ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(2) (2023), 287-300**
*(Dedicated to Professor V. P. Saxena on His $80^{th}$ Birth Anniversary Celebrations)*

**APPLICATION OF FIXED POINT THEOREM IN THE SOLUTION OF INTEGRO-DIFFERENTIAL EQUATIONS: A COMPLEX VALUED APPROACH***

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*(Received: February 23, 2023; In format: February 26, 2023; Revised : December 04, 2023; Accepted : December 06, 2023)*

DOI: https://doi.org/10.58250/jnanabha.2023.53235

**Abstract**

It's remarkable to note that Complex valued Integro-differential and integral type equations are currently intensifying the attention of appreciable researchers due to their comprehensive applications. Thus, this study is fully devoted to the application part of the complex valued controlled, double controlled metric $\eth_{\mathbb{C}}$. We introduce an extended version of the Fisher and Banach type contraction theorem and present some examples to sustain our results. As part of the main theorem's application, we address a common solution with uncertainty in two different folds as follows: [I] Applying the fractional Adams-Bashforth method to the (1.1) $FVI_dE$. [II] Applying it to the integral type equation (1.2) in the setting of the Extended complex valued metric space.

**2020 Mathematical Science Classification.**47H09,24H25,34A08,47H10.
**Keywords and Phrases:** Atangana Baleanu Fractional integral operator,fredholm Volterra integro differential equation*($FVI_dE$)*, Complex valued metric space(CVMS), Common fixed point, Cauchy sequence, Contractive condition and completeness.

## 1 Introduction

The terms calculus of integral equation and fractional calculus are introduced more than 10 decade back. These ten decade seems like a really big time but predominantly these topics are extensively gain new structures and effectively applied in different part of mathematics like fixed point theory, fuzzy theory and so on. Recently Atangana-Baleanu [1] studied new type of fractional derivative targeting non singular/local kernel. Subsequently in 2023 Shinde [33] gave complex valued version of existence and common solution for second order nonlinear boundary value problem using greens function along with another application of fixed point results for multivalued mapping in setting of CVMS. In 2017, Kumar et al. [19] studied a fractional non-linear biological model problem and its approximate solutions through Volterra Integral Equation. In 2019, Kumar [20] studied a class of two variable sequence of functions satisfying Abel's Integral equation and the phase shifts. in 2019 [20] H. Kumar given A class of two variables sequence of functions satisfying Abel's integral equation and the phase shifts. In literature we can see many generalizations of Atangana-Baleanu fractional derivative like $AB$-derivative [13], $AB$ derivative via $MHD$ channel flow [34], $AB_{RL}$ type [12], we can see more [8,9,11,16,17,18,21,22,26,29,31,32,34,35]. Here we recollecting the definition of Atangana-Baleanu fractional integral, Let $\omega \in (0,1]$ and integral define as,

$$ {}^{AB}_{s}D^{\omega}_{t}f(t) = \frac{(1-\omega)}{\zeta(\omega)}f(t) + \frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_s^t f(h)\frac{(t-h)^{\omega}}{(t-h)}dh. $$

where, $0 < t < s$ ; normalization function $\zeta(0) = \zeta(1) = 1$.

Subsequently, by applying fractional Adams Bashforth method to the (1.1) $FVI_dE$ in the setting of complex valued controlled metric we deal with following conditions,

$$ (1.1) \quad \tilde{\beth}_0 = \tilde{\beth}(0;\ell);{}^{ABC}_{0} D^{\omega}_{\hbar}\tilde{\beth}(\hbar;\ell) $$
$$ = \aleph(\hbar) + \Re(\hbar).\tilde{\beth}(\hbar,\ell) + \int_0^{\hbar} \mho_1(\hbar,\xi).\chi_1(\tilde{\beth}(\xi,\ell))d\xi + \int_0^1 \mho_2(\hbar,\xi)\chi_2(\tilde{\beth}(\xi,\ell))d\xi, $$

*Presented in $5^{\text{th}}$ International Conference of Vijñāna Parishand of India on Recent Advances in Computational Mathematics and Applied Sciences (IC-RA-CMAS) December 09-11, 2023 held at MRIIRS, Faridabad, Haryana, India.

