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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

INTEGER SOLUTION ANALYSIS FOR A DIOPHANTINE EQUATION WITH EXPONENTIALS

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Abstract

The exponential Diophantine equation is one of the distinctive types of Diophantine equations where the variables are expressed as exponents. For these equations, considerable excellent research has already been done. In this study, we try to solve the equations $3^\lambda + 103^\mu = \xi^2$, $3^\lambda + 181^\mu = \xi^2$, $3^\lambda + 193^\mu = \xi^2$.
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1 Introduction

For Elementry Number Theory, we may refer to [4, 6, 11].
A Diophantine equation is a polynomial equation with two or more unknowns and integer coefficients, with the only interesting solutions being those with integer coefficients. A Diophantine equation is called an exponential Diophantine equation if it contains an additional variable or variables that occur as exponents. There has been done some interested research work on these equation so far [1, 2, 3, 7].

The Exponential Diophantine equation of the form $p^x + q^y = z^2$ where p and q are distinct primes and $x, y$and z are non-negative integers has been the focus of a lot of research in recent years [5,10,12].

Recently, Pandichelvi and Vanaja [8] studied generating diophantine triples relating to figurate numbers with Thought-Provoking property. Also, Pandichelvi and Umamaheshwari [9] studied perceiving solutions for an exponential Diophantine equation linking safe and Sophie Germain primes. In the present paper, we shall discuss Integer solution Analysis for a Diophantine equation with exponentials.

2 Preliminaries

Proposition 2.1 ([6])**.** (3,2,2,3) *is a unique solution* (a, b, λ, μ) *for the Diophantine equation* $a^\lambda - b^\mu = \xi^2$ *where* a, b, λ, μ *are integers such that min* $(a, b, \lambda, \mu) > 1$.

Lemma 2.1. (1,0,2) *is a unique solution for the Diophantine equation* $3^\lambda + 1 = \xi^2$, *where* λ *and* ξ *are the non-negative integers.*

Proof. Let $\lambda, \xi \in \mathbb{N} \cup 0$. If $\lambda = 0$, then $\xi^2 = 2$, is not an integer solution. So consider $\lambda \geq 1$. Then $\xi^2 = 3^\lambda + 1 \geq 4 \Rightarrow \xi \geq 2$.

Consider $\xi^2 - 3^\lambda = 1$ then by proposition 2.1, λ must be equal to 1.Hence $\xi^2 = 4 \Rightarrow \xi = 2$.Therefore $(1, 0, 2)$ is a unique solution for $3^\lambda + 1 = \xi^2$. □

Lemma 2.2. *The Diophantine equation* $1 + (103)^\mu = \xi^2$ *has no non-negative integer solution.*

Proof. If $\mu = 0$, we obtain an irrational solution. So consider $\mu \geq 0$, then $\xi^2 = 1 + 103^\mu \geq 104 \Rightarrow \xi \geq 11$. Therefore the equation $\xi^2 - 103^\mu = 1$ is solvable only if $\mu = 1$, by proposition 2.1. But when $\mu = 1, \xi^2 = 104$,which is not a square number. Therefore there is no non-negative integer solution for the Diophantine equation $1 + 103^\mu = \xi^2$. □

Lemma 2.3. *The Diophantine equation* $1 + 181^\mu = \xi^2$ *has no non-negative integer solution.*

Proof. If $\mu = 0$, we obtain an irrational solution. So consider $\mu \geq 0$, then $\xi^2 = 1 + 181^\mu \geq 182 \Rightarrow \xi \geq 14$. Therefore the equation $\xi^2 - 181^\mu = 1$ is solvable only if $\mu = 1$, by proposition 2.1.But when $\mu = 1, \xi^2 = 182$,which is not a square number. Therefore there is no non-negative integer solution for the Diophantine equation $1 + 181^\mu = \xi^2$. □

Lemma 2.4. *The Diophantine equation* $1 + 193^\mu = \xi^2$ *has no non-negative integer solution.*

Proof. If $\mu = 0$, we obtain an irrational solution. So consider $\mu \geq 0$, then $\xi^2 = 1 + 193^\mu \geq 194 \Rightarrow \xi \geq 14$. Therefore the equation $\xi^2 - 193^\mu = 1$ is solvable only if $\mu = 1$, by proposition 2.1.But when $\mu = 1, \xi^2 = 194$,which is not a square number. Therefore there is no non-negative integer solution for the Diophantine equation $1 + 193^\mu = \xi^2$. □

