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INEQUALITIES INVOLVING GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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Abstract

Trigonometric and Hyperbolic inequalities, which have been obtained by C. Huygens[5], D. S. Mitrinovic[10] and many more, have attracted attention of several researchers. We offer several refinements and generalization of few trigonometric and hyperbolic inequalities involving tangent function, cotangent function, sine function, secant function and cosecant function. The established results are obtained with the aid of the Schwab-Borchardt mean.
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1 Introduction

It is well known from basic calculus that,

$$arcsin(x) = \int_0^x \frac{1}{(1-t^2)^{\frac{1}{2}}} dt, \quad 0 \le x \le 1$$

and

$$\frac{\pi}{2} = arcsin(1) = \int_0^1 \frac{1}{(1-t^2)^{\frac{1}{2}}} dt.$$

We can define the function sin on $[0, \frac{\pi}{2}]$ as the inverse of arcsine and extend it to $(-\infty, \infty)$.

Let, $p > 1$, we can generalize $arcsin(x)$ as

$$arcsin_p(x) = \int_0^x \frac{1}{(1-t^p)^{\frac{1}{p}}} dt, \quad 0 \le x \le 1$$

and

$$\frac{\pi_p}{2} = arcsin_p(1) = \int_0^1 \frac{1}{(1-t^p)^{\frac{1}{p}}} dt.$$

So, we define the π_p[4] function as,

$$\pi_p = 2\int_0^1 \frac{1}{(1-t^p)^{\frac{p+1}{p}}} dt. = 2\frac{\Gamma(\frac{p+1}{p})\Gamma(\frac{1}{p})}{\Gamma(\frac{2}{p})}.$$

The generalized sin function is the inverse of $\arcsin_p(x)$ defined on $[0, \frac{\pi_p}{2}]$. Now we can define generalized cosine function as the derivative of generalized sine function,

$$cos_p(x) = \frac{d}{dx} sin_p(x).$$

It is clear that,

$$cos_p(x) = (1 - sin_p(x)^p)^{\frac{1}{p}}, x \in \left[0, \frac{\pi_p}{2}\right],$$

and

$$\text{(1.1)} \qquad | \, sin_p(x) \, |^p + | \, cos_p(x) \, |^p = 1, x \in \mathbb{R}.$$

It is easy to prove that,

$$\frac{d}{dx}cos_p(x) = -cos_p(x)^{p-2}sin_p(x)^{p-1}, x \in \left[0, \frac{\pi_p}{2}\right].$$

The generalized tangent function is defined as,

$$tan_p(x) = \frac{sin_p(x)}{cos_p(x)}, x \in \mathbb{R} \setminus \{k\pi_p + \frac{\pi_p}{2} : k \in \mathbb{Z}\}.$$

It follows from the equation (1.1) that,

$$\frac{d}{dx}tan_p(x) = 1+ \mid tan_p(x) \mid^p, \quad x \in \left(\frac{-\pi_p}{2}, \frac{\pi_p}{2}\right).$$

Now we can define generalized inverse hyperbolic function as,

$$arcsinh_p(x) = \begin{cases} \int_0^x \frac{1}{(1+t^p)^{\frac{1}{p}}}dt & , x \in [0, \infty) \\ -arcsinh_p(-x) & , x \in (-\infty, 0]. \end{cases}$$

The inverse of $arcsinh_p(x)$ is called as the generalized hyperbolic sine function and it is denoted by $sinh_p(x)$. The generalized hyperbolic cosine function is defined as,

$$cosh_p(x) = \frac{d}{dx}sinh_p(x).$$

These definitions show that,

$$cosh_p(x)^p - \mid sinh_p(x) \mid^p = 1, \quad x \in \mathbb{R}$$

and

$$\frac{d}{dx}cosh_p(x) = cosh_p(x)^{2-p}sinh_p(x)^{p-1}, \quad x \geq 0.$$

The generalized Hyperbolic tangent function is defined as,

$$tanh_p(x) = \frac{sinh_p(x)}{cosh_p(x)},$$

and

$$\frac{d}{dx}tanh_p(x) = 1- \mid tanh_p(x) \mid^p .$$

It is clear that all these generalized functions coincide with the classical ones when $p = 2$ [7].
In recent years, the following two sided inequality for hyperbolic functions has attracted attention of several researchers,

$$(cosh(x))^{\frac{1}{3}} < \frac{sinh(x)}{x} < \frac{cosh(x)+2}{3}, x \neq 0. \tag{1.2}$$

