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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

RESULTS ON UNIQUENESS OF *L*-FUNCTIONS CONCERNING WEIGHTED SHARING

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Abstract

In this paper using the notion of weighted sharing,we consider the value distribution of a L- function and meromorphic function when certain type of difference-differential polynomials which share a small function and rational function and obtain some uniqueness results which extends recent results due to Hao and Chen[2].
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1 Introduction and preliminaries

In this paper, $\mathbb{C}$ denotes the complex plane and $\mathbb{N}$ denotes the set of natural numbers. Now, towards the end of twentieth century, a new class of Dirichlet series called the Selberg class was introduced by Atle Selberg[15]. The concept of L- function where L means Selberg class function with the Riemann Zeta function is the most speculative open world problem in today's pure mathematics. L- functions can be analytically continued as meromorphic functions in $\mathbb{C}$. A meromorphic function L is said to be an L-function in the Selberg class if it satisfies the following properties.

(i) $L(z)$ can be expressed as a Dirichlet series $L(z) = \sum_{m=1}^{\infty} a(m)/m^z$.
(ii) $|a(m)| = O(m^{\epsilon})$, for any $\epsilon > 0$.
(iii) There exists a non negetive integer n such that $(z-1)^n L(z)$ becomes an entire function of finite order.
(iv) Every L- function satisfies the functional equation
$$\lambda_L(z) = \omega\overline{\lambda_L(1-\overline{z})},$$
where
$$\lambda_L(z) = L(z)A^z \prod_{j=1}^{n} \Gamma(\eta_j z + v_j),$$
with positive real numbers A, η_j and complex numbers v_j, ω with $Re(v_j) \geq 0$ and $|\omega| = 1$.
(v) $L(z)$ satisfies $L(z) = \prod_p L_p(z)$, where $L_p(z) = \exp(\sum_{n=1}^{\infty} b(p^n)/p^{nz})$ with $b(p^n) = O(p^{n\theta})$ for some $\theta < 1/2$ and p denotes prime number.

If L satisfies (i)-(iv) then we say that L is an L-function in the extended Selberg class. In this paper, by an L-function we always mean an L-function in the extended Selberg class with $a(1) = 1$. Here we use the standard notations and definitions of the value distribution theory [3].

The Nevanlinna value distribution theory is an important area of research which has seen extensive work. It primarily focuses on the analysis of the distribution of solutions to the equation $f(z) = a$, where f is an entire or meromorphic function in $\mathbb{C}$. Let $\alpha \in \mathbb{C} \cup \{\infty\}$ and f, g be meromorphic functions in the complex plane. The set of all α- points of f with multiplicities not exceeding l is denoted by $E_{l)}(\alpha, f)(\overline{E}_{l)}(\alpha, f))$, where l is a positive integer and we consider(ignore) the multiplicities of the α- points. The hyper order $\rho_2(f)$ of f is defined by
$$\rho_2(f) = \overline{lim}_{r\to\infty} \frac{loglogT(r, f)}{logr}.$$

We denote $S(r, f)$ by any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$, outside a possible exceptional set of finite linear measure. We say that f and g share α CM if they have the set of α- points with the same multiplicities and if we do not consider the multiplicity then we say that f and g share α IM.

In general, for a meromorphic function $f(z)$, the quantity $m(r,f)$ denotes the proximity function of $f(z)$, while $N(r,f)$ denotes the counting function of poles of $f(z)$ whose multiplicities are taken into account(respectively $\overline{N}(r,f)$ denotes the reduced counting function when multiplicities are ignored). The Nevanlinna characteristic function of a meromorphic function f plays a very important role in the value distribution theory and it is denoted by $T(r,f)$. We have $T(r,f) = m(r,f) + N(r,f)$, which clearly shows that $T(r,f)$ is non-negative.