3 Main Results

Theorem 3.1. $(1, 0, 2)$ *is a unique solution for the Diophantine equation* $3^\lambda + 103^\mu = \xi^2$*, where* λ, μ *and* $\xi \in \mathbb{N} \cup 0$ *.*

Proof. Consider the integral value of μ into two cases as (i) μ is even and (ii) μ is odd. □

Case (i): Let μ be even. If $\mu = 0$, then by Lemma 2.1 $(1, 0, 2)$ is a solution. If $\mu = 2n, n \in \mathbb{N}$, the equation becomes $3^\lambda + 103^{2n} = \xi^2$.This can be written as $\xi^2 - 103^{2n} = 3^\lambda$.

$$\begin{aligned} &\Rightarrow (\xi + 103^n)(\xi - 103^n) = 3^{\alpha+\beta}, \text{ where } \alpha + \beta = \lambda \\ &\Rightarrow (\xi + 103^n) - (\xi - 103^n) = 3^\beta - 3^\alpha, \beta > \alpha \\ &\Rightarrow 2(103^n) = 3^\alpha(3^{\beta-\alpha} - 1)\ . \end{aligned}$$

$\alpha = 0$ is the only possible value. Therefore the equation becomes $2(103)^n = 3^\lambda - 1$. Adding -2 on both sides we obtain $-2 + 2(103^n) = 3^\lambda - 3$.This gives that $\lambda = 2$.
Thus $103^n = 4$, which is impossible.
Case (ii): Let μ be odd. Then $\mu = 2n + 1, n \in \mathbb{N} \cup 0$.Therefore the equation becomes $3^\lambda + 103^{2n+1} = \xi^2$.

$$\begin{aligned} &\Rightarrow 3^\lambda + 103(103)^{2n} = \xi^2. \\ &\Rightarrow 3^\lambda + 3(103)^{2n} = \xi^2 - 10^2(103)^{2n}. \\ &\Rightarrow 3(3^{(\lambda-1)} + (103)^{2n}) = (\xi + 10(103)^n))(\xi - 10(103)^n). \end{aligned}$$

We observe that ξ is even. i.e. the equation above can be written as

$$\begin{aligned} 3(3^{\lambda-1} + ((103)^{2n}) &= (2m + 10(103)^n))(2m - 10(103)^n)), \\ &= 4(m + 5(103)^n)(m - 5(103)^n). \end{aligned}$$

Now we have two possibilities: (i) $m = 3 + 5(103)^n$ (ii) $m = 3 - 5(103)^n$.
Subcase (i): $m = 3 + 5(103)^n$.
Then $3(3^{\lambda-1} + (103)^{2n})) = 4(3 + 5(103)^n + 5(103)^n)(3 + 5(103)^n - 5(103)^n)$.
$= 4(3)[3 + 10(103)^n)]$
$\Rightarrow 3^{\lambda-1} + 103^{2n} = 4(3 + 10(103)^n)$
$\Rightarrow 3^{\lambda-1} - 12 = 40(103^n) - 103^{2n}$
$\Rightarrow 3(3^{\lambda-2} - 4) = 103^n(40 - 103^n)$.
$n = 0$ is the only possible value. But $3^{\lambda-1} = 51$ is not solvable.
Subcase (ii): $m = 3 - 5(103)^n$.
Then $3(3^{\lambda-1} + 103^{2n}) = 4(3 - 5(103)^n + 5(103)^n)(3 - 5(103)^n - 5(103)^n)$
$= 4(3)[3 - 10(103)^n]$.
$\Rightarrow 3^{\lambda-1} + 103^{2n} = 4(3 - 10(103)^n)$
$\Rightarrow 3^{\lambda-1} - 12 = -40(103^n) - (103^{2n})$
$\Rightarrow 3(3^{\lambda-2} - 4) = -103^n(40 + (103^n))$.
$n = 0$ is the only possible value. But $3^{\lambda-1} = -29$ is not solvable.
Thus in any cases, we are never able to come up with a non-negative integral solution. Therefore $(1, 0, 2)$ is a unique solution for the Diophantine equation $3^\lambda + 103^\mu = \xi^2$.

