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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

OBSERVATION ON THE BIQUADRATIC EQUATION WITH FIVE UNKNOWNS $2(x-y)(x^3+y^3)+x^4-y^4=2(z^2-w^2)p^2$

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Abstract

This paper focuses on obtaining non-zero integer quintuples (x, y, z, w, p) satisfying the bi-quadratic equation with five unknowns given by $2(x-y)(x^3+y^3)+x^4-y^4=2(z^2-w^2)p^2$. Various patterns of solutions are obtained by reducing the given bi-quadratic equation to solvable ternary quadratic equation through employing linear transformations.
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1 Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular biquadratic Diophantine equation, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians, since antiquity. In this context , one may refer [1-31] for various problems on biquadratic equations with three ,four and five variables. This paper concerns with yet another problem of determining non-trivial integral solutions on the biquadratic equation with five unknowns given by $2(x-y)(x^3+y^3)+x^4-y^4=2(z^2-w^2)p^2$.

2 Method of Analysis

The Diophantine equation representing the biquadratic equation under consideration with five unknowns is given by

$$2(x-y)(x^3+y^3)+x^4-y^4=2(z^2-w^2)p^2. \tag{2.1}$$

Introducing the linear transformations

$$x=u+v, y=u-v, z=2u+v, w=2u-v \tag{2.2}$$

in (2.1) , we get

$$u^2+2v^2=p^2. \tag{2.3}$$

The above equation (2.3) is solved through different methods and thus, one obtains different patterns of distinct integer solutions to (2.1).

3 Patterns

Pattern 3.1

The most cited solutions to (2.3) are

$$v=2rs, u=2r^2-s^2 \tag{3.1}$$

and

$$p=2r^2+s^2. \tag{3.2}$$

Using (3.1) in (2.2) , we get

$$x=2r^2-s^2+2rs, y=2r^2-s^2-2rs, z=4r^2-2s^2+2rs, w=4r^2-2s^2-2rs. \tag{3.3}$$

Thus,(3.2) and (3.3) give the integer solutions to (2.1).

Pattern 3.2
Rewrite (2.3) as

$$u^2+2v^2=p^2*1. \tag{3.4}$$

Assume

$$p=a^2+2b^2. \tag{3.5}$$

Express integer 1 on the R.H.S of (3.4) as the product of complex conjugates as

$$1=\frac{(1+i2\sqrt{2})(1-i2\sqrt{2})}{9}. \tag{3.6}$$

Substituting (3.5) and (3.6) in (3.4) and employing the method of factorization we define

$$u+i\sqrt{2}v=\frac{(a+i\sqrt{2}b)^2(1+i2\sqrt{2})}{3},$$

from which, on equating the real and imaginary parts, we obtain

$$u=\frac{(a^2-2b^2-8ab)}{3},v=\frac{(2a^2-4b^2+2ab)}{3}. \tag{3.7}$$

As our interest is on finding integer solutions, replacing a by $3A$ and b by $3B$ in (3.5) and (3.7) and in view of (2.2) , the corresponding integer solutions to (2.1) are given by

$$x=3(3A^2-6B^2-6AB),y=3(-A^2+2B^2-10AB),$$
$$z=3(4A^2-8B^2-14AB),w=-54AB,p=9(A^2+2B^2).$$

Observation 3.2.1
Apart from (3.6), the integer 1 on the R.H.S. of (3.4) is expressed as

$$1=\frac{(7+i6\sqrt{2})(7-i6\sqrt{2})}{121}.$$

For this choice ,the corresponding integer solutions to (2.1) are obtained as

$$x=11(13A^2-26B^2-10AB),y=11(A^2-2B^2-38AB)$$
$$z=11(20A^2-40B^2-34AB),w=11(8A^2-16B^2-62AB),p=11(11A^2+22B^2).$$

Observation 3.2.2
It is worth to mention here that the integer 1 on the R.H.S. of (3.4) may be written in the general form as

$$1=\frac{(2p^2-q^2+i\sqrt{22}pq)(2p^2-q^2-i\sqrt{22}pq)}{(2p^2+q^2)^2}.$$

The repetition of the above process leads to a set of integer solutions to (2.1).

Pattern 3.3
Consider (2.3) as

$$p^2-2v^2=u^2*1. \tag{3.8}$$

Assume

$$u=a^2-2b^2. \tag{3.9}$$

Express integer 1 on the R.H.S of (3.8) as the product of irrational pairs as

$$1=(3+2\sqrt{2})(3-2\sqrt{2}). \tag{3.10}$$

Substituting (3.9) and (3.10) in (3.4) and employing the method of factorization, define

$$p+\sqrt{2}v=(a+\sqrt{2}b)^2(3+2\sqrt{2}).$$

from which, we get

$$v=2a^2+4b^2+6ab \tag{3.11}$$

and

$$p=3a^2+6b^2+8ab. \tag{3.12}$$

Using (3.9) and (3.11) in (2.2) , we write

$$x=3a^2+2b^2+6ab,y=-a^2-6b^2-6ab,z=4a^2+6ab,w=-8b^2-6ab. \tag{3.13}$$

Thus,(3.12) and (3.13) give the integer solutions to (2.1).

Observation 3.3.1
Apart from (3.10), the integer 1 on the R.H.S. of (3.8) is expressed as

$$1 = \frac{(11+6\sqrt{2})(11-6\sqrt{2})}{49}.$$

For this choice, the corresponding integer solutions to (2.1) are obtained as

$$x = 7(13A^2 - 2B^2 + 22AB), y = 7(A^2 - 26B^2 - 22AB),$$
$$z = 7(20A^2 - 16B^2 + 22AB), w = 7(8A^2 - 40B^2 - 22AB), p = 7(11A^2 + 22B^2 + 24AB).$$

Observation 3.3.2
It is worth to mention here that the integer 1 on the R.H.S. of (3.8) may be written in the general form as

$$1 = \frac{(2p^2 + q^2 + \sqrt{22}pq)(2p^2 + q^2 - \sqrt{22}pq)}{(2p^2 - q^2)^2}.$$

The repetition of the above process leads to a set of integer solutions to (2.1).

4 Conclusion

An attempt has been made to determine many non-zero distinct integer solutions to the considered fourth degree equation with five unknowns in the title. It is worth to mention that, in addition to the transformations presented in (2.2) ,one may also consider the transformations represented by

(4.1) $$x = u + v, y = u - v, z = u + 2v, w = u - 2v,$$

(4.2) $$x = u + v, y = u - v, z = 2uv + 1, w = 2uv - 1.$$

To conclude, the readers of this paper may search for integer solutions to other forms of homogeneous or non-homogeneous quinary bi-quadratic Diophantine equations.

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