (8)-(11)].

Further, we also show that due to Stirling numbers the relations (1.2) and (1.4) imply an important property as given by [17]

$$a(n,1) = \frac{1}{n}g_n = -\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k} f_k(n), \quad n \geq 1, \tag{1.5}$$

and we realize that applications of the results (1.1)-(1.5) for the cases $g_j = 1$ and $g_r = r$. We discuss these conditions in the next section on an application of Stirling numbers [6, 8], Lah numbers [1, 14] and then describe the harmonic numbers [26] and again derive the Truesdell's polynomials [3-5] and their discussions on generalizations.

2 Various properties of (1.5) and applications

In this section, we derive various results due to the formula (1.5) through following theorem:

Theorem 2.1. *Due to Stirling numbers, the results* (1.2) *and* (1.4) *imply the property given by*

$$a(n,1)=\frac{1}{n}g_n=-\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k}f_k(n),\quad n\geq 1. \tag{2.1}$$

Proof. Considering the results (1.2) and (1.4) we find that

$$\begin{aligned}\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k}f_k(n)&=\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k}\sum_{j=1}^{n}a(n,j)k^j,\\&=\sum_{j=1}^{n}a(n,j)\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{j-1},\\&=a(n,1)\sum_{k=1}^{n}(-1)^k\binom{n}{k}+\sum_{t=1}^{n-1}a(n,t+1)\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^t,\\&=-a(n,1)+(-1)^n n!\sum_{t=1}^{n-1}a(n,t+1)S_t^{[n]},\end{aligned} \tag{2.2}$$

but (1.4) gives $a(n,1)=\frac{1}{n}g_n$, and for the Stirling numbers of the second kind [1,10,15,21,23], we have that $S_t^{[n]}=0$, because $t<n$, therefore (2.2) implies (2.1) q.e.d. □

Corollary 2.1. *Applying the results* (1.1) *and* (1.5) *in the Theorem* 2.1 *and choosing* $g_j=j\geq 1$*, following relation holds true*

$$f_k(n)=\sum_{l=1}^{n}\frac{1}{l!}\binom{n-1}{l-1}k^l,\quad n\geq 1. \tag{2.3}$$

Proof. In (1.1), choosing $g_j=j\geq 1$, we have

$$f_k(n)=\frac{k}{n}\sum_{j=1}^{n}jf_k(n-j),\quad f_k(0)=1, \tag{2.4}$$

$$a(n,m)=\frac{1}{m!}\sum_{j=1}^{n-m}\binom{m}{j}\binom{n-m-1}{j-1}\overset{[17]}{=}\frac{1}{m!}\binom{n-1}{m-1}, \tag{2.5}$$

where in (2.5), we applied the following relation in terms of the Lah numbers $[1,14,15,23]$ as

$$B_{n-m,j}(1!,2!,\ldots,(n-m-j+1)!)=L_{n-m}^{[j]}=\frac{(n-m)!}{j!}\binom{n-m-1}{j-1}. \tag{2.6}$$

Hence making an appeal to the results (2.4)-(2.6), we find the relation

$$f_k(n)=\sum_{l=1}^{n}\frac{1}{l!}\binom{n-1}{l-1}k^l,\quad n\geq 1, \tag{2.7}$$

which verifies the relation (2.3). □

Corollary 2.2. *Applying the results* (1.1) *and* (1.5) *in the Theorem* 2.1*, for all* $j, g_j=1$*, following relations hold true*

$$f_k(n)=\frac{(-1)^n}{n!}\sum_{j=0}^{n}(-1)^jS_n^{(j)}k^j=\frac{1}{n!}(k)_n,\quad n\geq 1, \tag{2.8}$$

where $S_n^{(j)}$ *are the Stirling numbers* $\forall j=1,2,3,\ldots,n$.

Proof. In the results (1.1) and (1.4), setting $g_j = 1 \quad \forall j$, we have

$$f_k(n) = \frac{k}{n}\sum_{j=1}^{n} f_k(n-j), \quad f_k(0) = 1 \tag{2.9}$$

and

$$a(n,m) = \frac{(-1)^n}{m!}\sum_{l=1}^{n-m}\frac{(-1)^l l!}{(n-m+l)!}S^{(l)}_{n-m+l}\delta_{lm} = \frac{(-1)^{n-m}}{n!}S^{(m)}_n, \tag{2.10}$$

where the following identity [20] in terms of the Stirling numbers of the first kind [1, 20, 21, 23] was employed as

$$B_{n-m,j}\left(\frac{1!}{2},\frac{2!}{3},\ldots,\frac{(n-m-j+1)!}{n-m-j+2}\right) = (-1)^{n-m-j}(n-m)!\sum_{l=0}^{j}\frac{(-1)^l}{(j-l)!(n-m+l)!}S^{(l)}_{n-m+l}. \tag{2.11}$$

Hence, by the results (2.9)-(2.11), we obtain following identities

$$f_k(n) = \frac{(-1)^n}{n!}\sum_{j=0}^{n}(-1)^j S^{(j)}_n k^j \overset{[21]}{=} \frac{1}{n!}(k)_n, \quad n \geq 1, \tag{2.12}$$

such that $(k)_n = k(k+1)\cdots(k+n-1)$.

Finally, the identities in (2.12) give us the relations (2.8). $\square$

Thus the Corollary 2.2 implies an interesting recurrence relation for the ascending factorial function

$$(k)_n = (n-1)!k\sum_{j=1}^{n}\frac{1}{(n-j)!}(k)_{n-j}, \quad n,k \geq 1. \tag{2.13}$$

If we remember that $(n)_n = \frac{\Gamma(2n)}{\Gamma(n)} = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma\left(n+\frac{1}{2}\right)$, here given that $\Gamma(n+1) = n!, n \geq 1$.

Then due to the formula (2.13), we find the results

$$\sum_{j=1}^{n}\frac{1}{(n-j)!}(n)_{n-j} = \frac{1}{2}\frac{(2n)!}{(n!)^2} = \frac{1}{2}\binom{2n}{n}, \quad n \geq 1. \tag{2.14}$$

Furthermore, if we accept that in (2.8) the symbol k is a continuous variable, then we apply $\frac{d}{dk}$ to (2.13) and then we make $k = 1$ to deduce the following identity [26] involving harmonic numbers $[1, 10, 21, 23]$

$$\sum_{j=1}^{n} H_j = (n+1)H_n - n, \quad n \geq 1, \tag{2.15}$$

where in (2.15), we applied the expression

$$\left[\frac{d}{dx}(x)_m\right]_{x=1} = m!H_m. \tag{2.16}$$

Remark 2.1. *If F is the generating function of $f_k(n)$, then following convolution holds true*

$$\sum_{n=0}^{\infty} f_{k_3}(n)q^n = F^{k_3} = F^{k_1+k_2} = F^{k_1}F^{k_2} = \left(\sum_{j=0}^{\infty} f_{k_1}(j)q^j\right)\left(\sum_{l=0}^{\infty} f_{k_2}(l)q^l\right),$$

that is there exists

$$f_{k_3}(n) = \sum_{j=0}^{n} f_{k_1}(j)f_{k_2}(n-j), \quad k_3 = k_1 + k_2, \quad k_1, k_2 \geq 1, \tag{2.17}$$

which means that f_{k_3} is the Cauchy convolution of f_{k_1} with f_{k_2}.

3 Identities due to the formula (2.14)

The formula (2.14) has a great importance when we multiply it by a Beta function. Then we evaluate some of its identities and relations by employing the Beta function given by

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\quad \forall m,n>0, \tag{3.1}$$

and the gamma function defined by [25, p.19]

$$\Gamma(m)=\int_0^\infty e^{-t}t^{m-1}dt \quad \forall m>0, \tag{3.2}$$

Theorem 3.1. *$\forall n\geq 1$, by the formula (2.14) following identities hold*

$$\frac{\sqrt{\pi}\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)}\sum_{j=1}^{n}\frac{1}{(n-j)!}(n)_{n-j}=\frac{4^{\mathrm{n}}}{(2n+1)}=B\left(\frac{1}{2},n+1\right)\sum_{j=1}^{n}\frac{1}{(n-j)!}(n)_{n-j} \tag{3.3}$$

and

$$\frac{\sqrt{\pi}\mathrm{n}!}{2^{2\mathrm{n}+1}\left(n+\frac{1}{2}\right)!}\sum_{j=1}^{n}\frac{1}{(n-j)!}(n)_{n-j}=\frac{1}{2(2n+1)}=\left\{\sum_{j=1}^{n}\frac{1}{(n-j)!}(n)_{n-j}\right\}\int_0^1 x^n(1-x)^n dx. \tag{3.4}$$

Proof. Considering the formula (2.14) we find that

$$\left\{\sum_{j=1}^{n}\frac{1}{(n-j)!}(n)_{n-j}\right\}\int_0^1 x^n(1-x)^n dx=\frac{1}{2}\frac{(2n)!}{(n!)^2}\frac{\Gamma(n+1)\Gamma(n+1)}{(2n+1)\Gamma(2n+1)}=\frac{1}{2(2n+1)}. \tag{3.5}$$

Now, on making an appeal to well known Legendre duplication formula in the middle of the Eqn. (3.5), we obtain the identity (3.4).

Finally, by the Eqns. (3.1) and (3.4), we derive the identities in the Eqn. (3.3). □

4 Some of the extensions of the arithmetic function (1.1), their results and relations

In this section we introduce some extensions of the arithmetic function (1.1) and the identity (3.5). Then make their applications to derive some more other results connected to Bell polynomials [16, 17, 18] and the Riemann Zeta functions [13, 19, 24].

For the $g_j\forall j\geq 1$, given in (1.5), one of the extensions of (1.1) is taken by

$$f_k(n,t)=\frac{k}{n}\sum_{j=1}^{n}e^{g_j t}f_k(n-j),\quad f_k(0)=1,\quad n,k\geq 1. \tag{4.1}$$

Clearly, from (4.1) we have a relation with (1.1) as found by

$$\left.\frac{d}{dt}f_k(n,t)\right|_{t=0}=f_k(n),f_k(0)=1,n,k\geq 1. \tag{4.2}$$

Theorem 4.1. *Due to the extension (4.1), a formula exists as*

$$\left.\frac{d^n}{dt^n}f_k(n,t)\right|_{t=0}=\frac{k}{n}\sum_{j=1}^{n}a(j,1)f_k(n-j)j^n,\quad f_k(0)=1,\quad n,k\geq 1. \tag{4.3}$$

Proof. Operate (4.1) by the operator $\frac{d^n}{dt^n}$ to find that

$$\frac{d^n}{dt^n}f_k(n,t)=\frac{k}{n}\sum_{j=1}^{n}e^{g_j t}\left(\frac{g_j}{j}\right)^n f_k(n-j)j^n,\ \text{provided that } f_k(0)=1,n,k\geq 1. \tag{4.4}$$

Then in (4.4) apply the formula (1.5), to find that

$$\frac{d^n}{dt^n}f_k(n,t)=\frac{k}{n}\sum_{j=1}^{n}e^{g_j t}a(j,1)f_k(n-j)j^n,\ \text{provided that } f_k(0)=1,n,k\geq 1. \tag{4.5}$$

Finally, making an appeal to the result (4.5), we derive the result of (4.3). □

Theorem 4.2. *Due to the Theorem* 4.1*, there also exists another generating function*

(4.6)
$$f_k(n,t) = \frac{k}{n}\sum_{j=1}^{n} a(j,1) f_k(n-j)\frac{(jt)^n}{n!}, \quad f_k(0)=1, \quad n,k \geq 1.$$

Proof. Consider the Maclaurin series

(4.7)
$$f(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}f^{(n)}(0), f^{(n)}(0) = \left.\frac{d^n}{dt^n}f(t)\right|_{t=0},$$

where $f(t)$ possesses continuous derivative of all orders in the interval $[0,t]$. Then make an appeal to the formula (4.3) of the Theorem 4.1 to find the function (4.7). □

Theorem 4.3. *For the generalized Riemann Zeta function defined and studied by* [13, 19, 24]

(4.8)
$$\zeta(a,s) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}, \forall a.s \in \mathbb{C} \text{ and } \Re(a) > 0, \Re(s) > 1,$$

and

(4.9)
$$\zeta(a,s) = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\frac{e^{-at}}{(1-e^{-t})}dt \forall a.s \in \mathbb{C} \text{ and } \Re(a) > 0, \Re(s) > 0,$$

there exists following summation formulae

(4.10)
$$\sum_{n=0}^{\infty}\frac{\sqrt{\pi}\mathrm{n}!\left(n-\frac{1}{2}\right)!}{2^{2\mathrm{n}-1}\left\{\left(n+\frac{1}{2}\right)!\right\}}\sum_{j=1}^{n}\frac{1}{(n-j)!}(n)_{n-j} = \sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{2}\right)^2} = \zeta\left(\frac{1}{2},2\right),$$

(4.11)
$$\sum_{n=0}^{\infty}\frac{\sqrt{\pi}\mathrm{n}!\left(n-\frac{1}{2}\right)!}{2^{2\mathrm{n}-1}\left\{\left(n+\frac{1}{2}\right)!\right\}^2}\sum_{j=1}^{n}\frac{1}{(n-j)!}(n)_{n-j} = \int_0^{\infty}\left(\frac{t}{1-e^{-t}}\right)e^{-\frac{1}{2}t}dt.$$

Proof. Considering the results (3.3) and (3.4) and Making an appeal to the formulae of generalized Riemann Zeta function (4.8) and (4.9), we derive the formulae (4.10) and (4.11), respectively. □

5 Extensions in the coefficients $a(n,m)$ defined in (2.10) via sequence of Hankel type integral operators, to find different polynomials

The Hankel's contour integral is defined by [25, p. 219]

(5.1)
$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^u u^{-z}du, \sigma > 0, \Re(z) > 0, i = \sqrt{(-1)}.$$

Therefore to make extensions in the coefficients $a(n,m)$ given in the Eqn. (2.10), we introduce a sequence of Hankel type integral operators due to (5.1) and again apply the formula of the generating function for the Stirling numbers due to Riordan [22] (see also in (Chandel [6], Chandel and Yadava [8]) of first kind which is given by

(5.2)
$$S_n^{(k)} = \frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\begin{pmatrix} k \\ j \end{pmatrix} j^n.$$

Now from (2.10) considering the coefficients $a(n,m) \forall n,m \in \mathbb{N}\cup\{0\}$ as $a(n,m) = \frac{(-1)^{n-m}}{n!}S_n^{(m)}$ and in it applying (5.2), we find the formula of $a(n,m)$ consisting of sequence of Hankel's type contour integrals (5.1) in the form

(5.3)
$$a(n,m) = \frac{(-1)^n}{m!}\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}\frac{j^n}{n!} = \frac{(-1)^n}{m!}\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}\left\{\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{ju}u^{-(n+1)}du\right\}, \sigma > 0.$$

Due to (5.3), for exploring new ideas in the field of arithmetic functions and further extensions in these results, we define a sequence of Hankel type integral operators (5.1) in the form

(5.4)
$$K(j,n;\sigma)\{f\} = \frac{\Gamma(n+1)}{j^n}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{ju}u^{-(n+1)}f(u)du, \ \sigma > 0, \ f(0) = 1.$$

It is clear that when $f(u) \equiv 1$, then for $\sigma > 0$ the formulae (5.3) and (5.4) give us the relations with Bell coefficients

$$\frac{(-1)^n}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{j^n}{n!} \{K(j, n; \sigma) \{1\}\} = a(n,\ m) = a(n,\ m, 1), (\text{let}) . \tag{5.5}$$

Therefore in the formula (5.4) when we set $f^{\alpha,r}(u) = e^{(\alpha+(r-1)j)u}$ and $\sigma > 0$, we find

$$K(j, n, \alpha; \sigma) \{f^{\alpha,r}\} = \frac{\Gamma(n+1)}{j^n} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} u^{-(n+1)} e^{(\alpha+rj)u} du = \frac{\{(\alpha+rj)\}^{n+1}}{j^{n+1}}. \tag{5.6}$$

Further for $\sigma > 0$, making an application of the formulae (5.4) and (5.6), we get the coefficients of Bell polynomials in following generalized form

$$\begin{aligned} a(n,\ m, f^{\alpha,r}) &= \frac{(-1)^n}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{j^n}{n!} \{jK(j, n, \alpha; \sigma) \{f^{\alpha,r}\}\} \\ &= \frac{(-1)^n}{n!} \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \{\alpha + rj\}^{n+1} \\ &= \frac{(-1)^{n-m}}{n!} S^{\alpha}(n+1, m, r), \end{aligned} \tag{5.7}$$

where $f^{\alpha,r}(u) = e^{(\alpha+(r-1)j)u}$.

Here in (5.7), the generalized Stirling formula is given by Chandel and Yadava [8]

$$S^{\alpha}(n, m, r) = \frac{(-1)^m}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \{(\alpha+rj)\}^n . \tag{5.8}$$

Now making an appeal to (5.7), we obtain a generating function equivalent to the generating function due to Chandel and Yadava [8 , Eqn. (2.6)] as given by

$$\begin{aligned} (-1)^m \sum_{n=0}^{\infty} (-t)^n a(n-1,\ m, f^{\alpha,r}) &= \frac{(-1)^m}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{n=0}^{\infty} \frac{t^n}{n!} (\alpha+rj)^n \\ &= e^{\alpha t} \frac{(-1)^m}{m!} \sum_{j=0}^{m} (-m)_j \frac{(e^{rt})^j}{j!} = e^{\alpha t} \frac{(-1)^m}{m!} {}_1F_0\left(-m; -; e^{rt}\right) . \end{aligned} \tag{5.9}$$

Again considering the formula (5.7) we get Truesdell polynomials due to Chandel [3, 4, 5]

$$\begin{aligned} & (-1)^{m+n} n! \sum_{m=0}^{n} (-1)^m a(n-1,\ m, f^{\alpha,r}) p^r x^{rm} \\ &= \sum_{m=0}^{n} \frac{(-1)^m}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (\alpha+rj)^n p^r x^{rm} = T_n^{\alpha}(x, r, -p). \end{aligned} \tag{5.10}$$

6 Concluding remarks

In this article**,** a recurrence relation of the arithmetic function is considered to obtain the coefficients of Bell polynomials. Then we derive various results and relations connected with an ascending factorial function, Lah and Stirling numbers and to find a relation of harmonic numbers. To exploring of this work in multidisciplinary aspect, we make some of the extensions of the coefficients of Bell polynomials in terms of sequence of the Hankel type integral operators to derive generalized Stirling numbers and Truesdell's polynomials. We also derive the summation formulae in terms of Riemann Zeta function.

On the other hand, making an appeal to [7] in (5.7), we may introduce the coefficients of Bell polynomials into multivariable Truesdell's polynomials and then we may apply the techniques due to [7] to derive various results and generating functions.

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

ON COMPLEX VALUED FUZZY b - METRIC SPACE

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Abstract

In this paper, the notion of complex-valued fuzzy b-metric space is introduced. In this newly developed structure, we have established a sufficient condition for a sequence to be Cauchy. Moreover, under suitable conditions of contractive type, the existence and uniqueness of fixed points of self-maps are established in this structure. To demonstrate the validity of the hypothesis and the degree of generality of our results, some examples are also furnished.

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Keywords and Phrases: Complex valued fuzzy metric, complex valued fuzzy b-metric space, t-norm, Cauchys sequence, fixed point.

1 Introduction

In 1965, Zadeh [17] introduced the concept of fuzzy sets. Due to the widespread use of this concept in various fields, numerious authors have expansively developed the theory of fuzzy sets and its applications in variety of domain. Using the concept of fuzziness, Kramosil and Mechalek [9] introduced the notion of fuzzy metric space by generalizing the concept of probabilistic metric space. Grabiec [7] extended the well-known fixed point theorem of Banach[4] in complete fuzzy metric space in the sense of Kramosil and Michalek. In a paper, George and Vermani [6] modified the concept of fuzzy metric space and defined Hausdroff topology on fuzzy metric space. By observing weaker conditions of the triangle inequality, Bakhtin [2] and Czerwik [5] introduced the structure of b-metric space and generalized the Banach contraction principle. In this sequence, a relation between b-metric and fuzzy metric spaces has been studied by Hassanzadeh et al.[8]. On the other hand Sedghi et al.[15] introduced the notion of b-fuzzy metric spaces by weakening the triangle inequality. The concept of fuzzy b-metric space was first developed by Nadaban [11]. Recently, Mehmood et al. introduced the concept of extended fuzzy b-metric space [10].

In a paper, Buckley [3] introduced the fuzzy complex numbers and fuzzy complex analysis. After that many authors initiated work in fuzzy complex number by acknowledging the Buckleys work. In this series Ramot el al.[12] established the innovative concept of complex fuzzy sets. In this context, the range of membership function of complex fuzzy set is not limited to $[0, 1]$ as the membership function of traditional fuzzy set but, it extended to the unit circle in the complex plane. Then, here we see that the range of membership function of crisp set $\{0, 1\}$ is extended to the range of membership function of fuzzy set $[0, 1]$ and the range of membership function of fuzzy set $[0, 1]$ is extended the range of membership function of complex fuzzy set to the unit circle in complex plane.

In 2011, Azam et al.[1] defined a partial order $\precsim$ on set of complex numbers $\mathbb{C}$ for comparing the two complex numbers and introduced the concept of complex valued metric spaces. Also they obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions.

Recognizing the notion of complex valued fuzzy set of Ramot et al.[12], Sigh et al.[14] developed the structure of complex valued fuzzy metric spaces. They also established the complex valued fuzzy version of Banach contraction principle.

In a paper, Rao et al. [13] introduced the complex valued b-metric space and gave a common fixed point theorem for four maps in this structure. In this paper, we establish the structure of complex-valued fuzzy b-metric space along with its properties. Moreover, we establish a theorem of existence and uniqueness for a fixed point of self-map defined on this newly developed structure.

2 Preliminaries

Definition 2.1 ([14]). *A complex Fuzzy set S, defined on a universe of discourse U, is characterized by a membership function $\mu_s(x)$ that assigns every element $x \in U$, a complex valued grade of membership in S. The values $\mu_s(x)$ lie within the unit circle in the complex plane, and thus of the form*

$$\mu_s(x) = r_s(x).e^{i\omega_s(x)} \qquad (i = \sqrt{-1})$$

where $r_s(x)$ and $\omega_s(x)$ both are real valued, with $r_s(x) \in [0,1]$. The complex fuzzy set S, may be represented as the set of ordered pairs , given by

$$S = \{(x, \mu_s(x) \mid x \in U\}.$$

Clearly, each complex grade of membership is defined by an amplitude term $r_s(x)$ and a phase term $\omega_s(x)$. Notice that it is possible to represent any ordinary fuzzy set in terms of a complex fuzzy set. If any ordinary fuzzy set S is characterized by the real valued membership function $\lambda_s(x)$ where $x \in U$, then S can be transformed into complex fuzzy set by setting the amplitude terms $r_s(x)$ equal to $\lambda_s(x)$ and the phase term $\omega_s(x)$ equal to zero for all $x \in U$. Thus one can say that without a phase term, the complex fuzzy set effectively reduces to conventional fuzzy set.

Definition 2.2 ([1]). *Let $\mathbb{C}$ be the set of complex numbers and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as: $\alpha_1 \precsim \alpha_2 \Leftrightarrow Re(\alpha_1) \leq Re(\alpha_2)$, $Im(\alpha_1) \leq Im(\alpha_2)$. It follows that $\alpha_1 \precsim \alpha_2$ if one of the following conditions hold:*

(i) $Re(\alpha_1) = Re(\alpha_2)$ *and* $Im(\alpha_1) = Im(\alpha_2)$,
(ii) $Re(\alpha_1) < Re(\alpha_2)$ *and* $Im(\alpha_1) = Im(\alpha_2)$,
(iii) $Re(\alpha_1) = Re(\alpha_2)$ *and* $Im(\alpha_1) < Im(\alpha_2)$,
(iv) $Re(\alpha_1) < Re(\alpha_2)$ *and* $Im(\alpha_1) < Im(\alpha_2)$.

We write $\alpha_1 \precnsim \alpha_2$ if $\alpha_1 \neq \alpha_2$ and one of (ii), (iii) and (iv) is satisfied and we write $\alpha_1 \prec \alpha_2$ if only (iv) is satisfied.

Here we note the following condition trivially hold:

(i) If $0 \precsim \alpha_1 \precsim \alpha_2$ then $|\alpha_1| \leq |\alpha_2|$,
(ii) If $0 \precsim \alpha_1 \precnsim \alpha_2$ then $|\alpha_1| < |\alpha_2|$,
(iii) If $\alpha_1 \prec \alpha_2$ and $\alpha_2 \prec \alpha_3$ then $\alpha_1 \prec \alpha_3$,
(iv) If $a, b \in \mathbb{R}$ and $a \leq b$ then $a\alpha \precsim b\alpha$ for all $\alpha \in \mathbb{C}$,
(v) If $a, b \in \mathbb{R}$ and $0 \leq a \leq b$ then $\alpha_1 \precsim \alpha_2$ implies $a\alpha_1 \precsim b\alpha_2$.

Utilizing the concept due to Azam et al.[1] and the definition of max function by Verma et al.[16], Singh at al.[14] gave the similar definition of min function as follows

Definition 2.3 ([14]). *Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ and the partial order relation $\precsim$ is defined on $\mathbb{C}$. Then, min functions for complex numbers with partial order relations is defined as:*

(1) $\min\{\alpha_1, \alpha_2\} = \alpha_1 \Leftrightarrow \alpha_1 \precsim \alpha_2$,
(2) $\min\{\alpha_1, \alpha_2\} \precsim \alpha_3 \Rightarrow \alpha_1 \precsim \alpha_3$ *or* $\alpha_2 \precsim \alpha_3$.

Note 2.1. Throughout this paper the symbol $\leq$ or $\geq$ used in sense of real numbers while symbol $\precsim$ or $\succsim$ used in sense of complex numbers.

Definition 2.4 ([14]). *A binary operation $*: r_s e^{i\theta} \times r_s e^{i\theta} \to r_s e^{i\theta}$, where $r_s \in [0,1]$ and a fix $\theta \in [0, \frac{\pi}{2}]$, is called complex valued continuous t-norm if it satisfies the following conditions:*

(1) $*$ *is associative and commutative,*
(2) $*$ *is continuous,*
(3) $a * e^{i\theta} = a, \forall a \in r_s e^{i\theta}$ *where* $r_s \in [0,1]$,
(4) $a * b \precsim c * d$ *whenever* $a \precsim c$ *and* $b \precsim d$, *for all* $a, b, c, d \in r_s e^{i\theta}, r_s \in [0,1]$.

Definition 2.5. *Let $*$ be a complex valued continuous t-norm, let $*_n : r_s e^{i\theta} \to r_s e^{i\theta}$ where $n \in \mathbb{N}, r_s \in [0,1]$ be defined in the following way*

$$*_1(x) = x * x, \qquad *_{n+1}(x) = (*_n(x) * x) \qquad n \in \mathbb{N}, x \in r_s e^{i\theta}.$$

Each complex valued t-norm $*$ can be extended by associativity in a unique way to an n-aray operation taking for $(x_1, x_2, \ldots x_n) \in [r_s]^n e^{i\theta}$ where $r_s \in [0,1]$ the values

$$*_{i=1}^{1} x_i = x_1, *_{i=1}^{n} x_i = ((*_{i=1}^{n-1} x_i) * x_n) = (x_1 * x_2 * \cdots * x_n).$$

A complex valued t-norm $*$ can be extended to a countable infinite operation taking for any sequence $(x_n)_{n\in\mathbb{N}}$ from $r_s \in [0,1]$ the value

$$*_{i=1}^{\infty} x_i = \lim_{n\to\infty} *_{i=1}^{n} x_i.$$

The sequence $(*_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is nonincreasing and bounded from below, and hence the limit $*_{i=1}^{\infty} x_i$ exists.

Definition 2.6 ([14]). *The triplet $(X, M, *)$ is said to be complex valued fuzzy metric space if X is a non-empty set, $*$ is a complex valued t-norm and $M : X \times X \times (0,\infty) \to r_s e^{i\theta}$ is a complex valued fuzzy set, where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$, satisfying the following conditions:*

- $(CF1)$ $\quad M(x,y,t) \succ 0$,
- $(CF2)$ $\quad M(x,y,t) = e^{i\theta}$ *for all* $t > 0 \Leftrightarrow x = y$,
- $(CF3)$ $\quad M(x,y,t) = M(y,x,t)$,
- $(CF4)$ $\quad M(x,z,t+s) \succsim M(x,y,t) * M(y,z,s)$,
- $(CF5)$ $\quad M(x,y,.) : (0,\infty) \to r_s e^{i\theta}$ *is continuous,*

for all $x, y, z \in X, s, t > 0, r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$. Also $(M, *)$ is called a complex valued fuzzy metric.

Remark 2.1. *If we take $\theta = 0$ then complex valued fuzzy metric simply goes to real valued fuzzy metric.*

Now, in this paper we introduce the complex valued fuzzy b-metric space as.

Definition 2.7. *Let X be a non-empty set, $b \geq 1$ be a given real number, $*$ is a complex valued t-norm and $M : X \times X \times (0,\infty) \to r_s e^{i\theta}$ is a complex valued fuzzy set, where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$, satisfying the following conditions:*

- (CF_bM1) $\quad M(x,y,t) \succ 0$,
- (CF_bM2) $\quad M(x,y,t) = e^{i\theta}$ *for all* $t > 0 \Leftrightarrow x = y$,
- (CF_bM3) $\quad M(x,y,t) = M(y,x,t)$,
- (CF_bM4) $\quad M(x,z,t+s) \succsim M(x,y,\frac{t}{b}) * M(y,z,\frac{s}{b})$,
- (CF_bM5) $\quad M(x,y,.) : (0,\infty) \to r_s e^{i\theta}$ *is continuous and* $\lim\limits_{t\to\infty} M(x,y,t) = r_s e^{i\theta}$,

for all $x, y, z \in X, s, t > 0, r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$. A quadruple $(X, M, *, b)$ is said to be complex valued fuzzy b-metric space.

Remark 2.2. *The class of complex valued fuzzy b-metric spaces is effectively larger than that of complex valued fuzzy metric spaces[4], since a complex valued fuzzy b-metric is a complex valued fuzzy metric when $b = 1$.*

Example 2.1. Let $M(x,y,t) = e^{i\theta} e^{\frac{-d(x,y)}{t}}$, where d is a b-metric on X and $a * c = a.c$ for all $a, c \in r_s e^{i\theta}$. Then it is easy to show that $(X, M, *)$ is a complex valued fuzzy b- metric space. Obviously conditions from $(CF_bM1 - CF_bM3)$ of Definition 2.7 are satisfied. For each $x, y, z \in X$ we obtain

$$\begin{aligned} M(x,y,t+s) &= e^{i\theta} e^{\frac{-d(x,y)}{t+s}} \\ &\succsim e^{i\theta} e^{\frac{-b[d(x,z)+d(z,y)]}{t+s}} \\ &\succsim e^{i\theta} e^{-\frac{d(x,z)}{\frac{t}{b}}} . e^{-\frac{d(z,y)}{\frac{s}{b}}} \\ &= M\left(x, z, \frac{t}{b}\right) * M\left(z, y, \frac{s}{b}\right). \end{aligned}$$

So condition (CF_bM4) of Definition 2.7 holds and $(X, M, *)$ is a complex valued fuzzy b-metric space.

Example 2.2. Let $X = \mathbb{R}$. We define $a * c = a.c, \forall a, c \in r_s e^{i\theta}$, where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$. Furthermore for all $x, y \in X$ and $t \in (0,\infty)$, we define

$$M(x,y,t) = e^{i\theta} e^{\frac{-|x-y|^p}{t}},$$

where $p > 1$ is a real number. Then, $(X, M, *)$ is a complex valued fuzzy b-metric space with $b = 2^{p-1}$.

Example 2.3. Let $M(x,y,t) = e^{i\theta}\frac{t}{t+d(x,y)}$ where d is a b-metric on X and $a * c = a.c, \forall a,c \in r_s e^{i\theta}$. Then $M(x,y,*)$ is a complex valued fuzzy b-metric space.

Definition 2.8. *Let $b \geq 1$ be a given real number. A function $f\colon \mathbb{R} \to \mathbb{C}$ will be called b- non-decreasing if $t < s$ we will have $f(t) \precsim f(bs)$.*

Lemma 2.1. *The mapping $M(x,y,.)\colon [0,\infty) \to r_s e^{i\theta}$ is b-non decreasing for all $x,y \in X$.*

Proof. For some $0 < t < s$, we have

$$M(x,y,bs) \succsim M(x,y,t) * M(y,y,s-t) = M(x,y,t) * e^{i\theta} = M(x,y,t).$$

Therefore for all $x,y \in X, M(x,y,.)$ is b-non-decreasing.

Definition 2.9. *Let $(X,M,*)$ be a complex valued fuzzy b-metric space. We define an open ball $B(x,r,t)$ with centre $x \in X$ and radius $r \in \mathbb{C}, 0 \prec r \prec e^{i\theta}, t > 0$ as*

$$B(x,r,t) = \{y \in X\colon M(x,y,t) \succ e^{i\theta} - r\}, \text{ where } \theta \in \left[0, \frac{\pi}{2}\right].$$

Definition 2.10. *Let $(X,M,*)$ be a complex valued fuzzy b-metric space then*

(a) A sequence $\{x_n\}$ in X is said to be convergent to x in X, if and only if $\lim_{n\to\infty} M(x_n,x,t) = e^{i\theta}$ for any $n > 0$ and for all $t > 0$.

(b) A sequence $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n\to\infty} M(x_n,x_{n+m},t) = e^{i\theta}$ for any $m > 0$ and for all $t > 0$.

(c) The complex valued fuzzy b-metric space $(X,M,*)$ is called complete if every Cauchy sequence is convergent.

3 Main Results

Lemma 3.1. *Let $(X,M,*)$ be a complex valued fuzzy b-metric space such that $\lim_{t\to\infty} M(x_n,x_{n+m},t) = e^{i\theta}$ for all $x,y \in X$ if*

$$M(x,y,t) \succsim M\left(x,y,\frac{t}{\lambda}\right)$$

for all $x,y \in X, 0 < \lambda < 1, t \in (0,\infty)$ then $x = y$.

Proof. Suppose $\lambda \in (0,1)$ such that

$$M(x,y,\lambda t) \succsim M(x,y,t) \forall x,y \in X, t \in (0,\infty)$$

so that

$$M(x,y,t) \succsim M\left(x,y,\frac{t}{\lambda}\right).$$

On repeated application, we have

$$M(x,y,t) \succsim M\left(x,y,\frac{t}{\lambda^n}\right) \text{ for some positive integer } n.$$

On making $n \to \infty$, reduces to $M(x,y,t) \succsim e^{i\theta}$. This implies $M(x,y,t) = e^{i\theta}$. Thus by (CF_bM2), we have $x = y$.

Lemma 3.2. *Let $\{x_n\}$ be a sequence in a complex valued fuzzy b-metric space $(X,M,*)$. suppose that there exists $\lambda \in (0, \frac{1}{b})$ such that*

$$M(x_n,x_{n+1},t) \succsim M\left(x_{n-1},x_n,\frac{t}{\lambda}\right), n \in \mathbb{N}, t > 0 \tag{3.1}$$

and there exist $x_0, x_1 \in X$ and $v \in (0,1)$ such that

$$\lim_{n\to\infty} *_{i=n}^{\infty} M\left(x_0,x_1,\frac{t}{v^i}\right) = e^{i\theta} t > 0. \tag{3.2}$$

Then $\{x_n\}$ is a Cauchys sequence.

Proof. Let $\sigma \in (\lambda b, 1)$. Then the sum $\sum_{i=1}^{\infty} \sigma^i$ is convergent, and there exists $n_0 \in \mathbb{N}$ such that $\sum_{i=1}^{\infty} \sigma^i < 1$ for every $n > n_0$. Let $n > m > n_0$. Since M is b non-decreasing, by (CF_bM4) every $t > 0$

$$\begin{aligned} M(x_n, x_{n+m}, t) &\succsim M(x_n, x_{n+m}, \frac{t\sum_{i=n}^{n+m-1} \sigma^i}{b}) \\ &\succsim \left(M\Big(x_n, x_{n+1}, \frac{t\sigma^n}{b^2}\Big) * M\Big(x_{n+1}, x_{n+m}, \frac{t\sum_{i=n+1}^{n+m-1} \sigma^i}{b^2}\Big)\right) \\ &\succsim M\Big(x_n, x_{n+1}, \frac{t\sigma^n}{b^2}\Big) \\ &\quad * \left(M\Big(x_{n+1}, x_{n+2}, \frac{t\sigma^{n+1}}{b^3}\Big) * \cdots * M\Big(x_{n+m-1}, x_{n+m}, \frac{t\sigma^{n+m-1}}{b^m}\Big)\right). \end{aligned}$$

By (3.1) it follows that

$$M(x_n, x_{n+1}, t) \succsim M\Big(x_0, x_1, \frac{t}{\lambda^n}\Big), n \in \mathbb{N}, t > 0$$

and since $n > m$ and $b > 1$, we have

$$\begin{aligned} M(x_n, x_{n+m}, t) &\succsim M\Big(x_0, x_1, \frac{t\sigma^n}{b^2\lambda^n}\Big) \\ &\quad * \left(M\Big(x_0, x_1, \frac{t\sigma^{n+1}}{b^3\lambda^{n+1}}\Big) * \cdots * M\Big(x_0, x_1, \frac{t\sigma^{n+m-1}}{b^{m+1}\lambda^{n+m-1}}\Big)\right) \\ &\succsim *_{i=m}^{n+m-1} M\Big(x_0, x_1, \frac{t\sigma^i}{b^{i-n+2}\lambda^i}\Big) \\ &\succsim *_{i=m}^{n+m-1} M\Big(x_0, x_1, \frac{t\sigma^i}{b^i\lambda^i}\Big) \\ &\succsim *_{i=m}^{n+m-1} M\Big(x_0, x_1, \frac{t}{v^i}\Big) \text{ where } v = \frac{b\lambda}{\sigma} \end{aligned}$$

Since, $v \in (0, 1)$. By(3.2) it follows that $\{x_n\}$ is a Cauchys sequence.

Theorem 3.1. *Let $(X, M, *)$ be a complete complex valued fuzzy b-metric space and let $f\colon X \to X$. Suppose there exist $\lambda \in (0, \frac{1}{b})$ such that*

$$M(fx, fy, t) \succsim M\Big(x, y, \frac{t}{\lambda}\Big), \qquad x, y \in X, \quad t > 0 \tag{3.3}$$

and there exists $x_0 \in X$ and $v \in (0, 1)$ such that

$$\lim_{n\to\infty} *_{i=n}^{\infty} M\Big(x_0, x_1, \frac{t}{v^i}\Big) = e^{i\theta} \qquad t > 0. \tag{3.4}$$

Then f has a unique fixed point in X.

Proof. Let $x_0 \in X$ and $x_{n+1} = fx_n, n \in \mathbb{N}$. If we take $x = x_n$ and $y = x_{n-1}$ in (3.3) then we have

$$M(x_n, x_{n+1}, t) \succsim M\Big(x_{n-1}, x_n, \frac{t}{\lambda}\Big) \qquad n \in \mathbb{N}, \quad t > 0$$

By Lemma 3.2 it follows that $\{x_n\}$ is a Cauchys sequence. Since $(X, M, *)$ is complete there exist $x \in X$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} M(x_n, x, t) = e^{i\theta} \qquad t > 0. \tag{3.5}$$

Using condition (3.3) and (CF_bM4) we show that x is a fixed point of f.

$$\begin{aligned} M(fx, x, t) &\succsim \left(M\Big(fx, x_n, \frac{t}{2b}\Big) * M\Big(x_n, x, \frac{t}{2b}\Big)\right) \\ &\succsim \left(M\Big(fx_n, x_n, \frac{t}{2b}\Big) * M\Big(x_n, x, \frac{t}{2b}\Big)\right) \\ &\succsim \left(M\Big(x_{n+1}, x_n, \frac{t}{2b}\Big) * M\Big(x_n, x, \frac{t}{2b}\Big)\right) \\ &\succsim \left(M\Big(x_{n-1}, x_n, \frac{t}{2b\lambda}\Big) * M\Big(x_n, x, \frac{t}{2b}\Big)\right), \end{aligned}$$

for all $t > 0$, by (3.5) as $n \to \infty$, we get

$$M(fx, x, t) \succsim (e^{i\theta} * e^{i\theta}) = e^{i\theta}.$$

Suppose that x and y are fixed point for f. By (3.3)

$$M(x, y, t) = M(fx, fy, t) \succsim M\Big(x, y, \frac{t}{\lambda}\Big), \qquad t > 0.$$

Lemma 3.1 implies $x = y$.

Example 3.1. Let $X = [0,1]$ and $M(x, y, t) = e^{i\theta} e^{\frac{-|x-y|^2}{t}}$ be a complex valued fuzzy b-metric space with $b = 2$ and $\theta \in [0, \frac{\pi}{2}]$. Let $f(x) = kx$, where $k = \frac{1}{\sqrt{2}}$ and $x \in X$. Then

$$M(fx, fy, t) = e^{i\theta} e^{\frac{-k^2|x-y|^2}{t}} \succsim e^{i\theta} e^{\frac{-\lambda|x-y|^2}{t}} = M\Big(x, y, \frac{t}{\lambda}\Big), x, y \in X, t > 0. \tag{3.6}$$

For $k^2 < \lambda < \frac{1}{b}$. So, the condition of the Theorem 3.1 fulfilled and f has a unique fixed point in X.

Theorem 3.2. *Let $(X, M, *)$ be a complex valued complete fuzzy b-metric space and let $f \colon X \to X$. Suppose that there exist $\lambda \in (0, \frac{1}{b})$ such that*

$$M(fx, fy, t) \succsim \min\Big\{M\Big(x, y, \frac{t}{\lambda}\Big), M\Big(fx, x, \frac{t}{\lambda}\Big), M\Big(fy, y, \frac{t}{\lambda}\Big)\Big\} \tag{3.7}$$

for all $x, y \in X, t > 0$ and there exist $x_0 \in X$ and $v \in (0,1)$ such that

$$\lim_{n\to\infty} *_{i=n}^{\infty} M\Big(x_0, fx_0, \frac{t}{v^i}\Big) = e^{i\theta} \tag{3.8}$$

for all $t > 0$. Then f has a fixed point in X.

Proof. Let $x_0 \in X$ and $x_{n+1} = fx_n, n \in \mathbb{N}$. By 3.6 with $x = x_n$ and $y = x_{n-1}$ for every $n \in \mathbb{N}$ and every $t > 0$, we have

$$M(fx_n, fx_{n-1}, t) \succsim \min\Big\{M\Big(x_n, x_{n-1}, \frac{t}{\lambda}\Big), M\Big(fx_n, x_n, \frac{t}{\lambda}\Big), M\Big(fx_{n-1}, x_{n-1}, \frac{t}{\lambda}\Big)\Big\}$$
$$M(x_{n+1}, x_n, t) \succsim \min\Big\{M\Big(x_n, x_{n-1}, \frac{t}{\lambda}\Big), M\Big(x_{n+1}, x_n, \frac{t}{\lambda}\Big), M\Big(x_n, x_{n-1}, \frac{t}{\lambda}\Big)\Big\}$$
$$M(x_{n+1}, x_n, t) \succsim \min\Big\{M\Big(x_n, x_{n-1}, \frac{t}{\lambda}\Big), M\Big(x_{n+1}, x_n, \frac{t}{\lambda}\Big)\Big\}.$$

If $M(x_{n+1}, x_n, t) \succsim M\Big(x_{n+1}, x_n, \frac{t}{\lambda}\Big) n \in \mathbb{N}, t > 0$. By Lemma3.1 it follows that $x_n = x_{n+1}, n \in \mathbb{N}$. So,

$$M(x_{n+1}, x_n, t) \succsim M\Big(x_n, x_{n-1}, \frac{t}{\lambda}\Big), \qquad n \in \mathbb{N}, t > 0.$$

By Lemma 3.2 we have that $\{x_n\}$ is a Cauchys sequence. Hence there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} M(x, x_n, t) = e^{i\theta}, t > 0. \tag{3.9}$$

Now we prove that x is a fixed point of f. Let $\sigma_1 \in (\lambda b, 1)$ and $\sigma_2 = 1 - \sigma_1$. By (3.6)

$$\begin{aligned} M(fx, x, t) &\succsim \Big(M\Big(fx, fx_n, \frac{t\sigma_1}{b}\Big) * M\Big(fx_n, x, \frac{t\sigma_2}{b}\Big)\Big) \\ &\succsim \Big(\min\Big\{M\Big(x, x_n, \frac{t\sigma_1}{b\lambda}\Big), M\Big(x, fx, \frac{t\sigma_1}{b\lambda}\Big), M\Big(x_n, x_{n+1}, \frac{t\sigma_1}{b\lambda}\Big)\Big\} * M\Big(fx_n, x, \frac{t\sigma_2}{b}\Big)\Big) \end{aligned}$$

Taking $n \to \infty$ and using(3.8) we have

$$\begin{aligned} M(fx, x, t) &\succsim \min\Big\{e^{i\theta}, M\Big(x, fx, \frac{t\sigma_1}{b\lambda}\Big), e^{i\theta}\Big\} * e^{i\theta} \\ &\succsim \Big\{M\Big(x, fx, \frac{t\sigma_1}{b\lambda}\Big) * e^{i\theta}\Big\} = M\Big(x, fx, \frac{t}{v}\Big), t > 0, \end{aligned}$$

where $v = \frac{b\lambda}{\sigma_1} \in (0,1)$. Therefore,

$$M(fx, x, t) \succsim M\Big(x, fx, \frac{t}{v}\Big), t > 0,$$

By Lemma 3.1 it follows that $fx = x$. Suppose x and y are the fixed point for f, that is $fx = x$ and $fy = y$. By condition (3.6), we get

$$\begin{aligned} M(fx, fy, t) &\succsim \min\Big\{M\Big(x, y, \frac{t}{\lambda}\Big), M\Big(x, fx, \frac{t}{\lambda}\Big), M\Big(y, fy, \frac{t}{\lambda}\Big)\Big\} \\ &= \min\Big\{M\Big(x, y, \frac{t}{\lambda}\Big), e^{i\theta}, e^{i\theta}\Big\} = M\Big(x, y, \frac{t}{\lambda}\Big). \end{aligned}$$

For $t > 0$, by Lemma 3.1 it follows that $fx = fy$, that is $x = y$.