where, ${}_{0}^{ABC}D_{\hbar}^{\omega}$ ABC type of order $\omega$ such that $\tilde{\beth}(\hbar;\ell) = [\underline{\beth}(\hbar;\ell), \overline{\beth}(\hbar;\ell)]$; continuous function $\mho_1, \mho_2 : \nabla \times \nabla \to \mathbb{R}$, $\aleph : \nabla \to \mathbb{R}$; Lipschitz continuous function $\chi_1, \chi_2 : \nabla \to \mathbb{R}$; $L^z(\nabla, \mathbb{R})$ and $C^z(\nabla, \mathbb{R})$ are space of all continuous functions and the space of all Lebesgue integrable functions on $\nabla$, $\tilde{\beth}(\hbar;\ell) \in L^z(\nabla, \mathbb{R}) \cap C^z(\nabla, \mathbb{R})$. At the end we deal with following Integral type equation,

$$\Re_1(\hbar) - \curlywedge(\hbar) = \int_0^{\hbar} \chi(\hbar, \ell)\aleph(\ell, \Re_1 \ell)d\ell, \tag{1.2}$$

which has two bounded continuous function namely $\curlywedge(\hbar) : [0,1] \to \mathbb{R}$ and $\aleph(\hbar, \Re_1(\hbar)) : [0,1] \times \mathbb{R} \to \mathbb{R}$. The function $\chi : [0,1) \times [0,1) \to [0,\infty)$ with $\chi(\hbar, .) \in L^1[0,1]$ and $0 \leq \hbar \leq 1$. We successfully applied fixed point solution to above integral type equation. The novel approach has a promising uniqueness of solution in different fields, for more we can see[10, 11, 19, 22, 23, 24, 25, 26].

## 2 Preliminaries

Azam, Khan and Fisher [2] studied notion of complex valued metric and given important definition as follows,

**Definition 2.1.** *Consider a partial order $\precsim$ defined on a complex number($\mathbb{C}$), $\hbar \precsim \ell$ iff Real part of ($\hbar$) $\leq$ Real part of ($\ell$) ; Imaginary part of ($\hbar$) $\leq$ Imaginary part of ($\ell$). It follows, $\hbar \leq \ell$*

1. *Real part ($\hbar$)<Real part ($\ell$) ; Imaginary part ($\hbar$) < Imaginary part ($\ell$).*
2. *Real part ($\hbar$) = Real part ($\ell$) ; Imaginary part ($\hbar$) = Imaginary part ($\ell$).*
3. *Real part ($\hbar$) < Real part ($\ell$) ; Imaginary part ($\hbar$) = Imaginary part ($\ell$).*
4. *Real part ($\hbar$) = Real part ($\ell$) ; Imaginary part ($\hbar$) < Imaginary part ($\ell$).*

**Definition 2.2.** *Lets define the function $\eth_{\mathbb{C}} : \nabla \times \nabla \to \mathbb{C}$, where non empty set $\nabla$; the function $\psi, \zeta : \nabla \times \nabla \to [1, \infty)$ and $\mathbb{C}$ be the set of complex numbers. We define following condition for $\forall\ \hbar, \ell, \mu \in \nabla$ ,*
$\mathfrak{S}_1$ : *$\hbar = \ell$ if and only if $\eth_{\mathbb{C}}(\hbar, \ell) = 0$.*
$\mathfrak{S}_2$ : *$\eth_{\mathbb{C}}(\hbar, \ell) = \eth_{\mathbb{C}}(\ell, \hbar)$.*
$\mathfrak{S}_3$ : *Controlled triangle inequality- $\eth_{\mathbb{C}}(\hbar, \ell) \precsim \psi(\hbar, \mu)\eth_{\mathbb{C}}(\hbar, \mu) + \psi(\mu, \ell)\eth_{\mathbb{C}}(\mu, \ell)$.*
$\mathfrak{S}_4$ : *Extended triangle inequality- $\eth_{\mathbb{C}}(\hbar, \ell) \precsim \psi(\hbar, \ell)[\eth_{\mathbb{C}}(\hbar, \mu) + \eth_{\mathbb{C}}(\mu, \ell)]$.*
$\mathfrak{S}_5$ : *Double controlled triangle inequality- $\eth_{\mathbb{C}}(\hbar, \ell) \precsim \psi(\hbar, \mu)\eth_{\mathbb{C}}(\hbar, \mu) + \zeta(\mu, \ell)\eth_{\mathbb{C}}(\mu, \ell)$.*

**Definition 2.3.** *If $\eth_{\mathbb{C}}$ satisfied $\mathfrak{S}_1$, $\mathfrak{S}_2$ and $\mathfrak{S}_4$, then $\eth_{\mathbb{C}}$ is called complex valued extended metric and the pair $(\nabla, \eth_{\mathbb{C}})$ called complex valued extended metric space.*

**Definition 2.4.** *If $\eth_{\mathbb{C}}$ satisfied $\mathfrak{S}_1$, $\mathfrak{S}_2$ and $\mathfrak{S}_3$, then $\eth_{\mathbb{C}}$ is called complex valued controlled metric and the pair $(\nabla, \eth_{\mathbb{C}})$ called complex valued controlled metric space.*

**Definition 2.5.** *If $\eth_{\mathbb{C}}$ satisfied $\mathfrak{S}_1$, $\mathfrak{S}_2$ and $\mathfrak{S}_5$, then $\eth_{\mathbb{C}}$ is called complex valued double Controlled metric and the pair $(\nabla, \eth_{\mathbb{C}})$ called complex valued double Controlled metric space.*