We define same inequality in generalized hyperbolic form as follows,

$$(cosh_p(x))^{\frac{1}{3}} < \frac{sinh_p(x)}{x} < \frac{cosh_p(x)+2}{3}, x \neq 0. \tag{1.3}$$

The left inequality in (1.2) and (1.3) has been obtained by Lazarevic [23]. The counterpart of (1.2) and (1.3) for trigonometric functions are defined as,

$$(cos(\alpha))^{\frac{1}{3}} < \frac{sin(\alpha)}{\alpha} < \frac{cos(\alpha)+2}{3}, 0 < \alpha < \frac{\pi}{2}, \tag{1.4}$$

and

$$(cos_p(\alpha))^{\frac{1}{3}} < \frac{sin_p(\alpha)}{\alpha} < \frac{cos_p(\alpha)+2}{3}, 0 < \alpha < \frac{\pi_p}{2}. \tag{1.5}$$

Generalization of inequalities (1.2) and (1.3) to Jacobian elliptic functions are established in[12].
The left inequalities in (1.4) and (1.5) have been proved by Neumann [20] and Mitrinovic [10], while second inequality is due to Cusa Huygens [5]. Inequalities mentioned in (1.2), (1.3), (1.4) and (1.5) also have been obtained in [2, 15, 19]. For the recent research work in theory of inequalities for hyperbolic and trigonometric functions refer [6, 8, 9, 11, 13, 17, 18]. The goal of this paper is to derive inequalities involving hyperbolic and trigonometric functions. Most of them are the two-sided inequalities which are similar to inequalities (1.2), (1.3), (1.4) and (1.5). In section 2 we recall definition and basic properties of the Schwab-Borchardt mean. Definions of four particular bivariate means, which can be regarded as special case of the Schwab-Borchardt mean, are also included in this section. The main result is derived in section 3.

2 Definitions and Preliminaries

The geometric, arithmetic and the root mean square means of $a > 0$, $b > 0$ will be denoted by G, A and R respectively and they are defined as follows,

$$G = G(u, v) = \sqrt{uv}, A = A(u, v) = \frac{u + v}{2}, R = R(u, v) = \sqrt{\frac{u^2 + v^2}{2}}. \tag{2.1}$$

Other bivariate means used in the subsequent sections include the logarithmic mean which is defined as,

$$L = \left(\frac{a}{tanh_p^{-1}(a)}\right) A. \tag{2.2}$$

The first and second Seiffert means are,

$$P = \left(\frac{a}{sinh_p^{-1}(a)}\right) A. \tag{2.3}$$

and

$$T = \left(\frac{a}{tan_p^{-1}(a)}\right) A. \tag{2.4}$$

For this, one can refer to [8, 11, 18, 20]. Here,

$$a = \frac{u - v}{u + v}, u \neq v \tag{2.5}$$

Another mean introduced in [14], which is defined as follows,

$$M = \left(\frac{a}{sinh_p^{-1}(a)}\right) A. \tag{2.6}$$

It is known that, $G < L < P < A < M < T < R$[14].
All the bivariate means mentioned above are strict homogeneous of degree one and they are strictly increasing in each of it's variables. Let the letter W stand for one of these means. The homogeneity of W implies that,

$$W(u, v) = \sqrt{uv} W(e^x, e^{-x}), \text{ where, } x = \frac{1}{2} \ln\left(\frac{u}{v}\right).$$

It means L, P, T and M are special cases of Schwab-Borchardt mean for $u \geq 0, v > 0$.
This mean will be denoted by $SB(u, v) = SB$. The Schwab-Borchardt mean is the iterative mean. i.e.

$$SB = \lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n,$$

where,

$$u_0 = u, v_0 = v, u_{n+1} = \frac{u_n + v_n}{2}, v_{n+1} = \sqrt{u_{n+1} v_n}, n = 0, 1, 2, . \tag{2.7}$$