We can see a lot of work on uniqueness results with the help of Nevanlinna Theory. Recently people have raised great interest in difference analogues of Nevanlinna's theory and obtained many profound results. A number of papers have focused on value distribution and uniqueness of difference polynomials, which are analogues of Nevanlinna theory (see [11], [17]).

The value distribution of an L-function concerned with the distribution of the zeros of L and more generally, with the roots of the equation $L(s) = c$ for some $c \in \mathbb{C} \cup \{\infty\}$. Since L-functions are analytically continued as meromorphic functions, it is possible to study the value distribution and uniqueness outcomes between the L- functions and any arbitrary meromorphic functions (see [12], [13]).

We state the following standard definitions of Nevanlinna theory and it is important to note that all the definitions discussed also applies to the L-function.

In addition we need the following definitions.

Definition 1.1 ([6]). *Let f be a meromorphic function defined in the complex plane. Let n be a positive integer and $\alpha \in \mathbb{C} \cup \{\infty\}$. By $N(r,\alpha; f \mid\leq n)$ we denote the counting function of the α- points of f with multiplicity less than or equal to n and by $\overline{N}(r,\alpha; f \mid\leq n)$ the reduced counting function. Also by $N(r,\alpha; f \mid\geq n)$ we denote the counting function of the α- points of f with multiplicity greater than or equal to n and by $\overline{N}(r,\alpha; f \mid\geq n)$ the reduced counting function. We define*

$$N_n(r,\alpha; f) = \overline{N}(r,\alpha; f) + \overline{N}(r,\alpha; f \mid\geq 2) + ... + \overline{N}(r,\alpha; f \mid\geq n).$$

Definition 1.2 ([6]). *Let f be a meromorphic function defined in $\mathbb{C}$ and $p(z)$ be a small function of f or a rational function. Then we denote the notations by $N(r,p; f \mid\leq m), \overline{N}(r,p; f \mid\leq m), N(r,p; f \mid\geq m), \overline{N}(r,p; f \mid\geq m), N_m(r,p; f)$ etc, the counting functions $N(r,0; f-p \mid\leq m), \overline{N}(r,0; f-p \mid\leq m), N(r,0; f-p \mid\geq m), \overline{N}(r,0; f-p \mid\geq m), N_m(r,0; f-p)$ respectively.*

Definition 1.3 ([5]). *Let f and g be two meromorphic functions defined in the complex plane and n be an integer (≥ 0) or infinity. We denote by $E_n(\alpha; f)$ the set of all zeros of $f - \alpha$ and $\alpha \in \mathbb{C} \cup \{\infty\}$and a zero of multiplicity k is counted k times if $k \leq n$ and $n+1$ times if $k > n$, we say that f and g share α with weight n if $E_n(\alpha; f) = E_n(\alpha; g)$. We say that f and g share (α, n) to mean that f, g share α with weight n. Clearly f, g share α IM or CM if and only if f and g share $(\alpha, 0)$ or (α, ∞) respectively.*

Definition 1.4 ([9]). *Let f be a meromorphic function defined in the complex plane and $p(z)$ be a rational function or a small function of f. Then we denote by $E_{m)}(p; f), \overline{E}_{m)}(p; f)$ and $E_m(p; f)$ the sets $E_{m)}(r,0; f-p), \overline{E}_{m)}(r,0; f-p)$ and $E_m(r,0; f-p)$ respectively. We write f,g share (p,n) to mean that $f-p$ and $g-p$ share the value 0 with weight n. Clearly, if f,g share (p,n) then f,g share (p,m) for all integers m , $0 \leq m < n$. Also we note that f,g share p IM or CM if and only if f,g share $(p,0)$ or (p,∞) respectively.*

Definition 1.5 ([4]). *Let f and g be two non-constant meromorphic functions share a value α IM. Denote by $\overline{N}_*(r,\alpha; f, g)$ the counting function of the α-points of f and g with different multiplicities, where each α-point is counted only once.*