Theorem 3.2.$(1,0,2)$ *is a unique solution for the Diophantine equation* $3^{\lambda}+181^{\mu}=\xi^{2}$*, where* λ,μ *and* $\xi\in\mathbb{N}\cup 0$.

Proof. Consider the integral value of μ into two cases as (i) μ is even and (ii) μ is odd.

Case (i): Let μ be even. If $\mu=0$, then by Lemma 2.2 $(1,0,2)$ is a solution. If $\mu=2n, n\in\mathbb{N}$, the equation becomes $3^{\lambda}+181^{2n}=\xi^{2}$.This can be written as $\xi^{2}-181^{2n}=3^{\lambda}$.

$$\Rightarrow (\xi+181^{n})(\xi-181^{n})=3^{\alpha+\beta}, \text{ where } \alpha+\beta=\lambda$$
$$\Rightarrow (\xi+181^{n})-(\xi-103^{n})=3^{\beta}-3^{\alpha}, \beta>\alpha$$
$$\Rightarrow 2(181^{n})=3^{\alpha}(3^{\beta-\alpha}-1)\ .$$

$\alpha=0$ is the only possible value. Therefore the equation becomes $2(181)^{n}=3^{\lambda-1}$. Adding -2 on both sides we obtain $-2+2(181^{n})=3^{\lambda}-3$.This gives that $\lambda=2$.
Thus $181^{n}=4$, which is impossible.

Case (ii): Let μ be odd. Then $\mu=2n+1, n\in\mathbb{N}\cup 0$.Therefore the equation becomes $3^{\lambda}+181^{2n+1}=\xi^{2}$.

$$\Rightarrow 3^{\lambda}+181(181)^{2n}=\xi^{2}.$$
$$\Rightarrow 3^{\lambda}+3^{4}(181)^{2n}=\xi^{2}-10^{2}(181)^{2n}.$$
$$\Rightarrow 3^{u}(3^{\lambda-u}+3^{4-u}(181)^{2n})=(\xi+10(181)^{n}))(\xi-10(181)^{n}).$$

We observe that ξ is even. i.e. the equation above can be written as

$$3^{u}(3^{\lambda-u}+3^{4-u}(181)^{2n})=(2m+10(181)^{n}))(2m-10(181)^{n})).$$
$$=4(m+5(181)^{n})(m-5(181)^{n}).$$

Now we have two possibilities: (i) $m=3^{u}+5(181)^{n}$ (ii) $m=3^{u}-5(181)^{n}$.

Subcase (i): $m=3^{u}+5(181)^{n}$.
Then $3^{u}(3^{\lambda-u}+3^{4-u}(181)^{2n})=4(3^{u}+5(181)^{n}+5(181)^{n})(3^{u}+5(181)^{n}-5(181)^{n})$.
$=4(3^{u})[3^{u}+10(181^{n})]$.
$\Rightarrow (3^{\lambda-u}+3^{4-u}(181^{2n})=4(3^{u}+10(181^{n}))$.
$\Rightarrow 3^{4-u}(181^{2n})-40(181^{n})=3^{u}(4-3^{\lambda-2u})$
$\Rightarrow 181^{n}(3^{4-u}(181^{n})-40)=4.3^{u}-3^{\lambda-u}$.
$n=0$ is the only possible value.
$\Rightarrow 3^{4-u}-40=4.3^{u}-3^{\lambda-u}$
$\Rightarrow 3^{4}-40(3^{u})=3^{2u}(4-3^{\lambda-2u})$
$\Rightarrow 4((3^{2u}+10(3^{u})-20)=3^{\lambda}+1$,which is not a viable solution.

Subcase (ii): $m=3^{u}-5(181)^{n}$.
Then $3^{u}(3^{\lambda-u}+3^{4-u}(181)^{2n})=4(3^{u}-5(181)^{n}+5(181)^{n})(3^{u}-5(181)^{n}-5(181)^{n})$.
$=4(3^{u})[3^{u}-10(181^{n})]$.
$\Rightarrow (3^{\lambda-u}+3^{4-u}(181^{2n})=4(3^{u}-10(181^{n}))$
$\Rightarrow 3^{4-u}(181^{2n})+40(181^{n})=3^{u}(4-3^{\lambda-2u})$
$\Rightarrow 181^{n}(3^{4-u}(181^{n})+40)=4.3^{u}-3^{\lambda-u}$.
$n=0$ is the only possible value.
$\Rightarrow 3^{4-u}+40=4.3^{u}-3^{\lambda-u}$
$\Rightarrow 3^{4}+40(3^{u})=3^{2u}(4-3^{\lambda-2u})$
$\Rightarrow 4((3^{2u}-10(3^{u})-20)=3^{\lambda}+1$,which is not a viable solution.