Due to[1, 3], it is known that,

$$SB(u, v) = \begin{cases} \frac{\sqrt{v^2 - u^2}}{cos_p^{-1}(\frac{u}{v})}, & \text{if } u < v, \\ \frac{\sqrt{u^2 - v^2}}{cosh_p^{-1}(\frac{u}{v})}, & \text{if } v < u. \end{cases}$$

The mean SB is non symmetric, homogeneous of degree one and strictly increasing in it's variables. It has been shown in [14] that,

$$L = SB(A, B), P = SB(G, A), T = SB(A, R), M = SB(R, A). \tag{2.8}$$

The following two sided inequality is known [14],

$$(uv^2)^{\frac{1}{3}} < SB(u, v) < \frac{u + 2v}{3}. \tag{2.9}$$

using the in variance property, an invariant is a property of a mathematical object which remains unchanged after operations or transformations of a certain type are applied to the objects, it implies that,
$SB(u_n, v_n) = SB(u, v)$ see equation (2.6).
The previous inequality can be generalised as,

$$(u_n v_n^2)^{\frac{1}{3}} < SB(u, v) < \frac{u_n + 2v_n}{3}. \tag{2.10}$$

The sequence of left inequality is strictly increasing while sequence of right inequality is strictly decreasing provided $u \neq v$ [13, 15].

3 Main Result

Theorem 3.1. *Let $x \neq 0$, then,*

$$(cosh_p(x))^{\frac{2}{3}} < \frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))} < \frac{1+2cosh_p(x)}{3}, \tag{3.1}$$

$$((cosh_p(2x))^{\frac{1}{2}} cosh_p^2(x))^{\frac{1}{3}} < \frac{sinh_p(x)}{sinh_p^{-1}(tanh_p(x))} < \frac{(cosh_p(2x))^{\frac{1}{2}} + 2cosh_p(x)}{3}, \tag{3.2}$$

and

$$((cosh_p(2x))cosh_p(x))^{\frac{1}{3}} < \frac{sinh_p(x)}{tan_p^{-1}(tanh_p(x))} < \frac{2(cosh_p(2x))^{\frac{1}{2}} + 2cosh_p(x)}{3}. \tag{3.3}$$

Proof. For, $(u, v) = (e^x, e^{-x})$, we have,

$$G = 1, A = cosh_p(x), R = (cosh_p(2x))^{\frac{1}{2}}$$

and using equation(2.6), we get, $a = tanh_p(x)$.
Moreover using equation (2.8), (2.9) and (2.10), we obtain

$$P = \frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))}, M = \frac{sinh_p(x)}{sinh_p^{-1}(tanh_p(x))}, T = \frac{sinh_p(x)}{tan_p^{-1}(tanh_p(x))}. \tag{3.4}$$

For the proof of (3.1), we use (2.9) with $u = G$ and $v = A$ followed by the application of the second part of (2.8) and first formula of (3.4),we get

$$(GA^2)^{\frac{1}{3}} < SB(G, A) < \frac{G+2A}{3} \implies (GA^2)^{\frac{1}{3}} < P < \frac{G+2A}{3}, \tag{3.5}$$

$$\implies (cosh_p(x)^2)^{\frac{1}{3}} < \frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))} < \frac{1+2cosh_p(x)}{3} \tag{3.6}$$

$$\implies (cosh_p(x))^{\frac{2}{3}} < \frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))} < \frac{1+2cosh_p(x)}{3}. \tag{3.7}$$

Hence (3.1) is proved.
For the proof of (3.2) we use (2.9) with $u = R$ and $v = A$, followed by application of the fourth formula of equation (2.8),we get

$$(RA^2)^{\frac{1}{3}} < SB(R, A) < \frac{R+2A}{3} \implies ((cosh_p(2x))^{\frac{1}{2}} cosh_p^2(x))^{\frac{1}{3}} < M < \frac{(cosh_p(2x))^{\frac{1}{2}} + 2cosh_p(x)}{3}, \tag{3.8}$$

$$\implies ((cosh_p(2x))^{\frac{1}{2}} cosh_p^2(x))^{\frac{1}{3}} < \frac{sinh_p(x)}{sinh_p^{-1}(tanh_p(x))} < \frac{(cosh_p(2x))^{\frac{1}{2}} + 2cosh_p(x)}{3}. \tag{3.9}$$