**Example 2.1.** Lets define the function $\eth_{\mathbb{C}} : \nabla \times \nabla \to \mathbb{C}$ and the set $\nabla = \{2, 3, 1\}$ which has, $\eth_{\mathbb{C}}(2,3) = i$; $\eth_{\mathbb{C}}(1,2) = 2 + 4i$; $\eth_{\mathbb{C}}(3,2) = i$; $\eth_{\mathbb{C}}(2,1) = 2 + 4i$ $\eth_{\mathbb{C}}(1,1) = 0$; $\eth_{\mathbb{C}}(1,3) = 1 - i$; $\eth_{\mathbb{C}}(2,2) = 0$; $\eth_{\mathbb{C}}(3,1) = 1 - i$; $\eth_{\mathbb{C}}(3,3) = 0$. Again define $\zeta, \psi : \nabla \times \nabla \to [1, \infty)$ as
$\psi(2,3) = \psi(3,2) = \frac{8}{7}$, $\psi(1,2) = \psi(2,1) = 1$, $\psi(1,3) = \psi(3,1) = \frac{3}{2}$,
$\zeta(2,3) = \zeta(3,2) = \frac{9}{2}$, $\zeta(1,2) = \zeta(2,1) = \frac{7}{6}$, $\zeta(3,1) = \zeta(1,3) = 1$.

**Proposition 2.1.** *In above example we easily verify $\eth_{\mathbb{C}}$ is double controlled metric type but $\eth_{\mathbb{C}}$ is neither a complex valued extended metric nor a complex valued controlled metric.*

**Lemma 2.1.** *Suppose $(\nabla, \eth_{\mathbb{C}})$ be a $\eth_{\mathbb{C}}$ metric space. Then the sequence $\{\hbar_n\}$ in $\nabla$ is a cauchy sequence if and only if $\mid \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+s}) \mid \to 0$ as $n \to \infty$ where $s \in \mathbb{N}$.*

**Lemma 2.2.** *Suppose $(\nabla, \eth_{\mathbb{C}})$ be a $\eth_{\mathbb{C}}$ metric space. Then the sequence $\{\hbar_n\}$ in $\nabla$ Converges to $\hbar$ if and only if $\mid \eth_{\mathbb{C}}(\hbar_n, \hbar) \mid \to 0$ as $n \to \infty$.*

**Definition 2.6.** *Assume $\{\hbar_n\}$ be a sequence in a $\eth_{\mathbb{C}}$ metric space $(\nabla, \eth_{\mathbb{C}})$ and $\hbar \in \nabla$, then $(\nabla, \eth_{\mathbb{C}})$ is said to be a complete $\eth_{\mathbb{C}}$ metric space if every Cauchy sequence is convergent in $(\nabla, \eth_{\mathbb{C}})$.*

**Definition 2.7.** *Suppose $\{\hbar_n\}$ be a sequence in a $\eth_{\mathbb{C}}$ metric space $(\nabla, \eth_{\mathbb{C}})$ and $\hbar \in \nabla$, then $\hbar$ is a limit point of $\{\hbar_n\}$ if for every $\epsilon \in \mathbb{C}$ there exist $n_0 \in \mathbb{N}$ such that $\eth_{\mathbb{C}}(\{\hbar_n\}, n) \prec \epsilon, \forall n \succ n_0$ that is $\lim_{n\to\infty}, \hbar_n = n$.*

**Definition 2.8.** *Suppose $\{\hbar_n\}$ be a sequence in a $\eth_{\mathbb{C}}$ metric space $(\nabla, \eth_{\mathbb{C}})$ and $\hbar \in \nabla$, then $\{\hbar_n\}$ is a cauchy sequence if for any $\epsilon \in \mathbb{C}$ there exist $n_0 \in \mathbb{N}$ such that $\eth_{\mathbb{C}}(\hbar_n, \hbar_{n+s}) \prec \epsilon, \forall n \succ n_0$ and $s \in \mathbb{N}$ .*

**Remark 2.1** ([1])**.** *The left sided AB fractional integral of order $\omega \in (0,1]$ for a function $\tilde{\beth}$ is defined as*

$$^{AB}\text{ß}_0^{\omega}\tilde{\beth}(\hbar) = \frac{1}{\zeta(\omega)}[(1-\omega)\tilde{\beth}(\hbar) + \frac{\omega}{\Xi(\omega)}\int_0^{\hbar}\frac{(\hbar-\xi)^{\omega}}{(\hbar-\xi)}\tilde{\beth}(\xi)d\xi].$$

*where, we have continuous function $\tilde{\beth}(\hbar)$ on the interval $(0, b)$.*