Thus in any cases, we are never able to come up with a non-negative integral solution. Therefore $(1,0,2)$ is a unique solution for the Diophantine equation $3^{\lambda}+181^{\mu}=\xi^{2}$.

Theorem 3.2. $(1,0,2)$ *is a unique solution for the Diophantine equation* $3^{\lambda}+193^{\mu}=\xi^{2}$*, where* λ,μ *and* $\xi\in\mathbb{N}\cup 0$.

Proof. Consider the integral value of μ into two cases as (i) μ is even and (ii) μ is odd. □

Case (i): Let μ be even. If $\mu=0$, then by Lemma 2.1 $(1,0,2)$ is a solution. If $\mu=2n, n\in\mathbb{N}$, the equation becomes $3^{\lambda}+193^{2n}=\xi^{2}$.This can be written as $\xi^{2}-193^{2n}=3^{\lambda}$.

$$\Rightarrow (\xi+193^{n})(\xi-193^{n})=3^{\alpha+\beta}, \text{ where } \alpha+\beta=\lambda$$
$$\Rightarrow (\xi+193^{n})-(\xi-193^{n})=3^{\beta}-3^{\alpha}, \beta>\alpha$$
$$\Rightarrow 2(193^{n})=3^{\alpha}(3^{\beta-\alpha}-1)\ .$$

$\alpha = 0$ is the only possible value. Therefore the equation becomes $2(193)^n = 3^{\lambda-1}$. Adding -2 on both sides we obtain $-2 + 2(193^n) = 3^\lambda - 3$.This gives that $\lambda = 2$.
Thus $193^n = 4$, which is impossible.
Case (ii): Let μ be odd. Then $\mu = 2n+1, n \in \mathbb{N} \cup 0$.Therefore the equation becomes $3^\lambda + 193^{2n+1} = \xi^2$.

$$\Rightarrow 3^\lambda + 193(193)^{2n} = \xi^2.$$
$$\Rightarrow 3^\lambda + 3(193)^{2n} = \xi^2 - 14^2(103)^{2n}.$$
$$\Rightarrow 3(3^{(\lambda-1)} + (193)^{2n}) = (\xi + 14(103)^n))(\xi - 14(103)^n).$$

We observe that ξ is even. i.e. the equation above can be written as

$$3(3^{\lambda-1} + ((193)^{2n}) = (2m + 14(193)^n))(2m - 14(193)^n)).$$
$$= 4(m + 7(193)^n)(m - 7(193)^n).$$

Now we have two possibilities: (i) $m = 3 + 7(193)^n$ (ii) $m = 3 - 7(193)^n$.
Subcase (i): $m = 3 + 7(193)^n$.
Then $3(3^{\lambda-1} + (193)^{2n})) = 4(3 + 7(193)^n + 7(193)^n)(3 + 7(193)^n - 7(193)^n)$.
$= 4(3)[3 + 14(193)^n)]$
$\Rightarrow 3^{\lambda-1} + 193^{2n} = 4(3 + 14(193)^n)$
$\Rightarrow 3^{\lambda-1} - 12 = 56(193^n) - 193^{2n}$
$\Rightarrow 3(3^{\lambda-2} - 4) = 193^n(56 - 193^n)$.
$n = 0$ is the only possible value. But $3^{\lambda-1} = 69$ is not solvable.
Subcase (ii): $m = 3 - 7(193)^n$.
Then $3(3^{\lambda-1} + 193^{2n}) = 4(3 - 7(193)^n + 7(193)^n)(3 - 7(193)^n - 7(193)^n)$.
$= 4(3)[3 - 14(193)^n]$.
$\Rightarrow 3^{\lambda-1} + 193^{2n} = 4(3 - 14(193)^n)$
$\Rightarrow 3^{\lambda-1} - 12 = -56(193^n) - (193^{2n})$
$\Rightarrow 3(3^{\lambda-2} - 4) = 193^n(-56 + (193^n))$.
$n = 0$ is the only possible value. But $3^{\lambda-1} = -43$ is not solvable.
Thus in any cases, we are never able to come up with a non-negative integral solution. Therefore $(1, 0, 2)$ is a unique solution for the Diophantine equation $3^\lambda + 193^\mu = \xi^2$.