Definition 1.6 ([10]). *Let f and g be two non-constant meromorphic functions share a value α IM. Denote by $\overline{N}(r,\alpha; f \mid> g)$ the counting function of the α- points of f and g with multiplicities with respect to f is greater than the multiplicities with respect to g, where each α- points is counted only once.*

Definition 1.7 ([10]). *Let f and g be two non-constant meromorphic functions share a value α IM. We denote by $\overline{N}_E(r,\alpha; f, g \mid> m)$ the counting function of the α-points of f and g with multiplicities greater than m and the multiplicities with respect to f is equal to the multiplicities with respect to g, where each α-points is counted only once.*

In 2017, *Liu, Li and Yi* [8] *proved the following uniqueness theorems.*

Theorem A ([8]) *Let $j \geq 1$ and $k \geq 1$ be integers such that $j > 3k+6$. Also let L be an L-function and f be a non-constant meromorphic function. If $\{f^j\}^{(k)}$ and $\{L^j\}^{(k)}$ share $(1,\infty)$, then $f \equiv \alpha L$ for some non-constant α satisfying $\alpha^j = 1$.*

Theorem B ([8]). *Let $j \geq 1$ and $k \geq 1$ be integers such that $j > 3k+6$. Also let L be an L-function and f be a non-constant meromorphic function. If $\{f^j\}^{(k)}$ and $\{L^j\}^{(k)}$ share (z,∞), then $f \equiv \alpha L$ for some non-constant α satisfying $\alpha^j = 1$.*

In 2018, Hao and Chen $\big([2]\big)$ obtained the following uniqueness results on L -function.

Theorem C ([2]). *Let f be a non-constant meromorphic function and L be an L-function such that $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ share $(1,\infty)$ where $n,m,k \in Z^+$. If $n > m+3k+6$ and $k \geq 2$, then $f \equiv L$ or $f^n(f-1)^m \equiv L^n(L-1)^m$.*

Theorem D ([2]). *Let f be a non-constant meromorphic function and L be an L-function such that $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ share $(1,0)$ where $n,m,k \in Z^+$. If $n > 4m+7k+11$ and $k \geq 2$, then $f \equiv L$ or $f^n(f-1)^m \equiv L^n(L-1)^m$.*

Now it will be interesting to study the above Theorems A, B, C and D by considering more general form of difference-differential polynomial. The main motivation of this paper is the fact that the L- function where $L-$ function has only one possible pole at $s=1$ in $\mathbb{C}$.

Question 1.1. Can we consider rational or small function sharing in Theorem C and Theorem D?

Question 1.2. Can we take difference-differential polynomial of the form $[f^n(f-1)^m f(z+c)]^{(k)}$ and $[L^n(L-1)^m L(z+c)]^{(k)}$ in Theorem C and Theorem D?

In this paper, we try to find the possible answer of the above questions. The following are the main results of this paper.

2 Main Results

Theorem 2.1. *Let f be a transcendental meromorphic function and L be an L- function, n,k,m be positive integers. If $[f^n(f-1)^m f(z+c)]^{(k)}$ and $[L^n(L-1)^m L(z+c)]^{(k)}$ share $(\alpha(z),l)$ and f,L share $(\infty,0)$, where $\alpha(z)$ is a small function of f and L then*

(1) $l=0$ *and* $(n+m) > (5k+7)(m+2)+1$,

(2) $l=1$ *and* $(n+m) > \frac{1}{2}(5k+9)(m+2)+1$,

(3) $l \geq 2$ *and* $(n+m) > (2k+4)(m+2)+1$.

Then one of the following holds:

(i) $[f^n(f-1)^m f(z+c)]^{(k)} \equiv [L^n(L-1)^m L(z+c)]^{(k)}$,

(ii) $[f^n(f-1)^m f(z+c)]^{(k)}[L^n(L-1)^m L(z+c)]^{(k)} \equiv [\alpha(z)]^2$.