**Remark 2.2** ([7])**.** *Consider the map $\psi : \mathbb{R} \to \nabla$ satisfying following properties,*

- *The closure of Supp($\psi$) is compact.*
- *$\psi$- normal, Upper semi-continuous and convex.*

**Remark 2.3** ([7])**.** *The parametric interval of $\tilde{\psi}$ is given by,*

$$\tilde{\psi} = [\underline{\psi}(\beta), \overline{\psi}(\beta)] \text{ and } 0 \leq \beta \leq 1$$

- *With respect to $\beta$, $\underline{\psi}(\beta)$ is a left continuous and non-decreasing,*
- *$\forall\ \beta \in \nabla$, we have $\underline{\psi}(\beta) \leq \overline{\psi}(\beta)$,*
- *With respect to $\beta$, $\overline{\psi}(\beta)$ is a right continuous and non-decreasing.*

**Lemma 2.3.** *Let $(\nabla, \eth_{\mathbb{C}})$ be a complex valued controlled metric space. If the functional $\eth_{\mathbb{C}} : \nabla \times \nabla \to \mathbb{C}$ is continuous then limit of every convergent sequence is unique.*

**Lemma 2.4.** *Let $(\nabla, \eth_{\mathbb{C}})$ be a complex valued controlled metric space. If a sequence $\{\hbar_n\}$ in $\nabla$ is Cauchy sequence, such that $\hbar_n \neq \hbar_m$ when $m \neq n$. Then we say $\{\hbar_n\}$ converges at most one point.*

In this article, we present a new fixed point result under extended complex valued metric space with suitable examples, results and finally two folds of the application part.

## 3 Main Results

Moving towards the following Theorem and its hypothesis, we generalize some ideas via controlled, double controlled complex valued metric space.

**Theorem 3.1.** *Consider $(\nabla, \eth_{\mathbb{C}})$ be a complete $\eth_{\mathbb{C}}$ metric space. Suppose $\aleph = \frac{\eta}{(\Re^{\flat}-\mu)} < 1$ and*

$$\frac{1}{\aleph} > \sup_{1\leq m}\lim_{i\to\infty}\frac{\psi(\hbar_{i+1}, \hbar_{i+2})}{\psi(\hbar_i, \hbar_{i+1})}\zeta(\hbar_{i+1}, \hbar_m). \tag{3.1}$$

*For every $\hbar, \ell \in \nabla$ & $0 \prec \eth_{\mathbb{C}}(\hbar, \ell)$, we use $\mu, \lambda, \eta$ are non negative real numbers with $\mu + \lambda + \eta < 1$, $1 \leq \Re, \flat$ we choose $\hbar_n = \tilde{\beth}_2{}^n\hbar_0 \in \nabla$ for all $\hbar_0 \in \nabla$ then the map $\tilde{\beth}_1, \tilde{\beth}_2 : \nabla \to \nabla$ satisfying,*

$$\eth_{\mathbb{C}}(\tilde{\beth}_1\hbar, \tilde{\beth}_2\ell).\Re^{\flat} \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar, \tilde{\beth}_1\hbar)\eth_{\mathbb{C}}(\ell, \tilde{\beth}_2\ell)}{1+\eth_{\mathbb{C}}(\hbar, \ell)}\} + \lambda\{\frac{\eth_{\mathbb{C}}(\tilde{\beth}_1\hbar, \ell).\eth_{\mathbb{C}}(\tilde{\beth}_2\hbar, \hbar)}{1+\eth_{\mathbb{C}}(\hbar, \ell)}\} + \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\}, \tag{3.2}$$

*afterward Assume that, $\lim_{n\to\infty}\zeta(\hbar_n, \hbar)$ , $\lim_{n\to\infty}\psi(\hbar, \hbar_n)$ both are exist and finite, then $\tilde{\beth}_1$ and $\tilde{\beth}_2$ admits unique common fixed point.*

*Proof.* Suppose, $\hbar_0 \in \nabla$ be any arbitrary point. Let the sequence $\hbar_n = \tilde{\beth}_2{}^n\hbar_0 \in \nabla$ which satisfies hypothesis of theorem and we define it as,

$$\tilde{\beth}_1\hbar_{2n} = \hbar_{2n+1}\ ;\ \tilde{\beth}_2\hbar_{2n+1} = \hbar_{2n+2}, n = 0, 1, 2, ... \tag{3.3}$$

$$\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2}).\Re^{\flat} = \eth_{\mathbb{C}}(\tilde{\beth}_1\hbar_{2n}, \tilde{\beth}_2\hbar_{2n+1}).\Re^{\flat} \precsim$$