Example 3.2. Let $X = (0,2)$ with a b-metric d defined by

$$d(x,y) = |x-y|^2, \forall x, y \in X.$$

For all $x, y \in X$ and $t \in (0,\infty)$, we define

$$M(x,y,t) = e^{i\theta} e^{\frac{-(d(x,y))}{t}}.$$

Clearly $M(x,y,*)$ is complex valued complete fuzzy b-metric space with t-norm $*$ defined as $a*b = a.b$ where $a, b \in r_s e^{i\theta}$ for a fixed $\theta \in [0, \frac{\pi}{2}]$ and $r_s \in [0,1]$. Here $\lim_{t\to\infty} M(x,y,t) = e^{i\theta}$ for all $x, y \in X$. Then $M(x,y,t)$ is a complex valued fuzzy b-metric space with b=2. Define the map $f\colon X \to X$

$$f(x) = \begin{cases} 2-x, & x \in (0,1), \\ 1, & x \in [1,2). \end{cases}$$

Case 3.1. If $x, y \in [1,2)$ then, $M(fx, fy, t) = e^{i\theta}, t > 0$ and condition 3.6 are trivially satisfied.

Case 3.2. If $x \in [1,2)$ and $y \in (0,1)$, then for $\lambda(\frac{1}{4}, \frac{1}{2})$, we have

$$M(fx,fy,t) = e^{i\theta} e^{-\frac{|x-y|^2}{t}} = e^{i\theta} e^{-\frac{|1-y|^2}{t}} \succsim e^{i\theta} e^{-\frac{4\lambda|1-y|^2}{t}} = M\Big(fy, y, \frac{t}{\lambda}\Big).$$

Case 3.3. If $x \in (0,1)$ and $y \in [1,2)$, then for $\lambda \in (\frac{1}{4}, \frac{1}{2})$, we have

$$M(fx,fy,t) = e^{i\theta} e^{-\frac{|x-y|^2}{t}} = e^{i\theta} e^{-\frac{|1-x|^2}{t}} \succsim e^{i\theta} e^{-\frac{4\lambda|1-x|^2}{t}} = M\Big(fx, x, \frac{t}{\lambda}\Big).$$

Case 3.4. If $x, y \in (0,1)$, then for $\lambda \in (\frac{1}{4}, \frac{1}{2})$, we have

$$M(fx,fy,t) = e^{i\theta} e^{-\frac{|x-y|^2}{t}} = e^{i\theta} e^{-\frac{|1-y|^2}{t}} \succsim e^{i\theta} e^{-\frac{4\lambda|1-y|^2}{t}} = M\big(fy, y, \frac{t}{\lambda}\big), x > y, t > 0$$

and $M(fx,fy,t) \succsim M\Big(fx, x, \frac{t}{\lambda}\Big), x < y, t > 0$. So conditions (3.6) are satisfied for all $x, y \in X, t > 0$, and by Theorem 3.2 it follows that $x = 1$ is a unique fixed point for f.

4 Conclusion

In the present study, we defined a new concept of complex-valued fuzzy b-metric space. We also established the condition of being Cauchy and convergence in this newly developed space. Several allied aspects of complex-valued fuzzy b-metric space are also defined, which fortify the concept. In our main result, we obtained the Banach contraction principle in the "complex valued fuzzy b-metric space". For the sustainability of our result, we also furnished an example that satisfied our main result.

5 Open problem

It is the introduction of a phase term that makes the complex fuzzy set a distinctive and novel concept. Quantum mechanics allows an object to exhibit a wave-like nature associated with a phase term. Therefore, making use of this concept of complex valued fuzzy b metric space to show the existence of a fixed quantum state associated with quantum operations is an open problem.

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

FIVE SERIES EQUATIONS INVOLVING GENERALIZED BATEMAN k-FUNCTIONS

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Abstract

In this paper, the solution of five series equations involving generalized Bateman k-functions is obtained by reducing them to Fredholm integral equation of the second kind. The solution presented in this paper is obtained by employing the techniques of Narain and Lal [12] involving generalized Bateman k-functions by reducing them to the solution of a Fredholm inregral equation of second kind with different bounday conditions. Thus we have seen that Bateman k-functions are having interesting properties to solve double, triple, quadruple and five series equations as special functions. These solutions are very useful in Mathematical and Quantum Physics, Aero and Fluid Dynamics and Thermodynamics.

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Keywords and Phrases: Five Series equations, generalized Bateman k-functions, Fredholm integral equation.

1 Introduction

Chakraborty [3] discussed on generalization of Bateman functions. Erdélyi [7] has given tables of Integrals Transforms. Noble [13] presented formal solution of dual series equations involving Jacobi polynomials. Srivastava [15,16] and Srivastav [17] obtained solutions on dual series relations involving series of generalized Bateman k-functions and triple series equations involving series of Jacobi polynomials. Later on Lowndes [9, 10], Dwivedi and Trivedi [6] gave the solution of triple and quadruple series equations involving Jacobi polynomials. Chandel [2] solved a problem on Heat Conduction over the surface of a sphere by making an appeal to dual series equations involving Legendre polynomials by employing Mehler Dirichlet integrals [19, (2.6.20), (2.6.21)] and Fredholm integral equation. Srivastava [14] obtained solutions of a pair of dual series equations involving generalized Bateman k-functions. Narain, Singh and Lal [11] obtained solution of triple series equations involving generalized Bateman k-functions. Narain and Lal [12] gave a method for the solution of five Series equations by reducing them to Fredholm integral equations of the second kind. Dwivedi and Singh [7] gave the solution of some five series equations involving generalized Bateman k-functions by reducing them to simultaneous Fredholm integral equations. Recently Tripathi and Dixit [20] have obtained formal solution of four series equations involving generalized Bateman k-functions. Apelblat, Consiglia and Mainardi [1] in a recent survey expressed that Havlock(1925) and Bateman(1931) has introduced new functions as solutions of fluid dynamics problems. Recently, Shrivastava, Narain and Shrivastava [17] obtained solution of triple series equations involving generalized Laguerre polynomials. In this paper, we discuss the problem to obtain the solution of five series equations involving generalized Bateman K-functions, employing the technique due to Narain and Lal [12].

We shall obtain the solution of the following five series equations :

$$\sum_{n=0}^{\infty} D_n \Gamma(l+1+n) k_{2n}^{2l}\left(\frac{x}{2}\right) = 0; 0 < x < a, \tag{1.1}$$

$$\sum_{n=0}^{\infty} D_n \Gamma(l+m+n) k_{2n}^{2l}\left(\frac{x}{2}\right) = g(x); a < x < b, \tag{1.2}$$

$$\sum_{n=0}^{\infty} D_n \Gamma(l+1+n) k_{2n}^{2l}\left(\frac{x}{2}\right) = 0; b < x < c, \tag{1.3}$$

$$\sum_{n=0}^{\infty} D_n \Gamma(l+m+n) k_{2n}^{2l}\left(\frac{x}{2}\right) = h(x); c < x < d, \tag{1.4}$$

$$\sum_{n=0}^{\infty} D_n \Gamma(l+1+n) k_{2n}^{2l}\left(\frac{x}{2}\right) = 0; d < x < \infty, \tag{1.5}$$

where $l > -m, 0 < m < 1,\ k_{2n}^{2l}(x)$ is the generalized Bateman k-function as given by Chakrabarty ([2], 6), $g(x)$ and $h(x)$ are known functions. Solution is obtained by reducing them to Fredholm integral equation of the second kind.

2 Some Useful Results

From the orthogonality relation of Srivastava ([15],p.589, eq. (2.6)]), we have

$$\int_0^{\infty} x^{-2l-1} k_{2n}^{2l}(x) k_{2m}^{2l}(x) dx = 2^{2l} \frac{\Gamma(n-l)}{\Gamma(n+l+1)} \delta_{m,n} \tag{2.1}$$

where $\delta_{m,n}$ is Kronecker delta.

For $l > -\frac{1}{2}, 0 < m < 1$, it is easily shown by Erdélyi ([8], p.401 eq. (1); p.405 eq. (20)) that

$$\int_0^{y} (y-x)^{m-1} e^{+x} k_{2n}^{2l}(x) dx = \frac{\Gamma(m)}{2^m} e^{y} k_{2n+m}^{2l+m}(y), \tag{2.2}$$

$$\int_y^{\infty} (x-y)^{-m} x^{-l-1} e^{-x} k_{2n}^{2l}(x) dx = \frac{\Gamma(1-m)\Gamma(l+m+n)}{2^{\frac{(1-m)}{2}}\Gamma(l+1+n)} y^{(-l+\frac{m}{2}+\frac{1}{2})} e^{-y} \cdot k_{2n+m-1}^{2l+m-1}(y). \tag{2.3}$$

The following summation result can be easily established by using (2.1), (2.2) and (2.3):

$$S(x,u) = \sum_{n=0}^{\infty} \frac{\Gamma(l+m+n)}{2^{2l}\Gamma(n-l)} k_{2n}^{2l}(x) k_{2n}^{2l}(u), \tag{2.4}$$

$$\begin{aligned} S(x,u) &= \frac{e^{-x} x' \cdot 2^{\frac{3(1-m)}{2}}}{\{\Gamma(1-m)\}^2} \int_0^{r} E(y)(x-y)^{-m}(u-y)^{-m} dy, \\ &= \frac{e^{-x} \cdot x' \cdot 2^{\frac{3(1-m)}{2}}}{\{\Gamma(1-m)\}^2} S_r(x,u), \end{aligned} \tag{2.5}$$

where, $E(y) = e^{2y} \cdot y^{l+\frac{m}{2}+\frac{1}{2}}, r = \min(x,u)$.

3 Solution of Five Series Equations

To solve eqns. (1.1) to (1.5), we assume $\frac{x}{2} = X$ and

$$\begin{aligned} \sum_{n=0}^{\infty} D_n \Gamma(l+1+n) k_{2n}^{2l}(x) &= p(x), a < x < b \\ &= q(x), c < x < d. \end{aligned} \tag{3.1}$$

Using the orthogonality relation (2.1), with an appeal to (1.1), (1.3) and (3.1), we obtain

$$D_n = \frac{2^{-21}}{\Gamma(n-l)} \left\{ \int_a^b p(u) + \int_c^d q(u) \right\} u^{-2l-1} \cdot k_{2n}^{2l}(u) du. \tag{3.2}$$

After substituting the value of D_n in eqns. (1.2) and (1.4), and then interchanging the order of summation and integration, we find the equation

$$\begin{aligned} \int_a^b u^{-2l-1} p(u) S_u(X,u) du + \int_x^b u^{-2l-1} p(u) S_x(X,u) du + \int_c^d u^{-2l-1} q(u) S_x(X,u) du \\ = \frac{\{\Gamma(1-m)\}^2}{2^{\frac{3(1-m)}{2}}} e^{x} \cdot X^{-l} g(X)\ (a < x < b), \end{aligned} \tag{3.3}$$

$$(3.4)\quad \int_a^b u^{-2l-1}p(u)S_u(X,u)du+\int_c^x u^{-2l-1}q(u)S_x(X,u)du+\int_x^d u^{-2l-1}q(u)S_x(x,u)du$$
$$=\frac{\{\Gamma(1-m)\}^2}{2^{\frac{3(1-m)}{2}}}e^x, X^{-l}h(x),\ (c<x<d).$$

Inverting the order of integration in the above equations, we derive

$$(3.5)\qquad \int_a^x \frac{E(y)}{(x-y)^m}\left\{p_1(y)+\int_c^d \frac{u^{-2l-1}q(u)}{(u-y)^m}dudy=\frac{\{\Gamma(1-m)\}^2}{2^{\frac{3(1-m)}{2}}}e^x\cdot x^{l-1}g(x), a<x<b\right. ,$$

(3.6)
$$\int_c^x \frac{E(y)}{(x-y)^m}q_1(y)dy=\frac{\{\Gamma(1-m)\}^2}{2^{\frac{3(1-m)}{2}}}\cdot e^x\cdot x^{-1}h(x)-\int_a^b \frac{E(y)}{(x-y)^m}p_1(y)dy-\int_a^c \frac{E(y)dy}{(x-y)^m}\int_c^d \frac{u^{2l-1}q(u)du}{(u-y)^m},$$
$(c<x<d)$, where,

$$(3.7)\qquad \begin{cases} \text{(i) } p_1(y)=\int_y^b \frac{u^{-2l-1}p(u)}{(u-y)^m}du \\ \text{(ii) } q_1(y)=\int_y^d \frac{u^{-2l-1}q(u)}{(u-y)^m}du \end{cases}.$$

For $0<m<1$, we can solve Abel-type integral eqns. (3.5), (3.6) and (3.7) to obtain the equations

$$(3.8)\qquad E(y)p_1(y)=G(y)-E(y)\int_c^d \frac{u^{-2l-1}q(u)}{(u-y)^m}du,$$

$$(3.9)\quad E(y)q_1(y)=H(y)-\frac{\sin m\pi}{\pi(y-c)^{1-m}}\int_a^b \frac{(c-t)^{1-m}}{(y-t)}E(t)p_1(t)dt-\frac{\sin m\pi}{\pi(y-c)^{1-m}}\cdot$$
$$\times\int_a^c \frac{(c-t)^{1-m}}{(y-t)}E(t)dt\int_c^d \frac{u^{-2l-1}q(u)}{(u-t)^m}du,$$

$$(3.10)\qquad u^{-2l-1}p(u)=-\frac{\sin m\pi}{\pi}\frac{d}{du}\int_u^b \frac{p_1(y)du}{(y-u)^{1-m}}, a<u<b,$$

$$(3.11)\qquad u^{-2l-1}q(u)=-\frac{\sin m\pi}{\pi}\frac{d}{du}\int_u^d \frac{p_1(y)du}{(y-u)^{1-m}}, a<u<d,$$

where $G(y)$ and $H(y)$ are known functions given by

$$(3.12)\qquad \begin{cases} G(y)=\frac{\Gamma(1-m)}{2^{\frac{3(1-m)}{2}}\Gamma(m)}\frac{d}{dy}\int_a^y \frac{e^x\cdot x^{-1}h(x)dx}{(y-x)^{1-m}}, a<y<b \\ H(y)=\frac{\Gamma(1-m)}{2^{\frac{3(1-m)}{2}}\Gamma(m)}\frac{d}{dy}\int_c^y \frac{e^x\cdot x^{-1}h(x)dx}{(y-x)^{1-m}}, c<y<d \end{cases}.$$

From eqns. (3.8) and (3.10), we see that the functions $p(u)$ and $q(u)$ are related by the equation

$$(3.13)\qquad u^{-2l-1}p(u)=-\frac{\sin m\pi}{\pi}\frac{d}{du}\int_u^b \frac{G(y)dy}{E(y)(y-u)^{1-m}}+\frac{\sin m\pi}{\pi(b-u)^{1-m}}\int_c^d \frac{t^{-2l-1}(t-b)^{1-m}q(t)}{(u-t)}dt$$

where $a<u<b$.

Now

$$(3.14)\qquad \int_c^d \frac{u^{-2l-1}q(u)}{(u-y)^m}du=\frac{\sin m\pi}{\pi(c-y)^{m-1}}\int_c^d \frac{(t-c)^{m-1}q_1(t)dt}{(t-y)}.$$

Using this result together with eqn. (3.8), we see that eqn. (3.9) can be written in the form

$$(3.15)\qquad E(y)q_1(y)+\int_c^d q_1(x)T(x,y)dX=H(y)-\frac{\mathrm{sinm}\,\pi}{\pi(y-c)^{1-m}}\cdot\int_a^b (c-t)^{1-m}G(t)dt$$

where

$$(3.16)\qquad T(x,y)=\frac{\sin^2 m\pi}{\pi^2[(x-c)(y-c)]^{1-m}}\cdot\int_b^c \frac{E(t)(c-t)^{2(1-m)}}{(x-t)(y-t)}dt, c<y<d$$

is a symmetric kernel.

Eqn. (3.15) is a Fredholm integral equation of the second kind which determines $q_1(y).q(u)$ can be found from eqn. (3.11) and $p(u)$ from eqn. (3.13). Finally the coefficients D_n for $l>-\frac{1}{2}, 0<m<1$ are given by the eqn. (3.2).

If we replace X by $x/2$, we get the solution of equations (1.1) to (1.5) by eqn. (3.2).

In Particular if $a=0, b=a, c=b$ and $d\to\infty$ in equations (1.1) to (1.5), we get the solution of triple series equations considered by Dwivedi [4].

4 Conclusion

The generalized Bateman k-functions have been applied to solve the problems of different integral and series equations by many scholars like Srivastava [14], Srivastava[15], Dwivedi [5], Dwivedi and Trivedi [6], Narain, Singh and Lal [11], Narain and Lal [12] Dwivedi and Singh [7], Tripathi and Dixit [19] to solve pair of dual series, triple series, quadruple series, and some five series equations. The solution presented in this paper is obtained by employing the techniques of Narain and Lal involving generalized Bateman k-functions by reducing them to the solution of a Fredholm inregral equation of second kind with different bounday conditions. Thus we have seen that Bateman k-functions are having interesting properties to solve double, triple, quadruple and five series equations as special functions. These solutions are very useful in Mathematical and Quantum Physics, aero and Fluid Dynamics and Thermodynamics.

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

ACCEPTABLE STRINGS IN AN AUTOMATON

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Abstract

This paper is a presentation and discussion of two proofs of a theorem. The theorem is a statement about a specific type of sequence of inputs where all multiples of fixed denominations are accepted as inputs of an automaton. The theorem has interesting implications for accepted strings in a finite automaton in general setting. Here, we have examined different methods for proving the theorem. We present one analytic proof by using automata theory with graph theoretic concepts and another proof by utilizing the ordered partition of number theory. We also illustrate the results with the help of examples.

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Keywords and Phrases: Finite Automata, Acceptable Strings, Ordered Partition.

1 Introduction

An automaton (in plural Automata) is an abstract self-operating machine which follows a predetermined sequence of operation automatically gives an output from input. Here input may be energy, information, materials, etc. The system works without the intervention of man. Automata theory plays a major role in huge applied areas. The most significant areas include communication, transportation, health care, electronic banking, etc. Mainly finite automata are significant in many different areas, including Electrical Engineering, Linguistics, Computer Science, Philosophy, Biology, Mathematics, etc. In Computer science, automata widely used in text processing, Compilers, Software and hardware design, network protocol, etc. [6]. Many authors have done their work on the string of automaton for a long time. Yu et al. [5] presented symbolic string verification: An automata-based approach. Aydin et al. [1] had done their work on Automata-based model counting for string constraints. Most recently, Yue et al. [8] developed the language acceptability of finite automata based on theory of semi-tensor product of matrices. Dobronravov et al. [3] introduced the length of the shortest strings accepted by two-way finite automata. As a result of the techniques used in the aforementioned works, we are continuing our research on acceptable strings in an automaton. The originality of the paper is that we primarily provide many proofs utilizing entirely distinct methodologies.

2 Preliminaries

A *finite state automaton* consists of a finite set of states and a set of transitions from state to state that occurs on input symbols from a set of alphabets. An *alphabet* is a finite, non-empty set of symbols denoted by A, e.g. $A = \{0, 1\}$, the set of binary alphabet. A *string* (or word) is a finite sequence of symbols chosen from the set A, e.g. 01101, 01, 1, 0 are some strings over the set alphabet $A = \{0, 1\}$. A *Deterministic Finite Automata* can be formally defined as a 5-tuple $\sum = (Q, A, \delta, q_0^*, F)$ where $Q\,(\neq \phi)$ is a finite set of states, A is a finite non-empty set of inputs, $\delta : Q \times A \to Q$ is defined by $\delta\,(q_0^*, a) = q_1; q_0^*, q_1 \in Q, a \in A, q_0^*$, is the initial state, F is the set of final states and $F \subseteq Q$. A string x is *accepted* by finite state automata $\sum = (Q, A, \delta, q_0^*, F)$ if $\delta\,(q_0^*, x) = p$ for some $p \in F$. A final state is also called an *accepting state*. The initial state is denoted by an arrow mark and the final state is denoted by a double circle. The input is accepted when all input is read and match by transitions and the automaton is in a final state [6].

A finite-state automaton is a machine that constructs computing by reading a one-way read-only tape. The input is produced up of words written on the tape. The written words use a describe alphabet which is called the input alphabet and the words create a string. The Finite automata will be produced up of the input-output relations at every state and also the modifications of the states that will appear in receiving the input at a particular state. At the end of the process, it becomes visible whether the input is accepted or rejected by the automaton machine. Also, *Deterministic* refers to the distinctiveness of the computation. The finite automata are called deterministic finite automata if the machine reads an input string one symbol at a time [6, 8]. *Tree automata* are state machines. They deal with tree structures rather than the strings of the more traditional state machine [4].

In Combinatorics and Number theory, a partition of a positive integer n, also called an integer partition, is a way of writing n as a sum of natural numbers. If order matters, the sum becomes a composition or ordered partition. Thus, a composition or an *ordered partition* of an integer n is a way of writing n as the sum of a sequence of positive integers [2].

3 Main Results

In this section, we present our result with different proofs. Also we discuss some examples.

Theorem 3.1. *In a finite automata, if q_0 is the initial state, q_m is the final state with $m = nk$, where $m, n, k \in \mathbb{N}$, and automata accept inputs of denomination ln, where $1 \leq l \leq k$ then the number of acceptable strings in the automata is $2^{(k-1)}$.*

Proof. Let $l_s n, s \in \{1, 2, ..., r\}, 1 \leq r \leq k$ is a sequence of automaton acceptable inputs. Then, we have $nk = (l_1 + l_2 + ... + l_r)n$. i.e. $k = l_1 + l_2 + ... + l_r$. So, without loss of generality, let $n = 1$ be the lowest possible denomination accepted by the automaton. Hence $m = k$. Now, Consider the finite tree automata with states $\{q_0, q_1, ...q_k\}$ with a direct transition function from state q_i to state q_j if and only if $i < j$. q_l has stored value $l, 0 \leq l \leq k$. We get the final state when we reach the value $m = k$, i.e., when we reach state q_k. Now, going from state q_i to state $q_j, i < j$, takes us from value i to value j; i.e. input $(j - i)$ is added to the present value i.

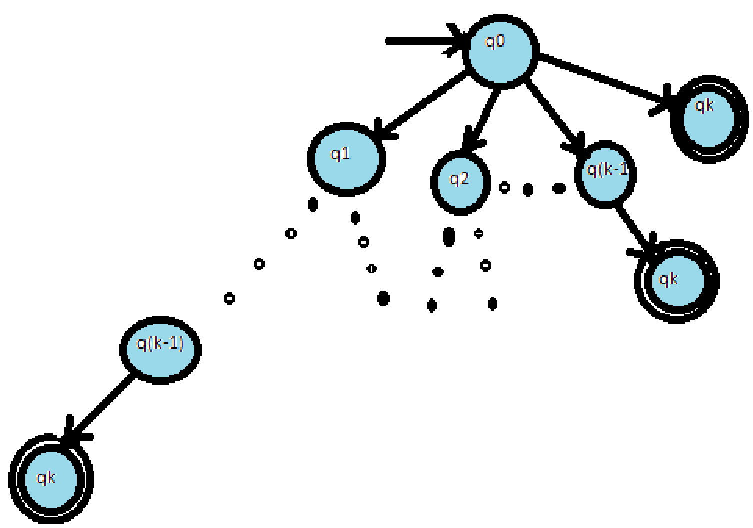

***Figure* 3.1**. A state diagram of a finite Tree automata

A path (transition sequence) from state q_0 to state q_k will be define an automaton acceptable strings. Eg. When $k = 3$ in the tree automaton below, we have $(q_0, q_1, q_2, q_3), (q_0, q_1, q_3), (q_0, q_3), (q_0, q_2, q_3)$ are acceptable strings.

Since q_0 is always the starting states and q_k the final states, we have each subset of $S = \{q_1, q_2, ..., q_{k-1}\}$ correspond to a unique (path) transition function from q_0 to q_k and vice-versa. Therefore, there are a total of $2^{|S|} = 2^{k-1}$ transition functions or paths from q_0 to q_k and therefore, there is 2^{k-1} acceptable strings of the finite automaton. $\square$

Alternate proof. If $l_s n, s \in \{1, 2, ..., r\}, 1 \leq r \leq k$ is an accepted sequence of inputs, then, we have $nk = (l_1 + l_2 + ... + l_r)n$. i.e. $k = l_1 + l_2 + ... + l_r$.
So, without loss of generality, let $n = 1$ be the lowest possible denomination accepted by the automaton. Hence we get $m = k$. Now $k = l_1 + l_2 + ... + l_r$. means $\{l_1, l_2, ..., l_r\}$ forms a partition of k.

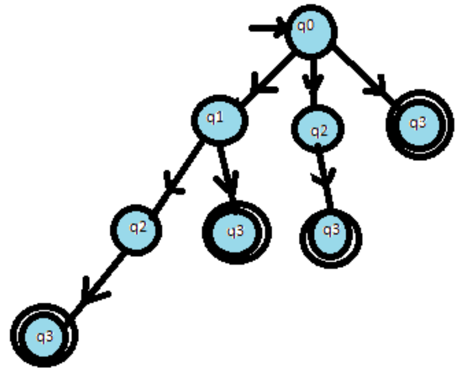

***Figure* 3.2**. A state diagram of a Tree automata with 3 states

Let N = No. of acceptable strings of the automaton.
We Claim that $N = 2^{k-1}$.

Clearly, any acceptable strings, or sequence of inputs form an ordered partition of m and vice-versa.Therefore, there are as many acceptable strings as there are ordered partition of m. Now $m = 1+1+...+1(k\text{-times of }1) = k$, there are $(k-1)$'$+'$ signs between the '$k'1s$. We define an operationdeleting a $+$ sign as replacing the '$1's$ joined by the $+$ signs to be deleted with their sum, keeping the remaining $+$ signs undisturbed. With this interpretation, each set of choice for '$+'$ signs to be deleted corresponds to a unique ordered partition of 'm and vice-versa. Eg. $m = 3, 3 = 1+1+1$, Choosing the 2nd '$+'$ sign correspond to the ordered partition $1+(1+1)$ i.e. $1+2$,and the ordered partition $2+1$ i.e. $(1+1)+1$ corresponds to choosing and deleting the 1st '$+'$ sign. Similarly choosing both the '$+'$ sign correspond to $(1+1+1) = 3$ and not choosing any of the $+$ sign corresponds to the partition $1+1+1$ of 3 etc. Since there are 2 choices for each $+$ sign, viz. to delete or not to delete, and there are a total of $(k-1)+$ signs, so there are 2^{k-1} such choices in total and as such there are 2^{k-1} ordered partition of m. Hence, $N = 2^{k-1}$.

Example 3.1. In a finite automata, if q_0 is the initial states, q_m is the final state with $n = 1, m = k = 4$, Allowed denomination i.e inputs are $S = 1, 2, 3, 4$. There are $2^{4-1} = 8$ accepted sequences of inputs by the

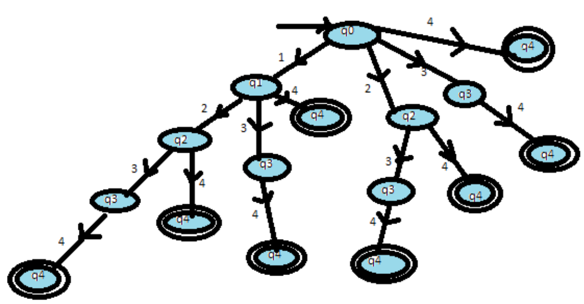

***Figure* 3.3**. A state diagram of a Tree automata with 4 states

finite automata. They are (1,2,3,4), (1,2,4), (1,3,4), (2,3,4), (2,4), (3,4), (1,4), and (4). These are obtained by traversing all the transition functions from the state q_0 to state q_4.

Alternatively, For $4 = 1+1+1+1$, there are four '1s and 3' $+$ signs. We define a function f_3 on all binary strings of length 3 to the ordered partition of 4. Now,
$f_3(000) = 1+1+1+1$,
$f_3(001) = 1+1+(1+1) = 1+1+2$,
$f_3(010) = 1+(1+1)+1 = 1+2+1$,
$f_3(011) = 1+(1+1+1) = 1+3$,
$f_3(100) = (1+1)+1+1 = 2+1+1$,

$f_3(101) = (1+1)+(1+1) = 2+2$,
$f_3(110) = (1+1+1)+1 = 3+1$,
$f_3(111) = (1+1+1+1) = 4$.
Corresponding to each binary string of length 3 we get an ordered partition of 4 which in turn corresponds to an acceptable sequence of inputs viz. $(1,2,3,4),(1,2,4),(1,3,4),(1,4),(2,3,4),(2,4),(3,4),(4)$ respectively and these are precisely 8 in numbers.

Example 3.2. In a finite automata, if q_0 is the initial states, q_m is the final state with $n=5, m=5k=20$, Allowed denomination are multiple of 5 i.e inputs are $S = 5,10,15,20$. There are $2^{4-1} = 8$ accepted

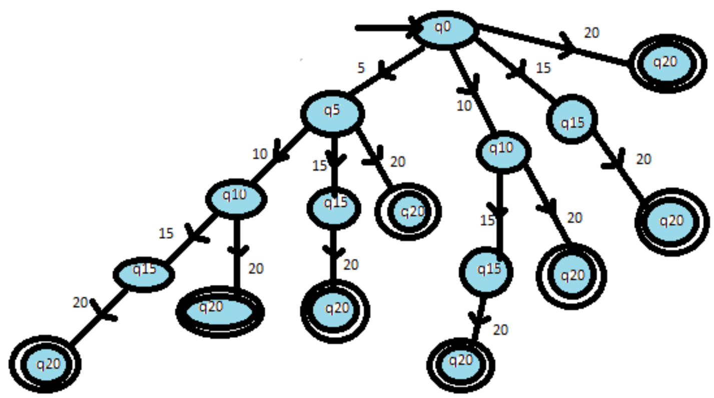

***Figure* 3.4**. A state diagram of a Tree automata with 4 states

sequences of inputs by the finite automata. They are (5,10,15,20), (5,10,20), (5,15,20), (5,20), (10,15,20), (10,20), (15,20), and (20). These are obtained by traversing all the transition functions from the state q_0 to state q_{20}.

Alternately, For $20 = 5+5+5+5$, there are four 5s and $3+$ signs. We define a function g_3 on all binary strings of length 3 to the ordered partition of 20.
Now,
$g_3(000) = 5+5+5+5$,
$g_3(001) = 5+5+(5+5) = 5+5+10$,
$g_3(010) = 5+(5+5)+5 = 5+10+5$,
$g_3(011) = 5+(5+5+5) = 5+15$,
$g_3(100) = (5+5)+5+5 = 10+5+5$,
$g_3(101) = (5+5)+(5+5) = 10+10$,
$g_3(110) = (5+5+5)+5 = 15+5$,
$g_3(111) = (5+5+5+5) = 20$.
Corresponding to each binary string of length 3 we get an ordered partition of 20 which in turn corresponds to an acceptable sequence of inputs (5,10,15,20), (5,10,20), (5,15,20), (5,20), (10,15,20), (10,20), (15,20), and (20) respectively and these are precisely 8 in numbers.
We note that in both the examples there are exactly 8 accepted sequence of inputs. This was bound to happen since we have $5k = 5l_1 + 5l_2 + \ldots + 5l_r \Leftrightarrow k = l_1 + l_2 + \ldots + l_r$.

We conclude with a final example.

Example 3.3. In a finite automata, if q_0 is the initial states, q_m is the final state with $n=1, m=k=5$, Allowed denomination i.e inputs are $S = 1,2,3,4,5$ There are $2^{5-1} = 16$ accepted sequences of inputs by the finite automata. They are (1,2,3,4,5), (1,2,3,5), (1,2,4,5), (1,3,4,5), (2,3,4,5), (1,2,5), (1,4,5), (3,4,5), (1,5), (4,5), (2,5), (3,5), (2,4,5), (1,3,5), (2,3,5),and (5). These are obtained by using transition functions from the state q_0 to state q_5.

Alternatively, For $5 = 1+1+1+1+1$, there are five 1s and 4 + signs. We define a function h_4 on all binary strings of length 4 to the ordered partition of 5. Now,

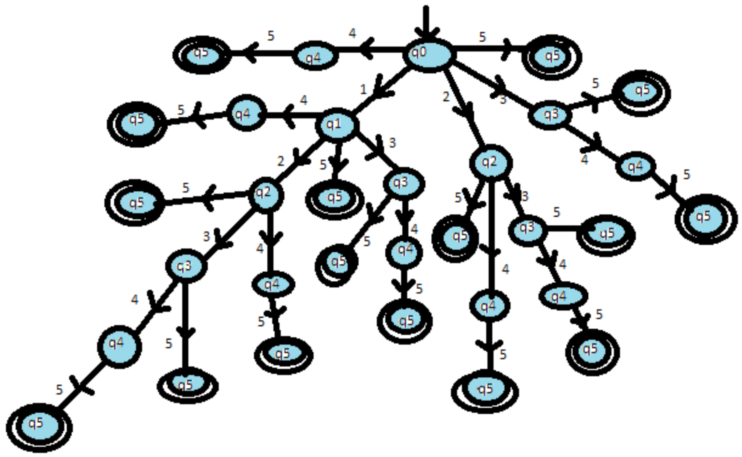

***Figure* 3.5**. A state diagram of a Tree automata with 5 states

$h_4(0000) = 1 + 1 + 1 + 1 + 1,$
$h_4(0001) = 1 + 1 + 1 + (1 + 1) = 1 + 1 + 1 + 2,$
$h_4(0010) = 1 + 1 + (1 + 1) + 1 = 1 + 1 + 2 + 1,$
$h_4(0011) = 1 + (1 + 1) + 1 + 1 = 1 + 2 + 1 + 1,$
$h_4(0100) = (1 + 1) + 1 + 1 + 1 = 2 + 1 + 1 + 1,$
$h_4(0101) = 1 + 1 + (1 + 1 + 1) = 1 + 1 + 3,$
$h_4(0110) = 1 + (1 + 1 + 1) + 1 = 1 + 3 + 1,$
$h_4(0111) = (1 + 1 + 1) + 1 + 1 = 3 + 1 + 1,$
$h_4(1000) = 1 + (1 + 1 + 1 + 1) = 1 + 4,$
$h_4(1001) = (1 + 1 + 1 + 1) + 1 = 4 + 1,$
$h_4(1010) = (1 + 1) + (1 + 1 + 1) = 2 + 3,$
$h_4(1011) = (1 + 1 + 1) + (1 + 1) = 3 + 2,$
$h_4(1100) = (1 + 1) + (1 + 1) + 1 = 2 + 2 + 1,$
$h_4(1101) = 1 + (1 + 1) + (1 + 1) = 1 + 2 + 2,$
$h_4(1110) = (1 + 1) + 1 + (1 + 1) = 2 + 1 + 2,$
$h_4(1111) = (1 + 1 + 1 + 1 + 1) = 5.$
Corresponding to each binary string of length 4 we get an ordered partition of 5 which in turn corresponds to an acceptable sequence of inputs viz. (1,2,3,4,5), (1,2,3,5), (1,2,4,5), (1,3,4,5), (2,3,4,5), (1,2,5), (1,4,5), (3,4,5), (1,5), (4,5), (2,5), (3,5), (2,4,5), (1,3,5), (2,3,5), and (5) respectively and these are precisely 16 in numbers.

4 Conclusion

We have presented two different proofs of a theorem in automata, each being different in its approach. We looked at an Graph theoretic proof using the concept of a tree, and our second proof used another very interesting concept of ordered partition or composition of numbers from Combinatorics and Number theory. Finally, we examplified the theorem with both the approaches.

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COEFFICIENT INEQUALITY FOR A CLASS OF ANALYTIC FUNCTIONS APPROACHING TO CLASS OF CONVEX FUNCTIONS IN THE LIMIT FORM AND CLASS OF STARLIKE FUNCTIONS DIRECTLY

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Abstract

We introduce a class of analytic functions and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to this class with special character that it tends to the class of convex functions as $\alpha \to \dfrac{\pi}{2}$.

2020 Mathematical Sciences Classification: 30C50
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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disc $\mathbb{E} = \{z : |z| < 1\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach [1, 2] proved that $|a_2| \leq 2$ for the functions $f(z)\mathcal{S}$. In 1923, Löwner [10] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$.

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was expected to try to find some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$**,** Fekete and Szegö [4] [8]used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$\left|a_3 - \mu a_2^2\right| = \begin{cases} 3 - 4\mu & if \mu \leq 0 \\ 1 + 2\exp\left(\dfrac{-2\mu}{1-\mu}\right) & if\ 0\ \leq \mu \leq 1 \\ 4\mu - 3 & if \mu \geq 1 \end{cases}. \tag{1.2}$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ [3, 9].

Let us define some subclasses of $\mathcal{S}$.

We denote by $\mathcal{S}^*$, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \ \in \ \mathcal{A}$$

and satisfying the condition

$$Re\ \left(\frac{zg\,(z)}{g\,(z)}\right) > 0,\ z \in \mathbb{E}. \tag{1.3}$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{A}$$

and satisfying the condition

$$Re\left(\frac{(zh'(z)}{h'(z)}\right) > 0, \ z \in \mathbb{E}. \tag{1.4}$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in \mathcal{S}^*$ such that

$$Re\ \left(\frac{zf'(z)}{g(z)}\right) > 0, \ z \in \mathbb{E}. \tag{1.5}$$

The class of close to convex functions is denoted by C and was introduced by Kaplan [7] and it was shown by him that all close to convex functions are univalent.

$$S^*\ (A,B) = \left\{ f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \ -1 \le B < A \le 1, z \in \mathbb{E} \right\}, \tag{1.6}$$

$$\mathcal{K}(A,B) = \left\{ f(z) \in \mathcal{A}; \frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}, \ -1 \le B < A \le 1, z \in \mathbb{E} \right\}. \tag{1.7}$$

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}\ (A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclass as

$$\left\{ f(z) \in \mathcal{A}; \left(\frac{zf'(z)}{f(z)}\right)^{\beta} + \tan\alpha \left(\frac{(zf'(z))'}{f'(z)}\right)^{1-\beta} \prec \left\{\frac{1+w(z)}{1-w(z)}\right\}^{\gamma}; z \in \mathbb{E} \right\}$$

and we shall denote this class as $\mathcal{K}S^*(\alpha,\beta\}$.

We shall deal with two subclasses of $S^*(f, f', \alpha, \beta)$ defined as follows in our next paper:

$$\mathcal{KS}^*(\alpha,\beta,A,B) = \left\{ f(z) \in \mathcal{A}; \left(\frac{zf'(z)}{f(z)}\right)^{\beta} + \tan\alpha \left(\frac{(zf'(z))'}{f'(z)}\right)^{1-\beta} \prec \frac{1+Az}{1+Bz}; z \in \mathbb{E} \right\}, \tag{1.8}$$

$$\mathcal{KS}^*(A,B,\alpha,\beta,\gamma) = \left\{ f(z) \in \mathcal{A}; \left(\frac{zf'(z)}{f(z)}\right)^{\beta} + \tan\alpha \left(\frac{(zf'(z))'}{f'(z)}\right)^{1-\beta} \prec \left\{\frac{1+Az}{1+Bz}\right\}^{\gamma}; z \in \mathbb{E} \right\}. \tag{1.9}$$

Several researchers established new subclasses using these classes and gave amazing results about coefficient inequality. [12], [15].

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination. *Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; z $\mathbb{E}$ and we write $f(z) \prec F(z)$.*[11]

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n, \ w(0) = 0, \ |w(z)| < 1. \tag{1.10}$$

It is known that

$$|d_1| \le 1, \ |d_2| \le 1 - |d_1|^2. \tag{1.11}$$

2 Preliminary Lemmas.

For $0 < c < 1$, we write $w(z) = \left(\dfrac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + \cdots. \tag{2.1}$$

3 Main Results

Theorem 3.1. Let $f(z) \in \mathcal{KS}^*(\alpha, \beta, \gamma)$
$|a_3 - \mu a_2^2| \leq$

$$\begin{cases} \frac{1}{\{\beta+2(1-\beta)\tan\alpha\}^2}\left[\frac{\{4(1-\beta)(\beta+2)\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}}{\{\beta+3(1-\beta)\tan\alpha\}} - 4\gamma^2\mu\right] & (3.1) \\ if \ \ \mu \leq \frac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}-\gamma\{(1-\alpha)\beta+2\alpha(1-\beta)\}}{4\{\beta+3(1-\beta)\tan\alpha\}} & \\ \frac{1}{3\alpha+\beta-4\alpha\beta} & (3.2) \\ if \ \ \frac{\{4\,(1-\beta)\,(\beta+2)\tan\alpha\ -\beta(\beta-3)\}\,\{\beta+2\,(1-\beta)\tan\alpha\ \}-\gamma\{(1-\alpha)\,\beta+2\alpha(1-\beta)\}^2}{4\,\{\beta+3\,(1-\beta)\tan\alpha\ \}} \leq & \\ \mu \leq \frac{\{4\,(1-\beta)\,(\beta+2)\tan\alpha\ -\beta(\beta-3)\}\,\{\beta+2\,(1-\beta)\tan\alpha\ \}+\gamma\{(1-\alpha)\,\beta+2\alpha(1-\beta)\}^2}{4\gamma^2\,\{\beta+3\,(1-\beta)\tan\alpha\ \}} & \\ \frac{1}{\{\beta+2\,(1-\beta)\tan\alpha\ \}^2}\left[4\gamma^2\mu - \frac{\{4\,(1-\beta)\,(\beta+2)\tan\alpha\ -\beta\,(\beta-3)\}\,\{\beta+2\,(1-\beta)\tan\alpha\ \}}{\{\beta+3\,(1-\beta)\tan\alpha\ \}}\right] & (3.3) \\ if \ \ \mu \geq \frac{\{4\,(1-\beta)\,(\beta+2)\tan\alpha\ -\beta\,(\beta-3)\}\,\{\beta+2\,(1-\beta)\tan\alpha\ \}+\gamma\{(1-\alpha)\,\beta+2\alpha\,(1-\beta)\}^2}{4\gamma^2\,\{\beta+3\,(1-\beta)\tan\alpha\ \}} & \end{cases}$$

The results are sharp.