Theorem 2.2. *Let f be a transcendental meromorphic function and L be an L- function, n,k,m be positive integers. If $[f^n(f-1)^m f(z+c)]^{(k)}$ and $[L^n(L-1)^m L(z+c)]^{(k)}$ share $(R(z),l)$ and f,L share $(\infty,0)$, where $R(z)$ is a rational function of f and L then*

(1) $l=0$ *and* $(n+m) > (5k+7)(m+2)+1$,

(2) $l=1$ *and* $(n+m) > \frac{1}{2}(5k+9)(m+2)+1$,

(3) $l \geq 2$ *and* $(n+m) > (2k+4)(m+2)+1$.

Then one of the following holds:

(i) $[f^n(f-1)^m f(z+c)]^{(k)} \equiv [L^n(L-1)^m L(z+c)]^{(k)}$,

(ii) $[f^n(f-1)^m f(z+c)]^{(k)}[L^n(L-1)^m L(z+c)]^{(k)} \equiv (R(z))^2$.

Example 2.1. Let us consider $L=\zeta$ and $f=-\zeta$, where ζ is Riemann zeta function which has a simple pole. By hypothesis of the theorem $F=[f^n(f-1)^m f(z+c)]^{(k)}$ and $L=[L^n(L-1)^m L(z+c)]^{(k)}$ share $(\alpha(z),l)$ and the conditions are satisfied for different weights $l=0, l=1$ and $l \geq 2$.

Remark 2.1. *Theorem 2.1 and Theorem 2.2 are the extension of Theorems A-D respectively.*

3 Auxiliary Lemmas

In this section, we present some necessary Lemmas.

Denote H by the following function.

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big).$$

Lemma 3.1 ([16]). *Let L be an L- function with degree q. Then*
$$T(r,L)=\frac{q}{\pi}rlogr+O(1).$$

Lemma 3.2 ([9]). *Let L be an L-function. Then*
$$N(r,\infty.L)=S(r,L)=O(logr).$$

Lemma 3.3 ([10]). *Let f be a non-constant meromorphic function and L be an L-function. If f and L share $(\infty,0)$ then*
$$\overline{N}(r,\infty;f)=\overline{N}(r,\infty;L)=S(r,L)=O(logr).$$

Lemma 3.4 ([21]). *Let $f(z)=\frac{\alpha_0+\alpha_1z+....+\alpha_nz^n}{\beta_0+\beta_1z+....+\beta_mz^m}$ be a non-constant rational function defined in the complex plane $\mathbb{C}$, where $\alpha_0,\alpha_1,....,\alpha_n(\neq 0)$ and $\beta_0,\beta_1,....,\beta_m(\neq 0)$ are complex constants. Then*
$$T(r,f)=max(m,n)logr+O(1).$$

Lemma 3.5 ([18]). *Let f be a transcendental meromorphic function of hyper order $\rho_2(f)<1$. Then for any $\alpha\in\mathbb{C}-\{0\}$.*
$$\begin{aligned}T(r,f(z+\alpha))&=T(r,f)+S(r,f),\\N(r,\infty;f(z+\alpha))&=N(r,\infty;f)+S(r,f),\\N(r,0;f(z+\alpha))&=N(r,0;f)+S(r,f).\end{aligned}$$

Lemma 3.6 ([14]). *Let F and G be two non-constant meromorphic functions sharing $(1,1)$ and $(\infty,0)$. If $H\not\equiv 0$, then*
$$\begin{aligned}T(r,F)\le{}&N_2(r,0;F)+N_2(r,0;G)+\frac{3}{2}\overline{N}(r,F)+\overline{N}(r,G)\\&+\overline{N}_*(r,\infty;F,G)+\frac{1}{2}\overline{N}(r,0;F)+S(r,F)+S(r,G).\\T(r,G)\le{}&N_2(r,0;F)+N_2(r,0;G)+\frac{3}{2}\overline{N}(r,G)+\overline{N}(r,F)\\&+\overline{N}_*(r,\infty;F,G)+\frac{1}{2}\overline{N}(r,0;G)+S(r,F)+S(r,G).\end{aligned}$$