$$\mu\{\frac{\eth_{\mathbb{C}}(\hbar_{2n}, \tilde{\beth}_1\hbar_{2n})\eth_{\mathbb{C}}(\hbar_{2n+1}, \tilde{\beth}_2\hbar_{2n+1})}{1+\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\}}\} + \lambda\{\frac{\eth_{\mathbb{C}}(\tilde{\beth}_1\hbar_{2n}, \hbar_{2n+1}).\eth_{\mathbb{C}}(\tilde{\beth}_2\hbar_{2n}, \hbar_{2n})}{1+\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\}}\} + \eta\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\}$$

$$\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2})\Re^{\flat} \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2})}{1+\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\}}\}+\lambda\{\frac{\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+1}).\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n})}{1+\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\}}\}+\eta\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\},$$

$$\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2}).\Re^{\flat} \precsim \mu\{\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2})\} + \eta\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\},$$

$$\eth_{\mathbb{C}}(\hbar_{2n+1}, \hbar_{2n+2}).(\Re^{\flat} - \mu) \precsim \eta\{\eth_{\mathbb{C}}(\hbar_{2n}, \hbar_{2n+1})\},$$

$$\eth_{\mathbb{C}}(\hbar_{2n+1},\hbar_{2n+2}) \precsim \frac{\eta}{(\Re^{\flat}-\mu)}\{\eth_{\mathbb{C}}(\hbar_{2n},\hbar_{2n+1})\},$$
$$\eth_{\mathbb{C}}(\hbar_{2n+1},\hbar_{2n+2}) \precsim \aleph.\{\eth_{\mathbb{C}}(\hbar_{2n},\hbar_{2n+1})\}.$$

Similarly, we get

$$\eth_{\mathbb{C}}(\hbar_{2n+2},\hbar_{2n+3}) \precsim \frac{\eta}{(\Re^{\flat}-\mu)}\{\eth_{\mathbb{C}}(\hbar_{2n+1},\hbar_{2n+2})\} \tag{3.4}$$
$$\eth_{\mathbb{C}}(\hbar_{2n+1},\hbar_{2n+2}) \precsim \aleph\{\eth_{\mathbb{C}}(\hbar_{2n},\hbar_{2n+1})\} \text{ where, } \aleph = \frac{\eta}{(\Re^{\flat}-\mu)} < 1$$
$$\mid \eth_{\mathbb{C}}(\hbar_n,\hbar_{n+1}) \mid \precsim \aleph \mid \{\eth_{\mathbb{C}}(\hbar_{n-1},\hbar_n)\} \mid,$$
$$\mid \eth_{\mathbb{C}}(\hbar_n,\hbar_{n+1}) \mid \precsim \aleph^2 \mid \{\eth_{\mathbb{C}}(\hbar_{n-2},\hbar_{n-1})\} \mid,$$
$$\mid \eth_{\mathbb{C}}(\hbar_n,\hbar_{n+1}) \mid \precsim \aleph^n \mid \{\eth_{\mathbb{C}}(\hbar_0,\hbar_1)\} \mid .$$