*Proof***.** By definition of $\mathcal{KS}^*(\alpha, \beta, \gamma)$, we have

$$f(z) \in \mathcal{A}; \left(\frac{zf'(z)}{f(z)}\right)^{\beta} + \tan\alpha\left(\frac{(zf'(z))'}{f'(z)}\right)^{1-\beta} = \left\{\frac{1+w(z)}{1-w(z)}\right\}^{\gamma}; w(z) \in \mathcal{U} \tag{3.4}$$

Expanding the series (2.1), we get

(3.5)

$$\left\{1+\beta a_2 z + (2\beta a_3 + \frac{\beta(\beta-3)}{2}a_2^2)z^2 + \ldots\right\} + \tan\alpha\ \{1+2(1-\beta)\,a_2 z + 2(1-\beta)(3a_3 - (\beta+2)a_2^2)z^2 + \ldots\}$$
$$= \left(1+2\gamma c_1 z + 2\gamma\left(c_2+\gamma {c_1}^2\right)z^2 + \ldots\right).$$

Identifying terms in (3.5), we get

$$a_2 = \frac{2\gamma}{\beta+2\,(1-\beta)\tan\alpha}\ c_1. \tag{3.6}$$

$$a_3 = \frac{\gamma}{\beta+3\,(1-\beta)\tan\alpha}\ c_2 + \frac{4\,(1-\beta)\,(\beta+2)\tan\alpha\ -\beta(\beta-3)}{\{\beta+3\,(1-\beta)\tan\alpha\ \}\,\{\beta+2\,(1-\beta)\tan\alpha\ \}}\gamma^2\ {c_1}^2. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$\begin{aligned} a_3 - \mu a_2^2 =& \frac{\gamma c_2}{\beta+3\,(1-\beta)\tan\alpha} \\ &+ \left[\frac{4\,(1-\beta)\,(\beta+2)\tan\alpha\ -\beta(\beta-3)}{\{\beta+3\,(1-\beta)\tan\alpha\ \}\,\{\beta+2\,(1-\beta)\tan\alpha\ \}} - \frac{4\gamma^2\mu}{\{\beta+2\,(1-\beta)\tan\alpha\ \}^2}\right]c_1^2. \end{aligned} \tag{3.8}$$

Taking absolute value and using Triangular inequality, (3.8) can be rewritten as

$$
\begin{aligned}
(3.9)\quad & |a_3-\mu a_2^2| \le \frac{\gamma|c_2|}{\beta+3\left(1-\beta\right)\tan\alpha} \\
& + \frac{1}{\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}^2}\left|\frac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}}{\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}}-4\gamma^2\mu\right||c_1^2|
\end{aligned}
$$

Using (1.9) in (3.6), simple calculations yield

$$
|a_3-\mu a_2^2| \le \frac{\gamma}{\beta+3\left(1-\beta\right)\tan\alpha}+\frac{1}{\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}^2} \tag{3.10}
$$

$$
\left[\left|\frac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}}{\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}}-4\gamma^2\mu\right|-\frac{\gamma\{(1-\alpha)\,\beta+2\alpha(1-\beta)\}^2}{\beta+3\left(1-\beta\right)\tan\alpha}\right]|c_1|^2
$$

***Case* I.** $\mu \le \dfrac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}}{4\gamma^2\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}}$. In this case, (3.10) can be rewritten as

$$
|a_3-\mu a_2^2| \le \frac{\gamma}{\beta+3\left(1-\beta\right)\tan\alpha}+\frac{1}{\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}^2} \tag{3.11}
$$

$$
\left[\frac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}-\gamma\{(1-\alpha)\,\beta+2\alpha(1-\beta)\}^2}{\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}}-4\mu\right]|c_1|^2.
$$

***Subcase* I (a).** $\mu \le \dfrac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}-\gamma\{(1-\alpha)\,\beta+2\alpha(1-\beta)\}^2}{4\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}}$.

Using (1.9), (3.8) becomes

(3.12)

$$
\left|a_3-\mu a_2^2\right| \le \frac{1}{\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}^2}\left[\frac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}}{\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}}-4\gamma^2\mu\right].
$$

***Subcase* I (b).** $\mu \ge \dfrac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}-\gamma\{(1-\alpha)\,\beta+2\alpha(1-\beta)\}^2}{4\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}}$.

We obtain from (3.8)

$$
\left|a_3-\mu a_2^2\right| \le \frac{\gamma}{\beta+3\left(1-\beta\right)\tan\alpha}. \tag{3.13}
$$

***Case* II.** $\mu \ge \dfrac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}}{4\gamma^2\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}}$

Preceding as in case I, we get

$$
\left|a_3-\mu a_2^2\right| \le \frac{1}{3\alpha+\beta-4\alpha\beta}+\frac{1}{\{(1-\alpha)\,\beta+2\alpha\,(1-\beta)\}^2} \tag{3.14}
$$

$$
\left[4\gamma^2\mu-\frac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta\left(\beta-3\right)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}+\gamma\{(1-\alpha)\,\beta+2\alpha\,(1-\beta)\}^2}{\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}}\right]|c_1|^2.
$$

***Subcase* II (a).**$\mu \le \dfrac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}+\gamma\{(1-\alpha)\,\beta+2\alpha(1-\beta)\}^2}{4\gamma^2\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}}$

(3) takes the form

$$
\left|a_3-\mu a_2^2\right| \le \frac{\gamma}{\beta+3\left(1-\beta\right)\tan\alpha} \tag{3.15}
$$

Combining subcase I (b) and subcase II (a), we obtain

(3.16) $$\left|a_3-\mu a_2^2\right| \le \frac{\gamma}{\beta+3\,(1-\beta)\tan\alpha} \text{ if}$$

$$\frac{\{4\,(1-\beta)\,(\beta+2)\tan\alpha\ -\beta(\beta-3)\}\,\{\beta+2\,(1-\beta)\tan\alpha\ \}\}-\gamma\{(1-\alpha)\,\beta+2\alpha(1-\beta)\}^2}{4\,\{\beta+3\,(1-\beta)\tan\alpha\ \}} \le$$

$$\mu \le \frac{\{4\,(1-\beta)\,(\beta+2)\tan\alpha\ -\beta(\beta-3)\}\,\{\beta+2\,(1-\beta)\tan\alpha\ \}\}+\gamma\{(1-\alpha)\,\beta+2\alpha(1-\beta)\}^2}{4\gamma^2\,\{\beta+3\,(1-\beta)\tan\alpha\ \}}$$

***Subcase* II (b).** $\mu\ \ge \dfrac{\{4\,(1-\beta)\,(\beta+2)\tan\alpha\ -\beta(\beta-3)\}\,\{\beta+2\,(1-\beta)\tan\alpha\ \}\}+\gamma\{(1-\alpha)\,\beta+2\alpha(1-\beta)\}^2}{4\gamma^2\,\{\beta+3\,(1-\beta)\tan\alpha\ \}}$

Preceding as in subcase I (a), we get

(3.17)

$$\left|a_3-\mu a_2^2\right| \le \frac{1}{\{\beta+2\,(1-\beta)\tan\alpha\ \}^2}\left[4\gamma^2\mu-\frac{\{4\,(1-\beta)\,(\beta+2)\tan\alpha\ -\beta(\beta-3)\}\,\{\beta+2\,(1-\beta)\tan\alpha\ \}\}}{\{\beta+3\,(1-\beta)\tan\alpha\ \}}\right].$$

Combining (3.9), (3.13) and (3.14), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$$f_1\,(z) = (1+az)^b,$$

where
$$a = \frac{2\gamma\,\{\beta+3(1-\beta)\tan a\}}{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan a\}^3-2\gamma}$$

and
$$b = \frac{\{4\,(1-\beta)\,(\beta+2)\tan\alpha\ -\beta(\beta-3)\}\,\{\beta+2\,(1-\beta)\tan\alpha\ \}\}^3-2\gamma}{\{\beta+3\,(1-\beta)\tan\alpha\ \}\,\{\beta+2\,(1-\beta)\tan\alpha\ \}\}}.$$

Extremal function for (3.2) is defined by $f_2\,(z) = z(1+cz^2)^d$,
where $c = \dfrac{\tan\alpha}{\beta+3\,(1-\beta)\tan\alpha}$ and $d = \dfrac{\gamma}{\tan\alpha}$.

Corollary 3.1. *Putting $\gamma=1, \beta=0$ and applying limit as $\alpha\to\frac{\pi}{2}$ in the theorem, we get*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} 1-\mu,\ if\ \mu\le\ 1; \\[2ex] \dfrac{1}{3}\ \ if\ \ 1\le\mu\le\dfrac{4}{3}; \\[2ex] \mu-1,\ if\ \mu\ge\ \dfrac{4}{3}. \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.

Corollary 3.2. *Putting $\alpha=0, \beta=1,\ \gamma=0$ in the theorem, we get*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} 3-4\mu\ if\ \mu\le\ \dfrac{1}{2}; \\[2ex] 1\ \ if\ \ \dfrac{1}{2}\le\mu\le 1; \\[2ex] 4\mu-3,\ if\ \mu\ge\ 1. \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent starlike functions.

Conclusion : A subclass of analytic functions which take a broad view of some well-known subclasses of analytic and univalent functions was demarcated. The better estimates for the Fekete-Szeg functional for the defined class were obtained along with extremal functions. The study combines existing results and attains new outcomes in geometric function theory. Forthcoming researches can be done to acquire the geometric properties.

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ON HARMONIOUS CHROMATIC NUMBER OF $C(T_{m,n})$, $M(T_{m,n})$, $C(L(T_{m,n}))$ AND $M(L(T_{m,n}))$

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Abstract

In this paper, we discuss the harmonious coloring and investigate the harmonious chromatic number of central and middle graph of tadpole graph and harmonious chromatic number of central and middle graph of line graph of tadpole graph, denoted by $\chi_H(C(T_{m,n}))$, $\chi_H(M(T_{m,n}))$ and $\chi_H(C(L(T_{m,n})))$, $\chi_H(M(L(T_{m,n})))$ respectively.

2020 Mathematical Sciences Classification. 05C15, 05C76
Keywords and Phrases: Harmonious coloring, Harmonious chromatic number, Central graph, Middle graph, Line graph and Tadpole graph.

1 Introduction

A proper vertex coloring of a graph G is a function $c : V(G) \longrightarrow \{1, 2, , k\}$ in which $c(u)$ and $c(v)$ are different for the adjacent vertices u and v and smallest number of colors are needed to color a graph G is called its chromatic number, and is often denoted $\chi(G)$. The Harmonious coloring [5, 6, 7, 9] of a simple graph G is proper vertex coloring in which no any two edges share the same color and minimum number of colors are to be used for harmonious coloring is known as the harmonious chromatic number, denoted by $\chi_H(G)$. For a graph $G = (V, E)$, subdividing each edge of the given graph G exactly once and joining all the non-adjacent vertices of it is the Central graph [3, 7] $C(G)$ of G and the middle graph $M(G)$ [8] is defined in such a way that the vertex set of $M(G)$ is $V(G) \cup E(G)$ and two vertices x, y of $M(G)$ are adjacent in $M(G)$) when one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G and the line Graph [4] of a simple graph G, denoted by $L(G)$ and defined in such a way that there exactly one vertex $v(e)$ in $L(G)$ for each edge e in G and for any two edges e and e' in G, $L(G)$ has an edge between $v(e)$ and $v(e')$, if and only if e and e' are incident with the same vertex in G. The (m, n)-tadpole graph [1, 2, 4] denoted by $T_{m,n}$ $(m \geq 3, n \geq 2)$ is obtained by joining cycle C_m and path P_n, with a bridge that consists $m + n$ vertices and $m + n$ edges.

2 Harmonious Chromatic Number of Tadpole Graph

Theorem 2.1. *For central graph of tadpole graph $T_{m,n}$, the harmonious chromatic number, $\chi_H(C(T_{m,n}) = 2m + n$.*

Proof. Let $T_{m,n}$ be a tadpole graph consisting $m + n$ vertices and $m + n$ edges. $V(T_{m,n}) = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ and $E(T_{m,n}) = \{u_1u_2, u_2u_3, ..., u_mu_1\} \cup \{u_1v_1, v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$. To get the central graph subdivide each edge of $T_{m,n}$ by the vertices u'_i and v'_j $(1 \leq i \leq m)(1 \leq j \leq n)$. $V(C(T_{m,n})) = \{u_i : 1 \leq i \leq m\} \cup \{u'_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\} \cup \{v'_j : 1 \leq j \leq n\}$. Coloring the vertices as follows; define coloring $c : V(C(T_{m,n})) \longrightarrow \{1, 2, 3, ..., (2m + n)\}$ by $c(u_i) = i$ $(1 \leq i \leq m)$, $c(u'_i) = m + i$ $(1 \leq i \leq m)$, $c(v_j) = 2m + j$ $(1 \leq j \leq n)$, $c(v'_j) = m + 1 + j$ $(1 \leq j \leq n)$.

Claim 2.1: c is proper; from above each $c(u_i)$ and $c(v_i)$ and its neighbors are assigned by different colors *i.e.* $c(u_i) \neq c(v_i)$, although $c(u'_i) = c(v_j)$, but these vertices are at least at a distance 2, which leads to proper coloring.

Claim 2.2; c is harmonious; it is clear that no two edges share the same color pair and we assign different colors on the vertices in such a way that they are at least at a distance 3. Therefore it is harmonious. *Claim* 2.3; c is minimum; all the vertices are colored by $2m+n$ colors, if we repeat (assign) any color on any vertex from these assigned colors, color pairs will be repeated which contradicts the harmonious coloring, therefore it is minimum. Hence the theorem. Figure 2.1 shows the central graph of $T_{4,3}$ for with coloring.

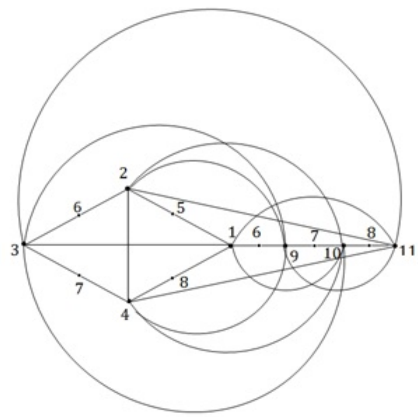

***Figure* 2.1:** $C(T_{4,3})$ with coloring, $\chi_H C(T_{4,3}) = 11$.

□

Theorem 2.2. *For middle graph of tadpole graph, $T_{m,n}$, the harmonious chromatic number, $\chi_H(M(T_{m,n})) = 2m+n$.*

Proof. Let $T_{m,n}$ be a tadpole graph consisting $m+n$ vertices and $m+n$ edges. $V(T_{m,n}) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\}$ and $E(T_{m,n}) = \{u_1u_2, u_2u_3, ..., u_mu_1\} \cup \{u_1v_1, v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$. For getting middle graph, let subdivide each edge of $T_{m,n}$ by the vertices u'_i and v'_j $(1 \le i \le m)(1 \le j \le n)$. $V(M(T_{m,n})) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{u'_i : 1 \le i \le m\} \cup \{v'_j : 1 \le j \le n\}$. Coloring the vertices as follows; define coloring $c : V(M(T_{m,n})) \longrightarrow \{1, 2, 3, ..., (2m+n)\}$ by $c(u_i) = i$ $(1 \le i \le m)$, $c(u'_i) = m+i$ $(1 \le i \le m)$, $c(v_j) = j+1$ $(1 \le j \le n)$, $c(v'_j) = 2m+j$ $(1 \le j \le n)$. For further proof follow Theorem 2.1. Figure 2.2 shows the middle graph of $T_{3,4}$ with coloring.

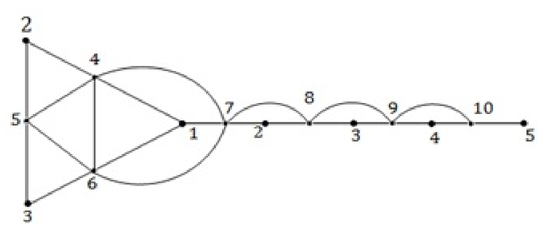

***Figure* 2.2:** $T_{3,4}$ with coloring, $\chi_H(M(T_{3,4})) = 10$.

□

3 Harmonious Chromatic Number of Line Graph of Tadpole Graph

Theorem 3.1. *For central graph of line graph of tadpole graph $L(T_{m,n})$, the harmonious chromatic number, $\chi_H(C(L(T_{m,n}))) = 2m+n+2$.*

Proof. Let $L(T_{m,n})$ be a line graph of tadpole graph consisting $m+n$ vertices and $m+n+1$ edges. $V(L(T_{m,n})) = \{x_i : 1 \le i \le m\} \cup \{y_j : 1 \le j \le n\}$ and $E(L(T_{m,n})) = \{x_1x_2, x_2x_3, ..., x_mx_1\} \cup \{x_1y_1, x_my_1, y_1y_2, y_2y_3, ..., y_{n-1}y_n\}$. Now to get the central graph subdivide each edge of $L(T_{m,n})$ by the vertices z_1, z_2, x'_i and y'_j $(1 \le i \le m)(1 \le j \le n-1)$. $V(C(L(T_{m,n}))) = \{z_1, z_2\} \cup \{x_i : 1 \le i \le m\} \cup \{x'_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{y'_j : 1 \le j \le n-1\}$. Coloring the vertices as follows; define coloring

$c : V(C(L(T_{m,n}))) \longrightarrow \{1, 2, 3, ..., (2m + n + 2)\}$ by $c(x_i) = i$ $(1 \le i \le m)$, $c(x_i') = m + i$ $(1 \le i \le m)$, $c(z_1) = 2m + 1$, $c(z_2) = 2m + 2$, $c(y_j) = 2m + 2 + j$ $(1 \le j \le n)$, $c(y_j') = m + j$ $(1 \le j \le n - 1)$. Figure 3.1 shows the central graph of $L(T_{3,4})$ with coloring. Now we proceed as done in Theorem 2.1.

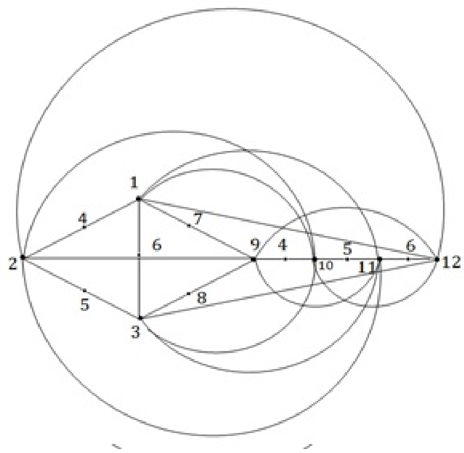

***Figure* 3.1:** $C(L(T_{3,4}))$ with coloring, $\chi_H C(L(T_{3,4})) = 12$.

□

Theorem 3.2. *For middle graph of line graph of tadpole graph $L(T_{m,n})$, the harmonious chromatic number, $\chi_H(M(L(T_{m,n}))) = 2m + n + 2$.*

Proof. Let $L(T_{m,n})$ be a line graph of tadpole graph consisting $m + n$ vertices and $m + n + 1$ edges. $V(L(T_{m,n})) = \{x_i : 1 \le i \le m\} \cup \{y_j : 1 \le j \le n\}$ and $E(L(T_{m,n})) = \{x_1x_2, x_2x_3, ..., x_mx_1\} \cup \{x_1y_1, x_my_1, y_1y_2, y_2y_3, ..., y_{n-1}y_n\}$. Now to get the middle graph subdivide each edge of $L(T_{m,n})$ by the vertices z_1, z_2, x_i' and y_j' $(1 \le i \le m)(1 \le j \le n - 1)$. $V(C(L(T_{m,n}))) = \{z_1, z_2\} \cup \{x_i : 1 \le i \le m\} \cup \{x_i' : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{y_j' : 1 \le j \le n - 1\}$. For further proof follow Theorem 2.1. Figure 3.2 shows the middle graph of $L(T_{3,4})$ with coloring.

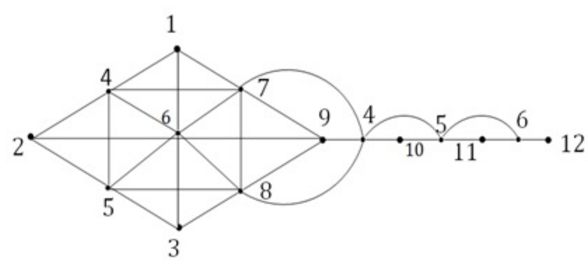

***Figure* 3.2:** $M(L(T_{3,4}))$ with coloring, $\chi_H M(L(T_{3,4})) = 12$.

□

4 Conclusion

In this paper, we investigate the harmonious chromatic number of central graph, middle graph and line graph of tadpole graph and we find that the harmonious chromatic number of central graph of line graph of tadpole graph is same as the harmonious chromatic number of middle graph of line graph of tadpole graph *i.e.* $\chi_H(C(L(T_{m,n}))) = \chi_H(M(L(T_{m,n})))$.

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COMMON FIXED POINT THEOREMS IN COMPLEX VALUED METRIC SPACES SATISFYING E.A. PROPERTY AND INTIMATE MAPPING

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Abstract

In this paper we prove the common fixed point theorems in complex valued metric spaces satisfying E.A. property and intimate mapping. Our result generalizes some recent results in the literature due to Azam et al.(2011) . Also we improve the results of Rajput & Singh(2014) satisfying E.A. property and Meena(2015) regarding intimate mapping. Some concepts have been taken from the results obtained by Choi et al.(2017) and Jebril et al.(2019) to improve our results. Also some examples are given to illustrate our obtained results.
2020 Mathematical Sciences Classification: : $47H10$, $54H25$
Keywords and Phrases: Complex valued metric space, common fixed point, E.A. property, intimate mapping, weakly compatible mapping.

1 Introduction, Definitions and Notations

The concept of fixed point theorem first introduced by Poincare & Miranda[15] in 1883. After that Brouwer[3] published his famous fixed point theorem in 1912. The theorem states that "If B is a closed unit ball in R^n and if $T : B \to B$ is continuous then T has a fixed point in B". In 1992 Banach[4] proved his famous fixed point theorem in which contraction principle is the main tools. Banach's fixed point theorem plays a major role in fixed point theory. It has applications in many branches of mathematics. Because of its usefulness, a lot of articles have been dedicated to the improvement and generalization of that result. Most of these generalizations have been made by considering different contractive type conditions in different spaces$\{cf.[5]-[21]\}$. In 2011, Azam et al.[2] made a generalization by introducing a complex valued metric space using some contractive type conditions.Very recently, Rajput & Singh [18] generalized this result by replacing the constants of contraction by some control functions. The purpose of this work is to obtain a common fixed point result for three self mappings in complex valued metric spaces which generalizes the results of [1] and improve the results of Rajput & Singh[18] satisfying E.A. property and Meena[14] regarding intimate mapping.

We write regular complex number as $z = x + iy$ where x and y are real numbers and $i^2 = -1$. Let $\mathbb{C}_1$ be the set of complex numbers and z_1 and $z_2 \in \mathbb{C}_1$. Define a partial order relation $\precsim$ on $\mathbb{C}_1$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re\,(z_2) \text{ and } Im\,(z_1) \leq Im\,(z_2)\,.$$

Thus $z_1 \precsim z_2$ if one of the following conditions is satisfied:

$(i)Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$, $(ii)Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$, $(iii)Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$, $(iv)Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of (ii), (iii) and (iv) is satiesfied and we write $z_1 \prec z_2$ if only (iv) is satisfied.

Taking this into account some fundamental properties of the partial order $\precsim$ on $\mathbb{C}_1$ as follows:

(1) If $0 \precsim z_1 \precsim z_2$ then $|z_1| < |z_2|$;
(2) If $z_1 \precsim z_2, z_2 \precsim z_3$ then $z_1 \precsim z_3$ and
(3) If $z_1 \precsim z_2$ and $0 < \lambda < 1$ is a real number then $\lambda z_1 \precsim z_2$.

Azam et al. defined the complex valued valued metric space in the following way:

Definition 1.1. *Let X be a nonempty set where as $\mathbb{C}_1$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}_1$ satisfies the following conditions:*
$(d_1) : 0 \precsim d(x, y)$,*for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;*
$(d_2) : d(x, y) = d(y, x)$ *for all $x, y \in X$;*
$(d_3) : d(x, y) \precsim d(x, z) + d(z, y)$,*for all $x, y, z \in X$.*
Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Definition 1.2 ([2])**.** *Let $\{x_n\}$ be sequence in X and $x \in X$. If for every $c \in \mathbb{C}_1$ with $0 \prec c$, there is an $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec c$ for all $n > n_0$ then x is called the limit of $\{x_n\}$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.*

Definition 1.3 ([2])**.** *If every Cauchy sequence is convergent in $\mathbb{C}_1$ then the space is called a complete complex valued metric space.*

Definition 1.4 ([6])**.** *Let f and g be two self-maps defined on a set X. Then f and g are said to be weakly compatible if they commute at their coincidence points.*

Definition 1.5 ([22])**.** *Let $T, S : X \to X$ be two self mappings of a bicomplex valued metric space (X, d). The pair (T, S) are said to satisfy E. A. property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.*

Definition 1.6 ([21])**.** *The self mappings $T,\ S : X \to X$ are said to satisfy the common limit in the range of S property ($CLRs$ property) if $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Sx$ for some $x \in X$.*

Definition 1.7 ([14])**.** *Let S and T be self maps on a bicomplex valued metric space (X, d). Then the pair $\{S, T\}$ is said to be $T-$intimate if and only if $\alpha d(TSz_n, Tz_n) \precsim \alpha d(SSz_n, Sz_n)$, where $\alpha = \limsup\{z_n\}$ or $\liminf\{z_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Tz_n = t$ for some t in X.*

Some common fixed point results are established by Rajput & Singh [18] for rational type contraction mapping in $\mathbb{C}_1$ on which they have proved the following theorem.

Theorem 1.1 ([18])**.** *Let (X, d) be a complex valued metric space and $A, B, S, T : X \to X$ be four self mappings satisfying the following conditions*
(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;
(ii) For all $x, y \in X$ and $0 < \alpha < 1$,

$$\begin{aligned} d(Ax, By) \quad &\precsim \quad \alpha \frac{[d(Ax, Sx)(dAx, Ty) + d(By, Ty)\, d(By, Sx)]}{d(Ax, Ty) + d(By, Sx)} \\ &\quad +\beta \frac{\left[\{d(Ax, Ty)\}^2 + \{d(By, Sx)\}^2\right]}{d(Ax, Ty) + d(By, Sx)}; \end{aligned}$$

(iii) The pairs (A, S) and (B, T) are weakly compatible and
(iv) The pair (A, S) or (B, T) satisfies E. A. property if the range of mappings $S(X)$ or $T(X)$ is closed subspace of X then A, B, S and T have a unique common fixed point in X.

Meena[14] investigated a common fixed point for intimate mappings in $\mathbb{C}_1$ as follows:

Theorem 1.2. *Let A, B, S and T be the four mappings from a complex valued metric space (X, d) into itself, such that*
(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;
(ii) For all $x, y \in X$,

$$d(Ax, By) \precsim \alpha d(Sx, Ty) + \beta \frac{d(Ax, Sx)\,.d(By, Ty)}{d(Ax, Ty) + d(Sx, By) + d(Sx, Ty)}$$

and $d(Ax, Ty) + d(Sx, By) + d(Sx, Ty) \neq 0$, where α, β are non-negative real numbers with $\alpha + \beta < 1$;
(iii) (A, S) is $S-$intimate and (B, T) is $T-$intimate and
(iv) $S(X)$ is complete.
Then A, B, S and T have a unique common fixed piont in X.

2 Main Results

In this section we prove some theorems and give some examples.

Theorem 2.1. *Let (X,d) be a complex valued metric space and $A,B,S,T : X \to X$ four self-mappings satisfying the conditions:*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(ii) for all $x,y \in X$,

$$d(Ax,By) \precsim \alpha d(Sx,Ty) + \beta \frac{d(Ax,Sx).d(By,Ty)}{[1+d(Sx,Ty)]} + \gamma \frac{d(Ty,By)[1+d(Sx,Ax)]}{[1+d(Sx,Ty)]}; \tag{2.1}$$

(iii) the pair (A,S) and (B,T) are weakly compatible;

(iv) one of the pair (A,S) or (B,T) satiesfies E.A. property.

If the range of one of the mapping $S(X)$ or $T(X)$ is a closed subspace of X then the mapping A,B,S and T have a unique common fixed point in X.

Proof. First we suppose that the pair (B,T) satisfies $E.A.$ property. Then by difinition there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Further since $B(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence, $\lim_{n\to\infty} Sy_n = t$. We claim that $\lim_{n\to\infty} Ay_n = t$. Let $\lim_{n\to\infty} Ay_n = t_1 \neq t$, then putting $x = y_n$, $y = x_n$ in condition (ii), we have

$$\begin{aligned} & d(Ay_n, Bx_n) \\ \precsim\ & \alpha d(Sy_n,Tx_n) + \beta \frac{d(Ay_n,Sy_n).d(Bx_n,Tx_n)}{[1+d(Sy_n,Tx_n)]} \\ & + \gamma \frac{d(Tx_n,Bx_n)[1+d(Sy_n,Ay_n)]}{[1+d(Sy_n,Tx_n)]}. \end{aligned}$$

or,

$$d(t_1,t) \precsim \alpha d(t,t) + \beta \frac{d(t_1,t).d(t,t)}{[1+d(t,t)]} + \gamma \frac{d(t,t)[1+d(t,t_1)]}{1+d(t,t)}.$$

Then $|d(t_1,t)| \leq 0$.

Hence $t_1 = t$ and this implies that $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = t$.

Now suppose that $S(X)$ is a closed subspace of X, then $t = Su$ for some $u \in X$. Subsequntly, we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = t = Su$.

We claim that $Au = Su$. For this we put $x = u$ and $y = x_n$ in contractive condition (ii). Then we have

$$\begin{aligned} d(Au,Bx_n) \ \precsim\ & \alpha d(Su,Tx_n) + \beta \frac{d(Au,Su).d(Bx_n,Tx_n)}{[1+d(Su,Tx_n)]} \\ & + \gamma \frac{d(Tx_n,Bx_n)[1+d(Su,Au)]}{[1+d(Su,Tx_n)]}. \end{aligned}$$

Taking $n \to \infty$, we get

$$d(Au,t) \precsim \alpha d(Su,t) + \beta \frac{d(Au,Su).d(t,t)}{[1+d(Su,t)]} + \gamma \frac{d(t,t)[1+d(Su,Au)]}{[1+d(Su,t)]}.$$

Then $|d(Au,t)| \leq 0$, which is a contradiction. Hence u is a coincident point of (A,S).

Now the weak compatibility of the pair (A,S) implies that $ASu = SAu$ or $At = St$.

On the other hand since $A(X) \subset T(X)$, there exists a v in X such that $Au = Tv$. Thus $Au = Su = Tv = t$. Now we show that v is a coincidence point of (B,T), i.e., $Bv = Tv = t$.

Putting $x = u$, $y = v$ in contractive condition (ii), we have

$$\begin{aligned} d(Au,Bv) \ \precsim\ & \alpha d(Su,Tv) + \beta \frac{d(Au,Su).d(Bv,Tv)}{[1+d(Su,Tv)]} \\ & + \gamma \frac{d(Tv,Bv)[1+d(Su,Au)]}{[1+d(Su,Tv)]}. \end{aligned}$$

or,

$$d(t, Bv) \precsim \alpha d(t,t) + \beta \frac{d(t,t) . d(Bv,t)}{[1 + d(t,t)]} + \gamma \frac{d(t,Bv)[1 + d(t,t)]}{[1 + d(t,t)]}.$$

or,

$$d(t, Bv) \precsim \gamma d(t, Bv).$$

This implies that $|d(t, Bv)| \leq 0$, which is a contradiction. Thus $Bv = t$.
Hence $Bv = Tv = t$ and v is a coincident point of B and T.
Further, the weak compatibility of the pair (B, T) implies that $BTv = TBv$, i.e., $Bt = Tt$. Therefore t is a common coincidence point of A, B, S and T.
Now we show that t is a common coincident point of A, B, S and T.
Putting $x = u$ and $y = t$ in contractive condition (ii), we get

$$\begin{aligned} d(t, Bt) &= d(Au, Bt) \precsim \alpha d(Su, Tt) + \beta \frac{d(Au, Su) . d(Bt, Tt)}{[1 + d(Su, Tt)]} \\ &\quad + \gamma \frac{d(Tt, Bt)[1 + d(Su, Au)]}{[1 + d(Su, Tt)]}. \end{aligned}$$

or,

$$|d(t, Bt)| \leq \alpha |d(t, Bt)| \, [\text{as } Bt = Tt \text{ and } Au = Su = Tv = t]$$

$$\text{or, } |d(t, Bt)| \leq 0,$$

which is a contradiction. Thus $Bt = t$.
Therefore, $At = Bt = St = Tt = t$.

Uniqueness:

For uniquness suppose t^* be another common fixed point of A, B, S and T.
Then we have $At^* = Bt^* = St^* = Tt^* = t^*$.
Therefore using (2.1) we get

$$\begin{aligned} d(At, Bt^*) &\precsim \alpha d(St, Tt^*) + \beta \frac{d(At, St) . d(Bt^*, Tt^*)}{[1 + d(St, Tt^*)]} + \gamma \frac{d(Tt^*, Bt^*)[1 + d(St, At)]}{[1 + d(St, Tt^*)]} \\ \text{or, } d(t, t^*) &\precsim \alpha d(t, t^*) + \beta \frac{d(t,t) . d(t^*, t^*)}{[1 + d(t, t^*)]} + \gamma \frac{d(t^*, t^*)[1 + d(t,t)]}{[1 + d(t, t^*)]} \\ \text{or, } d(t, t^*) &\precsim \alpha d(t, t^*). \end{aligned}$$

Hence,

$$|d(t, t^*)| \precsim \alpha |d(t, t^*)|,$$

which implies that $|d(t, t^*)| \leq 0$ i.e., $t = t^*$.
Therefore t is the unique common fixed point of A, B, S and T. □

Example 2.1. Let $X = [0, 1]$. We define the mapping $d : X \times X \to \mathbb{C}_1$ as follows

$$d(x, y) = (1 + i)|x - y|, \quad x, y \in X$$

Then (X, d) is a complex valued metric space.
Let $A, B, S, T : X \to X$ be defined by

$$Ax = \frac{x}{4},\ 0 \leq x \leq 1;\ Bx = \begin{cases} 0, & x \neq \frac{1}{2} \\ \frac{1}{8}, & x = \frac{1}{2}; \end{cases}$$

$$Tx = x,\ 0 \leq x < 1;\ Sx = \begin{cases} \frac{x}{4}, & 0 \leq x < 1 \\ \frac{1}{8}, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Clearly, $S(X)$ is closed and $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. We consider the sequence $\left\{x_n : x_n = \frac{1}{2} + \frac{1}{n+2}\right\}$ in X. Then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{8}$. So that the pair (A, S) satisfies the E.A. property. Thus all the conditions of Theorem 2.1 are satiefied and 0 is the unique common fixed point of A, B, S and T.

Theorem 2.2. *Let A, B, S and T be the four mapping from a complex valued metric space (X, d) into itself such that*

(i) $A(X) \subset T(X)$ and $B(X) \subset S(X)$;

(ii) $d(Ax, By) \precsim \alpha d(Sx, Ty) + \beta \frac{d(Ax,Sx).d(By,Ty)}{1+d(Sx,Ty)} + \gamma \frac{d(Ty,By).[1+d(Sx,Ax)]}{1+d(Sx,Ty)}$,
where α, β, γ are non negative real numbers with $\alpha + \beta + 2\gamma < 1$;

(iii) (A, S) is $S-$intimate and (B, T) is $T-$intimate;

(iv) $S(X)$ is complete.

Then A, B, S and T have a unique common fixed point in X.

Proof. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, therefore there exists a sequence $\{y_{2n}\}$ in X such that
$y_{2n} = Ax_{2n} = Tx_{2n+1}$.
$y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$.

Then using these conditions in contractive condition (ii), we get

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Ax_{2n}, Bx_{2n+1}) \\
&\precsim \alpha d(Sx_{2n}, Tx_{2n+1}) + \beta \frac{d(Ax_{2n}, Sx_{2n}).d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Sx_{2n}, Tx_{2n+1})} \\
&\quad + \gamma \frac{d(Tx_{2n+1}, Bx_{2n+1}).[1 + d(Sx_{2n}Ax_{2n})]}{1 + d(Sx_{2n}, Tx_{2n+1})} \\
&= \alpha d(y_{2n-1}, y_{2n}) + \beta \frac{d(y_{2n}, y_{2n-1}).d(y_{2n+1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} \\
&\quad + \gamma \frac{d(y_{2n}, y_{2n+1}).[1 + d(y_{2n-1}, y_{2n})]}{1 + d(y_{2n-1}, y_{2n})} \\
&\precsim \alpha d(y_{2n-1}, y_{2n}) + \beta d(y_{2n}, y_{2n+1}) + \gamma d(y_{2n}, y_{2n+1})
\end{aligned}$$

$$(1 - \beta - \gamma) d(y_{2n}, y_{2n+1}) \precsim \alpha d(y_{2n}, y_{2n-1})$$

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &\precsim \frac{\alpha}{(1 - \beta - \gamma)} d(y_{2n}, y_{2n-1}) \\
&\precsim \frac{\alpha + \gamma}{1 - \beta - \gamma} d(y_{2n}, y_{2n-1})
\end{aligned}$$

$$\text{or, } d(y_{2n}, y_{2n+1}) \precsim h d(y_{2n}, y_{2n-1}),$$

where $h = \frac{\alpha+\gamma}{1-\beta-\gamma} < 1$, as $\alpha + \beta + 2\gamma < 1$.

This implies that

$$|d(y_{2n}, y_{2n+1})| \precsim h |d(y_{2n}, y_{2n-1})|. \tag{2.2}$$

Similarly,

$$\begin{aligned}
d(y_{2n+2}, y_{2n+1}) &= d(Ax_{2n}, Bx_{2n+1}) \\
&\precsim \alpha d(Sx_{2n+2}, Tx_{2n+1}) + \beta \frac{d(Ax_{2n+2}, Sx_{2n+2}).d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Sx_{2n+2}, Tx_{2n+1})} \\
&\quad + \gamma \frac{d(Tx_{2n+1}, Bx_{2n+1}).[1 + d(Sx_{2n+2}Ax_{2n+2})]}{1 + d(Sx_{2n+2}, Tx_{2n+1})} \\
&= \alpha d(y_{2n+1}, y_{2n}) + \beta \frac{d(y_{2n+2}, y_{2n+1}).d(y_{2n+1}, y_{2n})}{1 + d(y_{2n+1}, y_{2n})} \\
&\quad + \gamma \frac{d(y_{2n+2}, y_{2n+1}).[1 + d(y_{2n+1}y_{2n})]}{1 + d(y_{2n+1}, y_{2n})} \\
&= \alpha d(y_{2n+1}, y_{2n}) + \beta \frac{d(y_{2n+2}, y_{2n+1}).d(y_{2n+1}, y_{2n})}{1 + d(y_{2n+1}, y_{2n})} \\
&\quad + \gamma \frac{d(y_{2n+2}, y_{2n+1}).}{1 + d(y_{2n+1}, y_{2n})} + \gamma \frac{[1 + d(y_{2n+1}y_{2n})].}{1 + d(y_{2n+1}, y_{2n})} \\
&\precsim \alpha d(y_{2n+1}, y_{2n}) + (\beta + \gamma) d(y_{2n+2}, y_{2n+1}) + \gamma d(y_{2n+1}, y_{2n})
\end{aligned}$$

$$\text{or, } d(y_{2n+2}, y_{2n+1}) \precsim \frac{\alpha+\gamma}{1-\beta-\gamma} d(y_{2n+1}, y_{2n})$$

$$\text{or, } d(y_{2n+2}, y_{2n+1}) \precsim h d(y_{2n+1}, y_{2n}),$$

where $h = \frac{\alpha+\gamma}{1-\beta-\gamma}$.

This implies that

$$|d(y_{2n+2}, y_{2n+1})| \leq h |d(y_{2n+1}, y_{2n})|. \tag{2.3}$$

Thus from (2.2) and (2.3) we can write

$$|d(y_{2n+2}, y_{2n+1})| \leq h |d(y_{2n+1}, y_{2n})| \leq ... \leq h^{n+1} |d(y_0, y_1)|.$$

So that for any $m > n$

$$\begin{aligned} |d(y_n, y_m)| &\leq |d(y_n, y_{n+1})| + |d(y_{n+1}, y_{n+2})| + ... + |d(y_{m-1}, y_m)| \\ &\leq h^n |d(y_0, y_1)| + h^{n-1} |d(y_0, y_1)| + ... + h^{m-1} |d(y_0, y_1)| \end{aligned}$$

$$\text{i.e., } |d(y_n, y_m)| \leq \frac{h^n}{1-h} |d(y_0, y_1)|,$$

which accounts to say that $\{y_n\}$ is a Cauchy sequence. i.e., $\{Sx_{2n}\}$ is Cauchy in $S(X)$, also $S(X)$ is complete, then $\{y_n\}$ converges to a point $p = Su$ for some $u \in X$.

Thus Ax_{2n}, Sx_{2n}, Bx_{2n+1}, $Tx_{2n+1} \to p$.

Now,

$$\begin{aligned} d(Au, Bx_{2n+1}) &\precsim \alpha d(Su, Tx_{2n+1}) + \beta \frac{d(Au, Su).d(Bx_{2n+1}Tx_{2n+1})}{1 + d(Su, Tx_{2n+1})} \\ &\quad + \gamma \frac{d(Tx_{2n+1}, Bx_{2n+1}).[1 + d(Su, Au)]}{1 + d(Su, Tx_{2n+1})}. \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$|d(Au, p)| \leq \alpha |d(Su, p)|.$$

Thus $|d(Au, p)| \leq 0$, [as $p = Su$] i.e., $Au = p = Su$.

Again $A(X) \subset T(X)$, therefore there exists $v \in X$ such that $Au = Tv = p$.

Now we cosider

$$d(p, Bv) = d(Au, Bv)$$

$$\precsim \alpha d(Su, Tv) + \beta \frac{d(Au, Su).d(Bv, Tv)}{1 + d(Su, Tv)} + \gamma \frac{d(Tv, Bv).[1 + d(Su, Au)]}{1 + d(Su, Tv)}$$

$$\text{or, } d(p, Bv) \leq \gamma d(p, Bv)$$

$$\text{or, } d(p, Bv) = 0,$$

which implies that $p = Bv = Tv = Au = Su$.

Now since $Au = Su = p$ and (A, S) is $S-$intimate,

Therefore we have

$$|d(Sp, p)| \leq |d(Ap, p)|. \tag{2.4}$$

Also

$$d(Ap, p) = d(Ap, Bv)$$

$$\precsim \alpha d(Sp, Tv) + \beta \frac{d(Ap, Sp).d(Bv, Tv)}{1 + d(Sp, Tv)} + \gamma \frac{d(Tv, Bv).[1 + d(Sp, Ap)]}{1 + d(Sp, Tv)}$$

$$\text{or, } |d(Ap, p)| \leq \alpha |d(Sp, p)| \leq \alpha |d(Ap, p)|.[\text{using } (2.4)]$$

Thus $|d(Ap, p)| = 0$, implies that $Ap = p$ and $Sp = p$. Similarly $Bp = Tp = p$.

Uniqueness:

Let us suppose that q be another common fixed point of A, B, S and T such that $p \neq q$.

Then

$$d(p, q) = d(Ap, Bq)$$

$$\precsim \quad \alpha d\left(Sp,Tq\right)+\beta\frac{d\left(Ap,Sp\right).d\left(Bq,Tq\right)}{1+d\left(Sp,Tq\right)}+\gamma\frac{d\left(Tq,Bq\right).\left[1+d\left(Sp,Ap\right)\right]}{1+d\left(Sp,Tq\right)}$$
$$\precsim \quad \alpha d\left(p,q\right),$$

which implies that $|d\left(p,q\right)|\leq\alpha\,|d\left(p,q\right)|\Rightarrow|d\left(p,q\right)|=0$, i.e., $p=q$.
This proves that the mappings A,B,S and T have a unique common fixed point. $\square$

Example 2.2. :Let $X=\mathbb{C}_1$ be the set of complex numbers. Define $d:X\times X\rightarrow\mathbb{C}_1$ by $d(z_1,z_2)=(1+i)\,|z_1-z_2|$ where $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ .Then (X,d) is a complete complex valued metric space. Define $A,B,S,T:X\rightarrow X$ as $Az=0, Bz=0$, $Sz=z$ and $Tz=\frac{z}{2}$. Clearly $A(X)\subset T(X)$ and $B(X)\subset S(X)$. Now consider the sequence $\{z_n=\frac{1}{n},n\in\mathbb{N}\}$ in $\mathbb{C}_1$, then $\lim\limits_{n\longrightarrow\infty}Az_n=\lim\limits_{n\longrightarrow\infty}Sz_n=0$. Also we have $\lim\limits_{n\longrightarrow\infty}d(SAz_n,Sz_n)\precsim\lim\limits_{n\longrightarrow\infty}d(AAz_n,Az_n)$. Thus the pair (A,S) is $S-$intimate. Again $\lim\limits_{n\longrightarrow\infty}d(TBz_n,Tz_n)\precsim\lim\limits_{n\longrightarrow\infty}$
$d(BBz_n,Bz_n)$ implies that the pair (B,T) is $T-$intimate. Therefore the mappings satisfies all the conditions of Theorem 2.2. Hence A,B,S and T have a unique common fixed point in X.

3 Future Prospect

In the line of the works as carried out in the paper one may think of the deduction of fixed point theorems using fuzzy metric, quasi metric, partial metric and other different types of metrics under the flavour of bicomplex analysis. This may be an active area of research to the future workers in this branch.

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

A NEW CLASS OF PÁL TYPE INTERPOLATION IN COMPLEX PLANE-II

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Abstract

In this paper, the author considered a new class for Pál type interpolation problems. They termed Pál type interpolation problems as $PTIP$. This new class for $PTIP$ is defined by omitting a non-zero complex node from the set of value nodes and simultaneously adding another complex node to the set of derivative nodes.

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1 Introduction

Lacunary Polynomial Interpolation is an extension of Hermite Interpolation. It comprises the matching of values and derivatives at certain points but does not insist that these points be consecutive. The author termed Lacunary Polynomial Interpolation as LPI. LPI problems are not always regular due to matching at non-consecutive derivatives.