Lemma 3.7 ([14]). *Let F and G be two non-constant meromorphic functions sharing $(1,0)$ and $(\infty,0)$. If $H\not\equiv 0$, then*
$$\begin{aligned}T(r,F)\le{}&N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,F)+2\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G)+2\overline{N}(r,0;F)\\&+\overline{N}(r,0;G)+S(r,F)+S(r,G).\\T(r,G)\le{}&N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,G)+2\overline{N}(r,F)+\overline{N}_*(r,\infty;F,G)+2\overline{N}(r,0;G)\\&+\overline{N}(r,0;F)+S(r,F)+S(r,G).\end{aligned}$$

Lemma 3.8 ([1]). *Let F and G be two non-constant meromorphic functions sharing $(1,l)$ and $(\infty,0)$ where $2\le l<\infty$ and $H\not\equiv 0$ then*
$$\begin{aligned}T(r,F)\le{}&N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G)-m(r,1,G)\\&-N_E(r,1;F|>3)-\overline{N}(r,1;G>F)+S(r,F)+S(r,G).\\T(r,G)\le{}&N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G)-m(r,1,F)\\&-N_E(r,1;G|>3)-\overline{N}(r,1;F>G)+S(r,F)+S(r,G).\end{aligned}$$

Lemma 3.9 ([20]). *Let F be a non-constant meromorphic function and k,p be two positive integers, then*
$$\begin{aligned}T(r,F^{(k)})&\le T(r,F)+k\overline{N}(r,\infty;F)+S(r,F),\\N_p(r,0;F^{(k)})&\le T(r,F^{(k)})-T(r,F)+N_{p+k}(r,0;F)+S(r,F),\\N_p(r,0;F^{(k)})&\le N_{p+k}(r,0;F)+k\overline{N}(r,\infty;F)+S(r,F),\\N(r,0;F^{(k)})&\le N(r,0;F)+k\overline{N}(r,\infty;F)+S(r,F).\end{aligned}$$

Lemma 3.10 ([20]). *Let f be a non-constant meromorphic function, define then polynomial $P(f) = a_0 + a_1 f + ... + a_n f^n$, where $a_0, ...a_n$ are complex constants and $a_n \neq 0$, then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 3.11 ([3]). *Let $f(z)$ be a meromorphic function and $a \in \mathbb{C}$. Then*

$$T\left(r, \frac{1}{f}\right) = T(r, f) + O(1).$$

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1).$$

Lemma 3.12. *Let f be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ and L be a $L-$function with $\rho_2(L) < 1$. Let $F_1 = [f^n(f-1)^m f(z+c)]$. where n, m are positive integers and c is a complex constant. Then*

$$(n + m - 1)T(r, f) \leq T(r, F_1) + S(r, f)$$

Proof. Since f is a meromorphic function, from Lemmas 3.5, 3.10, 3.11 we have

$$\begin{aligned}
(n + m + 1)T(r, f) &= T(r, f^{n+m+1}) + S(r, f) \\
&\leq T(r, f^n(f-1)^m f) + S(r, f) \\
&\leq T\left(r, \frac{F_1 f}{f(z+c)}\right) + S(r, f) \\
&\leq T(r, F_1) + T\left(r, \frac{f(z+c)}{f}\right) + S(r, f) \\
&\leq T(r, F_1) + m\left(r, \frac{f(z+c)}{f}\right) + N\left(r, \frac{f(z+c)}{f}\right) + S(r, f) \\
(n + m - 1)T(r, f) &\leq T(r, F_1) + S(r, f).
\end{aligned}$$