For every $n < m$, where $m, n \in \mathbb{N}$

$$\mid \eth_{\mathbb{C}}(\hbar_n,\hbar_m) \mid \precsim \psi(\hbar_n,\hbar_{n+1}) \mid \eth_{\mathbb{C}}(\hbar_n,\hbar_{n+1}) \mid + \zeta(\hbar_{n+1},\hbar_m) \mid \eth_{\mathbb{C}}(\hbar_{n+1},\hbar_m) \mid \tag{3.5}$$
$$\precsim \psi(\hbar_n,\hbar_{n+1}) \mid \eth_{\mathbb{C}}(\hbar_n,\hbar_{n+1}) \mid + \zeta(\hbar_{n+1},\hbar_m)[\psi(\hbar_{n+1},\hbar_{n+2}) \mid \eth_{\mathbb{C}}(\hbar_{n+1},\hbar_{n+2}) \mid + \zeta(\hbar_{n+2},\hbar_m) \mid \eth_{\mathbb{C}}(\hbar_{n+2},\hbar_m) \mid]$$
$$\precsim \psi(\hbar_n,\hbar_{n+1}) \mid \eth_{\mathbb{C}}(\hbar_n,\hbar_{n+1}) \mid + \zeta(\hbar_{n+1},\hbar_m)\psi(\hbar_{n+1},\hbar_{n+2}) \mid \eth_{\mathbb{C}}(\hbar_{n+1},\hbar_{n+2}) \mid + \zeta(\hbar_{n+1},\hbar_m)\zeta(\hbar_{n+2},\hbar_m) \mid \eth_{\mathbb{C}}(\hbar_{n+2},\hbar_m) \mid$$
$$\precsim \psi(\hbar_n,\hbar_{n+1}) \mid \eth_{\mathbb{C}}(\hbar_n,\hbar_{n+1}) \mid + \sum_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\zeta(\hbar_j,\hbar_m))\psi(\hbar_i,\hbar_{i+1}) \mid \eth_{\mathbb{C}}(\hbar_i,\hbar_{i+1}) \mid + \prod_{k=n+1}^{m-1}\zeta(\hbar_k,\hbar_m) \mid \eth_{\mathbb{C}}(\hbar_{m-1},\hbar_m) \mid$$
$$\precsim \psi(\hbar_n,\hbar_{n+1}).\aleph^n \mid \{\eth_{\mathbb{C}}(\hbar_0,\hbar_1)\} \mid + \sum_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\zeta(\hbar_j,\hbar_m))\psi(\hbar_i,\hbar_{i+1}).\aleph^i \mid \{\eth_{\mathbb{C}}(\hbar_0,\hbar_1)\} \mid$$
$$+ \prod_{i=n+1}^{m-1}\zeta(\hbar_i,\hbar_m).\aleph^{m-1} \mid \{\eth_{\mathbb{C}}(\hbar_0,\hbar_1)\} \mid$$
$$\precsim \psi(\hbar_n,\hbar_{n+1})\aleph^n \mid \{\eth_{\mathbb{C}}(\hbar_0,\hbar_1)\} \mid + \sum_{i=n+1}^{m-2}(\prod_{j=n+1}^{i}\zeta(\hbar_j,\hbar_m))\psi(\hbar_i,\hbar_{i+1})\aleph^i \mid \{\eth_{\mathbb{C}}(\hbar_0,\hbar_1)\} \mid +$$
$$\prod_{i=n+1}^{m-1}\zeta(\hbar_i,\hbar_m)\aleph^{m-1}\psi(\hbar_{m-1},\hbar_m) \mid \{\eth_{\mathbb{C}}(\hbar_0,\hbar_1)\} \mid$$
$$\precsim \psi(\hbar_n,\hbar_{n+1}).\aleph^n \mid \{\eth_{\mathbb{C}}(\hbar_0,\hbar_1)\} \mid + \sum_{i=n+1}^{m-1}(\prod_{j=n+1}^{i}\zeta(\hbar_j,\hbar_m))\psi(\hbar_i,\hbar_{i+1}).\aleph^i \mid \{\eth_{\mathbb{C}}(\hbar_0,\hbar_1)\} \mid$$
$$\precsim \psi(\hbar_n,\hbar_{n+1}).\aleph^n \mid \{\eth_{\mathbb{C}}(\hbar_0,\hbar_1)\} \mid + \sum_{i=n+1}^{m-1}(\prod_{j=0}^{i}\zeta(\hbar_j,\hbar_m))\psi(\hbar_i,\hbar_{i+1}).\aleph^i \mid \{\eth_{\mathbb{C}}(\hbar_0,\hbar_1)\} \mid .$$

Hence we write,

$$\mid \eth_{\mathbb{C}}(\hbar_n,\hbar_m) \mid \precsim \mid \eth_{\mathbb{C}}(\hbar_0,\hbar_1) \mid [\aleph^n.\psi(\hbar_n,\hbar_{n+1}) + (\mho_{m-1}-\mho_m)],$$

where, $\mho_\iota = \sum_{i=0}^{\iota}(\prod_{j=0}^{i}\zeta(\hbar_j,\hbar_m))\psi(\hbar_i,\hbar_{i+1})\aleph^i$.
As we have (3.1) and using ratio test we get limit of $\{\mho_n\}$ exists, so it is Cauchy. When we apply ratio test to following term and letting $m, n \to \infty$ in (3.6),

$$\omega_i = (\prod_{j=0}^{i}\zeta(\hbar_j,\hbar_m))\psi(\hbar_i,\hbar_{i+1}), \text{ and } \lim_{m,n\to\infty} \mid \eth_{\mathbb{C}}(\hbar_n,\hbar_m) \mid = 0, \tag{3.6}$$

which gives sequence $\{\hbar_n\}$ is Cauchy. Since $(\nabla, \eth_{\mathbb{C}})$ is Complete then $\exists F \in \nabla$ such that,

$$\lim_{m,n\to\infty} \mid \eth_{\mathbb{C}}(\hbar_n, F) \mid = 0. \tag{3.7}$$

Now, by triangle inequality,

$$| \eth_{\mathbb{C}}(\digamma, \hbar_{n+1}) | \precsim \psi(\digamma, \hbar_n) | \eth_{\mathbb{C}}(\digamma, \hbar_n) | + \zeta(\hbar_n, \hbar_{n+1}) | \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) | . \tag{3.8}$$

By Using (3.6) and (3.8) we finally get,

$$\lim_{n\to\infty} | \eth_{\mathbb{C}}(\digamma, \hbar_{n+1}) | = 0. \tag{3.9}$$