The study on LPI started with the evolution of Birkhoff interpolation. It is a finely honed theory on real nodes [9, 20]. LPI problems on non-uniformly distributed nodes received attention after the investigations of Brueck [1]. He studied non-uniformly distributed nodes on the unit disk, obtained by applying Mbius transform to the set of zeros of roots of unity. He defined the following polynomials;

$$v_n^{\alpha}(z) = (z+\alpha)^n - (1+\alpha z)^n, \tag{1.1}$$

$$w_n^{\alpha}(z) = (z+\alpha)^n + (1+\alpha z)^n. \tag{1.2}$$

where $0 < \alpha < 1$.

A revolution in the theory of LPI at special nodes was due to Pál [12]. He introduced a new type of interpolation on zeros of two different polynomials, referred as Pál type interpolation.
Let $A(z) \in \pi_m$ and $B(z) \in \pi_n$, where π_n be the set of polynomials of degree less than or equal to n with complex coefficients. For a given positive integer r the problem of $(0,r)$ Pál type interpolation i.e. $(0,r)$-$PTIP$ consists finding a polynomial $P(z) \in \pi_{m+n-1}$, that has prescribed values at m pairwise distinct nodes and prescribed value for r^{th} derivative at n pairwise distinct nodes. These m nodes are called value nodes, and n nodes are called derivative nodes.

The $(0,r)$-$PTIP$ on the pair $\{A(z), B(z)\}$ is regular if and only if any $P(z) \in \pi_{m+n-1}$ with the following sets of interpolation conditions:

$P(y_i) = 0;$ where $A(y_i) = 0;$ $i = 1, 2, \ldots, m,$

$P^{(r)}(z_j) = 0;$ where $B(z_j) = 0;$ $j = 1, 2, \ldots, n.$

implies that $P(z) \equiv 0$. Here the zeros of $A(z)$, $B(z)$ are assumed to be simple.

De Bruin [2, 4, 5], De Bruin et al. [3], De Bruin and Dikshit [6], Bokari et al. [7], Dikshit [8], Pathak [13], Mandloi and Pathak [10], Modi et al. [11], studied regularity of Pál type interpolation problems with some additional nodes.

De Bruin [2] evaluated regularity of incomplete Pál type interpolation on the zeros of polynomials given by (1.1) and (1.2). He omitted one or two real nodes from zeros of $w_n^{\alpha}(z)$ and/or $v_n^{\alpha}(z)$.

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The author [14, 15] investigated the regularity of Birkhoff interpolation in certain dimensions. They revisited $PTIP$ for the sets consisting of the zeros of polynomials with complex coefficients with some additional nodes. Also, they assessed the maximum number of nodes that can be added at value nodes to get regular $PTIP$ [16]. They studied the regularity of 'incomplete' $PTIP$ for several pairs, where they omitted real as well as complex nodes from zeros of certain polynomials [17, 18]. The author [19] defined a new class of Pál type interpolation obtained by adding a real node to one set of interpolation points and omitting a real node from another set of interpolation points.

In section 2, we consider $(0,1)$-$PTIP$, where we omit a non-zero complex node ζ from $v_{2n}^{\alpha}(z)$ and add $-\zeta$ to $w_n^{\alpha}(z)$ or $v_n^{\alpha}(z)$.

In section 3, We consider the polynomials $a_m(z) \in \pi_m$ and $b_n(z) \in \pi_n$ $(m \geq n)$ with simple zeros and take $A_m(z)$ and $B_n(z)$ as the sets of the zeros of these polynomials respectively with $B_n(z) \subseteq A_m(z)$. We assess the regularity of $(0,1)$-$PTIP$ and $(0,2)$- $PTIP$ by omitting a non-zero complex node ζ from $a_m(z)$ and adding $-\zeta$ to $b_n(z)$.

2 A new class of $PTIP$ on non-uniformly distributed nodes

Theorem 2.1. *Let $0 < \alpha < 1$, $n \geq 2$ then $(0,1)$-$PTIP$ on $\left\{\frac{v_{2n}^{\alpha}(z)}{(z-\zeta)}, (z+\zeta)w_n^{\alpha}(z)\right\}$ is regular, for $\pm\zeta \in v_{2n}^{\alpha}(z)$, $\pm\zeta \notin w_n^{\alpha}(z)$.*

Proof. Here, we have total $3n$ interpolation points.
The problem is to find a polynomial $P(z) \in \pi_{3n-1}$ with
$P(y_i) = 0$; y_i is zero of $\frac{v_{2n}^{\alpha}(z)}{(z-\zeta)}$; $i = 1, 2, \ldots, (2n-1)$,
$P'(-\zeta) = 0$,
$P'(z_j) = 0$; z_j is zero of $w_n^{\alpha}(z)$; $j = 1, 2, \ldots, n$.
Let $P(z) = \frac{v_{2n}^{\alpha}(z)}{(z-\zeta)} Q(z)$; where $Q(z) \in \pi_n$,
then $P(z) \in \pi_{3n-1}$.
The problem will be regular if $P(z) \equiv 0$.
As $P'(z_j) = 0$, we get

$$\frac{v_{2n}^{\alpha}(z_j)}{(z_j-\zeta)} Q'(z_j) + \left\{\frac{\{v_{2n}^{\alpha}(z_j)\}'}{(z_j-\zeta)} - \frac{v_{2n}^{\alpha}(z_j)}{(z_j-\zeta)^2}\right\} Q(z_j) = 0.$$

Also, $z_j \in w_n^{\alpha}(z) \subseteq v_{2n}^{\alpha}(z)$, thus we have

$$\frac{\{v_{2n}^{\alpha}(z_j)\}'}{(z_j-\zeta)} Q(z_j) = 0.$$

Since,

$$\{v_{2n}^{\alpha}(z_j)\}' = \frac{2n(1-\alpha^2)(z_j+\alpha)^{n-1}}{(1+\alpha z_j)} \neq 0,$$

Therefore,

$$Q(z_j) = 0.$$

Since z_j has n zeros, therefore

$$Q(z) = Cq_n(z). \tag{2.1}$$

Since,

$$P'(-\zeta) = 0,$$

Therefore,

$$\frac{\{v_{2n}^{\alpha}(-\zeta)\}'}{(2\zeta)} Q(-\zeta) = 0.$$

As,

$$\frac{\{v_{2n}^{\alpha}(-\zeta)\}'}{2\zeta} \neq 0,$$

we have

$$Q(-\zeta) = 0. \tag{2.2}$$

Equations (2.1), (2.2) and interpolatory conditions, give

$$C = 0.$$

Hence,

$$Q(z) \equiv 0.$$

□

Remark 2.1. *One of the essential conditions for the above result is $w_n^{\alpha}(z) \subseteq V_{2n}^{\alpha}(z)$. Since one more similar condition satisfies for the polynomials given by equations (1.1) and (1.2) viz. $v_n^{\alpha}(z) \subseteq V_{2n}^{\alpha}(z)$. Therefore, the following result must hold.*

Theorem 2.2. *Let $0 < \alpha < 1$, $n \geq 2$ then $(0,1)$-PTIP on $\left\{\frac{v_{2n}^{\alpha}(z)}{(z+\zeta)}, (z-\zeta)v_n^{\alpha}(z)\right\}$ is regular, for $\pm\zeta \in v_{2n}^{\alpha}(z)$, $\pm\zeta \notin v_n^{\alpha}(z)$.*

3 A new class of $PTIP$ on the zeros of the polynomials with complex coefficients

Theorem 3.1. *$(0,1)$-PTIP on $\left\{\frac{a_m(z)}{(z-\zeta)}, (z+\zeta)b_n(z)\right\}$, $m > n \geq 1$ for $\pm\zeta \in A_m(z), \pm\zeta \notin B_n(z)$ is regular.*

Proof. Here we have total $m + n$ interpolation points.
The problem is to find a polynomial $P(z) \in \pi_{m+n-1}$ with
$P(y_i) = 0$; where y_i is a zero of $\frac{a_m(z)}{(z-\zeta)}$; $i = 1, 2, \ldots, (m-1)$,
$P'(-\zeta) = 0$,
$P'(z_j) = 0$; where z_j is a zero of $b_n(z)$; $j = 1, 2, \ldots, n$.
Let $P(z) = \frac{a_m(z)}{(z-\zeta)}Q(z)$; where $Q(z) \in \pi_n$,
then $P(z) \in \pi_{m+n-1}$.
The problem will be regular if $P(z) \equiv 0$.
Now,
$P'(z_j) = 0$, we have

$$\frac{a_m(z_j)}{(z_j-\zeta)}Q'(z_j) + \left\{\frac{a'_m(z_j)}{(z_j-\zeta)} - \frac{a_m(z_j)}{(z_j-\zeta)^2}\right\}Q(z_j) = 0.$$

As $P'(-\zeta) = 0, -\zeta \in A_m(z)$ and $a_m(z)$ has simple zeros, the polynomial and its derivative cant vanish simultaneously

$$Q(-\zeta) = 0. \tag{3.1}$$

Also, $z_j \in B_n(z) \subseteq A_m(z)$, we have

$$\frac{a'_m(z_j)}{(z_j-\zeta)}Q(z_j) = 0.$$

Since $\zeta \notin B_n(z)$ and $a'_m(z_j) \neq 0$, therefore we get

$$Q(z_j) = 0.$$

As z_j has n zeros, thus we have

$$Q(z) = Cq_n(z). \tag{3.2}$$

Equations (3.1), (3.2) and interpolatory conditions, give

$$C = 0.$$

Hence,

$$Q(z) \equiv 0.$$

□

Theorem 3.2. *The $(0,2)$-PTIP on $\left\{\frac{a_m(z)}{(z-\zeta)}, (z+\zeta)b_n(z)\right\}$, $m > n \geq 1$ for $\pm\zeta \in A_m(z), \pm\zeta \notin B_n(z)$ is regular.*

Proof. Here we have total $m+n$ interpolation points.
The problem is to find a polynomial $P(z) \in \pi_{m+n-1}$ with
$P(y_i) = 0$; where y_i is a zero of $\frac{a_m(z)}{(z-\zeta)}$; $i = 1, 2, \ldots, (m-1)$,
$P''(-\zeta) = 0$,
$P''(z_j) = 0$; where z_j is a zero of $b_n(z)$; $j = 1, 2, \ldots, n$.
Let $P(z) = \frac{a_m(z)}{(z-\zeta)} Q(z)$; where $Q(z) \in \pi_n$,
then $P(z) \in \pi_{m+n-1}$.
The problem will be regular if $P(z) \equiv 0$.
Now,
$P''(z_j) = 0$,
Therefore,

$$\frac{a_m(z_j)}{(z_j-\zeta)} Q''(z_j) + 2\left\{\frac{a'_m(z_j)}{(z_j-\zeta)} - \frac{a_m(z_j)}{(z_j-\zeta)^2}\right\} Q'(z_j) + \left\{\frac{a''_m(z_j)}{(z_j-\zeta)} - 2\frac{a'_m(z_j)}{(z_j-\zeta)^2} + 2\frac{a_m(z_j)}{(z_j-\zeta)^3}\right\} Q(z_j) = 0.$$

Also $z_j \in B_n(z) \subseteq A_m(z)$ and $a_m(z)$ has simple zeros, the polynomial and its derivative cant vanish simultaneously, therefore we get

$$2(z_j-\zeta)a'_m(z_j)Q'(z_j) + \{(z_j-\zeta)a''_m(z_j) - 2a'_m(z_j)\} Q(z_j) = 0.$$

Since z_j has n zeros and $Q(z) \in \pi_n$, therefore the differential equation is given by

$$2(z-\zeta)a'_m(z)Q'(z) + \{(z-\zeta)a''_m(z) - 2a'_m(z)\} Q(z) = C(z+\zeta)b_n(z). \tag{3.3}$$

The integrating factor of the differential equation (3.3) is given by

$$\Phi(z) = \frac{\{a'_m(z)\}^{1/2}}{(z-\zeta)}.$$

The solution of the differential equation (3.3) is given by

$$\frac{\{a'_m(z)\}^{1/2}}{(z-\zeta)} Q(z) = C \int \frac{b_n(t)(t+\zeta)}{\{a'_m(z)\}^{1/2}(t-\zeta)^2} dt.$$

$$C \left. \frac{b_n(t)(t+\zeta)}{\{a'_m(z)\}^{1/2}(t-\zeta)^2} \right|_{t=\zeta} = 0 \Rightarrow C = 0.$$

Hence,

$$Q(z) \equiv 0$$

. ◻

4 Conclusion

The posed problems of $(0,1)$-$PTIP$ and $(0,2)$-$PTIP$ obtained by adding and omitting a non-zero complex node simultaneously are found to be regular on considered sets of value nodes and derivative nodes.

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SOME RESULTS OF THE GROWTH PROPERTIES OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES ON THE BASIS OF THEIR (p,q)th-RELATIVE GOL'DBERG ORDER AND (p,q)th-RELATIVE GOL'DBERG TYPE

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Abstract

Biswas [2] introduced the idea of (p,q)th-relative Gol'dberg order and (p,q)th-relative Gol'dberg type of an entire function of several complex variables. In this paper we want to establish some results of the growth analysis of entire function of several complex variables on the basis of their (p,q)-ψ relative Gol'dberg order and (p,q)-ψ relative Gol'dberg type of an entire function of several complex variables.
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1 Introduction

Let $\mathbb{C}^n$ and $\mathbb{R}^n$ respectively denotes the complex and real n-spaces. Also, let us indicate the point $(z_1, z_2, \ldots, z_n)$, $(m_1, m_2, \ldots, m_n)$ of $\mathbb{C}^n$ or $\mathbb{I}^n$ by their corresponding unsuffixed symbols z, m respectively where I denotes the set of non negative integers. The modulus of z, denoted by $|z|$, is defined as $|z| = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}$. If the coordinates of the vector m are non-negative integers, then z^n will denote $z_1^{m_1}, z_2^{m_2}, \ldots, z_n^{m_n}$ and $\|m\| = m_1 + m_2 + \cdots + m_n$. If $D \subseteq \mathbb{C}^n$ be an arbitrary bounded complex n-circular domain with center at the origin of coordinates, then for any entire function $f(z)$ on n-complex variables and $R > 0$, $M_{f,D}(R)$ may be defined as $M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$, where a point $z \in D_R$ iff $\frac{z}{R} \in D$. If $f(z)$ is non-constant, then $M_{f,D}(R)$ is strictly increasing and its inverse $M_{f,D}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists such that $\lim_{R\to\infty} M_{f,D}^{-1}(R) = \infty$. For $k \in \mathbb{N}$, we define $\exp^{[k]} R = \exp(\exp^{[k-1]} R)$ and $\log^{[k]} R = \log(\log^{[k-1]} R)$, where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} R = R$, $\log^{[-1]} R = \exp R$, $\exp^{[0]} R = R$ and $\exp^{[-1]} R = \log R$, where p and q always denote positive integers. Maji and Datta [9] introduced the definitions of (p,q)th-Gol'dberg order and (p,q)th-Gol'dberg lower order of an entire function $f(z)$ of n-complex variables, where $p \geq q$ in the following ways;

$$\rho_D^{(p,q)}(f) = \limsup_{R\to\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} R}, \tag{1.1}$$

and

$$\lambda_D^{(p,q)}(f) = \liminf_{R\to\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} R}. \tag{1.2}$$

For $p = 2$ and $q = 1$ the symbols $\rho_D^{(p,q)}(f)$ and $\lambda_D^{(p,q)}(f)$ are respectively denoted by $\rho_D(f)$ and $\lambda_D(f)$ which are actually classical growth indicators [7, 8]. However in the line of Gol'dberg [7, 8], it may be easily established that $\rho^{(p,q)}(f)$ and $\lambda^{(p,q)}(f)$ instead of $\rho_D^{(p,q)}(f)$ and $\lambda_D^{(p,q)}(f)$ respectively.

2 Definitions

Biswas [5] introduced the defintions of (p,q)-ψ order and (p,q)-ψ lower order of an entire function of n-complex variables.

Definition 2.1 ([5]). *Let $\psi(R):[0,+\infty)\rightarrow(0,+\infty)$ be a non-decreasing unbounded function. Then the (p,q)-ψ Gol'dberg order $\rho_D^{(p,q)}(f,\psi)$ and (p,q)-ψ Gol'dberg lower order $\lambda_D^{(p,q)}(f,\psi)$ of an entire function $f(z)$ of n-complex variables are defined as,*

$$\rho_D^{(p,q)}(f,\psi)=\limsup_{R\rightarrow\infty}\frac{\log^{[p]}M_{f,D}(R)}{\log^{[q]}\psi(R)}, \tag{2.1}$$

and

$$\lambda_D^{(p,q)}(f,\psi)=\liminf_{R\rightarrow\infty}\frac{\log^{[p]}M_{f,D}(R)}{\log^{[q]}\psi(R)}. \tag{2.2}$$

Definition 2.1 avoids the restriction for $p\geq q$. However, an entire function $f(z)$ for which $\rho_D^{(p,q)}(f,\psi)$ and $\lambda_D^{(p,\ q)}(f,\psi)$ are called regular (p,q)-ψ Gol'dberg growth. Otherwise, $f(z)$ is said to be irregular (p,q)-ψ Gol'dberg growth. For any non-decreasing unbounded function $\psi(R):[0,+\infty)\rightarrow(0,+\infty)$, if it is assumed that $\lim_{R\rightarrow+\infty}\frac{\log^{[q]}\psi(\alpha R)}{\log^{[q]}\psi(R)}=1$, for all $\alpha>0$, then one can easily verify that $\rho_D^{(p,q)}(f,\psi)$ and $\lambda_D^{(p,q)}(f,\psi)$ are independent of the choice of the domain D and use the symbols $\rho^{(p,q)}(f,\psi)$ and $\lambda^{(p,q)}(f,\psi)$ instead of $\rho_D^{(p,q)}(f,\psi)$ and $\lambda_D^{(p,q)}(f,\psi)$ respectively. Now for any two entire functions $f(z)$ and $g(z)$ of n-complex variables, Mondal and Roy [11] introduced the concept of relative Gol'dberg order of $f(z)$ with respect to $g(z)$ and relative Gol'berg lower order of $f(z)$ with respect to $g(z)$. For the (p,q)-ψ relative Goldberg order introduced by Biswas and Biswas [5] in the following definitions:

Definition 2.2 ([5]). *Let $\psi(R):[0,+\infty)\rightarrow(0,+\infty)$ be a non-decreasing unbounded function. Also, let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables. The (p,q)-ψ relative Gol'dberg order and the (p,q)-ψ relative Gol'dberg lower order of $f(z)$ with respect to $g(z)$ are defined as*

$$\rho_{g,D}^{(p,q)}(f,\psi)=\limsup_{R\rightarrow\infty}\frac{\log^{[p]}M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]}\psi(R)}, \tag{2.3}$$

and

$$\lambda_{g,D}^{(p,q)}(f,\psi)=\liminf_{R\rightarrow\infty}\frac{\log^{[p]}M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]}\psi(R)}. \tag{2.4}$$

Further an entire function $f(z)$ of n-complex variables for which $\rho_{g,D}^{(p,q)}(f,\psi)$ and $\lambda_{g,D}^{(p,q)}(f,\psi)$ are same, is called a function of (p,q)-ψ relative Gol'dberg growth with respect to an entire function $g(z)$ of n-complex variables. Otherwise, $f(z)$ is said to be irregular (p,q)-ψ relative Gol'dberg growth with respect to $g(z)$.

Definition 2.3 ([6]). *Let $f(z)$ and $g(z)$ be two entire functions of n-complex variables with the index-pair (m,q) and (m,p) respectively, where p, q, m are the integers such that $m\geq q+1\geq1$ and $m\geq p+1\geq1$, if $b<\rho_{g,D}^{(p,q)}(f,\psi)<+\infty$ and $\rho_{g,D}^{(p-1,q-1)}(f,\psi)$ is not a non-zero finite number, where $b=1$, if $p=q$, and $b=0$ for otherwise. Moreover, if $0<\rho_{g,D}^{(p,q)}(f,\psi)<\infty$,*

$$\begin{cases}\rho_{g,D}^{(p-n,q)}(f,\psi)=\infty & , \quad for \quad n<p;\\ \rho_{g,D}^{(p,q-n)}(f,\psi)=0 & , \quad for \quad n<q;\\ \rho_{g,D}^{(p+n,q+n)}(f,\psi)=1 & , \quad for \quad n=1,2,\ldots.\end{cases} \tag{2.5}$$

Similarly for $0<\lambda_{g,D}^{(p,q)}(f,\psi)<\infty$, then

$$\begin{cases}\lambda_{g,D}^{(p-n,q)}(f,\psi)=\infty & , \quad for \quad n<p;\\ \lambda_{g,D}^{(p,q-n)}(f,\psi)=0 & , \quad for \quad n<q;\\ \lambda_{g,D}^{(p+n,q+n)}(f,\psi)=1 & , \quad for \quad n=1,2,\ldots.\end{cases} \tag{2.6}$$

If $\psi(R)=R$ and $p\geq q$, then Definition 2.2 coincides with the definition of (p,q)-ψ relative Gol'dberg order and (p,q)-ψ relative Gol'dberg lower order introduced by T. Biswas and R. Biswas [6]. Consequently for $\psi(R)=R$ and $p\geq q$, Definition 2.3 reduces to the definition of index-pair (p,q) of an entire function with respect to another entire function of n-complex variables [3].

T. Biswas and C. Biswas [4] introduced the definition of (p,q)-ψ relative Gol'dberg type $\triangle_{g,D}^{(p,q)}(f,\psi)$ and (p,q)-ψ relative Gol'dberg lower type $\nabla_{g,D}^{(p,q)}(f,\psi)$, (p,q)-ψ relative Gol'dberg weak type $\overline{\triangle}_{g,D}^{(p,q)}(f,\psi)$ and the growth indicator $\overline{\nabla}_{g,D}^{(p,q)}(f,\psi)$ in the following ways;

Definition 2.4 ([4]). *Let $\psi(R) : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function. Also, let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables. The (p, q)-ψ relative Gol'dberg type and (p, q)-ψ relative Gol'dberg lower type of $f(z)$ with respect to $g(z)$ are defined as,*

$$\triangle_{g,D}^{(p,q)}(f,\psi) = \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left(\log^{[q-1]} \psi(R)\right)^{\rho_{g,D}^{(p,q)}(f,\psi)}}, \quad 0 < \rho_{g,D}^{(p,q)}(f,\psi) < +\infty, \tag{2.7}$$

and

$$\nabla_{g,D}^{(p,q)}(f,\psi) = \liminf_{R\to\infty} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left(\log^{[q-1]} \psi(R)\right)^{\rho_{g,D}^{(p,q)}(f,\psi)}}, \quad 0 < \rho_{g,D}^{(p,q)}(f,\psi) < +\infty. \tag{2.8}$$

Definition 2.5 ([4]). *Let $\psi(R) : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function. Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables. The relative (p, q)-ψ Gol'dberg weak type and the growth indicator of $f(z)$ with respect to $g(z)$ are defined as,*

$$\overline{\triangle}_{g,D}^{(p,q)}(f,\psi) = \liminf_{R\to\infty} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left(\log^{[q-1]} \psi(R)\right)^{\lambda_{g,D}^{(p,q)}(f,\psi)}}, \quad 0 < \lambda_{g,D}^{(p,q)}(f,\psi) < +\infty, \tag{2.9}$$

and

$$\overline{\nabla}_{g,D}^{(p,q)}(f,\psi) = \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left(\log^{[q-1]} \psi(R)\right)^{\lambda_{g,D}^{(p,q)}(f,\psi)}}, \quad 0 < \lambda_{g,D}^{(p,q)}(f,\psi) < +\infty. \tag{2.10}$$

During the past decades, the several authors [1, 2, 3, 5, 6, 10] made closed investigation on the growth properties of entire functions of n-complex variables using different growth indicator such as (p, q)-ψ relative order, (p, q)-ψ relative lower order etc.

3 Mains Results

Theorem 3.1. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \lambda_{h,D}^{(p,m)}(f,\psi) < \rho_{h,D}^{(p,m)}(f,\psi) < +\infty$, and $0 < \lambda_{k,D}^{(q,m)}(g,\psi) < \rho_{k,D}^{(q,m)}(g,\psi) < +\infty$, where p, q, m, are all positive integers.*
Then

$$\begin{aligned}\frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)} \le \liminf_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \le \frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)} \\ \le \limsup_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \le \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}.\end{aligned}$$

Proof. From the definition of $\lambda_{h,D}^{(p,m)}(f,\psi)$ and $\rho_{k,D}^{(q,m)}(g,\psi)$, we get for arbitrary positive $\epsilon > 0$ for all large values of R,

$$\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R)) \ge (\lambda_{h,D}^{(p,m)}(f,\psi) - \epsilon) \log^{[m]} \psi(R), \tag{3.1}$$

and

$$\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R)) \le (\rho_{k,D}^{(q,m)}(g,\psi) + \epsilon) \log^{[m]} \psi(R). \tag{3.2}$$

Now from (3.1) and (3.2), it follows that for all sufficiently large values of R

$$\frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \ge \frac{\lambda_{h,D}^{(p,m)}(f,\psi) - \epsilon}{\rho_{k,D}^{(q,m)}(g,\psi) + \epsilon}.$$

As $\epsilon > 0$ is arbitrary,

$$\liminf_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \ge \frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)}. \tag{3.3}$$

Again for a sequence of value of R tending to infinity,

$$\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R)) \leq (\lambda_{h,D}^{(p,m)}(f,\psi) + \epsilon) \log^{[m]} \psi(R), \tag{3.4}$$

and for all sufficiently large values of R,

$$\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R)) \geq (\lambda_{k,D}^{(q,m)}(g,\psi) - \epsilon) \log^{[m]} \psi(R). \tag{3.5}$$

Now from (3.4) and (3.5), we obtain for a sequence of values of R tending to infinity

$$\frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\lambda_{h,D}^{(p,m)}(f,\psi) + \epsilon}{\lambda_{k,D}^{(q,m)}(g,\psi) - \epsilon}.$$

As $\epsilon > 0$ is arbitrary,

$$\liminf_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}. \tag{3.6}$$

Also for all sufficient values of R,

$$\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R)) \leq (\lambda_{k,D}^{(q,m)}(g,\psi) + \epsilon) \log^{[m]} \psi(R). \tag{3.7}$$

Combining (3.1) and (3.7), we obtain for a sequence of values of R tending to infinity,

$$\frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{\lambda_{h,D}^{(p,m)}(f,\psi) - \epsilon}{\lambda_{k,D}^{(q,m)}(g,\psi) + \epsilon}.$$

As $\epsilon > 0$ is arbitrary,

$$\limsup_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}. \tag{3.8}$$

Also for all sufficiently large values of R,

$$\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R)) \leq (\rho_{h,D}^{(p,m)}(f,\psi) + \epsilon) \log^{[m]} \psi(R). \tag{3.9}$$

Now combining (3.5) and (3.9), for all sufficiently large values of R,

$$\frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\rho_{h,D}^{(p,m)}(f,\psi) + \epsilon}{\lambda_{k,D}^{(q,m)}(g,\psi) - \epsilon}.$$

Since ϵ is arbitrary

$$\limsup_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}. \tag{3.10}$$

Thus the theorem follows from (3.3), (3.6), (3.8) and (3.10). □

Theorem 3.2. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \rho_{h,D}^{(p,m)}(f,\psi) < +\infty$, and $0 < \rho_{k,D}^{(q,m)}(g,\psi) < +\infty$, where p, q, m, are all positive integers. Then*

$$\liminf_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)} \leq \limsup_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))}.$$

Proof. From the definition of $\rho_{k,D}^{(q,m)}(g,\psi)$, we get for a sequence of values of R tending to infinity,

$$\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R)) \geq (\rho_{k,D}^{(q,m)}(g,\psi) - \epsilon) \log^{[m]} \psi(R). \tag{3.11}$$

Now from (3.9) and (3.11), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\rho_{h,D}^{(p,m)}(f,\psi) + \epsilon}{\rho_{k,D}^{(q,m)}(g,\psi) - \epsilon}.$$

As $\epsilon > 0$ is arbitrary,

$$\liminf_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)}. \tag{3.12}$$

Also for all sufficiently large values of R ,

$$\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R)) \geq (\rho_{h,D}^{(q,m)}(f,\psi) - \epsilon) \log^{[m]} \psi(R). \tag{3.13}$$

Now from (3.2) and (3.13), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{\rho_{h,D}^{(p,m)}(f,\psi) - \epsilon}{\rho_{k,D}^{(q,m)}(g,\psi) + \epsilon}.$$

As $\epsilon > 0$ is arbitrary,

$$\limsup_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)}. \tag{3.14}$$

Thus the theorem follows from (3.12) and (3.14). □

Theorem 3.3. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \lambda_{h,D}^{(p,m)}(f,\psi) < \rho_{h,D}^{(p,m)}(f,\psi) < +\infty$ and $0 < \lambda_{k,D}^{(q,m)}(g,\psi) < \rho_{k,D}^{(q,m)}(g,\psi) < +\infty$, where p, q, m, are all positive integers.*
Then

$$\frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)} \leq \liminf_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \min\left\{\frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}, \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)}\right\}$$
$$\leq \max\left\{\frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}, \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)}\right\} \leq \limsup_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}.$$

Theorem 3.4. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \nabla_{h,D}^{(p,m)}(f,\psi) < \triangle_{h,D}^{(p,m)}(f,\psi) < +\infty$ and $0 < \nabla_{k,D}^{(q,m)}(g,\psi) < \triangle_{k,D}^{(q,m)}(g,\psi) < +\infty$ and $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$, where p, q, m are all positive integers then*

$$\frac{\nabla_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)} \leq \liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\nabla_{h,D}^{(p,m)}(f,\psi)}{\nabla_{k,D}^{(q,m)}(g,\psi)}$$
$$\leq \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\nabla_{k,D}^{(q,m)}(g,\psi)}.$$

Proof. From the definition of $\triangle_{k,D}^{(q,m)}(g,\psi)$ and $\nabla_{h,D}^{(p,m)}(f,\psi)$, we have for arbitrary $\epsilon > 0$ and for all sufficient large values of R,

$$\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R)) \geq (\nabla_{h,D}^{(p,m)}(f,\psi) - \epsilon)\{\log^{m-1} \psi(R)\}^{\rho_{h,D}^{(p,m)}(f,\psi)}, \tag{3.15}$$

and

$$\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R)) \leq (\triangle_{k,D}^{(q,m)}(g,\psi) + \epsilon)\{\log^{m-1} \psi(R)\}^{\rho_{k,D}^{(q,m)}(g,\psi)}. \tag{3.16}$$

Now using the condition $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$, combining (3.15) and (3.16), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{(\nabla_{h,D}^{(p,m)}(f,\psi) - \epsilon)}{(\triangle_{k,D}^{(q,m)}(g,\psi) + \epsilon)}.$$

As $\epsilon > 0$ is arbitrary,

$$\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{f,D}(R))} \geq \frac{\nabla_{h,D}^{(p,m)}(g,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)}. \tag{3.17}$$

Also, for all sufficient large values of R,

$$\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R)) \leq (\nabla_{h,D}^{(p,m)}(f,\psi) + \epsilon)\{\log^{m-1} \psi(R)\}^{\rho_{h,D}^{(p,m)}(f,\psi)}, \tag{3.18}$$

and for all sufficiently large values of R,

$$\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R)) \geq (\nabla_{k,D}^{(q,m)}(f,\psi) - \epsilon)\{\log^{m-1}\psi(R)\}^{\rho_{k,D}^{(q,m)}(g,\psi)}. \tag{3.19}$$

Using the condition $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$, combining (3.18) and (3.19), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{(\nabla_{h,D}^{(p,m)}(f,\psi) + \epsilon)}{(\nabla_{k,D}^{(q,m)}(g,\psi) - \epsilon)}.$$

As $\epsilon > 0$ is arbitrary,

$$\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\nabla_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)}. \tag{3.20}$$

Also for all sufficiently large values of R,

$$\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R)) \leq (\nabla_{k,D}^{(q,m)}(g,\psi) + \epsilon)\{\log^{m-1}\psi(R)\}^{\rho_{k,D}^{(q,m)}(g,\psi)}. \tag{3.21}$$

Using the condition $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$, combining (3.15) and (3.21), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{(\nabla_{h,D}^{(p,m)}(f,\psi) - \epsilon)}{(\nabla_{k,D}^{(q,m)}(g,\psi) + \epsilon)}.$$

As $\epsilon > 0$ is arbitrary,

$$\limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{f,D}(R))} \geq \frac{\nabla_{h,D}^{(p,m)}(f,\psi)}{\nabla_{k,D}^{(q,m)}(f,\psi)}. \tag{3.22}$$

Also for all sufficiently large values of R,

$$\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R)) \leq (\triangle_{h,D}^{(p,m)}(f,\psi) + \epsilon)\{\log^{m-1}\psi(R)\}^{\rho_{h,D}^{(p,m)}(f,\psi)}. \tag{3.23}$$

Using the condition $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$ and combining (3.19) and (3.23), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{f,D}(R))} \leq \frac{(\triangle_{h,D}^{(p,m)}(g,\psi) + \epsilon)}{(\nabla_{k,D}^{(q,m)}(g,\psi) - \epsilon)}.$$

As $\epsilon > 0$ is arbitrary,

$$\limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\nabla_{k,D}^{(q,m)}(g,\psi)}. \tag{3.24}$$

Thus the theorem follows from (3.17), (3.20), (3.22) and (3.24). □

Theorem 3.5. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \triangle_{h,D}^{(p,m)}(f,\psi) < +\infty$, and $0 < \triangle_{k,D}^{(q,m)}(g,\psi) < +\infty$ and $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$ where p, q, m are all positive integers.*
Then

$$\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)} \leq \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))}.$$

Proof. From the definition of $\triangle_{k,D}^{(q,m)}(g,\psi)$, we have for arbitrary ϵ and for all sufficient large values of R,

$$\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R)) \geq (\triangle_{k,D}^{(q,m)}(g,\psi) - \epsilon)\{\log^{m-1}\psi(R)\}^{\rho_{h,D}^{(q,m)}(g,\psi)}, \tag{3.25}$$

Using condition $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$ and combining (3.23) and (3.25), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\triangle_{h,D}^{(p,m)}(f,\psi) + \epsilon}{\triangle_{k,D}^{(q,m)}(g,\psi) - \epsilon}.$$

As $\epsilon > 0$ is arbitrary,

$$\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)}. \tag{3.26}$$

Also for all sufficiently large values of R,

$$\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R)) \geq (\triangle_{h,D}^{(p,m)}(f,\psi) - \epsilon)\{\log^{m-1} \psi(R)\}^{\rho_{h,D}^{(p,m)}(f,\psi)}. \tag{3.27}$$

Using condition $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$ and combining (3.16) and (3.27), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{\triangle_{h,D}^{(p,m)}(f,\psi) - \epsilon}{\triangle_{k,D}^{(q,m)}(g,\psi) + \epsilon}.$$

As $\epsilon > 0$ is arbitrary,

$$\limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)}. \tag{3.28}$$

Thus, the theorem follows from (3.26) and (3.28). □

Theorem 3.6. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \nabla_{h,D}^{(p,m)}(f,\psi) < \triangle_{h,D}^{(p,m)}(f,\psi) < +\infty$, and $0 < \nabla_{k,D}^{(q,m)}(g,\psi) < \triangle_{k,D}^{(q,m)}(g,\psi) < +\infty$ and $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$ where p, q, m are all positive integers.*
Then

$$\begin{aligned}\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} &\leq \min\left\{\frac{\nabla_{h,D}^{(p,m)}(f,\psi)}{\nabla_{k,D}^{(q,m)}(g,\psi)}, \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)}\right\}\\ &\leq \max\left\{\frac{\nabla_{h,D}^{(p,m)}(f,\psi)}{\nabla_{k,D}^{(q,m)}(g,\psi)}, \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)}\right\} \leq \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))}.\end{aligned}$$

Theorem 3.7. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \overline{\triangle}_{h,D}^{(p,m)}(f,\psi) < \overline{\nabla}_{h,D}^{(p,m)}(f,\psi) < +\infty$ and $0 < \overline{\triangle}_{k,D}^{(q,m)}(g,\psi) < \overline{\nabla}_{k,D}^{(q,m)}(g,\psi) < +\infty$ and $\lambda_{h,D}^{(p,m)}(f,\psi) = \lambda_{k,D}^{(q,m)}(g,\psi)$ where p, q, m are all positive integers.*
Then

$$\begin{aligned}\frac{\overline{\triangle}_{h,D}^{(p,m)}(f,\psi)}{\overline{\nabla}_{k,D}^{(q,m)}(g,\psi)} &\leq \liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\mathit{log}^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\overline{\triangle}_{h,D}^{(p,m)}(f,\psi)}{\overline{\triangle}_{k,D}^{(q,m)}(g,\psi)}\\ &\leq \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\overline{\nabla}_{h,D}^{(p,m)}(f,\psi)}{\overline{\triangle}_{k,D}^{(q,m)}(g,\psi)}.\end{aligned}$$

Similarly, in line with Theorem 3.8 and Theorem 3.9 and with help of Theorems 3.5 and 3.6, one may easily prove the following two theorems, and therefore their proofs are omitted.

Theorem 3.8. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \overline{\nabla}_{h,D}^{(p,m)}(f,\psi) < +\infty$, $0 < \overline{\nabla}_{k,D}^{(q,m)}(g,\psi) < +\infty$ and $\lambda_{h,D}^{(p,m)}(f,\psi) = \lambda_{k,D}^{(q,m)}(g,\psi)$ where p, q, m are all positive integers,*

$$\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\overline{\nabla}_{h,D}^{(p,m)}(f,\psi)}{\overline{\nabla}_{k,D}^{(q,m)}(g,\psi)} \leq \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))}.$$

Theorem 3.9. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \overline{\triangle}_{h,D}^{(p,m)}(f,\psi) < \overline{\nabla}_{h,D}^{(p,m)}(f,\psi) < +\infty$, $0 < \overline{\triangle}_{k,D}^{(q,m)}(g,\psi) < \overline{\nabla}_{k,D}^{(q,m)}(g,\psi) < +\infty$ and $\lambda_{h,D}^{(p,m)}(f,\psi) = \lambda_{k,D}^{(q,m)}(g,\psi)$ where p, q, m are all positive integers*

$$\begin{aligned}\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\mathit{log}^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} &\leq \min\left\{\frac{\overline{\triangle}_{h,D}^{(p,m)}(f,\psi)}{\overline{\triangle}_{k,D}^{(q,m)}(g,\psi)}, \frac{\overline{\nabla}_{h,D}^{(p,m)}(f,\psi)}{\overline{\nabla}_{k,D}^{(q,m)}(g,\psi)}\right\}\\ &\leq \max\left\{\frac{\overline{\triangle}_{h,D}^{(p,m)}(f,\psi)}{\overline{\triangle}_{k,D}^{(q,m)}(g,\psi)}, \frac{\overline{\nabla}_{h,D}^{(p,m)}(f,\psi)}{\overline{\nabla}_{k,D}^{(q,m)}(g,\psi)}\right\} \leq \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))}.\end{aligned}$$

4 Conclusion

In this paper, we want to establish some growth properties of entire function of n-complex variables on the basis of their of (p,q)-ψ relative Gol'dberg order, (p,q)-ψ relative Gol'dberg type, (p,q)-ψ relative Gol'dberg weak type and growth indicator where p, q are any positive integer.

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TAYLOR WAVELET APPROACH FOR THE SOLUTION OF THE FREDHOLM INTEGRO-DIFFERENTIAL EQUATION OF THE SECOND KIND

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Abstract

A Taylor wavelet technique is used to obtain the approximate solution of the Fredholm integro-differential equations *(IDEs)* of the second kind. Taylor wavelet method is based on an estimate of the unknown function involved in a given *IDEs* using the Taylor wavelet basis. The simplicity of the technique is a highly striking feature for the estimate of the unknown function. The applicability of the technique on various numerical problems shows the preciseness and usefulness of the technique. The suggested wavelets approach stands out for its simple operations, easy implementation, and accurate answers. A comparison is made with previous findings.
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Keywords and Phrases: Taylor wavelet, Operational integration matrix, Collocation points, Integro-differential equation, MATLAB.

1 Introduction

Integral equations play a crucial role in various branches of mathematics, engineering, and the sciences. These equations involve the unknown function as part of the integrand, and they arise in diverse fields due to their ability to model a wide range of phenomena. Integral equations can be solved using a variety of analytical and numerical methods. The choice of method often depends on the specific characteristics of the integral equation and the problem at hand. Problems involving heat transfer and fluid dynamics often lead to integro-differential equations. These equations can describe the distribution of temperature or fluid velocity in a given domain, accounting for both differential effects (diffusion or convection) and integral effects (boundary interactions). Here are some common techniques used to handle integral equations including the series solution method, Adomian decomposition method (ADM) and its modification, variational iteration method (VIM), Homotopy perturbation method, Nystrm method, and so on. On the other hand, recently Kumar, Chandel, and Srivastava [11], discussed a fractional non-linear biological model problem and its approximation. Further Kumar [12] discussed a class of two variable sequence of functions satisfying Able's integral equation and phase shifts. Take into account the Fredholm integro-differential equation *(IDEs)* of the following form:

$$\text{(1.1)}\qquad \begin{cases} \Lambda'(\rho) = h(\rho, \Lambda(\rho)) + \int_0^1 \kappa(\rho, \xi, \Lambda(\xi))d\xi, 0 \le \rho \le 1 \\ \Lambda(0) = \Lambda_0, \end{cases}$$

where $\Lambda(\rho)$ is the function(unknown) to be evaluated and $h(\rho, \Lambda(\rho)) \in L^2[0,1]$ and the kernel $\kappa(\rho, \xi) \in L^2(\mathbb{R})$ are known functions. In this paper, we present a new estimate solution for the Fredholm integro-differential equation *(IDEs)* (1.1). In this study, we investigate a fresh unified method for the solution of the integral equations, where the integral equations are expanded in terms of the Taylor wavelet with unique coefficients, leading to the reduction of the integral equations to algebraic equations.

A wavelet is a waveform that is localized in both time and frequency domains, and it is often used in signal processing and the analysis of time-varying signals. Wavelets have the property that they start from zero, oscillate, and then return to zero, and they come in different shapes and sizes. In the context of wavelets, the term "oscillation" refers to the repetitive pattern or behavior of the wavelet. The amplitude of a wavelet may indeed start at zero, rise or fall, and return to zero, and this property is often desirable in applications where a localized representation of a signal or function is needed. The ability of wavelets to capture both high and low-frequency components of a signal in a localized manner makes them well-suited for analyzing signals with complex and changing patterns. Multiresolution analysis (MRA), density, orthogonality, and compact support characteristics collectively make wavelets powerful tools in various fields, including signal processing, image compression, data analysis, and solving differential equations. The adaptability of wavelets to analyze signals at different scales, their ability to capture localized features, and their efficiency in representation and computation make them valuable tools in diverse scientific, engineering, and mathematical applications. Different types of wavelets are often chosen based on the specific requirements of the application at hand. Wavelet analysis is based on the concept of multiresolution analysis, meaning that it can represent a signal at different levels of detail or resolution. This is particularly useful for signals with non-stationary or time-varying characteristics.

Indeed, The use of orthogonal basis functions, especially in the form of orthogonal wavelet bases, has played a crucial role in the success and widespread adoption of wavelet analysis in numerical applications. The orthogonality property enhances the efficiency, stability, and accuracy of numerical procedures involving wavelet transformations. Wavelet basis functions lead to a sparse representation of signals. This sparsity often allows for a significant reduction in the number of coefficients needed to represent a signal accurately. As a result, the underlying problems, whether they are differential equations or integral equations, can be translated into a system of algebraic equations. This reduction simplifies the computational burden and facilitates efficient numerical solutions. The sparsity of wavelet representations makes computations more efficient. Since only a small subset of coefficients is required to represent a signal, algorithms based on wavelets can be computationally faster than methods that use non-sparse representations. This is particularly advantageous in applications such as signal processing and image compression. The reduction to algebraic equations, computational efficiency, ease of implementation, and the rapid combination of algorithms make wavelet basis functions a powerful tool in various applications, contributing to their widespread use in scientific, engineering, and computational domains. One of the important aspects of wavelet analysisis the ability to explicitly illustrate and represent other operators and functions through wavelet basis functions. When wavelets are used as basis functions, they provide a way to analyze and represent functions in a multi-resolution fashion, allowing for efficient and flexible representation of signals and operators. Other operators and functions can be explicitly illustrated using wavelets, after which they become apparent in continuous time as $\nu_{ij}(\rho)$ basis functions. A function's set contains a linearly independent set(basis) that spans the entire space. The basis of the set of functions generates all permissible functions, say $h(\rho)$ downsides

$$h(\rho) = \sum_{i,j} \beta_{ij}\nu_{ij}(\rho), \text{where } \ \beta_{i,j} = \langle h, \nu_{ij}\rangle.$$

The functions $\nu_{ij}(\rho)$ are built from a single mother wavelet $\nu(\rho) \in L^2[0,1]$, which is a small pulse, which is a unique property of the wavelet basis. The development of the integration's operational matrix is a critical job in Fredholm *(IDEs)* numerical findings. Many matrix approaches are mentioned in the literature, as mentioned here: *CAS* wavelet operational matrix [4], integral and integro-differential equations [18] and Abel's integral equations [16], Bernoulli wavelet [17], Legendre wavelets operational matrix [19], Haar wavelets operational matrix [7], Laguerre wavelets [10] and Hermite wavelet method for solving integro-differential equations [13]. This work investigates a unique operational integration matrix for the Taylor wavelet-assisted numerical solution of *IDEs*. Additionally, some wavelet-based numerical methods can be found in [22, 21].