□

4 Proof of the Main Results

Proof of Theorem 2.1. Let $F = \frac{F_1^{(k)}}{\alpha(z)}$ and $G = \frac{G_1^{(k)}}{\alpha(z)}$ where $F_1 = f^n(f-1)^m f(z+c)$ and $L_1 = L^n(L-1)^m L(z+c)$ respectively. Then F and G share $(1, l)$ and share $(\infty, 0)$ except for zeros and poles of $\alpha(z)$. Clearly by Lemma 3.1, L is a transcendental meromorphic function. We have by Lemmas 3.9 and 3.12

$$\begin{aligned}
N_2(r, 0; F) &\leq N_2(r, 0; F_1^{(k)}) + S(r, f) \\
&\leq T(r, F_1^{(k)}) - T(r, F_1) + N_{k+2}(r, 0; F_1) + S(r, f) \\
&\leq T\left(r, \frac{F_1^{(k)}}{\alpha(z)}\right) - (n + m - 1)T(r, f) + N_{k+2}(r, 0; F_1) + S(r, f). \qquad (4.1)
\end{aligned}$$

Hence from inequality (4.1), we get

$$(n + m - 1)T(r, f) \leq T(r, F) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f). \qquad (4.2)$$

Similarly,

$$(n + m - 1)T(r, L) \leq T(r, G) - N_2(r, 0; G) + N_{k+2}(r, 0; L_1) + S(r, f). \qquad (4.3)$$

Now we have to consider the following two cases.

Case 4.1. Let $H \not\equiv 0$. In this case we have to consider the following three subcases.

Subcase 4.1.1. Let $l = 0$. Hence by Lemmas 3.2, 3.3 and 3.7 and inequality (4.2) we have

$$\begin{aligned}
(n + m - 1)T(r, f) &\leq N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; G) + 3\overline{N}(r, \infty; F) + \overline{N}_*(r, \infty; F, G) \\
&\quad + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, L) \\
&\leq N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) - N_2(r, 0; F) \\
&\quad + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, L). \\
&\leq N_2(r, 0; F_1^{(k)}) + N_2(r, 0; L_1^{(k)}) + 2\overline{N}(r, 0; F_1^{(k)}) + \overline{N}(r, 0; L_1^{(k)}) - N_2(r, 0; F_1^{(k)})
\end{aligned}$$

$$
\begin{aligned}
&+ N_{k+2}(r,0;F_1) + S(r,f) + S(r,L).\\
&\le N_2(r,0;L_1^{(k)}) + 2\overline{N}(r,0;F_1^{(k)}) + \overline{N}(r,0;L_1^{(k)}) + N_{k+2}(r,0;F_1)\\
&\quad + S(r,f) + S(r,L).\\
&\le N_{k+2}(r,0;L_1) + 2N_{k+1}(r,0;F_1) + N_{k+1}(r,0;L_1) + N_{k+2}(r,0;F_1)\\
&\quad + S(r,f) + S(r,L).\\
&\le (3+2k)(m+2)T(r,L) + (3k+4)(m+2)T(r,f) + S(r,f) + S(r,L).
\end{aligned}
$$

Hence

$$(n+m-1)T(r,f) \le (3+2k)(m+2)T(r,L) + (3k+4)(m+2)T(r,f) + S(r,f) + S(r,L). \tag{4.4}$$

Similarly,

$$(n+m-1)T(r,L) \le (3+2k)(m+2)T(r,f) + (3k+4)(m+2)T(r,L) + S(r,f) + S(r,L). \tag{4.5}$$

From inequalities (4.4) and (4.5) we get

$$(n+m-1)[T(r,f)+T(r,L)] \le (7+5k)(m+2)[T(r,f)+T(r,L)] + S(r,f) + S(r,L). \tag{4.6}$$

which is a contradiction from (4.6) as $n+m > (7+5k)(m+2)+1$.