Now we claim $\digamma = \tilde{\beth}_1\digamma$,

$$| \eth_{\mathbb{C}}(\digamma, \tilde{\beth}_1\digamma) | \precsim \psi(\digamma, \hbar_{n+2}) | \eth_{\mathbb{C}}(\digamma, \hbar_{n+2}) | + \zeta(\hbar_{n+2}, \tilde{\beth}_1\digamma) | \eth_{\mathbb{C}}(\hbar_{n+2}, \tilde{\beth}_1\digamma) | \tag{3.10}$$

$$| \eth_{\mathbb{C}}(\digamma, \tilde{\beth}_1\digamma) | \precsim \psi(\digamma, \hbar_{n+2}) | \eth_{\mathbb{C}}(\digamma, \hbar_{n+2}) | + \zeta(\hbar_{n+2}, \tilde{\beth}_1\digamma) | \eth_{\mathbb{C}}(\tilde{\beth}_2\hbar_{n+1}, \tilde{\beth}_1\digamma) |$$

$$| \eth_{\mathbb{C}}(\digamma, \tilde{\beth}_1\digamma) | \precsim \psi(\digamma, \hbar_{n+2}) | \eth_{\mathbb{C}}(\digamma, \hbar_{n+2}) | + \zeta(\hbar_{n+2}, \tilde{\beth}_1\digamma).\Re^{\flat} | \eth_{\mathbb{C}}(\tilde{\beth}_1\digamma, \tilde{\beth}_2\hbar_{n+1}) |$$

$$\precsim \psi(\digamma, \hbar_{n+2}) | \eth_{\mathbb{C}}(\digamma, \hbar_{n+2}) | + \zeta(\hbar_{n+2}, \tilde{\beth}_1\digamma)\mu\{\frac{\eth_{\mathbb{C}}(\digamma, \tilde{\beth}_1\digamma)\eth_{\mathbb{C}}(\hbar_{n+1}, \tilde{\beth}_2\hbar_{n+1})}{1 + \eth_{\mathbb{C}}(\digamma, \hbar_{n+1})}\} +$$

$$\lambda\{\frac{\eth_{\mathbb{C}}(\tilde{\beth}_1\digamma, \hbar_{n+1}).\eth_{\mathbb{C}}(\tilde{\beth}_2\digamma, \digamma)}{1 + \eth_{\mathbb{C}}(\digamma, \hbar_{n+1})}\} + \eta\{\eth_{\mathbb{C}}(\digamma, \hbar_{n+1})\}.$$

We write this as,

$$\precsim \psi(\digamma, \hbar_{n+2}) | \eth_{\mathbb{C}}(\digamma, \hbar_{n+2}) | + \zeta(\hbar_{n+2}, \tilde{\beth}_1\digamma)\mu\{\frac{\eth_{\mathbb{C}}(\digamma, \tilde{\beth}_1\digamma)\eth_{\mathbb{C}}(\hbar_{n+1}, \hbar_{n+2})}{1 + \eth_{\mathbb{C}}(\digamma, \hbar_{n+1})}\} +$$

$$\lambda\{\frac{\eth_{\mathbb{C}}(\tilde{\beth}_1\digamma, \hbar_{n+1})\eth_{\mathbb{C}}(\tilde{\beth}_2\digamma, \digamma)}{1 + \eth_{\mathbb{C}}(\digamma, \hbar_{n+1})}\} + \eta\{\eth_{\mathbb{C}}(\digamma, \hbar_{n+1})\}.$$

Using (3.6),(3.7) and (3.8), we get

$$| \eth_{\mathbb{C}}(\digamma, \tilde{\beth}_1\digamma) | = 0.$$

Hence, $\tilde{\beth}_1$ admits fixed point $\digamma$. Subsequently we prove $\tilde{\beth}_2$ admits fixed point as $\digamma$. Now finally we have to work on Uniqueness property, that is $\tilde{\beth}_1$ and $\tilde{\beth}_2$ admits unique common fixed point.
On Contrary assume that $\digamma$ and $\digamma^*$ are two common fixed points of $\tilde{\beth}_1$ and $\tilde{\beth}_2$ & $\digamma \neq \digamma^*$.

$$\eth_{\mathbb{C}}(\digamma, \digamma^*).\Re^{\flat} = \eth_{\mathbb{C}}(\tilde{\beth}_1\digamma, \tilde{\beth}_2\digamma^*).\Re^{\flat} \tag{3.11}$$

$$\precsim \mu\{\frac{\eth_{\mathbb{C}}(\digamma, \tilde{\beth}_1\digamma)\eth_{\mathbb{C}}(\digamma^*, \tilde{\beth}_2\digamma^*)}{1 + \eth_{\mathbb{C}}(\digamma, \digamma^*)}\} + \lambda\{\frac{\eth_{\mathbb{C}}(\tilde{\beth}_1\digamma, \digamma^*).\eth_{\mathbb{C}}(\tilde{\beth}_2\digamma, \digamma)}{1 + \eth_{\mathbb{C}}(\digamma, \digamma^*)}\} + \eta\{\eth_{\mathbb{C}}(\digamma, \digamma^*)\}$$

$$\eth_{\mathbb{C}}(\digamma, \digamma^*).\Re^{\flat} \precsim \eta\{\eth_{\mathbb{C}}(\digamma, \digamma^*)\} \text{ which impies } \eth_{\mathbb{C}}(\digamma, \digamma^*).(\Re^{\flat} - \eta) \precsim 0.$$