2 Definitions and Preliminaries

2.1 Wavelets

A family of functions known as wavelets is generated by dilatating and translating a single function known as the mother wavelet. We have the following family of continuous wavelets when the dilation variable x and the translation variable y vary continuously.

$$\nu_{x,y}(\rho) = |x|^{-\frac{1}{2}}\nu\left(\frac{\rho - y}{x}\right), \quad x, y \in \mathbb{R}, \quad x \neq 0.$$

Discrete wavelets family is defined by restricting the variables x and y to discrete quantities like $x = x_0^{-\Theta}, y = \omega y_0 x_0^{-\Theta}, x_0 > 1$, and $y_0 > 1$, where ω and Θ are natural numbers.

$$\nu_{\Theta,\omega}(\rho) = \left|x_0^{-\Theta}\right|^{\frac{-1}{2}} \nu\left(\frac{\rho - \omega y_0 x_0^{-\Theta}}{x_0^{-\Theta}}\right) = |x_0|^{\frac{\Theta}{2}} \nu\left(x_0^{\Theta}\rho - \omega y_0\right),$$

where $\nu_{\Theta,\omega}(\rho)$ is a basis(wavelet basis) of $L^2(\mathbb{R})$.

2.2 Taylor Wavelets

Four arguments exist for Taylor wavelets $\nu_{\omega,r}(\rho) = \nu(\Theta, \hat{\omega}, r, \rho)$: $\hat{\omega} = \omega - 1, \omega = 1, 2, \ldots, 2^{\Theta-1}$. When Taylor polynomials have order r, we define them as follows on the range $[0, 1]$.

$$\nu_{\omega,r}(\rho) = \begin{cases} 2^{\frac{\Theta-1}{2}} \tilde{L}_r\left(2^{\Theta-1}\rho - \hat{\omega}\right), & \frac{\hat{\omega}}{2^{\Theta-1}} \leq \rho < \frac{\hat{\omega}+1}{2^{\Theta-1}} \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

with

$$\tilde{L}_r(\rho) = \sqrt{2r+1} L_r(\rho),$$

where $r = 0, 1, 2, \ldots, \mu - 1$ and $\omega = 1, 2, \ldots, 2^{\Theta-1}$. The coefficient $\sqrt{2r+1}$ is for normality, the dilation parameter is $a = 2^{-(\Theta-1)}$ and the translation parameter is $b = \hat{\omega} 2^{-(\Theta-1)}$. Here, $L_r(\rho)$ are the well-known Taylor polynomials of order r which form a complete basis over the interval $[0, 1]$, which are defined by $L_r(\rho) = \rho^r$[20].

2.3 Approximation Technique

Any function $\Lambda(\rho) \in L^2[0, 1]$ can be expressed in terms of the Taylor wavelet basis in the following way:

$$\Lambda(\rho) = \sum_{\omega=1}^{\infty} \sum_{r=0}^{\infty} g_{\omega,r} \nu_{\omega,r}(\rho)$$

and consider the truncated series of approximation for $\Lambda(\rho)$,

$$\Lambda(\rho) \simeq \sum_{\omega=1}^{2^{\Theta-1}} \sum_{r=0}^{\mu-1} g_{\omega,r} \nu_{\omega,r}(\rho) = E^{\intercal} \nu(\rho) = \Lambda_n(\rho) \tag{2.2}$$

where $\intercal$ indicates transposition, and $E, \nu(\rho)$ are $n \times 1$ $\left(n = 2^{\Theta-1}\mu\right)$ matrices given as

$$E = \left[g_{1,0}, g_{1,1}, \ldots, g_{1,\mu-1}, g_{2,0}, \ldots, g_{2,\mu-1}, \ldots, g_{2^{\Theta-1},\mu-1}, \ldots, g_{2^{\Theta-1},\mu-1}\right]^{\intercal}, \tag{2.3}$$

$$\nu(\rho) = \left[\nu_{1,0}(\rho), \nu_{1,1}(\rho), \ldots, \nu_{1,\mu-1}(\rho), \nu_{2,0}(\rho), \ldots, \nu_{2,\mu-1}(\rho), \ldots, \nu_{2^{\Theta-1},0}(\rho), \ldots, \nu_{2^{\Theta-1},\mu-1}(\rho)\right]^{\intercal}. \tag{2.4}$$

3 Convergence analysis

In this section, two new theorems have been established for the proposed method's convergence analysis as well as error estimation in the following form:

Theorem 3.1 *Let $\Lambda(\rho) \in L^2(\mathbb{R})$ be a continuous function on the interval $[0, 1)$ such that it is bounded by m i.e. $|\Lambda(\rho)| < m$, for every $\rho \in [0, 1)$. Then, the Taylor wavelet coefficients of $\Lambda(\rho)$ in Eq. (2.2) are bounded as:*

$$|g_{\omega,r}| < \frac{\lambda}{2^{\frac{\Theta-1}{2}}} m \frac{2}{2r+1},$$

where m is a constant and λ is given by

$$\lambda = \sqrt{2r+1}.$$

Proof. Using Taylor wavelets, any arbitrary function $\Lambda(\rho)$ can be approximated as:

$$\Lambda(\rho) \simeq \sum_{\omega=1}^{2^{\Theta-1}} \sum_{r=0}^{\mu-1} g_{\omega,r} \nu_{\omega,r}(\rho) = E^T \nu(\rho) = \Lambda_n(\rho), \tag{3.1}$$

where E and $\nu(\rho)$ are given in Eq. (2.3), (2.4) and the coefficients $g_{\omega,r}$ are determined as:

$$g_{\omega,r} = \langle \Lambda, \nu_{\omega,r} \rangle = \int_0^1 \Lambda(\rho) \nu_{\omega,r}(\rho) d\rho = 2^{\frac{\Theta-1}{2}} \sqrt{2r+1} \int_{\frac{\omega-1}{2^{\Theta-1}}}^{\frac{\omega}{2^{\Theta-1}}} \Lambda(\rho) L_r\left(2^{\Theta-1}\rho - \omega + 1\right) d\rho. \tag{3.2}$$

Using the definition of Taylor wavelets $\nu_{\omega,r}(\rho)$, we have

$$\nu_{\omega,r}(\rho) = 2^{\frac{\Theta-1}{2}} \sqrt{2r+1} L_r\left(2^{\Theta-1}\rho - \omega + 1\right), \quad \frac{\omega - 1}{2^{\Theta-1}} \leq \rho < \frac{\omega}{2^{\Theta-1}}. \tag{3.3}$$

Let $\lambda = \sqrt{2r+1}$. Let $2^{\Theta-1}\rho - \omega + 1 = x$, then Eq. (3.2) becomes

$$g_{\omega,r} = \frac{\lambda}{2^{\frac{\Theta-1}{2}}} \int_0^1 \Lambda\left(\frac{x + \omega - 1}{2^{\Theta-1}}\right) L_r(x) dx.$$

Therefore,

$$|g_{\omega,r}| \leq \frac{\lambda}{2^{\frac{\Theta-1}{2}}} \int_0^1 \left|\Lambda\left(\frac{x+\omega-1}{2^{\Theta-1}}\right)\right| |L_r(x)|\, dx. \tag{3.4}$$

Seeing the properties of Taylor polynomials, we can say that

$$\int_0^1 |L_r(v)|\, dv < \frac{2}{2r+1}, \quad r > 0. \tag{3.5}$$

Using the assumption $|\Lambda(\rho)| < m$ in Eqs. (3.5) and (3.4), we have

$$|g_{\omega,r}| < \frac{1}{2^{\frac{\Theta-1}{2}}} m \frac{2}{\sqrt{2r+1}}. \tag{3.6}$$

Thus the proof of Theorem 3.1 has been completed. Also, boundedness of the function implies absolutely convergent of the series $\Lambda(\rho) = \sum_{\omega=1}^{\infty}\sum_{r=0}^{\infty} g_{\omega,r}$. Hence the Taylor wavelet approximation of the function $\Lambda(\rho)$ is absolutely convergent.

Theorem 3.2 *Let $\Lambda(\rho) \in L^2(\mathbb{R})$ be a continuous function on the interval $[0,1)$ and $|\Lambda(\rho)| < m$ for every $\rho \in [0,1)$. Let $\Lambda^*(\rho) = \sum_{\omega=1}^{2^{\Theta-1}}\sum_{r=0}^{\mu-1} g_{\omega,r}\nu_{\omega,r}(\rho)$ be the Taylor wavelet series expansions where $g_{\omega,r}$, $\nu_{\omega,r}(\rho)$ be the Taylor wavelet coefficients and Taylor wavelet basis respectively. Then, the bound of the truncated error $e(\rho)$ is given as:*

$$\|e(\rho)\|_2 = \|\Lambda(\rho) - \Lambda^*(\rho)\| < \left(\sum_{\omega=2^{\Theta-1}+1}^{\infty}\sum_{r=0}^{\mu-1} g_{\omega,r}^2\right)^{\frac{1}{2}} + \left(\sum_{\omega=1}^{\infty}\sum_{r=\mu}^{\infty} g_{\omega,r}^2\right)^{\frac{1}{2}},$$

where,

$$g_{\omega,r} = \frac{\lambda}{2^{\frac{\Theta-1}{2}}} m \frac{2}{2r+1},\ \lambda = \sqrt{2r+1}.$$

Proof. Any function $\Lambda \in L^2[0,1)$ can be expanded in terms of Taylor wavelets as:

$$\Lambda(\rho) = \sum_{\omega=1}^{\infty}\sum_{r=0}^{\infty} g_{\omega,r}\nu_{\omega,r}(\rho).$$

If $\Lambda^*(\rho)$ is the expansion truncated by using Taylor wavelets, then the error obtained by truncating the above function can be computed as:

$$e(\rho) = \Lambda(\rho) - \Lambda^*(\rho) = \sum_{\omega=2^{\Theta-1}+1}^{\infty}\sum_{r=0}^{\mu-1} g_{\omega,r}\nu_{\omega,r}(\rho) + \sum_{\omega=1}^{\infty}\sum_{r=\mu}^{\infty} g_{\omega,r}\nu_{\omega,r}(\rho). \tag{3.7}$$

From Eq. (3.7), we can write

$$\begin{aligned}
\|e(\rho)\| &\leq \left\|\sum_{\omega=2^{\Theta-1}+1}^{\infty}\sum_{r=0}^{\mu-1} g_{\omega,r}\nu_{\omega,r}(\rho)\right\| + \left\|\sum_{\omega=1}^{\infty}\sum_{r=\mu}^{\infty} g_{\omega,r}\nu_{\omega,r}(\rho)\right\| \\
&= \left(\int_0^1 \left|\sum_{\omega=2^{\Theta-1}+1}^{\infty}\sum_{r=0}^{\mu-1} g_{\omega,r}\nu_{\omega,r}(\rho)\right|^2 d\rho\right)^{\frac{1}{2}} + \left(\int_0^1 \left|\sum_{\omega=1}^{\infty}\sum_{r=\mu}^{\infty} g_{\omega,r}\nu_{\omega,r}(\rho)\right|^2 d\rho\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{\omega=2^{\Theta-1}+1}^{\infty}\sum_{r=0}^{\mu-1} |g_{\omega,r}|^2 \int_0^1 |\nu_{\omega,r}(\rho)|^2\, d\rho\right)^{\frac{1}{2}} + \left(\sum_{\omega=1}^{\infty}\sum_{r=\mu}^{\infty} |g_{\omega,r}|^2 \int_0^1 |\nu_{\omega,r}(\rho)|^2\, d\rho\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.8}$$

From Theorem 3.1, using the property

$$|g_{\omega,r}| < \frac{\lambda}{2^{\frac{\Theta-1}{2}}} m \frac{2}{2r+1},$$

Eq. (3.8) reduces to

$$\|e(\rho)\|_2 < \left(\sum_{\omega=2^{\Theta-1}+1}^{\infty}\sum_{r=0}^{\mu-1}|g_{\omega,r}|^2\int_0^1|\nu_{\omega,r}(\rho)|^2\,d\rho\right)^{\frac{1}{2}} + \left(\sum_{\omega=1}^{\infty}\sum_{r=\mu}^{\infty}|g_{\omega,r}|^2\int_0^1|\nu_{\omega,r}(\rho)|^2\,d\rho\right)^{\frac{1}{2}}. \tag{3.9}$$

Let us define

$$d_{\omega,r} = \frac{\lambda}{2^{\frac{\Theta-1}{2}}}m\frac{2}{2r+1}. \tag{3.10}$$

Then from Eqs. (3.9) and (3.10), we get

$$\|e(\rho)\|_2 < \left(\sum_{\omega=2^{\Theta-1}+1}^{\infty}\sum_{r=0}^{\mu-1}|d_{\omega,r}|^2\int_0^1|\nu_{\omega,r}(\rho)|^2\,d\rho\right)^{\frac{1}{2}} + \left(\sum_{\omega=1}^{\infty}\sum_{r=\mu}^{\infty}|d_{\omega,r}|^2\int_0^1|\nu_{\omega,r}(\rho)|^2\,d\rho\right)^{\frac{1}{2}}. \tag{3.11}$$

Therefore,

$$\|e(\rho)\|_2 < \left(\sum_{\omega=2^{\Theta-1}+1}^{\infty}\sum_{r=0}^{\mu-1}d_{\omega,r}^2\int_0^1|\nu_{\omega,r}(\rho)|^2\,d\rho\right)^{\frac{1}{2}} + \left(\sum_{\omega=1}^{\infty}\sum_{r=\mu}^{\infty}d_{\omega,r}^2\int_0^1|\nu_{\omega,r}(\rho)|^2\,d\rho\right)^{\frac{1}{2}}. \tag{3.12}$$

By the definition of Taylor wavelets, we have

$$\nu_{\omega,r}^2(\rho) = 2^{\Theta-1}(2r+1)L_r^2\left(2^{\Theta-1}\rho-\omega+1\right), \quad \frac{\omega-1}{2^{\Theta-1}} \le \rho < \frac{\omega}{2^{\Theta-1}}. \tag{3.13}$$

Integrating Eq. (3.13) with respect to ρ, we get

$$\int_0^1 \nu_{\omega,r}^2(\rho) = 2^{\Theta-1}(2r+1)\int_{\frac{\omega-1}{2^{\Theta-1}}}^{\frac{\omega}{2^{\Theta-1}}} L_r^2\left(2^{\Theta-1}\rho-\omega+1\right)d\rho. \tag{3.14}$$

Let $2^{\Theta-1}\rho-\omega+1=u$, Eq. (3.14) becomes

$$\int_0^1 \nu_{\omega,r}^2(\rho)d\rho = (2r+1)\int_0^1 L_r^2(u)du. \tag{3.15}$$

But the standard definition of Taylor polynomial implies that,

$$\int_0^1 L_r^2(u)du = \int_0^1 u^{2r}du = \frac{1}{2r+1}. \tag{3.16}$$

Substituting Eq. (3.16) in Eq. (3.15), we get

$$\int_0^1 \nu_{\omega,r}^2(\rho)d\rho = 1.$$

Thus Theorem 3.2 has been proved. This theorem also implies consistency and stability of the approximation.

3.1 Integration Matrix of Taylor Wavelets

Let $\nu(\rho)$ be the vector consisting of Taylor wavelets defined in the previous section, then

$$I\nu(\rho) = S\nu(\rho),$$

where I and S are the integral operator and the $n\times n$ operational integration matrix , respectively for $n=2^{\Theta-1}\mu$. Using the Eq. (2.1) , formulation of the Taylor basis, for $\omega=1,\cdots,2^{\Theta-1}$ and $r=0,1,\cdots,\mu-1$, we have

$$\begin{aligned} I\left(\nu_{\omega,r}(\rho)\right) &= I\left(2^{\frac{\Theta-1}{2}}\tilde{L}_r\left(2^{\Theta-1}\rho-\hat{\omega}\right)\chi_{\left[\frac{\hat{\omega}}{2^{\Theta-1}},\frac{\hat{\omega}+1}{2^{\Theta-1}}\right]}(\rho)\right) \\ &= 2^{\frac{\Theta-1}{2}} I\left(\sqrt{2r+1}\left(2^{\Theta-1}\rho-\hat{\omega}\right)^r\chi_{\left[\frac{\hat{\omega}}{2^{\Theta-1}},\frac{\hat{\omega}+1}{2^{\Theta-1}}\right]}(\rho)\right), \end{aligned} \tag{3.17}$$

where $\chi_{\left[\frac{\hat{\omega}}{2^{\Theta-1}},\frac{\hat{\omega}+1}{2^{\Theta-1}}\right]}(\rho)$ is the characteristic function defined as

$$\chi_{\left[\frac{\hat{\omega}}{2^{\Theta-1}},\frac{\hat{\omega}+1}{2^{\Theta-1}}\right]}(\rho) = \begin{cases} 1, & \frac{\hat{\omega}}{2^{\Theta-1}} \le \rho \le \frac{\hat{\omega}+1}{2^{\Theta-1}} \\ 0, & \text{otherwise} \end{cases}$$

and $\hat{\omega} = \omega - 1$. For $\omega = 1$, Eq. (??) gives

$$I(\nu_{1,r}(\rho)) = 2^{\frac{\Theta-1}{2}}\sqrt{2r+1}2^{(\Theta-1)r} I\left(\rho^r \chi_{[\frac{\hat{\omega}}{2^{\Theta-1}},\frac{\hat{\omega}+1}{2^{\Theta-1}}]}(\rho)\right).$$

Therefore, we can obtain seven Taylor Wavelet basis as follows:

$$\begin{aligned}
\nu_{1,0}(\rho) &= 1,\\
\nu_{1,1}(\rho) &= \sqrt{3}\rho,\\
\nu_{1,2}(\rho) &= \sqrt{5}\rho^2,\\
\nu_{1,3}(\rho) &= \sqrt{7}\rho^3,\\
\nu_{1,4}(\rho) &= \sqrt{9}\rho^4,\\
\nu_{1,5}(\rho) &= \sqrt{11}\rho^5,\\
\nu_{1,6}(\rho) &= \sqrt{13}\rho^6,
\end{aligned}$$

for $\Theta = 1$ and $\mu = n = 7$.
Let $\nu(\rho) = [\nu_{1,0}(\rho), \nu_{1,1}(\rho), \nu_{1,2}(v), \nu_{1,3}(\rho), \nu_{1,4}(\rho), \nu_{1,5}(\rho), \nu_{1,6}(\rho)]$.

Now integrate each element of the above vector with respect to the variable ρ limit taking from 0 to ρ, then represent them as a linear combination of Taylor wavelet basis,

$$\begin{aligned}
\int_0^\rho \nu_{1,0}(\rho)d\rho &= \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\nu_7(\rho),\\
\int_0^\rho \nu_{1,1}(\rho)d\rho &= \begin{bmatrix} 0 & 0 & \frac{\sqrt{3}}{2\sqrt{5}} & 0 & 0 & 0 & 0 \end{bmatrix}\nu_7(\rho),\\
\int_0^\rho \nu_{1,2}(\rho)d\rho &= \begin{bmatrix} 0 & 0 & 0 & \frac{\sqrt{5}}{3\sqrt{7}} & 0 & 0 & 0 \end{bmatrix}\nu_7(\rho),\\
\int_0^\rho \nu_{1,3}(\rho)d\rho &= \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{4\sqrt{9}} & 0 & 0 \end{bmatrix}\nu_7(\rho),\\
\int_0^\rho \nu_{1,4}(\rho)d\rho &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{9}}{5\sqrt{11}} & 0 \end{bmatrix}\nu_7(\rho),\\
\int_0^\rho \nu_{1,5}(\rho)d\rho &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{11}}{6\sqrt{13}} \end{bmatrix}\nu_7(\rho),\\
\int_0^\rho \nu_{1,6}(\rho)d\rho &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\nu_7(\rho) + \frac{\sqrt{13}}{7\sqrt{15}}\overline{\nu_{1,7}}(\rho).
\end{aligned}.$$

Thus, we have $\int_0^\rho \nu(\rho)d\rho = S_{7\times7}\nu_7(\rho) + \overline{\nu_7}(\rho)$. where,

$$S_{7\times7} = \begin{bmatrix}
0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{3}}{2\sqrt{5}} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{\sqrt{5}}{3\sqrt{7}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{\sqrt{7}}{4\sqrt{9}} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{9}}{5\sqrt{11}} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{11}}{6\sqrt{13}}\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Here, S is defined to be the operational integration matrix of order 7×7, and

$$\overline{\nu_7}(\rho) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{\sqrt{13}}{7\sqrt{15}}\overline{\nu_{1,7}}(\rho) \end{bmatrix}.$$

This Operational Integration Matrix of Taylor Wavelet basis *(OIMTW)* can also be produced using the same technique for all other values of n.

4 Proposed method for solving the Integro-Diferential Equations *(IDEs)*

Let,

$$\Lambda'(\rho) = h(\rho, \Lambda) + \int_0^1 \kappa(\rho, \xi, \Lambda(\xi))d\xi \tag{4.1}$$

be the integro-differential equation having the initial condition $\Lambda(0) = \delta$. Here, $h(\rho, \Lambda)$ is a continuous function and δ is a constant.

Now we use the Taylor wavelet bases to approximate the highest derivative appearing in the given IDEs as follows,

$$\Lambda'(\rho) = E^{\intercal}\nu(\rho), \tag{4.2}$$

here $\Theta = 1$ and $r = 0, 1, \ldots \mu - 1$, where,

$$E^{\intercal} = [g_{1,0}, g_{1,1}, \ldots, g_{1,\mu-1}],$$
$$\nu(\rho) = [\nu_{1,0}, \nu_{1,1}, \ldots, \nu_{1,\mu-1}]^{\intercal}.$$

Apply integration on both sides of Eq. (4.2) concerning ρ limit from 0 to ρ.

$$\begin{aligned} \Lambda(\rho) &= \Lambda(0) + E^{\intercal}[S\nu(\rho) + \overline{\nu(\rho)}] \\ &= \delta + E^{\intercal}[S\nu(\rho) + \overline{\nu(\rho)}]. \end{aligned} \tag{4.3}$$

Substituting Eq. (4.2) and (4.3) in (4.1), we have

$$E^{\intercal}\nu(\rho) = h\left(\rho, \left(\delta + E^{\intercal}[S\nu(\rho) + \overline{\nu(\rho)}]\right)\right) + \int_0^1 \kappa\left(\rho, \xi, \left(\delta + E^{\intercal}[S\nu(\xi) + \overline{\nu(\xi)}]\right)\right) d\xi. \tag{4.4}$$

Collocate Eq. (4.4), n number of algebraic equations followed by using grid points $\rho_i = \frac{2i-1}{2n}$, where $i = 0, 1, 2, \ldots, n$. Solving these n equations with a suitable method provides Taylor wavelet coefficients. The required numerical solution will next be presented by substituting these coefficients in Eq.(4.3)

5 Quantitive Testings

We present several problems from the field and confirm the productivity as well as the accuracy of the findings to assess the method's efficacy:

$$\Upsilon = \|\Lambda_e(\rho_i) - \Lambda_a(\rho_i)\|_2 = \sqrt{\sum_{i=1}^{n} [\Lambda_e(\rho_i) - \Lambda_a(\rho_i)]^2},$$

where Λ_e is the approximate solution, Λ_a is the accurate solution and Υ is denoting the absolute error. The following illustrative problems show the presented method's efficiency, accuracy, and validity with respect to other existing methods.

Problem 5.1 Let

$$\begin{cases} \Lambda'(\rho) = \rho e^{\rho} + e^{\rho} - \rho + \int_0^1 \rho\Lambda(\xi)d\xi \\ \Lambda(0) = 0, \end{cases} \tag{5.1}$$

be the *IDEs* [2, 4, 5, 6, 8, 15, 20]. $\Lambda(\rho) = \rho e^{\rho}$ is the precise solution *(PS)* of this problem. When utilizing the present approach to solve Eq. (5.1), the accuracy of the current approach depends on the size n of the matrix of integration. Comparison between the solution of Taylor's method (current method) or *TS* and the precise solution is shown in Table 5.1, whereas absolute error *(AE)* is shown in Table 5.2. Finally, Table 5.3 shows the numerical findings obtained by the current approach and compared to one of the existing approaches

[4]. Table 5.4 compares the absolute inaccuracy of the current procedure to that of other methods. Fig. 5.1 describes a geometrical interpretation of numerical values of the current method with precise solutions. In Fig. 5.2, error variation for different values of n for the current approach can be observed.

***Table* 5.1:** Value comparison between the current approach *(TS)* and the accurate solution for the Problem 5.1

ρ	*TS at n=6*	*TS at n=7*	*TS at n=8*	*TS at n=9*	*Precise solution*
0	0	0	0	0	0
0.1	0.110517248729256	0.110517084061229	0.110517092019904	0.110517091798469	0.110517091807565
0.2	0.244280532997289	0.244280546432053	0.244280551607897	0.244280551621848	0.244280551632034
0.3	0.404957540507024	0.404957630933154	0.404957642172328	0.404957642254503	0.404957642272801
0.4	0.596729754231302	0.596729861627843	0.596729878676599	0.596729879030421	0.596729879056508
0.5	0.824360295875318	0.824360613332353	0.824360634706961	0.824360635310779	0.824360635350064
0.6	1.093270675744187	1.093271244789225	1.093271279256232	1.093271280183089	1.093271280234305
0.7	1.409626449015646	1.409626852859123	1.409626893661343	1.409626895155630	1.409626895229334
0.8	1.780431778417886	1.780432706086411	1.780432741677801	1.780432742722952	1.780432742793974
0.9	2.213632903312502	2.213642415993803	2.213642788219453	2.213642799701517	2.213642800041255
1	2.718230515182594	2.718278904461437	2.718281684837148	2.71828182217887	2.718281828459046

***Table* 5.2:** Error variation in the current approach for the Problem 5.2

ρ	*n=6*	*n=7*	*n=8*	*n=9*
0.0	0	0	0	0
0.1	1.56 e-07	0.77 e-8	2.12 e-10	0.90 e-11
0.2	0.18 e-07	0.51 e-8	0.24 e-10	1.01 e-11
0.3	1.01 e-07	1.13 e-8	1.00 e-10	1.82 e-11
0.4	1.24 e-07	1.74 e-8	3.79 e-10	2.60 e-11
0.5	3.39 e-07	2.20 e-8	6.43 e-10	3.92 e-11
0.6	6.04 e-07	3.54 e-8	9.78 e-10	5.12 e-11
0.7	4.46 e-07	4.23 e-8	1.56 e-9	7.37 e-11
0.8	9.64 e-07	3.67 e-8	1.11 e-9	7.10 e-11
0.9	9.89 e-06	3.84 e-7	1.18 e-8	3.39 e-10
1	5.13 e-05	2.92 e-6	1.43 e-7	6.28 e-09

***Table* 5.3:** The current method is compared to an existing method for the Problem 5.1.

ρ	*Precise solution*	*CAS in [4]*	*TS at* n=9	*CAS's AE [4]*	*AE of current approach*
0.1	0.110517091807565	0.00134917637	0.110517091798469	1.09 e-01	0.90 e-11
0.2	0.244280551632034	0.00115960044	0.244280551621848	2.43 e-01	1.01 e-11
0.3	0.404957642272801	0.00567152531	0.404957642254503	3.99 e-01	1.82 e-11
0.4	0.596729879056508	0.05931056450	0.596729879030421	5.37 e-01	2.60 e-11
0.5	0.824360635350064	0.01323307510	0.824360635310779	8.11 e-01	3.92 e-11
0.6	1.093271280234300	0.04392877200	1.093271280183089	1.05 e+00	5.12 e-11
0.7	1.409626895229330	0.01412016240	1.409626895155630	1.40 e+00	7.37 e-11
0.8	1.780432742793970	0.01345141170	1.780432742722952	1.77 e+00	7.10 e-11
0.9	2.213642800041250	0.01320452090	2.213642799701517	2.20 e+00	3.39 e-10

***Table* 5.4:** Comparison between the AE of the current approach with some other methods for the Problem 5.1.

ρ	*CAS's [4]*	*DPE's [5]*	*IHPM's [20]*	*SBM's [2]*	*HWM's [6]*	*HEK's [8]*	AE of TS
0.1	1.34 e-03	1.00 e-02	0.23 e-05	1.01 e-07	1.85 e-06	3.69 e-03	0.90 e-11
0.2	1.15 e-03	0.23 e-05	1.01 e-07	1.30 e-06	1.30 e-06	1.45 e-02	1.01 e-11
0.3	5.67 e-03	2.78 e-02	0.92 e-05	4.82 e-07	1.40 e-06	3.20 e-02	1.82 e-11
0.4	5.93 e-02	5.08 e-02	0.20 e-04	1.01 e-07	2.15 e-06	5.56 e-02	2.60 e-11
0.5	1.32 e-02	7.55 e-02	0.37 e-04	1.61 e-06	5.03 e-07	8.47 e-02	3.92 e-11
0.6	4.39 e-02	9.71 e-02	0.57 e-04	2.30 e-06	2.55 e-06	1.18 e-01	5.12 e-11
0.7	1.41 e-02	1.09 e-01	0.83 e-04	3.09 e-06	2.20 e-06	1.55 e-01	7.37 e-11
0.8	1.34 e-02	1.04 e-01	0.11 e-03	3.97 e-06	2.50 e-06	1.95 e-01	7.10 e-11
0.9	1.32 e-02	6.94 e-02	0.14 e-03	4.99 e-06	3.46 e-06	2.35 e-01	3.39 e-10

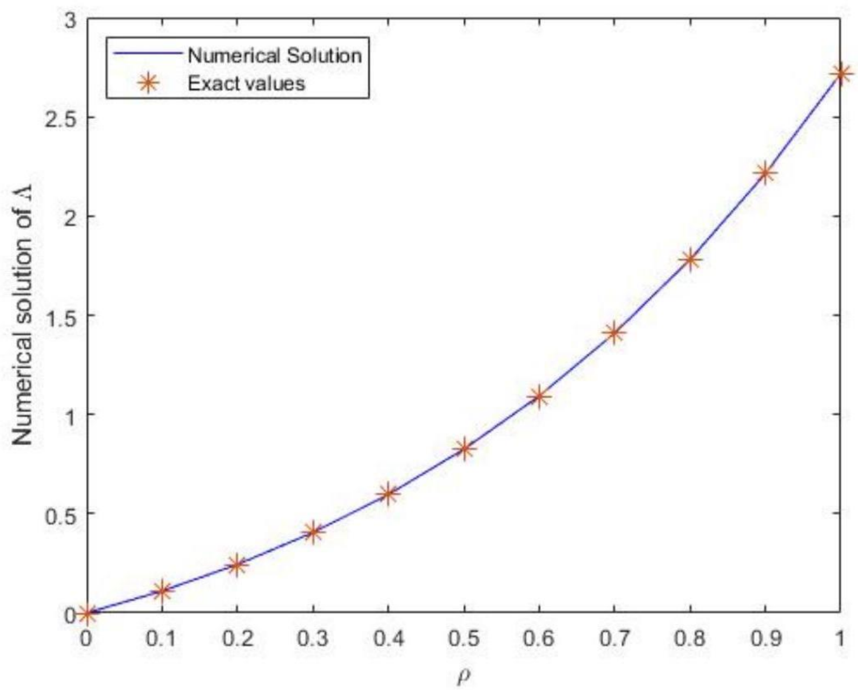

***Figure* 5.1:** Graph representation of the Approximate solution *(TS)* for n=6 and the Precise solution *(PS)* for the Problem 5.1.

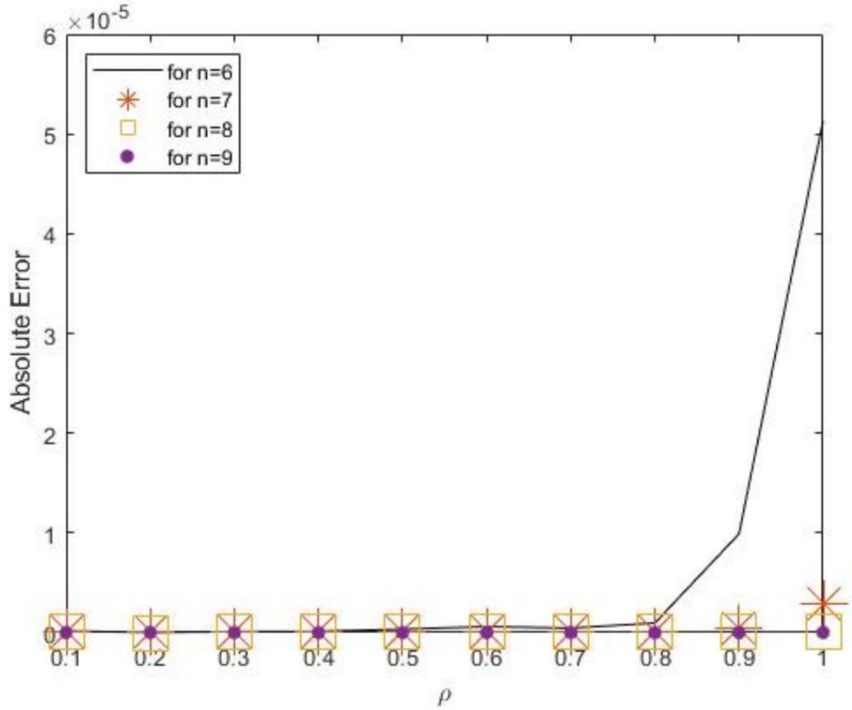

***Figure* 5.2:** Graphical comparison of AE of current method at $n = 6, 7, 8$ and 9 for the Problem 5.1.

Problem 5.2 Let

$$\begin{cases} \Lambda'(\rho) = 1 - \frac{1}{3}\rho + \int_0^1 \rho\xi\Lambda(\xi)d\xi \\ \Lambda(0) = 0. \end{cases} \tag{5.2}$$

be the *IDEs* [2, 4, 5, 6, 8, 15, 20]. $\Lambda(\rho) = \rho$ is the accurate solution of the problem 5.2. Applying the current method at n=10 for solving this problem, we have Ten Taylor wavelet coefficients as follows:

$$\begin{bmatrix} g_{1,0} = 1 \\ g_{1,1} = 0 \\ g_{1,2} = 0 \\ g_{1,3} = 0 \\ g_{1,4} = 0 \\ g_{1,5} = 0 \\ g_{1,6} = 0 \\ g_{1,7} = 0 \\ g_{1,8} = 0 \\ g_{1,9} = 0 \end{bmatrix}$$

The Taylor wavelet solution, which is identical to the analytic solution, is obtained by replacing these coefficients in Eq. (4.3). This problem demonstrates the method's efficiency, applicability, and validity. Numerical values of the solution of problem 5.2 as shown in Table 5.5 obtained in different existing methods and the current method *(TS)* along with the absolute error.

***Table* 5.5:** Numerical comparison of current solution with certain other existing methods for the Problem 5.2.

ρ	*PS*	*CAS [4]*	*TS*	*CAS's AE [4]*	*CAS in [4]*	*DPEM [5]*	*SAM [2]*	*HWM [6]*	*HEK [8]*	*AE of TS*
0.1	0.1	0.00021794238	0.1	9.98 e-02	2.17 e-04	1.60 e-03	3.79 e-06	1.60 e-06	1.50 e-03	0
0.2	0.2	0.00063854821	0.2	1.99 e-01	6.38 e-04	6.09 e-03	1.51 e-05	2.36 e-06	5.36 e-03	0
0.3	0.3	0.00077137049	0.3	2.99 e-01	7.91 e-04	1.32 e-02	3.41 e-05	2.26 e-06	1.06 e-02	0
0.4	0.4	0.02155860050	0.4	3.78 e-01	2.15 e-02	2.29 e-02	6.06 e-05	1.31 e-06	1.65 e-02	0
0.5	0.5	0.00499358429	0.5	4.95 e-01	4.99 e-03	3.51 e-02	9.47 e-05	4.85 e-07	2.21 e-02	0
0.6	0.6	0.02217288150	0.6	5.78 e-01	2.21 e-02	6.69 e-02	1.36 e-05	9.28 e-07	2.67 e-02	0
0.7	0.7	0.00010564545	0.7	7.00 e-01	1.05 e-04	7.12 e-02	1.85 e-04	1.48 e-06	2.95 e-02	0
0.8	0.8	0.00143233681	0.8	7.99 e-01	1.43 e-03	8.63 e-02	2.42 e-04	1.19 e-06	2.98 e-02	0
0.9	0.9	0.02077474610	0.9	8.79 e-01	2.07 e-02	1.08 e-01	3.06 e-04	5.40 e-08	2.71 e-02	0

Problem 5.3 Let

$$\begin{cases} \Lambda'(\rho) = \int_0^1 \sin[4\pi\rho + 2\pi\xi]\Lambda(\xi)d\xi + \Lambda(\rho) - 2\pi\sin(2\pi\rho) - \cos(2\pi\rho) - \frac{1}{2}\sin(4\pi\rho) \\ \Lambda(0) = 1 \end{cases} \tag{5.3}$$

be the *IDEs* [4], $\Lambda(\rho) = \cos(2\pi\rho)$ is the precise answer to this problem. Numerical values obtained in the current method *(TS)* and precise solution are shown in Table 5.6 and compared with an existing method after simplifying Eq. (5.3). The numerical answer is graphically compared to the precise solution in Fig. 5.3. The absolute error that occurred in this problem is compared with an existing method in Fig. 5.4.

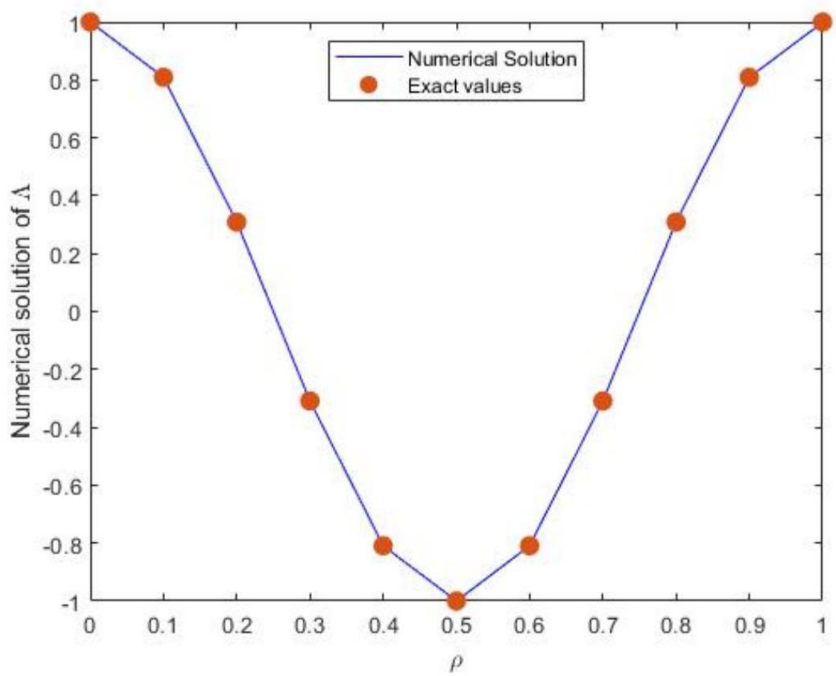

***Figure* 5.3:** Graphical representation of the *TS* for n=10 and the precise solution for the Problem 5.3.

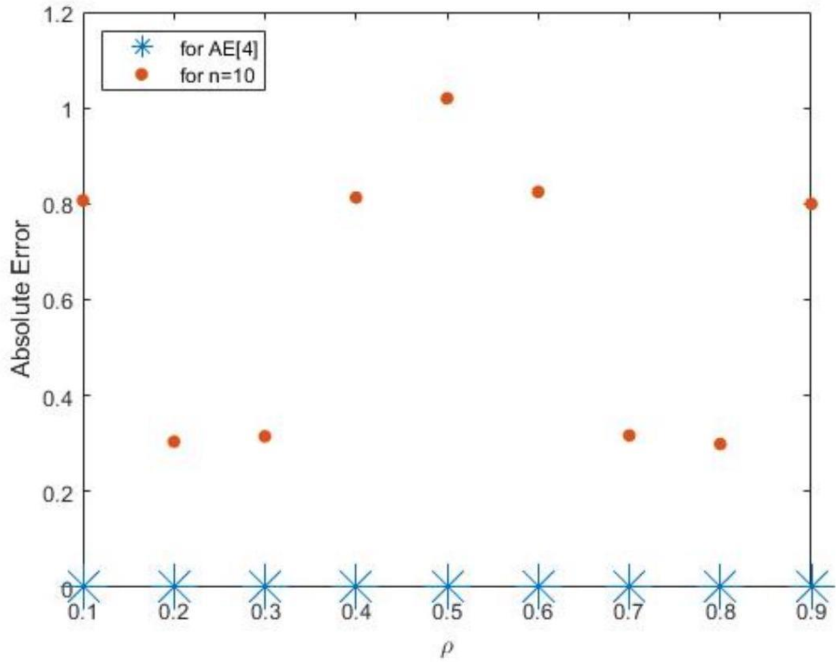

***Figure* 5.4:** Point diagram of absolute errors (AE) for the current method at n=10 and *CAS's AE*[4] for the Problem 5.3.

***Table* 5.6:** Current method is compared with an existing method for the Problem 5.3.

ρ	*PS*	*CAS method in [4]*	*TS at* n=10	*CAS's AE [4]*	*Absolute error of TS*
0.1	0.809016994374947	0.00240432854	0.809022044656112	8.07 e-01	5.05 e-06
0.2	0.309016994374947	0.00507123331	0.309016289100111	3.04 e-01	0.70 e-06
0.3	-0.309016994374948	0.00625477225	-0.309018658682028	3.15 e-01	1.66 e-06
0.4	-0.809016994374947	0.00387315246	-0.809014212286664	8.13 e-01	2.78 e-06
0.5	-1.000000000000000	0.01746016710	-0.999993298619171	1.02 e+00	6.70 e-06
0.6	-0.809016994374947	0.01584825600	-0.809012181620363	8.25 e-01	4.81 e-06
0.7	-0.309016994374948	0.00841721725	-0.309016273658540	3.17 e-01	0.72 e-06
0.8	0.309016994374947	0.00965467633	0.309025332212964	2.99 e-01	8.33 e-06
0.9	0.809016994374947	0.00948709579	0.809005478573982	8.00 e-01	1.15 e-05

Problem 5.4 Let,

$$\Lambda'(\rho) = \frac{5}{4} - \frac{1}{3}\rho^2 + \int_0^1 \left(\rho^2 - \xi\right) (\Lambda(\xi))^2 d\xi, \quad \Lambda(0) = 0, \tag{5.4}$$

be *IDEs* [14, 15].The nonlinear Eq. (5.4) has a precise solution as $\Lambda(\rho) = \rho$. Using the current approach at $n = 3$ to simplify Eq. (5.4), we have Taylor wavelet approximation coefficients as $g_{1,0} = 1, g_{1,1} = 0, g_{1,2} = 0$, obtained by using Newton's iterative technique (taking an initial guess (1,0,0)) in *MATLAB*. Now using these coefficients in Eq.(4.3), We find a solution that is identical to the precise solution. This problem demonstrates the effectiveness of our method. Numerical values and Absolute errors of the current approach are placed in Table 5.6 and can be compared with the precise solution and different known methods [14, 15].

***Table* 5.7:** The current method is compared with some other existing methods for the Problem 5.4.

ρ	*PS*	*TS*	*RHFs [15] at* k = 32	*RHF's AE [15]*	*B-spline's AE* [14] *at* N = 27	*Absolute error of TS*
0.1	0.1	0.1	0.10002	2.0 e-05	1.94 e-14	0
0.2	0.2	0.2	0.20008	8.0 e-05	3.83 e-14	0
0.3	0.3	0.3	0.30007	7.0 e-05	5.05 e-14	0
0.4	0.4	0.4	0.40008	8.0 e-05	6.77 e-14	0
0.5	0.5	0.5	0.50001	1.0 e-05	8.37 e-14	0
0.6	0.6	0.6	0.60001	1.0 e-05	9.30 e-14	0
0.7	0.7	0.7	0.70002	2.0 e-05	10.5 e-14	0
0.8	0.8	0.8	0.80008	8.0 e-05	11.4 e-14	0
0.9	0.9	0.9	0.89991	9.0 e-05	12.1 e-14	0

6 Conclusion

From the above analysis, it is concluded that the Taylor wavelet-based collocation method is much better efficient compared to many different existing methods for numerically solving *IDEs* in numerous disciplines of science and engineering. To get more accurate results, the number of collocation points can be increased.