Subcase 4.1.2. Let $l = 1$. Hence by Lemmas 3.2, 3.3 and 3.6 and inequality (4.2) we have

$$
\begin{aligned}
(n+m-1)T(r,f) &\le N_2(r,0;G) + \frac{3}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) + \frac{1}{2}\overline{N}(r,0;F)\\
&\quad + N_{k+2}(r,0;F_1) + S(r,f) + S(r,L).\\
&\le N_2(r,0;L_1^{(k)}) + \frac{1}{2}N_{k+1}(r,0;F_1) + \overline{N}(r,0;L_1^{(k)}) + N_{k+2}(r,0;F_1)\\
&\quad + S(r,f) + S(r,L).\\
&\le (k+2)(m+2)T(r,L) + \frac{1}{2}(3k+5)(m+2)T(r,f)\\
&\quad + S(r,f) + S(r,L).
\end{aligned}
$$

Hence

$$(n+m-1)T(r,f) \le (k+2)(m+2)T(r,L) + \frac{1}{2}(3k+5)(m+2)T(r,f) + S(r,f) + S(r,L). \tag{4.7}$$

Similarly

$$(n+m-1)T(r,L) \le (k+2)(m+2)T(r,f) + \frac{1}{2}(3k+5)(m+2)T(r,L) + S(r,f) + S(r,L). \tag{4.8}$$

From inequalities (4.7) and (4.8) we arrive at a contradiction as $(n+m) > \frac{1}{2}(5k+9)(m+2)+1$.

Subcase 4.1.3. Let $2 \le l < 1$. Hence by Lemmas 3.2, 3.3 and 3.8 and inequality (4.2)

$$
\begin{aligned}
(n+m-1)T(r,L) &\le N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_*(r,\infty;F,G)\\
&\quad - m(r,1,F) - N_E(r,1;G| > 3) - \overline{N}(r,1;F > G) + S(r,f) + S(r,L)\\
&\le N_2(r,0;F_1^{(k)}) + N_2(r,0;L_1^{(k)}) + S(r,f) + S(r,L)\\
&\le (k+2)(m+2)T(r,f) + (k+2)(m+2)T(r,L) + S(r,f) + S(r,L).
\end{aligned}
$$

$$(n+m-1)T(r,L) \le (k+2)(m+2)T(r,f) + (k+2)(m+2)T(r,L) + S(r,f) + S(r,L). \tag{4.9}$$

Similarly

$$(n+m-1)T(r,f) \le (k+2)(m+2)T(r,L) + (k+2)(m+2)T(r,f) + S(r,f) + S(r,L). \tag{4.10}$$

From inequalities (4.9) and (4.10) we arrive at a contradiction as $l \ge 2$ and $(n+m) > (2k+4)(m+2)+1$.

Case 4.2. Let $H \equiv 0$. Then

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big) \equiv 0.$$

Integrating both sides we get

$$F - 1 = \frac{G-1}{b - c(G-1)}, \tag{4.11}$$

where $b \neq 0$ and c are constants. Now we have to consider the following subcases.

Subcase 4.2.1. Let $c = 0$. Then from (4.11) we have

$$F - 1 = \frac{G-1}{b}. \tag{4.12}$$

If $b \neq 1$ then from (4.12)

$$\overline{N}(r, 0; F) = \overline{N}(r, 1-b; G). \tag{4.13}$$

By Lemmas 3.2 and 3.9, using Second Fundamental Theorem of Nevanlinna and from inequality (4.3) we have

$$\begin{aligned}(n+m-1)T(r,L) &\le T(r,G) - N_2(r,0;G) + N_{k+2}(r,0;L_1) + S(r,L)\\ &\le \overline{N}(r,0;G) + \overline{N}(r,1-b;G) + \overline{N}(r,\infty;G) - N_2(r,0;G)\\ &\quad + N_{k+2}(r,0;L_1) + S(r,L).\\ &\le \overline{N}(r,0;G) + \overline{N}(r,0;F) - N_2(r,0;G) + N_{k+2}(r,0;L_1) + S(r,L)\\ &\le \overline{N}(r,0;F_1^{(k)}) + \overline{N}(r,0;L_1^{(k)}) + N_{k+2}(r,0;L_1) + S(r,L)\\ &\le N_{k+1}(r,0;F_1) + N_{k+1}(r,0;L_1) + N_{k+2}(r,0;L_1) + S(r,L)\\ &\le (2k+3)(m+2)T(r,L) + (k+1)(m+2)T(r,f)\\ &\quad + S(r,f) + S(r,L).\end{aligned}$$