Corollary 3.1. *The Diophantine equation $3^\lambda + 103^\mu = \psi^4$has no non-negative integer solution.*

Proof. λ, μ and ψ be non-negative integers such that $3^\lambda + 103^\mu = \psi^4$. Let $\xi = \psi^2$. Then by theorem 3.1, $3^\lambda + 103^\mu = \xi^2$ has a unique solution $(1, 0, 2)$.
That is $\psi^2 = 2 \Rightarrow \psi = \sqrt{2}$, which is impossible as $\psi \in \mathbb{N} \cup 0$. Therefore $3^\lambda + 103^\mu = \psi^4$ has no non-negative integer solution. ☐

Corollary 3.2. Corollary 3.2. *The Diophantine equation $9^\psi + 103^\mu = \xi^2$ has no non-negative integer solution.*

Proof. μ, ψ, and ξ be non-negative integers such that $9^\psi + 103^\mu = \xi^2$. Let $\lambda = 2\psi$. Then by theorem 3.1, $3^\lambda + 103\mu = \xi^2$ has a unique solution $(1, 0, 2)$, which shows that $\lambda = 2 \Rightarrow \psi = \frac{1}{2} \in \mathbb{Q}$. Therefore $9^\psi + 103^\mu = \xi^2$ has no non-negative integer solution.
Corollary 3.3.*The Diophantine equation $3^\lambda + 181^\mu = \psi^4$has no non-negative integer solution.*
Proof. λ, μ and ψ be non-negative integers such that $3^\lambda + 181^\mu = \psi^4$. Let $\xi = \psi^2$. Then by theorem 3.2, $3^\lambda + 181^\mu = \xi^2$ has a unique solution $(1, 0, 2)$.
That is $\psi^2 = 2 \Rightarrow \psi = \sqrt{2}$, which is impossible as $\psi \in \mathbb{N} \cup 0$. Therefore $3^\lambda + 181^\mu = \psi^4$ has no non-negative integer solution. ☐

Corollary 3.3. *The Diophantine equation $9^\psi + 181^\mu = \xi^2$ has no non-negative integer solution.*

Proof. μ, ψ, and ξ be non-negative integers such that $9^\psi + 181^\mu = \xi^2$. Let $\lambda = 2\psi$. Then by theorem 3.2, $3^\lambda + 181\mu = \xi^2$ has a unique solution $(1, 0, 2)$, which shows that $\lambda = 2 \Rightarrow \psi = \frac{1}{2} \in \mathbb{Q}$. Therefore $9^\psi + 181^\mu = \xi^2$ has no non-negative integer solution.
Corollary 3.5.*The Diophantine equation $3^\lambda + 193^\mu = \psi^4$has no non-negative integer solution.*

Proof. λ, μ and ψ be non-negative integers such that $3^\lambda + 193^\mu = \psi^4$. Let $\xi = \psi^2$. Then by theorem 3.3, $3^\lambda + 193^\mu = \xi^2$ has a unique solution $(1, 0, 2)$.
That is $\psi^2 = 2 \Rightarrow \psi = \sqrt{2}$, which is impossible as $\psi \in \mathbb{N} \cup 0$. Therefore $3^\lambda + 193^\mu = \psi^4$ has no non-negative integer solution. □

Corollary 3.4. *The Diophantine equation* $9^\psi + 193^\mu = \xi^2$ *has no non-negative integer solution.*

Proof. μ, ψ, and ξ be non-negative integers such that $9^\psi + 193^\mu = \xi^2$. Let $\lambda = 2\psi$. Then by theorem 3.3, $3^\lambda + 193\mu = \xi^2$ has a unique solution $(1, 0, 2)$, which shows that $\lambda = 2 \Rightarrow \psi = \frac{1}{2} \in \mathbb{Q}$. Therefore $9^\psi + 193^\mu = \xi^2$ has no non-negative integer solution. □

4 Conclusion

. In this paper, we have shown the solutions of the Diophantine equations of several primes. One can find the solutions of the Exponential Diophantine equations using other primes.
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