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

APPLICATION OF FIXED POINT THEOREM IN THE SOLUTION OF INTEGRO-DIFFERENTIAL EQUATIONS: A COMPLEX VALUED APPROACH*

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Abstract

It's remarkable to note that Complex valued Integro-differential and integral type equations are currently intensifying the attention of appreciable researchers due to their comprehensive applications. Thus, this study is fully devoted to the application part of the complex valued controlled, double controlled metric $\eth_{\mathbb{C}}$. We introduce an extended version of the Fisher and Banach type contraction theorem and present some examples to sustain our results. As part of the main theorem's application, we address a common solution with uncertainty in two different folds as follows: [I] Applying the fractional Adams-Bashforth method to the (1.1) FVI_dE. [II] Applying it to the integral type equation (1.2) in the setting of the Extended complex valued metric space.
2020 Mathematical Science Classification.47H09,24H25,34A08,47H10.
Keywords and Phrases: Atangana Baleanu Fractional integral operator,fredholm Volterra integro differential equation*(FVI_dE)*, Complex valued metric space(CVMS), Common fixed point, Cauchy sequence, Contractive condition and completeness.

1 Introduction

The terms calculus of integral equation and fractional calculus are introduced more than 10 decade back. These ten decade seems like a really big time but predominantly these topics are extensively gain new structures and effectively applied in different part of mathematics like fixed point theory, fuzzy theory and so on. Recently Atangana-Baleanu [1] studied new type of fractional derivative targeting non singular/local kernel. Subsequently in 2023 Shinde [33] gave complex valued version of existence and common solution for second order nonlinear boundary value problem using greens function along with another application of fixed point results for multivalued mapping in setting of CVMS. In 2017, Kumar et al. [19] studied a fractional non-linear biological model problem and its approximate solutions through Volterra Integral Equation. In 2019, Kumar [20] studied a class of two variable sequence of functions satisfying Abel's Integral equation and the phase shifts. in 2019 [20] H. Kumar given A class of two variables sequence of functions satisfying Abel's integral equation and the phase shifts. In literature we can see many generalizations of Atangana-Baleanu fractional derivative like AB-derivative [13], AB derivative via MHD channel flow [34], AB_{RL} type [12], we can see more [8,9,11,16,17,18,21,22,26,29,31,32,34,35]. Here we recollecting the definition of Atangana-Baleanu fractional integral, Let $\omega \in (0,1]$ and integral define as,

$$ {}^{AB}_{s}D^{\omega}_{t} f(t) = \frac{(1-\omega)}{\zeta(\omega)} f(t) + \frac{\omega}{\Xi(\omega)\zeta(\omega)} \int_s^t f(h) \frac{(t-h)^{\omega}}{(t-h)} dh. $$

where, $0 < t < s$; normalization function $\zeta(0) = \zeta(1) = 1$.

Subsequently, by applying fractional Adams Bashforth method to the (1.1) FVI_dE in the setting of complex valued controlled metric we deal with following conditions,

$$ (1.1) \quad \tilde{\beth}_0 = \tilde{\beth}(0;\ell); {}^{ABC}_{0} D^{\omega}_{\hbar} \tilde{\beth}(\hbar;\ell) $$
$$ = \aleph(\hbar) + \Re(\hbar).\tilde{\beth}(\hbar,\ell) + \int_0^{\hbar} \mho_1(\hbar,\xi).\chi_1(\tilde{\beth}(\xi,\ell))d\xi + \int_0^1 \mho_2(\hbar,\xi)\chi_2(\tilde{\beth}(\xi,\ell))d\xi, $$

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where, ${}_{0}^{ABC}D_{\hbar}^{\omega}$ ABC type of order ω such that $\tilde{\beth}(\hbar;\ell) = [\underline{\beth}(\hbar;\ell), \overline{\beth}(\hbar;\ell)]$; continuous function $\mho_1, \mho_2 : \nabla\times\nabla \to \mathbb{R}$, $\aleph : \nabla \to \mathbb{R}$; Lipschitz continuous function $\chi_1, \chi_2 : \nabla \to \mathbb{R}$; $L^z(\nabla,\mathbb{R})$ and $C^z(\nabla,\mathbb{R})$ are space of all continuous functions and the space of all Lebesgue integrable functions on ∇, $\tilde{\beth}(\hbar;\ell) \in L^z(\nabla,\mathbb{R}) \cap C^z(\nabla,\mathbb{R})$. At the end we deal with following Integral type equation,

$$\Re_1(\hbar) - \curlywedge(\hbar) = \int_0^{\hbar} \chi(\hbar,\ell)\aleph(\ell,\Re_1\ell)d\ell, \tag{1.2}$$

which has two bounded continuous function namely $\curlywedge(\hbar) : [0,1] \to \mathbb{R}$ and $\aleph(\hbar, \Re_1(\hbar)) : [0,1] \times \mathbb{R} \to \mathbb{R}$. The function $\chi : [0,1) \times [0,1) \to [0,\infty)$ with $\chi(\hbar,.) \in L^1[0,1]$ and $0 \le \hbar \le 1$. We successfully applied fixed point solution to above integral type equation. The novel approach has a promising uniqueness of solution in different fields, for more we can see[10, 11, 19, 22, 23, 24, 25, 26].

2 Preliminaries

Azam, Khan and Fisher [2] studied notion of complex valued metric and given important definition as follows,

Definition 2.1. *Consider a partial order $\precsim$ defined on a complex number($\mathbb{C}$), $\hbar \precsim \ell$ iff Real part of ($\hbar$) $\le$ Real part of (ℓ) ; Imaginary part of ($\hbar$) $\le$ Imaginary part of (ℓ). It follows, $\hbar \le \ell$*

1. *Real part ($\hbar$)<Real part (ℓ) ; Imaginary part ($\hbar$) < Imaginary part (ℓ).*
2. *Real part ($\hbar$) = Real part (ℓ) ; Imaginary part ($\hbar$) = Imaginary part (ℓ).*
3. *Real part ($\hbar$) < Real part (ℓ) ; Imaginary part ($\hbar$) = Imaginary part (ℓ).*
4. *Real part ($\hbar$) = Real part (ℓ) ; Imaginary part ($\hbar$) < Imaginary part (ℓ).*

Definition 2.2. *Lets define the function $\eth_{\mathbb{C}} : \nabla \times \nabla \to \mathbb{C}$, where non empty set ∇; the function $\psi, \zeta : \nabla \times \nabla \to [1,\infty)$ and $\mathbb{C}$ be the set of complex numbers. We define following condition for $\forall\ \hbar, \ell, \mu \in \nabla$,*
$\mathfrak{S}_1$: *$\hbar = \ell$ if and only if $\eth_{\mathbb{C}}(\hbar,\ell) = 0$.*
$\mathfrak{S}_2$: *$\eth_{\mathbb{C}}(\hbar,\ell) = \eth_{\mathbb{C}}(\ell,\hbar)$.*
$\mathfrak{S}_3$: *Controlled triangle inequality- $\eth_{\mathbb{C}}(\hbar,\ell) \precsim \psi(\hbar,\mu)\eth_{\mathbb{C}}(\hbar,\mu) + \psi(\mu,\ell)\eth_{\mathbb{C}}(\mu,\ell)$.*
$\mathfrak{S}_4$: *Extended triangle inequality- $\eth_{\mathbb{C}}(\hbar,\ell) \precsim \psi(\hbar,\ell)[\eth_{\mathbb{C}}(\hbar,\mu) + \eth_{\mathbb{C}}(\mu,\ell)]$.*
$\mathfrak{S}_5$: *Double controlled triangle inequality- $\eth_{\mathbb{C}}(\hbar,\ell) \precsim \psi(\hbar,\mu)\eth_{\mathbb{C}}(\hbar,\mu) + \zeta(\mu,\ell)\eth_{\mathbb{C}}(\mu,\ell)$.*

Definition 2.3. *If $\eth_{\mathbb{C}}$ satisfied $\mathfrak{S}_1$, $\mathfrak{S}_2$ and $\mathfrak{S}_4$, then $\eth_{\mathbb{C}}$ is called complex valued extended metric and the pair $(\nabla, \eth_{\mathbb{C}})$ called complex valued extended metric space.*

Definition 2.4. *If $\eth_{\mathbb{C}}$ satisfied $\mathfrak{S}_1$, $\mathfrak{S}_2$ and $\mathfrak{S}_3$, then $\eth_{\mathbb{C}}$ is called complex valued controlled metric and the pair $(\nabla, \eth_{\mathbb{C}})$ called complex valued controlled metric space.*

Definition 2.5. *If $\eth_{\mathbb{C}}$ satisfied $\mathfrak{S}_1$, $\mathfrak{S}_2$ and $\mathfrak{S}_5$, then $\eth_{\mathbb{C}}$ is called complex valued double Controlled metric and the pair $(\nabla, \eth_{\mathbb{C}})$ called complex valued double Controlled metric space.*

Example 2.1. Lets define the function $\eth_{\mathbb{C}} : \nabla \times \nabla \to \mathbb{C}$ and the set $\nabla = \{2,3,1\}$ which has, $\eth_{\mathbb{C}}(2,3) = i$; $\eth_{\mathbb{C}}(1,2) = 2+4i$; $\eth_{\mathbb{C}}(3,2) = i$; $\eth_{\mathbb{C}}(2,1) = 2+4i$ $\eth_{\mathbb{C}}(1,1) = 0$; $\eth_{\mathbb{C}}(1,3) = 1-i$; $\eth_{\mathbb{C}}(2,2) = 0$; $\eth_{\mathbb{C}}(3,1) = 1-i$; $\eth_{\mathbb{C}}(3,3) = 0$. Again define $\zeta, \psi : \nabla \times \nabla \to [1,\infty)$ as
$\psi(2,3) = \psi(3,2) = \frac{8}{7}$, $\psi(1,2) = \psi(2,1) = 1$, $\psi(1,3) = \psi(3,1) = \frac{3}{2}$,
$\zeta(2,3) = \zeta(3,2) = \frac{9}{2}$, $\zeta(1,2) = \zeta(2,1) = \frac{7}{6}$, $\zeta(3,1) = \zeta(1,3) = 1$.

Proposition 2.1. *In above example we easily verify $\eth_{\mathbb{C}}$ is double controlled metric type but $\eth_{\mathbb{C}}$ is neither a complex valued extended metric nor a complex valued controlled metric.*

Lemma 2.1. *Suppose $(\nabla, \eth_{\mathbb{C}})$ be a $\eth_{\mathbb{C}}$ metric space. Then the sequence $\{\hbar_n\}$ in ∇ is a cauchy sequence if and only if $|\ \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+s})\ | \to 0$ as $n \to \infty$ where $s \in \mathbb{N}$.*

Lemma 2.2. *Suppose $(\nabla, \eth_{\mathbb{C}})$ be a $\eth_{\mathbb{C}}$ metric space. Then the sequence $\{\hbar_n\}$ in ∇ Converges to $\hbar$ if and only if $|\ \eth_{\mathbb{C}}(\hbar_n, \hbar)\ | \to 0$ as $n \to \infty$.*

Definition 2.6. *Assume $\{\hbar_n\}$ be a sequence in a $\eth_{\mathbb{C}}$ metric space $(\nabla, \eth_{\mathbb{C}})$ and $\hbar \in \nabla$, then $(\nabla, \eth_{\mathbb{C}})$ is said to be a complete $\eth_{\mathbb{C}}$ metric space if every Cauchy sequence is convergent in $(\nabla, \eth_{\mathbb{C}})$.*

Definition 2.7. *Suppose $\{\hbar_n\}$ be a sequence in a $\eth_{\mathbb{C}}$ metric space $(\nabla, \eth_{\mathbb{C}})$ and $\hbar \in \nabla$, then $\hbar$ is a limit point of $\{\hbar_n\}$ if for every $\epsilon \in \mathbb{C}$ there exist $n_0 \in \mathbb{N}$ such that $\eth_{\mathbb{C}}(\{\hbar_n\}, n) \prec \epsilon, \forall n \succ n_0$ that is $\lim_{n\to\infty}, \hbar_n = n$.*

Definition 2.8. *Suppose $\{\hbar_n\}$ be a sequence in a $\eth_{\mathbb{C}}$ metric space $(\nabla, \eth_{\mathbb{C}})$ and $\hbar \in \nabla$, then $\{\hbar_n\}$ is a cauchy sequence if for any $\epsilon \in \mathbb{C}$ there exist $n_0 \in \mathbb{N}$ such that $\eth_{\mathbb{C}}(\hbar_n, \hbar_{n+s}) \prec \epsilon, \forall n \succ n_0$ and $s \in \mathbb{N}$.*

Remark 2.1 ([1])**.** *The left sided AB fractional integral of order $\omega \in (0, 1]$ for a function $\tilde{\gimel}$ is defined as*

$$^{AB}\text{ß}_0^{\omega}\tilde{\gimel}(\hbar) = \frac{1}{\zeta(\omega)}[(1-\omega)\tilde{\gimel}(\hbar) + \frac{\omega}{\Xi(\omega)}\int_0^{\hbar}\frac{(\hbar-\xi)^{\omega}}{(\hbar-\xi)}\tilde{\gimel}(\xi)d\xi].$$

where, we have continuous function $\tilde{\gimel}(\hbar)$ on the interval $(0, b)$.

Remark 2.2 ([7])**.** *Consider the map $\psi : \mathbb{R} \to \nabla$ satisfying following properties,*

- *The closure of Supp(ψ) is compact.*
- *ψ- normal, Upper semi-continuous and convex.*

Remark 2.3 ([7])**.** *The parametric interval of $\tilde{\psi}$ is given by,*

$$\tilde{\psi} = [\underline{\psi}(\beta), \overline{\psi}(\beta)] \text{ and } 0 \leq \beta \leq 1$$

- *With respect to β, $\underline{\psi}(\beta)$ is a left continuous and non-decreasing,*
- *$\forall\ \beta \in \nabla$, we have $\underline{\psi}(\beta) \leq \overline{\psi}(\beta)$,*
- *With respect to β, $\overline{\psi}(\beta)$ is a right continuous and non-decreasing.*

Lemma 2.3. *Let $(\nabla, \eth_{\mathbb{C}})$ be a complex valued controlled metric space. If the functional $\eth_{\mathbb{C}} : \nabla \times \nabla \to \mathbb{C}$ is continuous then limit of every convergent sequence is unique.*

Lemma 2.4. *Let $(\nabla, \eth_{\mathbb{C}})$ be a complex valued controlled metric space. If a sequence $\{\hbar_n\}$ in ∇ is Cauchy sequence, such that $\hbar_n \neq \hbar_m$ when $m \neq n$. Then we say $\{\hbar_n\}$ converges at most one point.*

In this article, we present a new fixed point result under extended complex valued metric space with suitable examples, results and finally two folds of the application part.

3 Main Results

Moving towards the following Theorem and its hypothesis, we generalize some ideas via controlled, double controlled complex valued metric space.

Theorem 3.1. *Consider $(\nabla, \eth_{\mathbb{C}})$ be a complete $\eth_{\mathbb{C}}$ metric space. Suppose $\aleph = \frac{\eta}{(\Re^{\flat}-\mu)} < 1$ and*

$$\frac{1}{\aleph} > \sup_{1\leq m}\lim_{i\to\infty}\frac{\psi(\hbar_{i+1}, \hbar_{i+2})}{\psi(\hbar_i, \hbar_{i+1})}\zeta(\hbar_{i+1}, \hbar_m). \tag{3.1}$$

For every $\hbar, \ell \in \nabla$ & $0 \prec \eth_{\mathbb{C}}(\hbar, \ell)$, we use μ, λ, η are non negative real numbers with $\mu + \lambda + \eta < 1$, $1 \leq \Re, \flat$ we choose $\hbar_n = \tilde{\gimel}_2{}^n\hbar_0 \in \nabla$ for all $\hbar_0 \in \nabla$ then the map $\tilde{\gimel}_1, \tilde{\gimel}_2 : \nabla \to \nabla$ satisfying,

$$\eth_{\mathbb{C}}(\tilde{\gimel}_1\hbar, \tilde{\gimel}_2\ell).\Re^{\flat} \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar, \tilde{\gimel}_1\hbar)\eth_{\mathbb{C}}(\ell, \tilde{\gimel}_2\ell)}{1+\eth_{\mathbb{C}}(\hbar, \ell)}\} + \lambda\{\frac{\eth_{\mathbb{C}}(\tilde{\gimel}_1\hbar, \ell).\eth_{\mathbb{C}}(\tilde{\gimel}_2\hbar, \hbar)}{1+\eth_{\mathbb{C}}(\hbar, \ell)}\} + \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\}, \tag{3.2}$$

afterward Assume that, $\lim_{n\to\infty}\zeta(\hbar_n, \hbar)$, $\lim_{n\to\infty}\psi(\hbar, \hbar_n)$ both are exist and finite, then $\tilde{\gimel}_1$ and $\tilde{\gimel}_2$ admits unique common fixed point.

Proof. Suppose, $\hbar_0 \in \nabla$ be any arbitrary point. Let the sequence $\hbar_n = \tilde{\gimel}_2{}^n\hbar_0 \in \nabla$ which satisfies hypothesis of theorem and we define it as,

$$\tilde{\gimel}_1\hbar_{2n} = \hbar_{2n+1}\ ;\ \tilde{\gimel}_2\hbar_{2n+1} = \hbar_{2n+2}, n = 0, 1, 2, ... \tag{3.3}$$

$$\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2}).\Re^{\flat} = \eth_{\mathbb{C}}(\tilde{\gimel}_1\hbar_{2n}, \tilde{\gimel}_2\hbar_{2n+1}).\Re^{\flat} \precsim$$

$$\mu\{\frac{\eth_{\mathbb{C}}(\hbar_{2n}, \tilde{\gimel}_1\hbar_{2n})\eth_{\mathbb{C}}(\hbar_{2n+1}, \tilde{\gimel}_2\hbar_{2n+1})}{1+\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\}}\} + \lambda\{\frac{\eth_{\mathbb{C}}(\tilde{\gimel}_1\hbar_{2n}, \hbar_{2n+1}).\eth_{\mathbb{C}}(\tilde{\gimel}_2\hbar_{2n}, \hbar_{2n})}{1+\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\}}\} + \eta\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\}$$

$$\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2})\Re^{\flat} \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2})}{1+\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\}}\} + \lambda\{\frac{\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+1}).\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n})}{1+\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\}}\} + \eta\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\},$$

$$\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2}).\Re^{\flat} \precsim \mu\{\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2})\} + \eta\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\},$$

$$\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2}).(\Re^{\flat} - \mu) \precsim \eta\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\},$$

$$\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2}) \precsim \frac{\eta}{(\Re^{\flat} - \mu)}\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\},$$

$$\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2}) \precsim \aleph.\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\}.$$

Similarly, we get

$$\eth_{\mathbb{C}}(\hbar_{2n+2}, \hbar_{2n+3}) \precsim \frac{\eta}{(\Re^{\flat} - \mu)}\{\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2})\} \tag{3.4}$$

$$\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2}) \precsim \aleph\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\} \text{ where, } \aleph = \frac{\eta}{(\Re^{\flat} - \mu)} < 1$$

$$\mid \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) \mid \precsim \aleph \mid \{\eth_{\mathbb{C}}(\hbar_{n-1}, \hbar_n)\} \mid,$$

$$\mid \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) \mid \precsim \aleph^2 \mid \{\eth_{\mathbb{C}}(\hbar_{n-2}, \hbar_{n-1})\} \mid,$$

$$\mid \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) \mid \precsim \aleph^n \mid \{\eth_{\mathbb{C}}(\hbar_0, \hbar_1)\} \mid .$$

For every $n < m$, where $m, n \in \mathbb{N}$

$$\mid \eth_{\mathbb{C}}(\hbar_n, \hbar_m) \mid \precsim \psi(\hbar_n, \hbar_{n+1}) \mid \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) \mid + \zeta(\hbar_{n+1}, \hbar_m) \mid \eth_{\mathbb{C}}(\hbar_{n+1}, \hbar_m) \mid \tag{3.5}$$

$$\precsim \psi(\hbar_n, \hbar_{n+1}) \mid \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) \mid + \zeta(\hbar_{n+1}, \hbar_m)[\psi(\hbar_{n+1}, \hbar_{n+2}) \mid \eth_{\mathbb{C}}(\hbar_{n+1}, \hbar_{n+2}) \mid + \zeta(\hbar_{n+2}, \hbar_m) \mid \eth_{\mathbb{C}}(\hbar_{n+2}, \hbar_m) \mid]$$

$$\precsim \psi(\hbar_n, \hbar_{n+1}) \mid \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) \mid + \zeta(\hbar_{n+1}, \hbar_m)\psi(\hbar_{n+1}, \hbar_{n+2}) \mid \eth_{\mathbb{C}}(\hbar_{n+1}, \hbar_{n+2}) \mid + \zeta(\hbar_{n+1}, \hbar_m)\zeta(\hbar_{n+2}, \hbar_m) \mid \eth_{\mathbb{C}}(\hbar_{n+2}, \hbar_m) \mid$$

$$\precsim \psi(\hbar_n, \hbar_{n+1}) \mid \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) \mid + \sum_{i=n+1}^{m-2} (\prod_{j=n+1}^{i} \zeta(\hbar_j, \hbar_m))\psi(\hbar_i, \hbar_{i+1}) \mid \eth_{\mathbb{C}}(\hbar_i, \hbar_{i+1}) \mid + \prod_{k=n+1}^{m-1} \zeta(\hbar_k, \hbar_m) \mid \eth_{\mathbb{C}}(\hbar_{m-1}, \hbar_m) \mid$$

$$\precsim \psi(\hbar_n, \hbar_{n+1}).\aleph^n \mid \{\eth_{\mathbb{C}}(\hbar_0, \hbar_1)\} \mid + \sum_{i=n+1}^{m-2} (\prod_{j=n+1}^{i} \zeta(\hbar_j, \hbar_m))\psi(\hbar_i, \hbar_{i+1}).\aleph^i \mid \{\eth_{\mathbb{C}}(\hbar_0, \hbar_1)\} \mid$$

$$+ \prod_{i=n+1}^{m-1} \zeta(\hbar_i, \hbar_m).\aleph^{m-1} \mid \{\eth_{\mathbb{C}}(\hbar_0, \hbar_1)\} \mid$$

$$\precsim \psi(\hbar_n, \hbar_{n+1})\aleph^n \mid \{\eth_{\mathbb{C}}(\hbar_0, \hbar_1)\} \mid + \sum_{i=n+1}^{m-2} (\prod_{j=n+1}^{i} \zeta(\hbar_j, \hbar_m))\psi(\hbar_i, \hbar_{i+1})\aleph^i \mid \{\eth_{\mathbb{C}}(\hbar_0, \hbar_1)\} \mid +$$

$$\prod_{i=n+1}^{m-1} \zeta(\hbar_i, \hbar_m)\aleph^{m-1}\psi(\hbar_{m-1}, \hbar_m) \mid \{\eth_{\mathbb{C}}(\hbar_0, \hbar_1)\} \mid$$

$$\precsim \psi(\hbar_n, \hbar_{n+1}).\aleph^n \mid \{\eth_{\mathbb{C}}(\hbar_0, \hbar_1)\} \mid + \sum_{i=n+1}^{m-1} (\prod_{j=n+1}^{i} \zeta(\hbar_j, \hbar_m))\psi(\hbar_i, \hbar_{i+1}).\aleph^i \mid \{\eth_{\mathbb{C}}(\hbar_0, \hbar_1)\} \mid$$

$$\precsim \psi(\hbar_n, \hbar_{n+1}).\aleph^n \mid \{\eth_{\mathbb{C}}(\hbar_0, \hbar_1)\} \mid + \sum_{i=n+1}^{m-1} (\prod_{j=0}^{i} \zeta(\hbar_j, \hbar_m))\psi(\hbar_i, \hbar_{i+1}).\aleph^i \mid \{\eth_{\mathbb{C}}(\hbar_0, \hbar_1)\} \mid .$$

Hence we write,

$$\mid \eth_{\mathbb{C}}(\hbar_n, \hbar_m) \mid \precsim \mid \eth_{\mathbb{C}}(\hbar_0, \hbar_1) \mid [\aleph^n.\psi(\hbar_n, \hbar_{n+1}) + (\mho_{m-1} - \mho_m)],$$

where, $\mho_\iota = \sum_{i=0}^{\iota}(\prod_{j=0}^{i} \zeta(\hbar_j, \hbar_m))\psi(\hbar_i, \hbar_{i+1})\aleph^i$.
As we have (3.1) and using ratio test we get limit of $\{\mho_n\}$ exists, so it is Cauchy. When we apply ratio test to following term and letting $m, n \to \infty$ in (3.6),

$$\omega_i = (\prod_{j=0}^{i} \zeta(\hbar_j, \hbar_m))\psi(\hbar_i, \hbar_{i+1}), \text{ and } \lim_{m,n\to\infty} \mid \eth_{\mathbb{C}}(\hbar_n, \hbar_m) \mid = 0, \tag{3.6}$$

which gives sequence $\{\hbar_n\}$ is Cauchy. Since $(\nabla, \eth_{\mathbb{C}})$ is Complete then $\exists \digamma \in \nabla$ such that,

$$\lim_{m,n\to\infty} \mid \eth_{\mathbb{C}}(\hbar_n, \digamma) \mid = 0. \tag{3.7}$$

Now, by triangle inequality,

$$|\eth_{\mathbb{C}}(F,\hbar_{n+1})| \precsim \psi(F,\hbar_n)\,|\eth_{\mathbb{C}}(F,\hbar_n)| + \zeta(\hbar_n,\hbar_{n+1})\,|\eth_{\mathbb{C}}(\hbar_n,\hbar_{n+1})|\,. \tag{3.8}$$

By Using (3.6) and (3.8) we finally get,

$$\lim_{n\to\infty} |\eth_{\mathbb{C}}(F,\hbar_{n+1})| = 0. \tag{3.9}$$

Now we claim $F = \tilde{\beth}_1 F$,

$$\begin{aligned}
|\eth_{\mathbb{C}}(F,\tilde{\beth}_1F)| &\precsim \psi(F,\hbar_{n+2})\,|\eth_{\mathbb{C}}(F,\hbar_{n+2})| + \zeta(\hbar_{n+2},\tilde{\beth}_1F)\,|\eth_{\mathbb{C}}(\hbar_{n+2},\tilde{\beth}_1F)| \\
|\eth_{\mathbb{C}}(F,\tilde{\beth}_1F)| &\precsim \psi(F,\hbar_{n+2})\,|\eth_{\mathbb{C}}(F,\hbar_{n+2})| + \zeta(\hbar_{n+2},\tilde{\beth}_1F)\,|\eth_{\mathbb{C}}(\tilde{\beth}_2\hbar_{n+1},\tilde{\beth}_1F)| \\
|\eth_{\mathbb{C}}(F,\tilde{\beth}_1F)| &\precsim \psi(F,\hbar_{n+2})\,|\eth_{\mathbb{C}}(F,\hbar_{n+2})| + \zeta(\hbar_{n+2},\tilde{\beth}_1F).\Re^{\flat}\,|\eth_{\mathbb{C}}(\tilde{\beth}_1F,\tilde{\beth}_2\hbar_{n+1})| \\
&\precsim \psi(F,\hbar_{n+2})\,|\eth_{\mathbb{C}}(F,\hbar_{n+2})| + \zeta(\hbar_{n+2},\tilde{\beth}_1F)\mu\{\frac{\eth_{\mathbb{C}}(F,\tilde{\beth}_1F)\eth_{\mathbb{C}}(\hbar_{n+1},\tilde{\beth}_2\hbar_{n+1})}{1+\eth_{\mathbb{C}}(F,\hbar_{n+1})}\}+ \\
&\lambda\{\frac{\eth_{\mathbb{C}}(\tilde{\beth}_1F,\hbar_{n+1}).\eth_{\mathbb{C}}(\tilde{\beth}_2F,F)}{1+\eth_{\mathbb{C}}(F,\hbar_{n+1})}\} + \eta\{\eth_{\mathbb{C}}(F,\hbar_{n+1})\}.
\end{aligned} \tag{3.10}$$

We write this as,

$$\begin{aligned}
&\precsim \psi(F,\hbar_{n+2})\,|\eth_{\mathbb{C}}(F,\hbar_{n+2})| + \zeta(\hbar_{n+2},\tilde{\beth}_1F)\mu\{\frac{\eth_{\mathbb{C}}(F,\tilde{\beth}_1F)\eth_{\mathbb{C}}(\hbar_{n+1},\hbar_{n+2})}{1+\eth_{\mathbb{C}}(F,\hbar_{n+1})}\}+ \\
&\lambda\{\frac{\eth_{\mathbb{C}}(\tilde{\beth}_1F,\hbar_{n+1})\eth_{\mathbb{C}}(\tilde{\beth}_2F,F)}{1+\eth_{\mathbb{C}}(F,\hbar_{n+1})}\} + \eta\{\eth_{\mathbb{C}}(F,\hbar_{n+1})\}.
\end{aligned}$$

Using (3.6),(3.7) and (3.8), we get

$$|\eth_{\mathbb{C}}(F,\tilde{\beth}_1F)| = 0.$$

Hence, $\tilde{\beth}_1$ admits fixed point F. Subsequently we prove $\tilde{\beth}_2$ admits fixed point as F. Now finally we have to work on Uniqueness property, that is $\tilde{\beth}_1$ and $\tilde{\beth}_2$ admits unique common fixed point.
On Contrary assume that F and F^* are two common fixed points of $\tilde{\beth}_1$ and $\tilde{\beth}_2$ & $F \neq F^*$.

$$\begin{aligned}
\eth_{\mathbb{C}}(F,F^*).\Re^{\flat} &= \eth_{\mathbb{C}}(\tilde{\beth}_1F,\tilde{\beth}_2F^*).\Re^{\flat} \\
&\precsim \mu\{\frac{\eth_{\mathbb{C}}(F,\tilde{\beth}_1F)\eth_{\mathbb{C}}(F^*,\tilde{\beth}_2F^*)}{1+\eth_{\mathbb{C}}(F,F^*)}\} + \lambda\{\frac{\eth_{\mathbb{C}}(\tilde{\beth}_1F,F^*).\eth_{\mathbb{C}}(\tilde{\beth}_2F,F)}{1+\eth_{\mathbb{C}}(F,F^*)}\} + \eta\{\eth_{\mathbb{C}}(F,F^*)\}
\end{aligned} \tag{3.11}$$

$$\eth_{\mathbb{C}}(F,F^*).\Re^{\flat} \precsim \eta\{\eth_{\mathbb{C}}(F,F^*)\} \text{ which impies } \eth_{\mathbb{C}}(F,F^*).(\Re^{\flat}-\eta) \precsim 0.$$

Hence we get, $\eth_{\mathbb{C}}(F,F^*) = 0$ which is the contradiction to our assumption. Thus $F = F^*$, $\tilde{\beth}_1$ and $\tilde{\beth}_2$ admits unique common fixed point. □

If we assume $\tilde{\beth}_1$ & $\tilde{\beth}_2$ are equal and which is equal to $\tilde{\beth}$ along with we include map $\tilde{\beth} : \nabla \to \nabla$ be a continuous mapping; $\Re, \flat = 1$ & $\lambda, \mu = 0$ then Theorem 3.1 reduces to following result,

Theorem 3.2. *Consider $(\nabla, \eth_{\mathbb{C}})$ be a Complete $\eth_{\mathbb{C}}$ metric space. Suppose*

$$\frac{1}{\eta} > \sup_{1\leq m} \lim_{i\to\infty} \frac{\psi(\hbar_{i+1},\hbar_{i+2})}{\psi(\hbar_i,\hbar_{i+1})}\zeta(\hbar_{i+1},\hbar_m). \tag{3.12}$$

For every $\hbar, \ell \in \nabla$ & $0 \prec \eth_{\mathbb{C}}(\hbar,\ell)$, we use η non negative real numbers with $0 < \eta < 1$, we choose $\hbar_n = \tilde{\beth}^n\hbar_0 \in \nabla$ for all $\hbar_0 \in \nabla$ then the map $\tilde{\beth} : \nabla \to \nabla$ be a continuous mapping such that,

$$\eth_{\mathbb{C}}(\tilde{\beth}\hbar,\tilde{\beth}\ell) \precsim \eta\{\eth_{\mathbb{C}}(\hbar,\ell)\}, \tag{3.13}$$

afterward assume that, $\lim_{n\to\infty}\zeta(\hbar_n,\hbar)$, $\lim_{n\to\infty}\psi(\hbar,\hbar_n)$ both exist and finite, then $\tilde{\beth}$ admits unique common fixed point.

Proof. Consider $\hbar_n = \{\tilde{\beth}^n \hbar_0\}$ and by Using inequalities (3.13),

$$\eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) \precsim \eta \eth_{\mathbb{C}}(\hbar_{n-1}, \hbar_n) \precsim ... \precsim \eta \eth_{\mathbb{C}}(\hbar_0, \hbar_1), \forall n \geq 0$$

for every $m > n$, where $m, n \in \mathbb{N}$

$$\eth_{\mathbb{C}}(\hbar_n, \hbar_m) \precsim \psi(\hbar_n, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) + \psi(\hbar_{n+1}, \hbar_m)\eth_{\mathbb{C}}(\hbar_{n+1}, \hbar_m)$$

$$\precsim \psi(\hbar_n, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) + \psi(\hbar_{n+1}, \hbar_m)\psi(\hbar_{n+1}, \hbar_{n+2})\eth_{\mathbb{C}}(\hbar_{n+1}, \hbar_{n+2}) + \psi(\hbar_{n+1}, \hbar_m)\psi(\hbar_{n+2}, \hbar_m)\eth_{\mathbb{C}}(\hbar_{n+2}, \hbar_m)$$

$$\precsim \psi(\hbar_n, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) + \psi(\hbar_{n+1}, \hbar_m)\psi(\hbar_{n+1}, \hbar_{n+2})\eth_{\mathbb{C}}(\hbar_{n+1}, \hbar_{n+2}) + \psi(\hbar_{n+1}, \hbar_m)$$

$$\psi(\hbar_{n+2}, \hbar_m)\psi(\hbar_{n+3}, \hbar_m)\psi(\hbar_{n+3}, \hbar_m) \precsim ...$$

$$\precsim \psi(\hbar_n, \hbar_{n+1})\eta^n \eth_{\mathbb{C}}(\hbar_0, \hbar_1) + \sum_{i=n+1}^{m-2} \prod_{j=n+1}^{i} \psi(\hbar_j, \hbar_m)\psi(\hbar_i, \hbar_{n+1})\eta^i \eth_{\mathbb{C}}(\hbar_0, \hbar_1) + \prod_{k=n+1}^{m-1} \psi(\hbar_k, \hbar_m)\eta^{m-1}\eth_{\mathbb{C}}(\hbar_0, \hbar_1)$$

If we follow same steps given in main Theorem 3.1, we get

$$\precsim \psi(\hbar_n, \hbar_{n+1})\eta^n \eth_{\mathbb{C}}(\hbar_0, \hbar_1) + \sum_{i=n+1}^{m-1} \prod_{j=0}^{i} \zeta(\hbar_j, \hbar_m)\psi(\hbar_i, \hbar_{i+1})\eta^i \eth_{\mathbb{C}}(\hbar_0, \hbar_1)$$

Let,

$$\mho_\iota = \sum_{i=0}^{\iota} \prod_{j=0}^{\iota} \psi(\hbar_j, \hbar_m)\psi(\hbar_i, \hbar_{i+1})\eta^i.$$

$$\eth_{\mathbb{C}}(\hbar_n, \hbar_m) \precsim \eth_{\mathbb{C}}(\hbar_0, \hbar_1)[\eta^n \psi(\hbar_n, \hbar_{n+1}) + (\mho_{m-1}, \mho_n)]. \tag{3.14}$$

By using ratio test and (3.12), $\lim_{m,n\to\infty} \mho_n$ exists which implies sequence $\{\mho_n\}$ is Cauchy. Applying $\lim_{m,n\to\infty}$ to (3.14), we get

$$\lim_{m,n\to\infty} \eth_{\mathbb{C}}(\hbar_n, \hbar_m) = 0. \tag{3.15}$$

As we know $\{\hbar_n\}$ is Cauchy in complete $\eth_{\mathbb{C}}$ -metric space, then we say that $\{\hbar_n\}$ is converges to a point $\hbar^* \in \nabla$. Now next part $\hbar^*$ is fixed point of $\tilde{\beth}$. We use definition of continuity of $\tilde{\beth}$,

$$\hbar^* = \lim_{n\to\infty} \hbar_{n+1} = \lim_{n\to\infty} \tilde{\beth}\hbar_n = \tilde{\beth}(\lim_{n\to\infty} \hbar_n) = \tilde{\beth}\hbar^*$$

and finally remaining part is uniqueness of fixed point. On contrary we assume $\tilde{\beth}$ has two fixed point say F and F^*,

$$\eth_{\mathbb{C}}(F, F^*) = \eth_{\mathbb{C}}(\tilde{\beth}F, \tilde{\beth}F^*) \precsim \psi \eth_{\mathbb{C}}(F, F^*),$$

which holds only when $\eth_{\mathbb{C}}(F, F^*) = 0$ and Hence it finally gives uniqueness of fixed point. □

If we assume $\tilde{\beth}_1$ & $\tilde{\beth}_2$ are equal and which is equal to $\tilde{\beth}$ along with we avoid map $\tilde{\beth} : \nabla \to \nabla$ is continuous; $\Re, \flat = 1$ & $\lambda, \mu = 0$ then Theorem 3.1 reduces to following result:

Theorem 3.3. *Consider* $(\nabla, \eth_{\mathbb{C}})$ *be a complete* $\eth_{\mathbb{C}}$ *metric space. Suppose*

$$\frac{1}{\eta} > \sup_{1\leq m} \lim_{i\to\infty} \frac{\psi(\hbar_{i+1}, \hbar_{i+2})}{\psi(\hbar_i, \hbar_{i+1})} \zeta(\hbar_{i+1}, \hbar_m). \tag{3.16}$$

For every $\hbar, \ell \in \nabla$ & $0 \prec \eth_{\mathbb{C}}(\hbar, \ell)$, *we use* η *non negative real numbers with* $0 < \eta < 1$, *we choose* $\hbar_n = \tilde{\beth}^n \hbar_0 \in \nabla$ *for all* $\hbar_0 \in \nabla$ *then the map* $\tilde{\beth} : \nabla \to \nabla$ *be a mapping such that,*

$$\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) \precsim \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\}, \tag{3.17}$$

afterward assume that, $\lim_{n\to\infty} \zeta(\hbar_n, \hbar)$, $\lim_{n\to\infty} \psi(\hbar, \hbar_n)$ *both exist and finite, then* $\tilde{\beth}$ *admits unique common fixed point.*

Proof. If we follow similar steps like Theorem 3.2 we can easily get the Cauchy sequence $\{\hbar_n\}$ under $\eth_{\mathbb{C}}$-metric space $(\nabla, \eth_{\mathbb{C}})$. Subsequently we say $\{\hbar_n\}$ converges to $\hbar^* \in \nabla$. We shall prove $\tilde{\beth}$ admits $\hbar^*$ as a fixed point, we consider the triangle inequality of complex valued controlled metric space,

$$\eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) \precsim \psi(\hbar^*, \hbar_n)\eth_{\mathbb{C}}(\hbar^*, \hbar_n) + \psi(\hbar_n, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}).$$

with the help of Statement (b) of Theorem 3.3, we write

$$\lim_{n\to\infty} \eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) = 0. \tag{3.18}$$

Again by (3.17) and triangle inequality, we get

$$\eth_{\mathbb{C}}(\hbar^*, \tilde{\beth}\hbar^*) \precsim \psi(\hbar^*, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) + \psi(\hbar_{n+1}, \tilde{\beth}\hbar^*)\eth_{\mathbb{C}}(\hbar_{n+1}, \tilde{\beth}\hbar^*)$$
$$\precsim \psi(\hbar^*, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) + \eta\psi(\hbar_{n+1}, \tilde{\beth}\hbar^*)\eth_{\mathbb{C}}(\hbar_n, \tilde{\beth}\hbar^*).$$

Letting $\lim_{n\to\infty}$ and Statement of Theorem 3.3, we get $\eth_{\mathbb{C}}(\hbar^*, \tilde{\beth}\hbar^*) = 0$, Hence proved. □

We use following example to verify above results:

Example 3.1. Let $\eth_{\mathbb{C}} : \nabla \times \nabla \to \mathbb{C}$ be a symmetric metric. Suppose $\nabla = \{1, 2, 0\}$ and $\eth_{\mathbb{C}}(1, 2) = \eth_{\mathbb{C}}(0, 1) = 1 + i$ & $\eth_{\mathbb{C}}(0, 2) = 4 + 4i$ again function $\psi : \nabla \times \nabla \to [1, \infty)$ is symmetric and

$$\psi(1,1) = \tfrac{4}{3}, \psi(2,2) = \tfrac{6}{5}, \psi(1,2) = \tfrac{5}{4}.$$
$$\psi(0,2) = \tfrac{4}{3}, \psi(0,1) = \tfrac{3}{2}, \psi(0,0) = 2.$$

It's easy to verify $\eth_{\mathbb{C}}$ is a metric space, Suppose self map $\tilde{\beth}$ follows $\tilde{\beth}(2) = \tilde{\beth}(1) = \tilde{\beth}(0) = 0$ & use $\eta = \frac{2}{5}$ and we clearly see that (3.17) holds for $\hbar_0 \in \nabla$ then condition (3.16) is satisfied. We follow the following cases to verify hypothesis of Theorem 3.3,

Case I. If $\hbar = 1$, $\ell = 2$ then,
$\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) = \eth_{\mathbb{C}}(\tilde{\beth}1, \tilde{\beth}2) = \eth_{\mathbb{C}}(2, 2) = 0 \precsim \frac{2}{5}(1 + i) = \eta\eth_{\mathbb{C}}(1, 2) = \eta\eth_{\mathbb{C}}(\hbar, \ell)$.

Case II. If $\hbar = 0$, $\ell = 1$ then,
$\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) = \eth_{\mathbb{C}}(\tilde{\beth}0, \tilde{\beth}1) = \eth_{\mathbb{C}}(2, 2) = 0 \precsim \frac{2}{5}(1 + i) = \eta\eth_{\mathbb{C}}(0, 1) = \eta\eth_{\mathbb{C}}(\hbar, \ell)$.

Case III.If $\hbar = 0$, $\ell = 2$ then,
$\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) = \eth_{\mathbb{C}}(\tilde{\beth}0, \tilde{\beth}2) = \eth_{\mathbb{C}}(2, 2) = 0 \precsim \frac{2}{5}(4 + 4i) = \eta\eth_{\mathbb{C}}(0, 2) = \eta\eth_{\mathbb{C}}(\hbar, \ell)$

Case IV. If $\hbar = 0$, $\ell = 0$; $\hbar = 1$, $\ell = 1$; $\hbar = 2$, $\ell = 2$ then, the results hold good. Then we say that $\tilde{\beth}$ admits a unique fixed point as $\hbar^* = 0$.

If we assume $\tilde{\beth}_1$ and $\tilde{\beth}_2$ are equal and which is equal to $\tilde{\beth}$; $\Re, \flat = 1$ & $\lambda = 0$ then Theorem 3.1 reduces to following result,

Theorem 3.4. *Consider* $(\nabla, \eth_{\mathbb{C}})$ *be a Complete* $\eth_{\mathbb{C}}$ *metric space. Suppose* $\aleph = \frac{\eta}{(1-\mu)} < 1$ *and*

$$\frac{1}{\aleph} > \sup_{1 \leq m} \lim_{i \to \infty} \frac{\psi(\hbar_{i+1}, \hbar_{i+2})}{\psi(\hbar_i, \hbar_{i+1})} \zeta(\hbar_{i+1}, \hbar_m). \tag{3.19}$$

For every $\hbar, \ell \in \nabla$ & $0 \prec \eth_{\mathbb{C}}(\hbar, \ell)$, *we use* μ, η *are non negative real numbers with* $0 \leq \eta < 1$, $0 \leq \mu < 1$ *we choose* $\hbar_n = \tilde{\beth}^n \hbar_0 \in \nabla$ *for all* $\hbar_0 \in \nabla$ *then the map* $\tilde{\beth} : \nabla \to \nabla$ *be a Continuous map satisfying,*

$$\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar, \tilde{\beth}\hbar)\eth_{\mathbb{C}}(\ell, \tilde{\beth}\ell)}{1 + \eth_{\mathbb{C}}(\hbar, \ell)}\} + \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\} \tag{3.20}$$

afterward Assume that, $\lim_{n\to\infty} \zeta(\hbar_n, \hbar)$, $\lim_{n\to\infty} \psi(\hbar, \hbar_n)$ *both are exist and finite, then* $\tilde{\beth}$ *admits unique common fixed point.*

Proof. The proof of the above result is similar to Theorem 3.1 therefore we omit it. □

Proposition 3.1. *Above results gives generalization of* [10] *D.Lateef rational functions result, Fisher type, under Complex valued double controlled metric space* [11].