Hence

$$(n+m-1)T(r,L) \le (2k+3)(m+2)T(r,L) + (k+1)(m+2)T(r,f) + S(r,f) + S(r,L). \tag{4.14}$$

Similarly

$$(n+m-1)T(r,f) \le (2k+3)(m+2)T(r,f) + (k+1)(m+2)T(r,L) + S(r,f) + S(r,L). \tag{4.15}$$

From the inequalities (4.14) and (4.15) we arrive at a contradiction as $n + m > (3k+4)(m+2)+1$.

Hence $b = 1$ and therefore we get from (4.12)

$$[f^n(f-1)^m f(z+c)]^{(k)} \equiv [L^n(L-1)^m L(z+c)]^{(k)}.$$

Subcase 4.2.2. Let $c \neq 0$ and $b = -c$.

If $c = 1$, then from (4.11) we have $FG \equiv 1$. Hence

$$[f^n(f-1)^m f(z+c)]^{(k)}[L^n(L-1)^m L(z+c)]^{(k)} = [\alpha(z)]^2.$$

If $c \neq 1$, then from (4.11) we have,

$$\frac{1}{F} = \frac{-cG}{(1-c)G-1}.$$

Hence $\overline{N}(r,0;F) = N(r, \frac{1}{1-c}; G)$.

Now proceeding as in subcase (4.2.1), we arrive at a contradiction. If $c = 1$, then from (4.11) we have

$$F \equiv \frac{-b}{G-b-1}. \tag{4.16}$$

Hence by Lemma 3.3 we have from (4.16)

$$\overline{N}(r, b+1; G) = \overline{N}(r, F) = \overline{N}(r, f) + S(r,L) = S(r,L).$$

Now proceeding as in subcase (4.2.1), we arrive at a contradiction. If $c \neq 1$, then from (4.11) we have

$$F - \left(1 - \frac{1}{c}\right) \equiv \frac{-b}{c^2\left(G - \frac{b+c}{c}\right)}.$$

Therefore by Lemma 3.3 we have

$$\overline{N}\left(r, \frac{b+c}{c}; G\right) = \overline{N}(r,F) = \overline{N}(r,f) + S(r,L) = S(r,L).$$

Hence proceeding as in subcase (4.2.1) we arrive at a contradiction.

This completes the proof of the Theorem 2.1.

Proof of Theorem 2.2. Since f and L are transcendental meromorphic function and $R(z)$ is a rational function therefore $R(z)$ is a small function of f and L. Thus, Theorem 2.2 can be proved in a similar way as Theorem 2.1.

5 Conclusion

We have investigate the value distribution of a L- function and an arbitrary meromorphic function using the concept of weighted sharing when certain type of difference-differential polynomials $f^n(f-1)^m f(z+c)$ and $L^n(L-1)^m L(z+c)$ share a small and rational function. L- functions can be analytically continued as meromorphic functions in $\mathbb{C}$ and it has only one possible pole at $s=1$ in $\mathbb{C}$ is the main concept of this paper. Our results extends earlier results due to Hao and Chen.

6 Open Questions

1. Can the condition for n in Theorem 2.1 and Theorem 2.2 be still reduced?
2. Can the difference polynomials in Theorems 2.1 - 2.2 be replaced by difference polynomials of the form $f^n P(f)\Delta_c f$ by using weakly weighted sharing and truncated weighted sharing?

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