Hence (3.2) proved. For the proof of (3.3) we use (2.9) with $u = A$ and $v = R$, we have

$$(AR^2)^{\frac{1}{3}} < SB(A, R) < \frac{A+2R}{3}, \implies (AR^2)^{\frac{1}{3}} < T < \frac{A+2R}{3},$$

$$\implies (cosh_p(x)(cosh_p(2x)^{\frac{1}{2}})^2)^{\frac{1}{3}} < \frac{sinh_p(x)}{tan_p^{-1}(tanh_p(x))} < \frac{2(cosh_p(2x))^{\frac{1}{2}} + 2cosh_p(x)}{3}$$

$$\implies ((cosh_p(2x))cosh_p(x))^{\frac{1}{3}} < \frac{sinh_p(x)}{tan_p^{-1}(tanh_p(x))} < \frac{2(cosh_p(2x))^{\frac{1}{2}} + 2cosh_p(x)}{3}.$$

Hence (3.3) is proved. □

Inequality (1.2) is established using the method utilized in the proof of theorem 3.1. Let, $u = A, v = G$, using (2.1) and (2.7), we obtain

Theorem 3.2. *Let $u = A, v = G$, using (2.1) and (2.7), we obtain*

$$SB(A,G) = L = \frac{A}{tanh_p^{-1}(A)}A = \frac{sinh_p(x)}{x}.$$

Equation (2.9) gives the required result as follows,

$$(AG^2)^{\frac{1}{3}} < SB(A,G) < \frac{A+2G}{3}.$$

Let, $G = 1$ *and* $A = cosh_p(x)$,

$$\implies (cosh_p(x))^{\frac{1}{3}} < \frac{sinh_p(x)}{x} < \frac{cosh_p(x)+2}{3}.$$

Let us define new variable α *as,*

$$tanh_p(x) = sin\alpha. \tag{3.10}$$

This implies that,

$$sinh_p(x) = tan\alpha, cosh_p(x) = sec\alpha, x = tanh_p^{-1}(sin\alpha). \tag{3.11}$$

Equation (1.5) is verified using (3.10) and (3.11). For $x \neq 0$,

$$1 < \left(\frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))}\right)\left(\frac{tanh_p(x)}{x}\right) \tag{3.12}$$

and for $0 < \alpha < \frac{\pi_p}{2}$,

$$1 < \left(\frac{sin_p(\alpha)}{tanh_p^{-1}(sin_p(\alpha))}\right)\left(\frac{tan_p(\alpha)}{\alpha}\right). \tag{3.13}$$

Proof. Using left inequality of equation (2.9), let $u = A$ and $v = G$, we have

$$(AG^2)^{\frac{1}{3}} < SB(A,G) \implies (AG^2)^{\frac{1}{3}} < L. \tag{3.14}$$

Similarly, let $u = G$ and $v = A$ in (2.9), we obtain

$$(GA^2)^{\frac{1}{3}} < SB(G,A) \implies (GA^2)^{\frac{1}{3}} < P. \tag{3.15}$$

Multiplying (3.14) and (3.15), we get

$$(AG^2)^{\frac{1}{3}}(GA^2)^{\frac{1}{3}} < PL \implies AG < PL. \tag{3.16}$$

Let, $(u,v) = (e^x, e^{-x})$ and

$$A = cosh_p(x), G = 1, P = \frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))}, L = \frac{sinh_p(x)}{x}.$$

Therefore (3.16) gives,

$$cosh_p(x) < \left(\frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))}\right)\left(\frac{sinh_p(x)}{x}\right) \implies 1 < \left(\frac{sinh_p(x)}{sin_p^{-1}(tanh_p(x))}\right)\left(\frac{tanh_p(x)}{x}\right).$$

Hence (3.12) proved. Inequality (3.13) follows from (3.12) by using transformations (3.4) and (3.10). □

4 Conclusion

Using Schwab-Borchardt mean, both the refinement of generalized trigonometric and hyperbolic function is verified.

Conflict of Interest

Authors declare that, there is no conflict of interest.

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