Suppose that $\tilde{\beth}_1$ & $\tilde{\beth}_2$ are equal and which is equal to $\tilde{\beth}$ along with we map $\tilde{\beth} : \nabla \to \nabla$ is not continuous; $\Re, \flat = 1$ & $\lambda = 0$ then Theorem 3.1 reduces to following result,

Theorem 3.5. *Consider* $(\nabla, \eth_{\mathbb{C}})$ *be a Complete* $\eth_{\mathbb{C}}$ *metric space. Suppose* $\aleph = \frac{\eta}{(1-\mu)} < 1$ *and*

$$\frac{1}{\aleph} > \sup_{1 \leq m} \lim_{i \to \infty} \frac{\psi(\hbar_{i+1}, \hbar_{i+2})}{\psi(\hbar_i, \hbar_{i+1})} \zeta(\hbar_{i+1}, \hbar_m). \tag{3.21}$$

For every $\hbar, \ell \in \nabla$ *and* $0 \prec \eth_{\mathbb{C}}(\hbar, \ell)$, *we use* μ, η *are non negative real numbers with* $0 \leq \eta < 1$, $0 \leq \mu < 1$ *we choose* $\hbar_n = \tilde{\beth}^n \hbar_0 \in \nabla$ *for all* $\hbar_0 \in \nabla$ *then the map* $\tilde{\beth} : \nabla \to \nabla$ *be a mapping such that,*

$$\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar, \tilde{\beth}\hbar)\eth_{\mathbb{C}}(\ell, \tilde{\beth}\ell)}{1 + \eth_{\mathbb{C}}(\hbar, \ell)}\} + \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\}. \tag{3.22}$$

afterward assume that, $\lim_{n\to\infty} \zeta(\hbar_n, \hbar)$, $\lim_{n\to\infty} \psi(\hbar, \hbar_n)$ *both are exist and finite, then* $\tilde{\beth}$ *admits unique common fixed point.*

Proof. If we follow similar steps like Theorem 3.1 we can easily get the Cauchy sequence $\{\hbar_n\}$ under $\eth_{\mathbb{C}}$-metric space $(\nabla, \eth_{\mathbb{C}})$. Subsequently we say $\{\hbar_n\}$ converges to $\hbar^* \in \nabla$. We shall prove $\tilde{\beth}$ admits $\hbar^*$ as a fixed point. Lets consider the triangle inequality of complex valued controlled metric space,

$$\eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) \precsim \psi(\hbar^*, \hbar_n)\eth_{\mathbb{C}}(\hbar^*, \hbar_n) + \psi(\hbar_n, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}).$$

with the help of Statement of Theorem 3.5, we write

$$\lim_{n\to\infty} \eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) = 0. \tag{3.23}$$

Again by inequalities (3.22) and triangle inequality, we get

$$\eth_{\mathbb{C}}(\hbar^*, \tilde{\beth}\hbar^*) \precsim \psi(\hbar^*, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) + \psi(\hbar_{n+1}, \tilde{\beth}\hbar^*)\eth_{\mathbb{C}}(\hbar_{n+1}, \tilde{\beth}\hbar^*)$$

$$\precsim \psi(\hbar^*, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) + \eta\psi(\hbar_{n+1}, \tilde{\beth}\hbar^*)\mu\{\frac{\eth_{\mathbb{C}}(\hbar_n, \tilde{\beth}\hbar_n)\eth_{\mathbb{C}}(\hbar^*, \tilde{\beth}\hbar^*)}{1+\eth_{\mathbb{C}}(\hbar_n, \hbar^*)}\} + \eta\eth_{\mathbb{C}}(\hbar_n, \hbar^*)$$

$$\precsim \psi(\hbar^*, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) + \eta\psi(\hbar_{n+1}, \tilde{\beth}\hbar^*)\mu\{\frac{\eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar^*, \tilde{\beth}\hbar^*)}{1+\eth_{\mathbb{C}}(\hbar_n, \hbar^*)}\} + \eta\eth_{\mathbb{C}}(\hbar_n, \hbar^*).$$

Letting $\lim_{n\to\infty}$ and Statement of Theorem 3.5, we get $\eth_{\mathbb{C}}(\hbar^*, \tilde{\beth}\hbar^*) = 0$, Hence proved. □

Lets verify above result through the following example.

Example 3.2. Let $\eth_{\mathbb{C}} : \nabla \times \nabla \to \mathbb{C}$ be a symmetric metric. Suppose $\nabla = \{1, 2, 0\}$ and $\eth_{\mathbb{C}}(1, 2) = \eth_{\mathbb{C}}(0, 1) = 1 + i$ & $\eth_{\mathbb{C}}(0, 2) = 4 + 4i$ again function $\psi : \nabla \times \nabla \to [1, \infty)$ is symmetric and

$$\psi(1, 1) = \tfrac{7}{3}, \psi(2, 2) = \tfrac{9}{5}, \psi(1, 2) = 2;\ \psi(0, 2) = \tfrac{7}{3}, \psi(0, 1) = 3, \psi(0, 0) = 5$$

It's easy to verify $\eth_{\mathbb{C}}$ is a metric space, Suppose self map $\tilde{\beth}$ follows $\tilde{\beth}(2) = \tilde{\beth}(1) = \tilde{\beth}(0) = 1$ & use $\mu, \eta = \frac{2}{5}$ and we clearly see that (3.20) holds for $\hbar_0 \in \nabla$ then condition (3.19) is satisfied. We follow the following cases to verify hypothesis of Theorem 3.5,

Case I.) If $\hbar = 1$, $\ell = 2$ then,
$\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) = 0 \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar, \tilde{\beth}\hbar)\eth_{\mathbb{C}}(\ell, \tilde{\beth}\ell)}{1+\eth_{\mathbb{C}}(\hbar, \ell)}\} + \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\}$.

Case II. If $\hbar = 0$, $\ell = 1$ then,
$\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) = 0 \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar, \tilde{\beth}\hbar)\eth_{\mathbb{C}}(\ell, \tilde{\beth}\ell)}{1+\eth_{\mathbb{C}}(\hbar, \ell)}\} + \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\}$.

Case III.If $\hbar = 0$, $\ell = 2$ then,
$\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) = 0 \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar, \tilde{\beth}\hbar)\eth_{\mathbb{C}}(\ell, \tilde{\beth}\ell)}{1+\eth_{\mathbb{C}}(\hbar, \ell)}\} + \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\}$.

Case IV. If $\hbar = 0$, $\ell = 0$; $\hbar = 1$, $\ell = 1$; $\hbar = 2$, $\ell = 2$ then $\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) = 0$, results hold good. Then we say that $\tilde{\beth}$ admits a unique fixed point as $\hbar^* = 1$.

4 Application of the Main theorem

We divide application part of main Theorem in to two following folds,

4.1 Application Part I

In this part we would like to introduce the notion of Existence and unique fixed point solution in the context of fractional FVI_dE. By applying fractional Adams Bashforth method to the (1.1) FVI_dE,

$$\tilde{\beth}_0 = \tilde{\beth}(0; \ell) \text{ and } {}_0^{ABC}D_{\hbar}^{\omega}\tilde{\beth}(\hbar; \ell) = \aleph(\hbar) + \Re(\hbar).\tilde{\beth}(\hbar, \ell) + \int_0^{\hbar} \mho_1(\hbar, \xi).\chi_1(\tilde{\beth}(\xi, \ell))d\xi + \int_0^1 \mho_2(\hbar, \xi)\chi_2(\tilde{\beth}(\xi, \ell))d\xi$$

in the setting of complex valued controlled metric space we prove following application part.

4.1.1 Application to fractional Fredholm Volterra integro differential equation.

We consider the following hypothesis,

1. $\Re$ and $\aleph$ both function are continuous,
2. $$|\ \eth(\tilde{\beth}_1(\hbar; \ell), \tilde{\beth}_2(\hbar; \ell))\ |\ .\alpha_1 \geq |\ (\eth(\chi_1(\tilde{\beth}_1(\hbar; \ell))), (\chi_1(\tilde{\beth}_2(\hbar; \ell))))\ |, \forall\tilde{\beth}_1, \tilde{\beth}_2 \in C^z(\nabla), \alpha_1, \alpha_2 > 0,$$
$$|\ \eth(\tilde{\beth}_1(\hbar; \ell), \tilde{\beth}_2(\hbar; \ell))\ |\ .\alpha_2 \geq |\ (\eth(\chi_2(\tilde{\beth}_1(\hbar; \ell))), (\chi_2(\tilde{\beth}_2(\hbar; \ell))))\ |, \forall\tilde{\beth}_1, \tilde{\beth}_2 \in C^z(\nabla), \alpha_1, \alpha_2 > 0. \tag{4.1}$$
3. For the function $\mho_1^*$ and $\mho_2^*$,
$$\mho_1^* < \infty \Rightarrow \mho_1^* = \sup_{\hbar\in\nabla} \int_0^{\hbar} |\ \mho_1(\hbar, \xi)\ |\ d\xi \text{ and } \mho_2^* < \infty \Rightarrow \mho_2^* = \sup_{\hbar\in\nabla} \int_0^{\hbar} |\ \mho_2(\hbar, \xi)\ |\ d\xi, \tag{4.2}$$

$C^z(\nabla, \mathbb{R})$ be the space of all continuous functions $\beth : \nabla \to \mathbb{R}$ which has $\|\beth\|_\infty = max\{|\ \beth(\rho)\ |: \forall\rho \in \nabla\}$ then $(C^z(\nabla, \mathbb{R}), \|.\|_\infty)$ is banach space.

Theorem 4.1. *Suppose* (4.1),(4.2) *and* (1) *are satisfied. If*

$$\Psi_1 = [\frac{\zeta\omega.[\Xi(\omega+1).(1-\omega)+\omega]}{\zeta^2(\omega).\Xi(\omega+1)}]_{\|\Re\|_\infty<1}. \tag{4.3}$$

Then above problem (1.1) *FVI_dE admits at least one solution* $\tilde{\daleth}(\hbar,\ell)$.

Before starting our proof we go through following result;

$$0<\omega\le 1 \text{ and } \tilde{\daleth}(\hbar,\ell)-\tilde{\daleth}_0 = \frac{1}{\zeta(\omega)}[(1-\omega)\tilde{\aleph}(\hbar,\ell)+\frac{\omega}{\Xi(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}\tilde{\aleph}(\xi,\ell)d\xi], \tag{4.4}$$

which is the solution of, $\tilde{\daleth}_0=\tilde{\daleth}(0;\ell)$ and $\tilde{\aleph}(\hbar,\ell)={}_0^{ABC}D_\hbar^\omega\ \tilde{\daleth}(\hbar;\ell)$. Applying the operator $({}_0^{AB}\ B_\hbar^\omega)$ to above equation, $({}_0^{AB}\ B_\hbar^\omega)\ \tilde{\aleph}(\hbar,\ell)=({}_0^{AB}\ B_\hbar^\omega)\ {}_0^{ABC}D_\hbar^\omega\ \tilde{\daleth}(\hbar;\ell)$. Hence we write (4.4) as,

$$\tilde{\daleth}(\hbar,\ell)-\tilde{\daleth}(0;\ell) = \frac{1}{\zeta(\omega)}[(1-\omega)\tilde{\aleph}(\hbar,\ell)+\frac{\omega}{\Xi(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}\tilde{\aleph}(\xi,\ell)d\xi].$$

Proof. As we know that,

$$\tilde{\daleth}(\hbar,\ell)-\tilde{\daleth}(0;\ell) = \frac{1}{\zeta(\omega)}[(1-\omega)\tilde{\aleph}(\hbar,\ell)+\frac{\omega}{\Xi(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}\tilde{\aleph}(\xi,\ell)d\xi]. \tag{4.5}$$

We write main equation (1.1) as,

$$\tilde{\aleph}(\hbar,\ell)-\aleph(\hbar)=\Re(\hbar).\tilde{\daleth}(\hbar,\ell)+\int_0^\hbar \mho_1(\hbar,\xi).\chi_1(\tilde{\daleth}(\xi,\ell))d\xi+\int_0^1 \mho_2(\hbar,\xi).\chi_2(\tilde{\daleth}(\xi,\ell))d\xi.$$

Similarly, we write

$$\tilde{\aleph}(\xi,\ell)-\aleph(\xi)=\Re(\xi).\tilde{\daleth}(\xi,\ell)+\int_0^\xi \mho_1(\xi,\digamma).\chi_1(\tilde{\daleth}(\digamma,\ell))d\digamma+\int_0^1 \mho_2(\xi,\digamma).\chi_2(\tilde{\daleth}(\digamma,\ell))d\digamma.$$

Applying above two equations in (4.5), we get

$$\begin{aligned}\tilde{\daleth}(\hbar,\ell)-\tilde{\daleth}(0;\ell) &= \frac{(1-\omega)}{\zeta(\omega)}[\aleph(\hbar)+\Re(\hbar).\tilde{\daleth}(\hbar,\ell)+\int_0^\hbar \mho_1(\hbar,\xi).\chi_1(\tilde{\daleth}(\xi,\ell))d\xi+\int_0^1 \mho_2(\hbar,\xi)\chi_2(\tilde{\daleth}(\xi,\ell))d\xi]\\ &+\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}[\aleph(\xi)+\Re(\xi)\tilde{\daleth}(\xi,\ell)+\int_0^\xi \mho_1(\xi,\digamma)\chi_1(\tilde{\daleth}(\digamma,\ell))d\digamma+\int_0^1 \mho_2(\xi,\digamma)\chi_2(\tilde{\daleth}(\digamma,\ell))d\digamma]d\xi.\end{aligned}$$

Now lets use operator Υ in above equation,

$$\begin{aligned}\Upsilon\tilde{\daleth}(\hbar,\ell)-\tilde{\daleth}(0;\ell) &= \frac{(1-\omega)}{\zeta(\omega)}[\aleph(\hbar)+\Re(\hbar).\tilde{\daleth}(\hbar,\ell)+\int_0^\hbar \mho_1(\hbar,\xi).\chi_1(\tilde{\daleth}(\xi,\ell))d\xi+\int_0^1 \mho_2(\hbar,\xi)\chi_2(\tilde{\daleth}(\xi,\ell))d\xi]\\ &+\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}[\aleph(\xi)+\Re(\xi).\tilde{\daleth}(\xi,\ell)+\int_0^\xi \mho_1(\xi,\digamma)\chi_1(\tilde{\daleth}(\digamma,\ell))d\digamma+\int_0^1 \mho_2(\xi,\digamma)\chi_2(\tilde{\daleth}(\digamma,\ell))d\digamma]d\xi.\end{aligned}$$

Here we claim, operator Υ admits fixed point and we defined it as,

$$\Upsilon : L^z(\nabla,\mathbb{R})\cap C^z(\nabla,\mathbb{R})\to L^z(\nabla,\mathbb{R})\cap C^z(\nabla,\mathbb{R})$$

So, we divide our proof into following folds, **Firstly**, we show χ_1,χ_2 continuous which finally gives Υ is continuous. Suppose $\{\tilde{\daleth}_n\}$ be a sequence such that $\tilde{\daleth}_n\to\tilde{\daleth}$ in $C(\nabla,\mathbb{R}^z)$. Then $\hbar\in\nabla$ we get,

$$\begin{aligned}&|\ \eth(\Upsilon\tilde{\daleth}_n(\hbar,\ell),\Upsilon\tilde{\daleth}(\hbar,\ell))\ |\le \tilde{\daleth}_n(0;\ell)+\frac{(1-\omega)}{\zeta(\omega)}[\aleph(\hbar)+\Re(\hbar)\tilde{\daleth}_n(\hbar,\ell)+\int_0^\hbar \mho_1(\hbar,\xi)\chi_1(\tilde{\daleth}_n(\xi,\ell))d\xi+\int_0^1 \mho_2(\hbar,\xi)\\ &\chi_2(\tilde{\daleth}_n(\xi,\ell))d\xi]+\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}[\aleph(\xi)+\Re(\xi)\tilde{\daleth}_n(\xi,\ell)+\int_0^\xi \mho_1(\xi,\digamma)\chi_1(\tilde{\daleth}_n(\digamma,\ell))d\digamma+\int_0^1 \mho_2(\xi,\digamma)\\ &\chi_2(\tilde{\daleth}_n(\digamma,\ell))d\digamma]d\xi-[\tilde{\daleth}(0;\ell)+\frac{(1-\omega)}{\zeta(\omega)}[\aleph(\hbar)+\Re(\hbar)\tilde{\daleth}(\hbar,\ell)+\int_0^\hbar \mho_1(\hbar,\xi)\chi_1(\tilde{\daleth}(\xi,\ell))d\xi+\int_0^1 \mho_2(\hbar,\xi)\chi_2(\tilde{\daleth}(\xi,\ell))d\xi]\\ &+\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}[\aleph(\xi)+\Re(\xi)\tilde{\daleth}(\xi,\ell)+\int_0^\xi \mho_1(\xi,\digamma)\chi_1(\tilde{\daleth}(\digamma,\ell))d\digamma+\int_0^1 \mho_2(\xi,\digamma)\chi_2(\tilde{\daleth}(\digamma,\ell))d\digamma]d\xi].\end{aligned}$$

$$\leq \frac{(1-\omega)}{\zeta(\omega)}[|\ \Re(\hbar)\ ||\ \tilde{\beth}_n(\hbar,\ell) - \tilde{\beth}(\hbar,\ell)\ | + \int_0^{\hbar} |\ \mho_1(\hbar,\xi)\ ||\ \chi_1(\tilde{\beth}_n(\xi,\ell)) - \chi_1(\tilde{\beth}(\xi,\ell))\ |\ d\xi + \int_0^1 |\ \mho_2(\xi,F)\ ||\ \chi_2(\tilde{\beth}_n(F,\ell)) - \chi_2$$

$$(\tilde{\beth}(F,\ell))\ |\ dF] + \frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^{\hbar}\frac{(\hbar-\xi)^{\omega}}{(\hbar-\xi)}[|\ \Re(\xi)\ ||\ \tilde{\beth}_n(\xi,\ell) - \tilde{\beth}(\xi,\ell)\ | + \int_0^{\xi} |\ \mho_1(\xi,F)\ ||\ \chi_1(\tilde{\beth}_n(F,\ell)) - \chi_1(\tilde{\beth}(F,\ell))\ |\ dF]d\xi$$

Apply supremum then,

$$\|\eth(\Upsilon\tilde{\beth}_n(\hbar,\ell) - \Upsilon\tilde{\beth}(\hbar,\ell))\|_\infty \leq \frac{(1-\omega)}{\zeta(\omega)}[\|\aleph\|_\infty + \|\tilde{\beth}_n - \tilde{\beth}\|_\infty + \mho_1^*\|\chi_1(\tilde{\beth}_n) - \chi_1(\tilde{\beth})\|_\infty + \mho_2^*\|\chi_2(\tilde{\beth}_n) - \chi_2(\tilde{\beth})\|_\infty]$$

$$+\frac{\omega\hbar^{\omega}}{\Xi(\omega+1)\zeta(\omega)}[\|\aleph\|_\infty + \|\tilde{\beth}_n - \tilde{\beth}\|_\infty + \mho_1^*\|\chi_1(\tilde{\beth}_n) - \chi_1(\tilde{\beth})\|_\infty + \mho_2^*\|\chi_2(\tilde{\beth}_n) - \chi_2(\tilde{\beth})\|_\infty].$$

$$\leq \frac{(1-\omega)}{\zeta(\omega)} + \frac{\omega\hbar^{\omega}}{\Xi(\omega+1)\zeta(\omega)}[\|\aleph\|_\infty + \|\tilde{\beth}_n - \tilde{\beth}\|_\infty + \mho_1^*\|\chi_1(\tilde{\beth}_n) - \chi_1(\tilde{\beth})\|_\infty + \mho_2^*\|\chi_2(\tilde{\beth}_n) - \chi_2(\tilde{\beth})\|_\infty]$$

$$\|\eth(\Upsilon\tilde{\beth}_n(\hbar,\ell) - \Upsilon\tilde{\beth}(\hbar,\ell))\|_\infty \to 0, when \tilde{\beth}_n \to \tilde{\beth},$$

which finally gives that Υ is continuous. **Secondly** we work on compactness property for Υ and then completely continuous. Let $\mathfrak{C}_R = \{\tilde{\beth} \in C(\nabla, \mathbb{R}^z) : \|\tilde{\beth}\|_\infty \leq R\}$ be a convex, closed and bounded set with, $\frac{\Psi_2}{1-\Psi_1} \leq R$ and we define $\lambda_j = \sup_{\tilde{\beth}\in\nabla\times[0,R]} \chi_j(\tilde{\beth}(\xi,\ell)+1), j = 1, 2, ..$

$$\Psi_2 = |\ \tilde{\beth}_0\ | + [\frac{\zeta\omega.[\Xi(\omega+1).(1-\omega)+\omega]}{\zeta^2(\omega).\Xi(\omega+1)}]_{[\|\aleph\|_\infty + \mho_1^*\lambda_1 + \mho_2^*\lambda_2]}, \tag{4.6}$$

$$\tilde{\beth} \in \mathfrak{C}_R, |\ \Upsilon\tilde{\beth}(\hbar,\ell)\ | - |\ \tilde{\beth}(0;\ell)\ | \leq \frac{(1-\omega)}{\zeta(\omega)}[|\ \aleph(\hbar)\ | + |\ \Re(\hbar)\ ||\ \tilde{\beth}(\hbar,\ell)\ | + \int_0^{\hbar} |\ \mho_1(\hbar,\xi)\ ||\ \chi_1(\tilde{\beth}(\xi,\ell))\ |\ d\xi +$$

$$\int_0^1 |\ \mho_2(\hbar,\xi)\ ||\ \chi_2(\tilde{\beth}(\xi,\ell))\ |\ d\xi] + \frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^{\hbar}\frac{(\hbar-\xi)^{\omega}}{(\hbar-\xi)}[|\ \aleph(\xi)\ | + |\ \Re(\xi)\ ||\ \tilde{\beth}(\xi,\ell)\ | + \int_0^{\xi} |\ \mho_1(\xi,F)\ |$$

$$|\ \chi_1(\tilde{\beth}(F,\ell))\ |\ dF + \int_0^1 |\ \mho_2(\xi,F)\ ||\ \chi_2(\tilde{\beth}(F,\ell))\ |\ dF]d\xi,$$

$$|\ \Upsilon\tilde{\beth}(\hbar,\ell)\ | - |\ \tilde{\beth}(0;\ell)\ | \leq \frac{(1-\omega)}{\zeta(\omega)}[|\ \aleph(\hbar)\ | + |\ \Re(\hbar)\ ||\ \tilde{\beth}(\hbar,\ell)\ | + \mho_1^*\lambda_1 + \mho_2^*\lambda_2] + \frac{\omega\hbar^{\omega}}{\Xi(\omega+1)\zeta(\omega)}$$

$$[|\ \aleph(\xi)\ | + |\ \Re(\xi)\ ||\ \tilde{\beth}(\xi,\ell)\ | + \mho_1^*\lambda_1 + \mho_2^*\lambda_2].$$

Take Supremum on both side,

$$\|\Upsilon\tilde{\beth}\|_\infty - |\ \tilde{\beth}(0;\ell)\ | \leq \frac{(1-\omega)}{\zeta(\omega)} + \frac{\omega}{\Xi(\omega+1)\zeta(\omega)}[\|\aleph\|_\infty + \|\Re\|_\infty R + \mho_1^*\lambda_1 + \mho_2^*\lambda_2], \tag{4.7}$$

$$\leq [\frac{\zeta\omega[\Xi(\omega+1)(1-\omega)+\omega]}{\zeta^2(\omega)\Xi(\omega+1)}][\|\aleph\|_\infty + \mho_1^*\lambda_1 + \mho_2^*\lambda_2] + [\frac{\zeta\omega[\Xi(\omega+1)(1-\omega)+\omega]}{\zeta^2(\omega)\Xi(\omega+1)}]\|\Re\|_\infty R$$

$$\leq \Psi_1 R + \Psi_2 \leq R.$$

It gives that Υ is uniformly bounded. Now our next claim is that Υ is equicontinuous. Let $\hbar_1 < \hbar_2$,

$$|\ \eth(\Upsilon\tilde{\beth}(\hbar_2,\ell), \Upsilon\tilde{\beth}(\hbar_1,\ell))\ | = |\ \frac{(1-\omega)}{\zeta(\omega)}[\aleph(\hbar_2) + \Re(\hbar_2).\tilde{\beth}(\hbar_2,\ell) + \int_0^{\hbar_2}\mho_1(\hbar_2,\xi)\chi_1(\tilde{\beth}(\xi,\ell))d\xi + \int_0^1 \mho_2(\hbar_2,\xi)$$

$$\chi_2(\tilde{\beth}(\xi,\ell))d\xi] + \frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^{\hbar_2}\frac{(\hbar_2-\xi)^{\omega}}{(\hbar_2-\xi)}[\aleph(\xi) + \Re(\xi)\tilde{\beth}(\xi,\ell) + \int_0^{\xi}\mho_1(\xi,F)\chi_1(\tilde{\beth}(F,\ell))dF + \int_0^1 \mho_2(\xi,F)$$

$$\chi_2(\tilde{\beth}(F,\ell))dF]d\xi - [\frac{(1-\omega)}{\zeta(\omega)}[\aleph(\hbar_1) + \Re(\hbar_1)\tilde{\beth}(\hbar_1,\ell) + \int_0^{\hbar_1}\mho_1(\hbar_1,\xi).\chi_1(\tilde{\beth}(\xi,\ell))d\xi + \int_0^1 \mho_2(\hbar_1,\xi)\chi_2(\tilde{\beth}(\xi,\ell))d\xi]$$

$$+\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^{\hbar_1}\frac{(\hbar_1-\xi)^{\omega}}{(\hbar_1-\xi)}[\aleph(\xi) + \Re(\xi).\tilde{\beth}(\xi,\ell) + \int_0^{\xi}\mho_1(\xi,F)\chi_1(\tilde{\beth}(F,\ell))dF + \int_0^1 \mho_2(\xi,F).\chi_2(\tilde{\beth}(F,\ell))dF]d\xi]\ |$$

$$\leq \frac{(1-\omega)}{\zeta(\omega)}(|\ \aleph(\hbar_2) - \aleph(\hbar_1)\ | + |\ \Re(\hbar_2).\tilde{\beth}(\hbar_2,\ell) - \Re(\hbar_1).\tilde{\beth}(\hbar_1,\ell)\ | + \int_0^{\hbar_1}(\mho_1(\hbar_2,\xi) - \mho_1(\hbar_1,\xi))\chi_1(\tilde{\beth}(\xi,\ell))d\xi$$

$$+\int_{\hbar_1}^{\hbar_2}\mho_1(\hbar_2,\xi)\chi_1(\tilde{\beth}(\xi,\ell))d\xi+\int_0^1(\mho_2(\hbar_2,\xi)-\mho_2(\hbar_2,\xi))\chi_2(\tilde{\beth}(\xi,\ell))d\xi+\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^{\hbar_1}(\frac{(\hbar_2-\xi)^\omega}{(\hbar_2-\xi)}-\frac{(\hbar_1-\xi)^\omega}{(\hbar_1-\xi)})$$

$$[\Re(\xi).\tilde{\beth}(\xi,\ell)+\int_0^\xi\mho_1(\xi,F)\chi_1(\tilde{\beth}(F,\ell))dF+\int_0^1\mho_2(\xi,F).\chi_2(\tilde{\beth}(F,\ell))dF]d\xi\int_{\hbar_1}^{\hbar_2}\frac{(\hbar_2-\xi)^\omega}{(\hbar_2-\xi)}$$

$$[\Re(\xi).\tilde{\beth}(\xi,\ell)+\int_0^\xi\mho_1(\xi,F)\chi_1(\tilde{\beth}(F,\ell))dF+\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^1\mho_2(\xi,F).\chi_2(\tilde{\beth}(F,\ell))dF]d\xi$$

$=\mathfrak{S}+\mathfrak{T}+\mathfrak{U}$, where

$$\mathfrak{S}=\frac{(1-\omega)}{\zeta(\omega)}(|\aleph(\hbar_2)-\aleph(\hbar_1)|+|\Re(\hbar_2).\tilde{\beth}(\hbar_2,\ell)-\Re(\hbar_1).\tilde{\beth}(\hbar_1,\ell)|+\int_0^{\hbar_1}(\mho_1(\hbar_2,\xi)-\mho_1(\hbar_1,\xi))$$
$$\chi_1(\tilde{\beth}(\xi,\ell))d\xi+\int_{\hbar_1}^{\hbar_2}\mho_1(\hbar_2,\xi)\chi_1(\tilde{\beth}(\xi,\ell))d\xi+\int_0^1(\mho_2(\hbar_2,\xi)-\mho_2(\hbar_2,\xi))\chi_2(\tilde{\beth}(\xi,\ell))d\xi. \tag{4.8}$$

If we use $\hbar_2\to\hbar_1$ then $\mathfrak{S}\to 0$. Again for $\mathfrak{T}$ we write ,

$$\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^{\hbar_1}(\frac{(\hbar_2-\xi)^\omega}{(\hbar_2-\xi)}-\frac{(\hbar_1-\xi)^\omega}{(\hbar_1-\xi)})[\Re(\xi)\tilde{\beth}(\xi,\ell)+\int_0^\xi\mho_1(\xi,F)\chi_1(\tilde{\beth}(F,\ell))dF+\int_0^1\mho_2(\xi,F)\chi_2(\tilde{\beth}(F,\ell))dF]d\xi$$
$$\leq(\|\Re\|_\infty R+\mho_1^*\lambda_1+\mho_2^*\lambda_2)\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^{\hbar_1}(\frac{(\hbar_2-\xi)^\omega}{(\hbar_2-\xi)}-\frac{(\hbar_1-\xi)^\omega}{(\hbar_1-\xi)})d\xi$$

$$\mathfrak{T}\leq((\hbar_2-\hbar_1)^\omega-(\hbar_2)^\omega+(\hbar_1)^\omega)\frac{(\|\Re\|_\infty R+\mho_1^*\lambda_1+\mho_2^*\lambda_2)\omega}{\Xi(\omega+1)\zeta(\omega)}. \tag{4.9}$$

If we use $\hbar_2\to\hbar_1$ then $\mathfrak{T}\to 0$. Again similar for $\mathfrak{U}$,

$$\mathfrak{U}=\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_{\hbar_1}^{\hbar_2}\frac{(\hbar_2-\xi)^\omega}{(\hbar_2-\xi)}[\Re(\xi).\tilde{\beth}(\xi,\ell)+\int_0^\xi\mho_1(\xi,F)\chi_1(\tilde{\beth}(F,\ell))dF+\int_0^1\mho_2(\xi,F)\chi_2(\tilde{\beth}(F,\ell))dF]d\xi$$
$$\leq(\|\Re\|_\infty R+\mho_1^*\lambda_1+\mho_2^*\lambda_2)\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_{\hbar_1}^{\hbar_2}(\frac{(\hbar_2-\xi)^\omega}{(\hbar_2-\xi)})d\xi$$

$$\mathfrak{U}\leq((\hbar_2-\hbar_1)^\omega)\frac{(\|\Re\|_\infty R+\mho_1^*\lambda_1+\mho_2^*\lambda_2)\omega}{\Xi(\omega+1)\zeta(\omega)}. \tag{4.10}$$

and hence, If we use $\hbar_2\to\hbar_1$ then $\mathfrak{U}\to 0$. By using above condition of $\mathfrak{U}$ of (4.10) , $\mathfrak{S}$ of (4.9) and $\mathfrak{T}$ of (4.9),

$$\|\eth(\Upsilon\tilde{\beth}(\hbar_2,\ell),\Upsilon\tilde{\beth}(\hbar_1,\ell))\|_\infty\to 0,$$

as $\hbar_2\to\hbar_1$, which implies Υ is equicontinuous. With the help of Arzelà-Ascoli theorem, we say Υ is completely continuous, as we have Υ is compact in $C(\nabla,\mathbb{R}^z)$.

Thirdly we deduce Υ admits at least one fixed point in ∇. Suppose, $\tilde{\beth}(\hbar,\ell)\in\text{ß}$. Then $\tilde{\beth}(\hbar,\ell)=\flat\Upsilon\tilde{\beth}(\hbar,\ell)$ Set $\text{ß}=\{\tilde{\beth}(\hbar,\ell)\in C(\nabla,\mathbb{R}^z):\tilde{\beth}(\hbar,\ell)=\flat\Upsilon\tilde{\beth}(\hbar,\ell),0<\flat<1\}$ bounded. Now for $0\leq\hbar\leq 1$,

$$|\tilde{\beth}(\hbar,\ell)|=|\flat\Upsilon\tilde{\beth}(\hbar,\ell)|\leq|\Upsilon\tilde{\beth}(\hbar,\ell)|\leq|\tilde{\beth}(0;\ell)|+(\frac{(1-\omega)}{\zeta(\omega)}+\frac{\omega}{\Xi(\omega+1)\zeta(\omega)})[\|\aleph\|_\infty+\|\Re\|_\infty R+\mho_1^*\lambda_1+\mho_2^*\lambda_2]$$
$$\leq|\tilde{\beth}(0;\ell)|+[\frac{\zeta\omega[\Xi(\omega+1)(1-\omega)+\omega]}{\zeta^2(\omega)\Xi(\omega+1)}][\|\aleph\|_\infty+\mho_1^*\lambda_1+\mho_2^*\lambda_2]+[\frac{\zeta\omega[\Xi(\omega+1)(1-\omega)+\omega]}{\zeta^2(\omega)\Xi(\omega+1)}].$$

$\|\Re\|_\infty R\leq\Psi_2R+\Psi_2$. By equation (4.3), we get $\Psi_2R+\Psi_2\leq R$. Hence we say ß is bounded, as we given in our main Theorem 3.2 we conclude Υ admits at least one fixed point. Hence, we deduce that Problem (1.1) admits at least one fixed point solution in ∇. **Lastly** We work on the uniqueness of solution for our FVI_dE (1.1). We have to show here Υ admits unique solution. For that we consider $\tilde{\beth}_1(\hbar,\ell)$,$\tilde{\beth}_2(\hbar,\ell)\in C(\nabla,\mathbb{R}^z)$,

$$\curlywedge=(\frac{\zeta\omega.[\Xi(\omega+1).(1-\omega)+\omega\hbar^\omega]}{\zeta^2(\omega).\Xi(\omega+1)})[\|\Re\|_\infty+\mho_1^*c_1+\mho_2^*c_2]<1, \tag{4.11}$$

$$|\eth(\Upsilon\tilde{\beth}_1(\hbar,\ell),\Upsilon\tilde{\beth}_2(\hbar,\ell))|\leq\frac{(1-\omega)}{\zeta(\omega)}[|\Re(\hbar)||\tilde{\beth}_1(\hbar,\ell)-\tilde{\beth}_2(\hbar,\ell)|+\int_0^\hbar|\mho_1(\hbar,\xi)||\chi_1(\tilde{\beth}_1(\xi,\ell))-\chi_1(\tilde{\beth}_2(\xi,\ell))|d\xi$$

$$+\int_0^1 \mid \mho_2(\hbar,\xi) \mid\mid \chi_2(\tilde{\beth}_1(\xi,\ell)) - \chi_2(\tilde{\beth}_2(\xi,\ell)) \mid d\xi] + \frac{\omega}{\Xi(\omega)\zeta(\omega)} \int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)} [\mid \Re(\xi) \mid\mid \tilde{\beth}_1(\xi,\ell) - \tilde{\beth}_2(\xi,\ell) \mid +$$

$$\int_0^\xi \mid \mho_1(\xi,F) \mid\mid \chi_1(\tilde{\beth}_1(F,\ell)) - \chi_1(\tilde{\beth}_2(F,\ell)) \mid dF + \int_0^1 \mid \mho_2(\xi,F) \mid\mid \chi_2(\tilde{\beth}_1(F,\ell)) - \chi_2(\tilde{\beth}_2(F,\ell)) \mid dF]d\xi.$$

Apply supremum both sides, we get

$$\|\eth(\Upsilon\tilde{\beth}_1(\hbar,\ell), \Upsilon\tilde{\beth}_2(\hbar,\ell))\|_\infty \leq \frac{(1-\omega)}{\zeta(\omega)}[\|\Re\|_\infty + \mho_1^* c_1 + \mho_2^* c_2] + \|\tilde{\beth}_1 - \tilde{\beth}_2\|_\infty + \frac{\omega\hbar^\omega}{\Xi(\omega+1)\zeta(\omega)}[\|\Re\|_\infty +$$

$$\mho_1^* c_1 + \mho_2^* c_2] + \|\tilde{\beth}_1 - \tilde{\beth}_2\|_\infty = (\frac{(1-\omega)}{\zeta(\omega)} + \frac{\omega\hbar^\omega}{\Xi(\omega+1)\zeta(\omega)})[\|\Re\|_\infty + \mho_1^* c_1 + \mho_2^* c_2] + \|\tilde{\beth}_1 - \tilde{\beth}_2\|_\infty$$

$$= (\frac{\zeta\omega.[\Xi(\omega+1).(1-\omega) + \omega\hbar^\omega]}{\zeta^2(\omega).\Xi(\omega+1)})[\|\Re\|_\infty + \mho_1^* c_1 + \mho_2^* c_2] + \|\tilde{\beth}_1 - \tilde{\beth}_2\|_\infty.$$

So, By equation (4.11), we write $\|\eth(\Upsilon\tilde{\beth}_1(\hbar,\ell), \Upsilon\tilde{\beth}_2(\hbar,\ell))\|_\infty \leq \curlywedge\|\tilde{\beth}_1 - \tilde{\beth}_2\|_\infty$, which shows that Υ is a Contraction map. Thus using Theorem 3.2, Υ admits a unique fixed point solution and hence we say system (1.1) admits a unique solution $\tilde{\beth}(\hbar,\ell)$. □

4.2 Application Part II

We consider the integral type of equation (1.2), which has two bounded continuous function namely $\curlywedge(\hbar) : [0,1] \to \mathbb{R}$ and $\aleph(\hbar, \Re_1(\hbar)) : [0,1] \times \mathbb{R} \to \mathbb{R}$. The function $\chi : [0,1) \times [0,1) \to [0,\infty)$ with $\chi(\hbar,.) \in L^1[0,1]$ and $0 \leq \hbar \leq 1$. Here we present Theorem 4.2 for existence and common solution to the equation (1.2).

4.2.1 Application to the integral type equation

Theorem 4.2. *Suppose,*

I) *The continuous function,* $\curlywedge(\hbar) : [0,1] \to \mathbb{R}$ *and* $\aleph(\hbar, \Re_1(\hbar)) : [0,1] \times \mathbb{R} \to \mathbb{R}$*. Let* $\tilde{\beth} : \nabla \times \nabla$ *be an operator of,*

$$\nabla\Re_1(\hbar) - \curlywedge(\hbar) = \int_0^\hbar \chi(\hbar,\ell)\aleph(\ell, \Re_1\ell)d\ell \tag{4.12}$$

II) $\mid \aleph(\hbar, \Re_1(\hbar)) - \aleph(\hbar, \Re_2(\hbar)) \mid \leq \frac{1}{F e^{iF\hbar}} \mid \Re_1(\hbar) - \Re_2(\hbar) \mid$ *for* $\forall\ \Re_1, \Re_2 \in \nabla$ *&* $1 < F \leq \frac{1}{\eta}$*;* $0 < \eta < 1$.

III) *The function* $\chi : [0,1) \times [0,1) \to [0,\infty)$ *with* $\chi(\hbar,.) \in L^1[0,1]$ *and* $0 \leq \hbar \leq 1$*;*

$$1 \geq \| \int_0^\hbar \chi(\hbar,\ell)d\ell \|, \tag{4.13}$$

where, $\nabla = C([0,1], \mathbb{R})$ *be real valued continuous function on* $[0,1]$ *and* $\Re_1(\hbar) \in \nabla$ *then* (1.2) *admits unique solution.*

Proof. Let the mapping, $\psi(\hbar) : \nabla \times \nabla \to [1,\infty)$ defined as,

$$\psi(\hbar) = \begin{cases} F + \max\{\Re_1(\hbar), \Re_2(\hbar)\}, & \text{Otherwise} \\ \\ 1, & \text{if } \Re_1, \Re_2 \in [0,1]. \end{cases}$$

Assume $\eth_{\mathbb{C}} : \nabla \times \nabla \to \mathbb{C}$ be a complex valued $\eth_{\mathbb{C}}$ metric space,

$$\eth_{\mathbb{C}}(\Re_1, \Re_2) = \|\Re_1\|_\infty = \sup_{0 \leq \hbar \leq 1} \mid \Re_1(\hbar) \mid e^{-iF\hbar},$$

where, $\nabla = C([0,1], \mathbb{R})$, $1 < F \leq \frac{1}{\eta}; 0 < \eta < 1$ and $(i)^2 = -1$. Here its easy to say $(\nabla, \eth_{\mathbb{C}})$ is complete complex valued $\eth_{\mathbb{C}}$ metric space. Main integral type equation (1.2) can be again resumed to find the element $\hbar^* \in \nabla$ which gives fixed point for $\tilde{\beth}$, Now

$$\mid \tilde{\beth}\Re_1(\hbar) - \tilde{\beth}\Re_2(\hbar) \mid \leq \mid \int_0^\hbar [\chi(\hbar,\ell)\aleph(\ell,\Re_1\ell) - \chi(\hbar,\ell)\aleph(\ell,\Re_2\ell)] \mid d\ell \leq \mid \int_0^\hbar \mid \chi(\hbar,\ell)[\aleph(\ell,\Re_1\ell) - \aleph(\ell,\Re_2\ell)] \mid d\ell$$

$$\leq (\int_0^\hbar \chi(\hbar,\ell)d\ell) \int_0^\hbar \mid [\aleph(\ell,\Re_1\ell) - \aleph(\ell,\Re_2\ell)] \mid d\ell \leq \frac{1}{\text{ß} e^{iF\hbar}}(\int_0^\hbar \chi(\hbar,\ell)d\ell) \int_0^\hbar \mid [\Re_1\ell - \Re_2\ell] \mid d\ell$$

$$= \frac{e^{iF\hbar}}{\text{ß} e^{iF\hbar}e^{-iF\hbar}}(\int_0^\hbar \chi(\hbar,\ell)e^{-iF\hbar}d\ell) \int_0^\hbar \mid [\Re_1\ell - \Re_2\ell] \mid e^{-iF\hbar}d\ell.$$

Apply Supremum to both side, we get

$$[\sup_{0\leq\hbar\leq 1} \mid \tilde{\beth}\Re_1(\hbar) - \tilde{\beth}\Re_2(\hbar) \mid e^{-iF\hbar}] \leq \frac{1}{F}(\int_0^{\hbar} \sup_{0\leq\hbar\leq 1} \chi(\hbar,\ell)e^{-iF\hbar}d\ell)[\sup_{0\leq\hbar\leq 1} \mid [\Re_1\hbar - \Re_2\hbar] \mid e^{-iF\hbar}d\ell].$$

with the help of (4.2) and II, we get

$$\eth_{\mathbb{C}}(\tilde{\beth}\Re_1, \tilde{\beth}\Re_2) = \|\tilde{\beth}\Re_1 - \tilde{\beth}\Re_2\|_\infty \leq \frac{1}{F}\|\Re_1 - \Re_2\|_\infty = \frac{1}{F}\eth_{\mathbb{C}}(\Re_1, \Re_2).$$

We can check easily both cases of $\psi(\Re_1, \Re_2)$ when $0 \leq \Re_1 \leq 1$; $0 \leq \Re_2 \leq 1$ or else (3.13) true. Hence for $0 < \frac{1}{F} < 1$, all hypothesis of Theorem 3.2 hold true, which finally gives that (1.2) admits unique solution. □

5 Conclusion

To study and contribute to worldly problems we consider the concept of controlled, double controlled metric in the setting of Extended complex valued metric space. Afterwards, we present our paper in three folds as, **Firstly**, we introduce fixed point theorem which is the extended version of famous results from literature, namely Fisher and Banach [16] contraction type results along with some examples to sustain our results. **Secondly** with the help of ABC fractional derivative (1.1), we introduced common fixed point Theorem 4.1 for FVI_dE and its unique fixed point solution. **Thirdly** we introduced a fixed point solution to the integral type equation (1.2) in $\eth_{\mathbb{C}}$ metric as the application part of main results.

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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

INEQUALITIES INVOLVING GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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Abstract

Trigonometric and Hyperbolic inequalities, which have been obtained by C. Huygens[5], D. S. Mitrinovic[10] and many more, have attracted attention of several researchers. We offer several refinements and generalization of few trigonometric and hyperbolic inequalities involving tangent function, cotangent function, sine function, secant function and cosecant function. The established results are obtained with the aid of the Schwab-Borchardt mean.
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Keywords and Phrases: Hyperbolic function, Trigonometric function.

1 Introduction

It is well known from basic calculus that,

$$arcsin(x) = \int_0^x \frac{1}{(1-t^2)^{\frac{1}{2}}}dt, \quad 0 \le x \le 1$$

and

$$\frac{\pi}{2} = arcsin(1) = \int_0^1 \frac{1}{(1-t^2)^{\frac{1}{2}}}dt.$$

We can define the function sin on $[0, \frac{\pi}{2}]$ as the inverse of arcsine and extend it to $(-\infty, \infty)$.
Let, $p > 1$, we can generalize $arcsin(x)$ as

$$arcsin_p(x) = \int_0^x \frac{1}{(1-t^p)^{\frac{1}{p}}}dt, \quad 0 \le x \le 1$$

and

$$\frac{\pi_p}{2} = arcsin_p(1) = \int_0^1 \frac{1}{(1-t^p)^{\frac{1}{p}}}dt.$$

So, we define the π_p[4] function as,

$$\pi_p = 2\int_0^1 \frac{1}{(1-t^p)^{\frac{p+1}{p}}}dt. = 2\frac{\Gamma(\frac{p+1}{p})\Gamma(\frac{1}{p})}{\Gamma(\frac{2}{p})}.$$

The generalized sin function is the inverse of $\arcsin_p(x)$ defined on $[0, \frac{\pi_p}{2}]$. Now we can define generalized cosine function as the derivative of generalized sine function,

$$cos_p(x) = \frac{d}{dx}sin_p(x).$$

It is clear that,

$$cos_p(x) = (1 - sin_p(x)^p)^{\frac{1}{p}}, x \in \left[0, \frac{\pi_p}{2}\right],$$

and

$$| \, sin_p(x) \, |^p + | \, cos_p(x) \, |^p = 1, x \in \mathbb{R}. \tag{1.1}$$

It is easy to prove that,

$$\frac{d}{dx}cos_p(x) = -cos_p(x)^{p-2}sin_p(x)^{p-1}, x \in \left[0, \frac{\pi_p}{2}\right].$$

The generalized tangent function is defined as,

$$tan_p(x) = \frac{sin_p(x)}{cos_p(x)}, x \in \mathbb{R} \setminus \{k\pi_p + \frac{\pi_p}{2} : k \in \mathbb{Z}\}.$$

It follows from the equation (1.1) that,

$$\frac{d}{dx}tan_p(x) = 1 + \mid tan_p(x) \mid^p, \quad x \in \left(\frac{-\pi_p}{2}, \frac{\pi_p}{2}\right).$$

Now we can define generalized inverse hyperbolic function as,

$$arcsinh_p(x) = \begin{cases} \int_0^x \frac{1}{(1+t^p)^{\frac{1}{p}}}dt & , x \in [0, \infty) \\ -arcsinh_p(-x) & , x \in (-\infty, 0]. \end{cases}$$

The inverse of $arcsinh_p(x)$ is called as the generalized hyperbolic sine function and it is denoted by $sinh_p(x)$. The generalized hyperbolic cosine function is defined as,

$$cosh_p(x) = \frac{d}{dx}sinh_p(x).$$

These definitions show that,

$$cosh_p(x)^p - \mid sinh_p(x) \mid^p = 1, \quad x \in \mathbb{R}$$

and

$$\frac{d}{dx}cosh_p(x) = cosh_p(x)^{2-p}sinh_p(x)^{p-1}, \quad x \geq 0.$$

The generalized Hyperbolic tangent function is defined as,

$$tanh_p(x) = \frac{sinh_p(x)}{cosh_p(x)},$$

and

$$\frac{d}{dx}tanh_p(x) = 1 - \mid tanh_p(x) \mid^p .$$

It is clear that all these generalized functions coincide with the classical ones when $p = 2$ [7].
In recent years, the following two sided inequality for hyperbolic functions has attracted attention of several researchers,

$$(cosh(x))^{\frac{1}{3}} < \frac{sinh(x)}{x} < \frac{cosh(x)+2}{3}, x \neq 0. \tag{1.2}$$

We define same inequality in generalized hyperbolic form as follows,

$$(cosh_p(x))^{\frac{1}{3}} < \frac{sinh_p(x)}{x} < \frac{cosh_p(x)+2}{3}, x \neq 0. \tag{1.3}$$

The left inequality in (1.2) and (1.3) has been obtained by Lazarevic [23]. The counterpart of (1.2) and (1.3) for trigonometric functions are defined as,

$$(cos(\alpha))^{\frac{1}{3}} < \frac{sin(\alpha)}{\alpha} < \frac{cos(\alpha)+2}{3}, 0 < \alpha < \frac{\pi}{2}, \tag{1.4}$$

and

$$(cos_p(\alpha))^{\frac{1}{3}} < \frac{sin_p(\alpha)}{\alpha} < \frac{cos_p(\alpha)+2}{3}, 0 < \alpha < \frac{\pi_p}{2}. \tag{1.5}$$

Generalization of inequalities (1.2) and (1.3) to Jacobian elliptic functions are established in[12].
The left inequalities in (1.4) and (1.5) have been proved by Neumann [20] and Mitrinovic [10], while second inequality is due to Cusa Huygens [5]. Inequalities mentioned in (1.2), (1.3), (1.4) and (1.5) also have been obtained in [2, 15, 19]. For the recent research work in theory of inequalities for hyperbolic and trigonometric functions refer [6, 8, 9, 11, 13, 17, 18]. The goal of this paper is to derive inequalities involving hyperbolic and trigonometric functions. Most of them are the two-sided inequalities which are similar to inequalities (1.2), (1.3), (1.4) and (1.5). In section 2 we recall definition and basic properties of the Schwab-Borchardt mean. Definions of four particular bivariate means, which can be regarded as special case of the Schwab-Borchardt mean, are also included in this section. The main result is derived in section 3.

2 Definitions and Preliminaries

The geometric, arithmetic and the root mean square means of $a > 0$, $b > 0$ will be denoted by G, A and R respectively and they are defined as follows,

$$G = G(u, v) = \sqrt{uv}, A = A(u, v) = \frac{u + v}{2}, R = R(u, v) = \sqrt{\frac{u^2 + v^2}{2}}. \tag{2.1}$$

Other bivariate means used in the subsequent sections include the logarithmic mean which is defined as,

$$L = \left(\frac{a}{tanh_p^{-1}(a)}\right) A. \tag{2.2}$$

The first and second Seiffert means are,

$$P = \left(\frac{a}{sinh_p^{-1}(a)}\right) A. \tag{2.3}$$

and

$$T = \left(\frac{a}{tan_p^{-1}(a)}\right) A. \tag{2.4}$$

For this, one can refer to [8, 11, 18, 20]. Here,

$$a = \frac{u - v}{u + v}, u \neq v \tag{2.5}$$

Another mean introduced in [14], which is defined as follows,

$$M = \left(\frac{a}{sinh_p^{-1}(a)}\right) A. \tag{2.6}$$

It is known that, $G < L < P < A < M < T < R$[14].
All the bivariate means mentioned above are strict homogeneous of degree one and they are strictly increasing in each of it's variables. Let the letter W stand for one of these means. The homogeneity of W implies that,

$$W(u, v) = \sqrt{uv} W(e^x, e^{-x}), \text{ where, } x = \frac{1}{2} \ln\left(\frac{u}{v}\right).$$

It means L, P, T and M are special cases of Schwab-Borchardt mean for $u \geq 0, v > 0$.
This mean will be denoted by $SB(u, v) = SB$. The Schwab-Borchardt mean is the iterative mean. i.e.

$$SB = \lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n,$$

where,

$$u_0 = u, v_0 = v, u_{n+1} = \frac{u_n + v_n}{2}, v_{n+1} = \sqrt{u_{n+1} v_n}, n = 0, 1, 2, . \tag{2.7}$$

Due to[1, 3], it is known that,

$$SB(u, v) = \begin{cases} \frac{\sqrt{v^2 - u^2}}{cos_p^{-1}(\frac{u}{v})}, & \text{if } u < v, \\ \frac{\sqrt{u^2 - v^2}}{cosh_p^{-1}(\frac{u}{v})}, & \text{if } v < u. \end{cases}$$

The mean SB is non symmetric, homogeneous of degree one and strictly increasing in it's variables. It has been shown in [14] that,

$$L = SB(A, B), P = SB(G, A), T = SB(A, R), M = SB(R, A). \tag{2.8}$$

The following two sided inequality is known [14],

$$(uv^2)^{\frac{1}{3}} < SB(u, v) < \frac{u + 2v}{3}. \tag{2.9}$$

using the in variance property, an invariant is a property of a mathematical object which remains unchanged after operations or transformations of a certain type are applied to the objects, it implies that,
$SB(u_n, v_n) = SB(u, v)$ see equation (2.6).
The previous inequality can be generalised as,

$$(u_n v_n^2)^{\frac{1}{3}} < SB(u, v) < \frac{u_n + 2v_n}{3}. \tag{2.10}$$

The sequence of left inequality is strictly increasing while sequence of right inequality is strictly decreasing provided $u \neq v$ [13, 15].

3 Main Result

Theorem 3.1. *Let $x \neq 0$, then,*

$$(cosh_p(x))^{\frac{2}{3}} < \frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))} < \frac{1+2cosh_p(x)}{3}, \tag{3.1}$$

$$((cosh_p(2x))^{\frac{1}{2}}cosh_p^2(x))^{\frac{1}{3}} < \frac{sinh_p(x)}{sinh_p^{-1}(tanh_p(x))} < \frac{(cosh_p(2x))^{\frac{1}{2}}+2cosh_p(x)}{3}, \tag{3.2}$$

and

$$((cosh_p(2x))cosh_p(x))^{\frac{1}{3}} < \frac{sinh_p(x)}{tan_p^{-1}(tanh_p(x))} < \frac{2(cosh_p(2x))^{\frac{1}{2}}+2cosh_p(x)}{3}. \tag{3.3}$$

Proof. For, $(u,v)=(e^x,e^{-x})$, we have,

$$G=1, A=cosh_p(x), R=(cosh_p(2x))^{\frac{1}{2}}$$

and using equation(2.6), we get, $a=tanh_p(x)$.
Moreover using equation (2.8), (2.9) and (2.10), we obtain

$$P=\frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))}, M=\frac{sinh_p(x)}{sinh_p^{-1}(tanh_p(x))}, T=\frac{sinh_p(x)}{tan_p^{-1}(tanh_p(x))}. \tag{3.4}$$

For the proof of (3.1), we use (2.9) with $u=G$ and $v=A$ followed by the application of the second part of (2.8) and first formula of (3.4),we get

$$(GA^2)^{\frac{1}{3}} < SB(G,A) < \frac{G+2A}{3} \implies (GA^2)^{\frac{1}{3}} < P < \frac{G+2A}{3}, \tag{3.5}$$

$$\implies (cosh_p(x)^2)^{\frac{1}{3}} < \frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))} < \frac{1+2cosh_p(x)}{3} \tag{3.6}$$

$$\implies (cosh_p(x))^{\frac{2}{3}} < \frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))} < \frac{1+2cosh_p(x)}{3}. \tag{3.7}$$

Hence (3.1) is proved.
For the proof of (3.2) we use (2.9) with $u=R$ and $v=A$, followed by application of the fourth formula of equation (2.8),we get

$$(RA^2)^{\frac{1}{3}} < SB(R,A) < \frac{R+2A}{3} \implies ((cosh_p(2x))^{\frac{1}{2}}cosh_p^2(x))^{\frac{1}{3}} < M < \frac{(cosh_p(2x))^{\frac{1}{2}}+2cosh_p(x)}{3}, \tag{3.8}$$

$$\implies ((cosh_p(2x))^{\frac{1}{2}}cosh_p^2(x))^{\frac{1}{3}} < \frac{sinh_p(x)}{sinh_p^{-1}(tanh_p(x))} < \frac{(cosh_p(2x))^{\frac{1}{2}}+2cosh_p(x)}{3}. \tag{3.9}$$

Hence (3.2) proved. For the proof of (3.3) we use (2.9) with $u=A$ and $v=R$, we have

$$(AR^2)^{\frac{1}{3}} < SB(A,R) < \frac{A+2R}{3}, \implies (AR^2)^{\frac{1}{3}} < T < \frac{A+2R}{3},$$

$$\implies (cosh_p(x)(cosh_p(2x)^{\frac{1}{2}})^2)^{\frac{1}{3}} < \frac{sinh_p(x)}{tan_p^{-1}(tanh_p(x))} < \frac{2(cosh_p(2x))^{\frac{1}{2}}+2cosh_p(x)}{3}$$

$$\implies ((cosh_p(2x))cosh_p(x))^{\frac{1}{3}} < \frac{sinh_p(x)}{tan_p^{-1}(tanh_p(x))} < \frac{2(cosh_p(2x))^{\frac{1}{2}}+2cosh_p(x)}{3}.$$

Hence (3.3) is proved. □

Inequality (1.2) is established using the method utilized in the proof of theorem 3.1. Let, $u=A, v=G$, using (2.1) and (2.7), we obtain

Theorem 3.2. *Let $u = A, v = G$, using (2.1) and (2.7), we obtain*

$$SB(A,G) = L = \frac{A}{tanh_p^{-1}(A)}A = \frac{sinh_p(x)}{x}.$$

Equation (2.9) gives the required result as follows,

$$(AG^2)^{\frac{1}{3}} < SB(A,G) < \frac{A+2G}{3}.$$

Let, $G = 1$ and $A = cosh_p(x)$,

$$\implies (cosh_p(x))^{\frac{1}{3}} < \frac{sinh_p(x)}{x} < \frac{cosh_p(x)+2}{3}.$$

Let us define new variable α as,

$$tanh_p(x) = sin\alpha. \tag{3.10}$$

This implies that,

$$sinh_p(x) = tan\alpha, cosh_p(x) = sec\alpha, x = tanh_p^{-1}(sin\alpha). \tag{3.11}$$

Equation (1.5) is verified using (3.10) and (3.11). For $x \neq 0$,

$$1 < \left(\frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))}\right)\left(\frac{tanh_p(x)}{x}\right) \tag{3.12}$$

and for $0 < \alpha < \frac{\pi_p}{2}$,

$$1 < \left(\frac{sin_p(\alpha)}{tanh_p^{-1}(sin_p(\alpha))}\right)\left(\frac{tan_p(\alpha)}{\alpha}\right). \tag{3.13}$$

Proof. Using left inequality of equation (2.9), let $u = A$ and $v = G$, we have

$$(AG^2)^{\frac{1}{3}} < SB(A,G) \implies (AG^2)^{\frac{1}{3}} < L. \tag{3.14}$$

Similarly, let $u = G$ and $v = A$ in (2.9), we obtain

$$(GA^2)^{\frac{1}{3}} < SB(G,A) \implies (GA^2)^{\frac{1}{3}} < P. \tag{3.15}$$

Multiplying (3.14) and (3.15), we get

$$(AG^2)^{\frac{1}{3}}(GA^2)^{\frac{1}{3}} < PL \implies AG < PL. \tag{3.16}$$

Let, $(u,v) = (e^x, e^{-x})$ and

$$A = cosh_p(x), G = 1, P = \frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))}, L = \frac{sinh_p(x)}{x}.$$

Therefore (3.16) gives,

$$cosh_p(x) < \left(\frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))}\right)\left(\frac{sinh_p(x)}{x}\right) \implies 1 < \left(\frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))}\right)\left(\frac{tanh_p(x)}{x}\right).$$

Hence (3.12) proved. Inequality (3.13) follows from (3.12) by using transformations (3.4) and (3.10). □

4 Conclusion

Using Schwab-Borchardt mean, both the refinement of generalized trigonometric and hyperbolic function is verified.

Conflict of Interest

Authors declare that, there is no conflict of interest.

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RESULTS ON UNIQUENESS OF L-FUNCTIONS CONCERNING WEIGHTED SHARING

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Abstract

In this paper using the notion of weighted sharing,we consider the value distribution of a L- function and meromorphic function when certain type of difference-differential polynomials which share a small function and rational function and obtain some uniqueness results which extends recent results due to Hao and Chen[2].

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1 Introduction and preliminaries

In this paper, $\mathbb{C}$ denotes the complex plane and $\mathbb{N}$ denotes the set of natural numbers. Now, towards the end of twentieth century, a new class of Dirichlet series called the Selberg class was introduced by Atle Selberg[15]. The concept of L- function where L means Selberg class function with the Riemann Zeta function is the most speculative open world problem in today's pure mathematics. L- functions can be analytically continued as meromorphic functions in $\mathbb{C}$. A meromorphic function L is said to be an L-function in the Selberg class if it satisfies the following properties.

(i) $L(z)$ can be expressed as a Dirichlet series $L(z) = \sum_{m=1}^{\infty} a(m)/m^z$.
(ii) $|a(m)| = O(m^{\epsilon})$, for any $\epsilon > 0$.
(iii) There exists a non negetive integer n such that $(z-1)^n L(z)$ becomes an entire function of finite order.
(iv) Every L- function satisfies the functional equation

$$\lambda_L(z) = \omega\overline{\lambda_L(1-\overline{z})},$$

where

$$\lambda_L(z) = L(z)A^z \prod_{j=1}^{n} \Gamma(\eta_j z + v_j),$$

with positive real numbers A, η_j and complex numbers v_j, ω with $Re(v_j) \geq 0$ and $|\omega| = 1$.
(v) $L(z)$ satisfies $L(z) = \prod_p L_p(z)$, where $L_p(z) = \exp(\sum_{n=1}^{\infty} b(p^n)/p^{nz})$ with $b(p^n) = O(p^{n\theta})$ for some $\theta < 1/2$ and p denotes prime number.

If L satisfies (i)-(iv) then we say that L is an L-function in the extended Selberg class. In this paper, by an L-function we always mean an L-function in the extended Selberg class with $a(1) = 1$. Here we use the standard notations and definitions of the value distribution theory [3].

The Nevanlinna value distribution theory is an important area of research which has seen extensive work. It primarily focuses on the analysis of the distribution of solutions to the equation $f(z) = a$, where f is an entire or meromorphic function in $\mathbb{C}$. Let $\alpha \in \mathbb{C} \cup \{\infty\}$ and f, g be meromorphic functions in the complex plane. The set of all α- points of f with multiplicities not exceeding l is denoted by $E_{l)}(\alpha, f)(\overline{E}_{l)}(\alpha, f))$, where l is a positive integer and we consider(ignore) the multiplicities of the α- points. The hyper order $\rho_2(f)$ of f is defined by

$$\rho_2(f) = \overline{lim}_{r\to\infty} \frac{loglogT(r,f)}{logr}.$$

We denote $S(r, f)$ by any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$, outside a possible exceptional set of finite linear measure. We say that f and g share α CM if they have the set of α- points with the same multiplicities and if we do not consider the multiplicity then we say that f and g share α IM.

In general, for a meromorphic function $f(z)$, the quantity $m(r, f)$ denotes the proximity function of $f(z)$, while $N(r, f)$ denotes the counting function of poles of $f(z)$ whose multiplicities are taken into account(respectively $\overline{N}(r, f)$ denotes the reduced counting function when multiplicities are ignored). The Nevanlinna characteristic function of a meromorphic function f plays a very important role in the value distribution theory and it is denoted by $T(r, f)$. We have $T(r, f) = m(r, f) + N(r, f)$, which clearly shows that $T(r, f)$ is non-negative.

We can see a lot of work on uniqueness results with the help of Nevanlinna Theory. Recently people have raised great interest in difference analogues of Nevanlinna's theory and obtained many profound results. A number of papers have focused on value distribution and uniqueness of difference polynomials, which are analogues of Nevanlinna theory (see [11], [17]).

The value distribution of an L-function concerned with the distribution of the zeros of L and more generally, with the roots of the equation $L(s) = c$ for some $c \in \mathbb{C} \cup \{\infty\}$. Since L-functions are analytically continued as meromorphic functions, it is possible to study the value distribution and uniqueness outcomes between the L- functions and any arbitrary meromorphic functions (see [12], [13]).

We state the following standard definitions of Nevanlinna theory and it is important to note that all the definitions discussed also applies to the L-function.

In addition we need the following definitions.

Definition 1.1 ([6]). *Let f be a meromorphic function defined in the complex plane. Let n be a positive integer and $\alpha \in \mathbb{C} \cup \{\infty\}$. By $N(r, \alpha; f \mid\leq n)$ we denote the counting function of the α- points of f with multiplicity less than or equal to n and by $\overline{N}(r, \alpha; f \mid\leq n)$ the reduced counting function. Also by $N(r, \alpha; f \mid\geq n)$ we denote the counting function of the α- points of f with multiplicity greater than or equal to n and by $\overline{N}(r, \alpha; f \mid\geq n)$ the reduced counting function. We define*

$$N_n(r, \alpha; f) = \overline{N}(r, \alpha; f) + \overline{N}(r, \alpha; f \mid\geq 2) + ... + \overline{N}(r, \alpha; f \mid\geq n).$$

Definition 1.2 ([6]). *Let f be a meromorphic function defined in $\mathbb{C}$ and $p(z)$ be a small function of f or a rational function. Then we denote the notations by $N(r, p; f \mid\leq m), \overline{N}(r, p; f \mid\leq m), N(r, p; f \mid\geq m), \overline{N}(r, p; f \mid\geq m), N_m(r, p; f)$ etc, the counting functions $N(r, 0; f-p \mid\leq m), \overline{N}(r, 0; f-p \mid\leq m), N(r, 0; f-p \mid\geq m), \overline{N}(r, 0; f-p \mid\geq m), N_m(r, 0; f-p)$ respectively.*

Definition 1.3 ([5]). *Let f and g be two meromorphic functions defined in the complex plane and n be an integer (≥ 0) or infinity. We denote by $E_n(\alpha; f)$ the set of all zeros of $f - \alpha$ and $\alpha \in \mathbb{C} \cup \{\infty\}$and a zero of multiplicity k is counted k times if $k \leq n$ and $n+1$ times if $k > n$, we say that f and g share α with weight n if $E_n(\alpha; f) = E_n(\alpha; g)$. We say that f and g share (α, n) to mean that f, g share α with weight n. Clearly f, g share α IM or CM if and only if f and g share $(\alpha, 0)$ or (α, ∞) respectively.*

Definition 1.4 ([9]). *Let f be a meromorphic function defined in the complex plane and $p(z)$ be a rational function or a small function of f. Then we denote by $E_{m)}(p; f), \overline{E}_{m)}(p; f)$ and $E_m(p; f)$ the sets $E_{m)}(r, 0; f-p), \overline{E}_{m)}(r, 0; f-p)$ and $E_m(r, 0; f-p)$ respectively. We write f,g share (p, n) to mean that $f-p$ and $g-p$ share the value 0 with weight n. Clearly, if f,g share (p, n) then f,g share (p, m) for all integers m , $0 \leq m < n$. Also we note that f,g share p IM or CM if and only if f,g share $(p, 0)$ or (p, ∞) respectively.*

Definition 1.5 ([4]). *Let f and g be two non-constant meromorphic functions share a value α IM. Denote by $\overline{N}_*(r, \alpha; f, g)$ the counting function of the α-points of f and g with different multiplicities, where each α-point is counted only once.*

Definition 1.6 ([10]). *Let f and g be two non-constant meromorphic functions share a value α IM. Denote by $\overline{N}(r, \alpha; f \mid> g)$ the counting function of the α- points of f and g with multiplicities with respect to f is greater than the multiplicities with respect to g, where each α- points is counted only once.*

Definition 1.7 ([10]). *Let f and g be two non-constant meromorphic functions share a value α IM. We denote by $\overline{N}_E(r, \alpha; f, g \mid> m)$ the counting function of the α-points of f and g with multiplicities greater than m and the multiplicities with respect to f is equal to the multiplicities with respect to g, where each α-points is counted only once.*

In 2017, Liu, Li and Yi [8] proved the following uniqueness theorems.

Theorem A ([8]) *Let $j \geq 1$ and $k \geq 1$ be integers such that $j > 3k + 6$. Also let L be an L-function and f be a non-constant meromorphic function. If $\{f^j\}^{(k)}$ and $\{L^j\}^{(k)}$ share $(1, \infty)$, then $f \equiv \alpha L$ for some non-constant α satisfying $\alpha^j = 1$.*

Theorem B ([8]). *Let $j \geq 1$ and $k \geq 1$ be integers such that $j > 3k + 6$. Also let L be an L-function and f be a non-constant meromorphic function. If $\{f^j\}^{(k)}$ and $\{L^j\}^{(k)}$ share (z, ∞), then $f \equiv \alpha L$ for some non-constant α satisfying $\alpha^j = 1$.*

In 2018, Hao and Chen $\big([2]\big)$ obtained the following uniqueness results on L -function.

Theorem C ([2]). *Let f be a non-constant meromorphic function and L be an L-function such that $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ share $(1, \infty)$ where $n, m, k \in Z^+$. If $n > m + 3k + 6$ and $k \geq 2$, then $f \equiv L$ or $f^n(f-1)^m \equiv L^n(L-1)^m$.*

Theorem D ([2]). *Let f be a non-constant meromorphic function and L be an L-function such that $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ share $(1, 0)$ where $n, m, k \in Z^+$. If $n > 4m + 7k + 11$ and $k \geq 2$, then $f \equiv L$ or $f^n(f-1)^m \equiv L^n(L-1)^m$.*

Now it will be interesting to study the above Theorems A, B, C and D by considering more general form of difference-differential polynomial. The main motivation of this paper is the fact that the L- function where $L-$ function has only one possible pole at $s = 1$ in $\mathbb{C}$.

Question 1.1. Can we consider rational or small function sharing in Theorem C and Theorem D?

Question 1.2. Can we take difference-differential polynomial of the form $[f^n(f-1)^m f(z+c)]^{(k)}$ and $[L^n(L-1)^m L(z+c)]^{(k)}$ in Theorem C and Theorem D?

In this paper, we try to find the possible answer of the above questions. The following are the main results of this paper.

2 Main Results

Theorem 2.1. *Let f be a transcendental meromorphic function and L be an L- function, n, k, m be positive integers. If $[f^n(f-1)^m f(z+c)]^{(k)}$ and $[L^n(L-1)^m L(z+c)]^{(k)}$ share $(\alpha(z), l)$ and f, L share $(\infty, 0)$, where $\alpha(z)$ is a small function of f and L then*

(1) $l = 0$ *and* $(n + m) > (5k + 7)(m + 2) + 1$,

(2) $l = 1$ *and* $(n + m) > \frac{1}{2}(5k + 9)(m + 2) + 1$,

(3) $l \geq 2$ *and* $(n + m) > (2k + 4)(m + 2) + 1$.

Then one of the following holds:

(i) $[f^n(f-1)^m f(z+c)]^{(k)} \equiv [L^n(L-1)^m L(z+c)]^{(k)}$,

(ii) $[f^n(f-1)^m f(z+c)]^{(k)}[L^n(L-1)^m L(z+c)]^{(k)} \equiv [\alpha(z)]^2$.

Theorem 2.2. *Let f be a transcendental meromorphic function and L be an L- function, n, k, m be positive integers. If $[f^n(f-1)^m f(z+c)]^{(k)}$ and $[L^n(L-1)^m L(z+c)]^{(k)}$ share $(R(z), l)$ and f, L share $(\infty, 0)$, where $R(z)$ is a rational function of f and L then*

(1) $l = 0$ *and* $(n + m) > (5k + 7)(m + 2) + 1$,

(2) $l = 1$ *and* $(n + m) > \frac{1}{2}(5k + 9)(m + 2) + 1$,

(3) $l \geq 2$ *and* $(n + m) > (2k + 4)(m + 2) + 1$.

Then one of the following holds:

(i) $[f^n(f-1)^m f(z+c)]^{(k)} \equiv [L^n(L-1)^m L(z+c)]^{(k)}$,

(ii) $[f^n(f-1)^m f(z+c)]^{(k)}[L^n(L-1)^m L(z+c)]^{(k)} \equiv (R(z))^2$.

Example 2.1. Let us consider $L = \zeta$ and $f = -\zeta$, where ζ is Riemann zeta function which has a simple pole. By hypothesis of the theorem $F = [f^n(f-1)^m f(z+c)]^{(k)}$ and $L = [L^n(L-1)^m L(z+c)]^{(k)}$ share $(\alpha(z), l)$ and the conditions are satisfied for different weights $l = 0, l = 1$ and $l \geq 2$.

Remark 2.1. *Theorem 2.1 and Theorem 2.2 are the extension of Theorems A-D respectively.*

3 Auxiliary Lemmas

In this section, we present some necessary Lemmas.

Denote H by the following function.

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big).$$

Lemma 3.1 ([16]). *Let L be an L- function with degree q. Then*

$$T(r,L)=\frac{q}{\pi}rlogr+O(1).$$

Lemma 3.2 ([9]). *Let L be an L-function. Then*

$$N(r,\infty .L)=S(r,L)=O(logr).$$

Lemma 3.3 ([10]). *Let f be a non-constant meromorphic function and L be an L-function. If f and L share $(\infty,0)$ then*

$$\overline{N}(r,\infty;f)=\overline{N}(r,\infty;L)=S(r,L)=O(logr).$$

Lemma 3.4 ([21]). *Let $f(z)=\frac{\alpha_0+\alpha_1 z+....+\alpha_n z^n}{\beta_0+\beta_1 z+....+\beta_m z^m}$ be a non-constant rational function defined in the complex plane $\mathbb{C}$, where $\alpha_0,\alpha_1,....,\alpha_n(\neq 0)$ and $\beta_0,\beta_1,....,\beta_m(\neq 0)$ are complex constants. Then*

$$T(r,f)=max(m,n)logr+O(1).$$

Lemma 3.5 ([18]). *Let f be a transcendental meromorphic function of hyper order $\rho_2(f)<1$. Then for any $\alpha\in\mathbb{C}-\{0\}$.*

$$\begin{aligned}T(r,f(z+\alpha))&=T(r,f)+S(r,f),\\ N(r,\infty;f(z+\alpha))&=N(r,\infty;f)+S(r,f),\\ N(r,0;f(z+\alpha))&=N(r,0;f)+S(r,f).\end{aligned}$$

Lemma 3.6 ([14]). *Let F and G be two non-constant meromorphic functions sharing $(1,1)$ and $(\infty,0)$. If $H\not\equiv 0$, then*

$$\begin{aligned}T(r,F)\le{}& N_2(r,0;F)+N_2(r,0;G)+\frac{3}{2}\overline{N}(r,F)+\overline{N}(r,G)\\ &+\overline{N}_*(r,\infty;F,G)+\frac{1}{2}\overline{N}(r,0;F)+S(r,F)+S(r,G).\\ T(r,G)\le{}& N_2(r,0;F)+N_2(r,0;G)+\frac{3}{2}\overline{N}(r,G)+\overline{N}(r,F)\\ &+\overline{N}_*(r,\infty;F,G)+\frac{1}{2}\overline{N}(r,0;G)+S(r,F)+S(r,G).\end{aligned}$$

Lemma 3.7 ([14]). *Let F and G be two non-constant meromorphic functions sharing $(1,0)$ and $(\infty,0)$. If $H\not\equiv 0$, then*

$$\begin{aligned}T(r,F)\le{}& N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,F)+2\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G)+2\overline{N}(r,0;F)\\ &+\overline{N}(r,0;G)+S(r,F)+S(r,G).\\ T(r,G)\le{}& N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,G)+2\overline{N}(r,F)+\overline{N}_*(r,\infty;F,G)+2\overline{N}(r,0;G)\\ &+\overline{N}(r,0;F)+S(r,F)+S(r,G).\end{aligned}$$

Lemma 3.8 ([1]). *Let F and G be two non-constant meromorphic functions sharing $(1,l)$ and $(\infty,0)$ where $2\le l<\infty$ and $H\not\equiv 0$ then*

$$\begin{aligned}T(r,F)\le{}& N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G)-m(r,1,G)\\ &-N_E(r,1;F|>3)-\overline{N}(r,1;G>F)+S(r,F)+S(r,G).\\ T(r,G)\le{}& N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G)-m(r,1,F)\\ &-N_E(r,1;G|>3)-\overline{N}(r,1;F>G)+S(r,F)+S(r,G).\end{aligned}$$

Lemma 3.9 ([20]). *Let F be a non-constant meromorphic function and k,p be two positive integers, then*

$$\begin{aligned}T(r,F^{(k)})&\le T(r,F)+k\overline{N}(r,\infty;F)+S(r,F),\\ N_p(r,0;F^{(k)})&\le T(r,F^{(k)})-T(r,F)+N_{p+k}(r,0;F)+S(r,F),\\ N_p(r,0;F^{(k)})&\le N_{p+k}(r,0;F)+k\overline{N}(r,\infty;F)+S(r,F),\\ N(r,0;F^{(k)})&\le N(r,0;F)+k\overline{N}(r,\infty;F)+S(r,F).\end{aligned}$$

Lemma 3.10 ([20]). *Let f be a non-constant meromorphic function, define then polynomial $P(f) = a_0 + a_1 f + ... + a_n f^n$, where $a_0, ...a_n$ are complex constants and $a_n \neq 0$, then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 3.11 ([3]). *Let $f(z)$ be a meromorphic function and $a \in \mathbb{C}$. Then*

$$T\left(r, \frac{1}{f}\right) = T(r, f) + O(1).$$

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1).$$

Lemma 3.12. *Let f be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ and L be a $L-$function with $\rho_2(L) < 1$. Let $F_1 = [f^n(f-1)^m f(z+c)]$. where n, m are positive integers and c is a complex constant. Then*

$$(n + m - 1)T(r, f) \leq T(r, F_1) + S(r, f)$$

Proof. Since f is a meromorphic function, from Lemmas 3.5, 3.10, 3.11 we have

$$\begin{aligned}(n + m + 1)T(r, f) &= T(r, f^{n+m+1}) + S(r, f)\\ &\leq T(r, f^n(f-1)^m f) + S(r, f)\\ &\leq T\left(r, \frac{F_1 f}{f(z+c)}\right) + S(r, f)\\ &\leq T(r, F_1) + T\left(r, \frac{f(z+c)}{f}\right) + S(r, f)\\ &\leq T(r, F_1) + m\left(r, \frac{f(z+c)}{f}\right) + N\left(r, \frac{f(z+c)}{f}\right) + S(r, f)\\ (n + m - 1)T(r, f) &\leq T(r, F_1) + S(r, f).\end{aligned}$$

□

4 Proof of the Main Results

Proof of Theorem 2.1. Let $F = \frac{F_1^{(k)}}{\alpha(z)}$ and $G = \frac{G_1^{(k)}}{\alpha(z)}$ where $F_1 = f^n(f-1)^m f(z+c)$ and $L_1 = L^n(L-1)^m L(z+c)$ respectively. Then F and G share $(1, l)$ and share $(\infty, 0)$ except for zeros and poles of $\alpha(z)$. Clearly by Lemma 3.1, L is a transcendental meromorphic function. We have by Lemmas 3.9 and 3.12

$$\begin{aligned}N_2(r, 0; F) &\leq N_2(r, 0; F_1^{(k)}) + S(r, f)\\ &\leq T(r, F_1^{(k)}) - T(r, F_1) + N_{k+2}(r, 0; F_1) + S(r, f)\\ &\leq T\left(r, \frac{F_1^{(k)}}{\alpha(z)}\right) - (n + m - 1)T(r, f) + N_{k+2}(r, 0; F_1) + S(r, f).\end{aligned} \tag{4.1}$$

Hence from inequality (4.1), we get

$$(n + m - 1)T(r, f) \leq T(r, F) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f). \tag{4.2}$$

Similarly,

$$(n + m - 1)T(r, L) \leq T(r, G) - N_2(r, 0; G) + N_{k+2}(r, 0; L_1) + S(r, f). \tag{4.3}$$

Now we have to consider the following two cases.

Case 4.1. Let $H \not\equiv 0$. In this case we have to consider the following three subcases.

Subcase 4.1.1. Let $l = 0$. Hence by Lemmas 3.2, 3.3 and 3.7 and inequality (4.2) we have

$$\begin{aligned}(n + m - 1)T(r, f) &\leq N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; G) + 3\overline{N}(r, \infty; F) + \overline{N}_*(r, \infty; F, G)\\ &\quad + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, L)\\ &\leq N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) - N_2(r, 0; F)\\ &\qquad + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, L).\\ &\leq N_2(r, 0; F_1^{(k)}) + N_2(r, 0; L_1^{(k)}) + 2\overline{N}(r, 0; F_1^{(k)}) + \overline{N}(r, 0; L_1^{(k)}) - N_2(r, 0; F_1^{(k)})\end{aligned}$$

$$+ N_{k+2}(r,0;F_1) + S(r,f) + S(r,L).$$
$$\le N_2(r,0;L_1^{(k)}) + 2\overline{N}(r,0;F_1^{(k)}) + \overline{N}(r,0;L_1^{(k)}) + N_{k+2}(r,0;F_1)$$
$$+ S(r,f) + S(r,L).$$
$$\le N_{k+2}(r,0;L_1) + 2N_{k+1}(r,0;F_1) + N_{k+1}(r,0;L_1) + N_{k+2}(r,0;F_1)$$
$$+ S(r,f) + S(r,L).$$
$$\le (3+2k)(m+2)T(r,L) + (3k+4)(m+2)T(r,f) + S(r,f) + S(r,L).$$

Hence

$$(n+m-1)T(r,f) \le (3+2k)(m+2)T(r,L) + (3k+4)(m+2)T(r,f) + S(r,f) + S(r,L). \tag{4.4}$$

Similarly,

$$(n+m-1)T(r,L) \le (3+2k)(m+2)T(r,f) + (3k+4)(m+2)T(r,L) + S(r,f) + S(r,L). \tag{4.5}$$

From inequalities (4.4) and (4.5) we get

$$(n+m-1)[T(r,f)+T(r,L)] \le (7+5k)(m+2)[T(r,f)+T(r,L)] + S(r,f) + S(r,L). \tag{4.6}$$

which is a contradiction from (4.6) as $n+m > (7+5k)(m+2)+1$.

Subcase 4.1.2. Let $l=1$. Hence by Lemmas 3.2, 3.3 and 3.6 and inequality (4.2) we have

$$\begin{aligned}(n+m-1)T(r,f) \le\ & N_2(r,0;G) + \frac{3}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) + \frac{1}{2}\overline{N}(r,0;F)\\
& + N_{k+2}(r,0;F_1) + S(r,f) + S(r,L).\\
\le\ & N_2(r,0;L_1^{(k)}) + \frac{1}{2}N_{k+1}(r,0;F_1) + \overline{N}(r,0;L_1^{(k)}) + N_{k+2}(r,0;F_1)\\
& + S(r,f) + S(r,L).\\
\le\ & (k+2)(m+2)T(r,L) + \frac{1}{2}(3k+5)(m+2)T(r,f)\\
& + S(r,f) + S(r,L).\end{aligned}$$

Hence

$$(n+m-1)T(r,f) \le (k+2)(m+2)T(r,L) + \frac{1}{2}(3k+5)(m+2)T(r,f) + S(r,f) + S(r,L). \tag{4.7}$$

Similarly

$$(n+m-1)T(r,L) \le (k+2)(m+2)T(r,f) + \frac{1}{2}(3k+5)(m+2)T(r,L) + S(r,f) + S(r,L). \tag{4.8}$$

From inequalities (4.7) and (4.8) we arrive at a contradiction as $(n+m) > \frac{1}{2}(5k+9)(m+2)+1$.

Subcase 4.1.3. Let $2 \le l < 1$. Hence by Lemmas 3.2, 3.3 and 3.8 and inequality (4.2)

$$\begin{aligned}(n+m-1)T(r,L) \le\ & N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_*(r,\infty;F,G)\\
& - m(r,1,F) - N_E(r,1;G| > 3) - \overline{N}(r,1;F > G) + S(r,f) + S(r,L)\\
\le\ & N_2(r,0;F_1^{(k)}) + N_2(r,0;L_1^{(k)}) + S(r,f) + S(r,L)\\
\le\ & (k+2)(m+2)T(r,f) + (k+2)(m+2)T(r,L) + S(r,f) + S(r,L).\end{aligned}$$

$$(n+m-1)T(r,L) \le (k+2)(m+2)T(r,f) + (k+2)(m+2)T(r,L) + S(r,f) + S(r,L). \tag{4.9}$$

Similarly

$$(n+m-1)T(r,f) \le (k+2)(m+2)T(r,L) + (k+2)(m+2)T(r,f) + S(r,f) + S(r,L). \tag{4.10}$$

From inequalities (4.9) and (4.10) we arrive at a contradiction as $l \ge 2$ and $(n+m) > (2k+4)(m+2)+1$.

Case 4.2. Let $H \equiv 0$. Then

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \equiv 0.$$

Integrating both sides we get

$$F - 1 = \frac{G-1}{b - c(G-1)}, \tag{4.11}$$

where $b \ne 0$ and c are constants. Now we have to consider the following subcases.

Subcase 4.2.1. Let $c = 0$. Then from (4.11) we have

$$F - 1 = \frac{G-1}{b}. \tag{4.12}$$

If $b \neq 1$ then from (4.12)

$$\overline{N}(r,0;F) = \overline{N}(r,1-b;G). \tag{4.13}$$

By Lemmas 3.2 and 3.9, using Second Fundamental Theorem of Nevanlinna and from inequality (4.3) we have

$$\begin{aligned}(n+m-1)T(r,L) &\leq T(r,G) - N_2(r,0;G) + N_{k+2}(r,0;L_1) + S(r,L)\\ &\leq \overline{N}(r,0;G) + \overline{N}(r,1-b;G) + \overline{N}(r,\infty;G) - N_2(r,0;G)\\ &\quad + N_{k+2}(r,0;L_1) + S(r,L).\\ &\leq \overline{N}(r,0;G) + \overline{N}(r,0;F) - N_2(r,0;G) + N_{k+2}(r,0;L_1) + S(r,L)\\ &\leq \overline{N}(r,0;F_1^{(k)}) + \overline{N}(r,0;L_1^{(k)}) + N_{k+2}(r,0;L_1) + S(r,L)\\ &\leq N_{k+1}(r,0;F_1) + N_{k+1}(r,0;L_1) + N_{k+2}(r,0;L_1) + S(r,L)\\ &\leq (2k+3)(m+2)T(r,L) + (k+1)(m+2)T(r,f)\\ &\quad + S(r,f) + S(r,L).\end{aligned}$$

Hence

$$(n+m-1)T(r,L) \leq (2k+3)(m+2)T(r,L) + (k+1)(m+2)T(r,f) + S(r,f) + S(r,L). \tag{4.14}$$

Similarly

$$(n+m-1)T(r,f) \leq (2k+3)(m+2)T(r,f) + (k+1)(m+2)T(r,L) + S(r,f) + S(r,L). \tag{4.15}$$

From the inequalities (4.14) and (4.15) we arrive at a contradiction as $n + m > (3k+4)(m+2) + 1$.
Hence $b = 1$ and therefore we get from (4.12)

$$[f^n(f-1)^m f(z+c)]^{(k)} \equiv [L^n(L-1)^m L(z+c)]^{(k)}.$$

Subcase 4.2.2. Let $c \neq 0$ and $b = -c$.

If $c = 1$, then from (4.11) we have $FG \equiv 1$. Hence

$$[f^n(f-1)^m f(z+c)]^{(k)}[L^n(L-1)^m L(z+c)]^{(k)} = [\alpha(z)]^2.$$

If $c \neq 1$, then from (4.11) we have,

$$\frac{1}{F} = \frac{-cG}{(1-c)G-1}.$$

Hence $\overline{N}(r,0;F) = N(r,\frac{1}{1-c};G)$.

Now proceeding as in subcase (4.2.1), we arrive at a contradiction. If $c = 1$, then from (4.11) we have

$$F \equiv \frac{-b}{G-b-1}. \tag{4.16}$$

Hence by Lemma 3.3 we have from (4.16)

$$\overline{N}(r,b+1;G) = \overline{N}(r,F) = \overline{N}(r,f) + S(r,L) = S(r,L).$$

Now proceeding as in subcase (4.2.1), we arrive at a contradiction. If $c \neq 1$, then from (4.11) we have

$$F - \left(1 - \frac{1}{c}\right) \equiv \frac{-b}{c^2\left(G - \frac{b+c}{c}\right)}.$$

Therefore by Lemma 3.3 we have

$$\overline{N}\left(r,\frac{b+c}{c};G\right) = \overline{N}(r,F) = \overline{N}(r,f) + S(r,L) = S(r,L).$$

Hence proceeding as in subcase (4.2.1) we arrive at a contradiction.

This completes the proof of the Theorem 2.1.

Proof of Theorem 2.2. Since f and L are transcendental meromorphic function and $R(z)$ is a rational function therefore $R(z)$ is a small function of f and L. Thus, Theorem 2.2 can be proved in a similar way as Theorem 2.1.

5 Conclusion

We have investigate the value distribution of a L- function and an arbitrary meromorphic function using the concept of weighted sharing when certain type of difference-differential polynomials $f^n(f-1)^m f(z+c)$ and $L^n(L-1)^m L(z+c)$ share a small and rational function. L- functions can be analytically continued as meromorphic functions in $\mathbb{C}$ and it has only one possible pole at $s=1$ in $\mathbb{C}$ is the main concept of this paper. Our results extends earlier results due to Hao and Chen.

6 Open Questions

1. Can the condition for n in Theorem 2.1 and Theorem 2.2 be still reduced?
2. Can the difference polynomials in Theorems 2.1 - 2.2 be replaced by difference polynomials of the form $f^n P(f)\Delta_c f$ by using weakly weighted sharing and truncated weighted sharing?

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