Hence we get, $\eth_{\mathbb{C}}(\digamma, \digamma^*) = 0$ which is the contradiction to our assumption. Thus $\digamma = \digamma^*$, $\tilde{\beth}_1$ and $\tilde{\beth}_2$ admits unique common fixed point. □

If we assume $\tilde{\beth}_1$ & $\tilde{\beth}_2$ are equal and which is equal to $\tilde{\beth}$ along with we include map $\tilde{\beth} : \nabla \to \nabla$ be a continuous mapping; $\Re, \flat = 1$ & $\lambda, \mu = 0$ then Theorem 3.1 reduces to following result,

**Theorem 3.2.** *Consider $(\nabla, \eth_{\mathbb{C}})$ be a Complete $\eth_{\mathbb{C}}$ metric space. Suppose*

$$\frac{1}{\eta} > \sup_{1\leq m} \lim_{i\to\infty} \frac{\psi(\hbar_{i+1}, \hbar_{i+2})}{\psi(\hbar_i, \hbar_{i+1})}\zeta(\hbar_{i+1}, \hbar_m). \tag{3.12}$$

*For every $\hbar, \ell \in \nabla$ & $0 \prec \eth_{\mathbb{C}}(\hbar, \ell)$, we use $\eta$ non negative real numbers with $0 < \eta < 1$, we choose $\hbar_n = \tilde{\beth}^n\hbar_0 \in \nabla$ for all $\hbar_0 \in \nabla$ then the map $\tilde{\beth} : \nabla \to \nabla$ be a continuous mapping such that,*

$$\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) \precsim \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\}, \tag{3.13}$$

*afterward assume that, $\lim_{n\to\infty} \zeta(\hbar_n, \hbar)$ , $\lim_{n\to\infty} \psi(\hbar, \hbar_n)$ both exist and finite, then $\tilde{\beth}$ admits unique common fixed point.*

*Proof.* Consider $\hbar_n = \{\tilde{\beth}^n \hbar_0\}$ and by Using inequalities (3.13),

$$\eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) \precsim \eta \eth_{\mathbb{C}}(\hbar_{n-1}, \hbar_n) \precsim ... \precsim \eta \eth_{\mathbb{C}}(\hbar_0, \hbar_1), \forall n \geq 0$$

for every $m > n$, where $m, n \in \mathbb{N}$

$$\eth_{\mathbb{C}}(\hbar_n, \hbar_m) \precsim \psi(\hbar_n, \hbar_{n+1}) \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) + \psi(\hbar_{n+1}, \hbar_m) \eth_{\mathbb{C}}(\hbar_{n+1}, \hbar_m)$$

$$\precsim \psi(\hbar_n, \hbar_{n+1}) \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) + \psi(\hbar_{n+1}, \hbar_m) \psi(\hbar_{n+1}, \hbar_{n+2}) \eth_{\mathbb{C}}(\hbar_{n+1}, \hbar_{n+2}) + \psi(\hbar_{n+1}, \hbar_m) \psi(\hbar_{n+2}, \hbar_m) \eth_{\mathbb{C}}(\hbar_{n+2}, \hbar_m)$$

$$\precsim \psi(\hbar_n, \hbar_{n+1}) \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}) + \psi(\hbar_{n+1}, \hbar_m) \psi(\hbar_{n+1}, \hbar_{n+2}) \eth_{\mathbb{C}}(\hbar_{n+1}, \hbar_{n+2}) + \psi(\hbar_{n+1}, \hbar_m)$$

$$\psi(\hbar_{n+2}, \hbar_m) \psi(\hbar_{n+3}, \hbar_m) \psi(\hbar_{n+3}, \hbar_m) \precsim ...$$

$$\precsim \psi(\hbar_n, \hbar_{n+1}) \eta^n \eth_{\mathbb{C}}(\hbar_0, \hbar_1) + \sum_{i=n+1}^{m-2} \prod_{j=n+1}^{i} \psi(\hbar_j, \hbar_m) \psi(\hbar_i, \hbar_{n+1}) \eta^i \eth_{\mathbb{C}}(\hbar_0, \hbar_1) + \prod_{k=n+1}^{m-1} \psi(\hbar_k, \hbar_m) \eta^{m-1} \eth_{\mathbb{C}}(\hbar_0, \hbar_1)$$

If we follow same steps given in main Theorem 3.1, we get

$$\precsim \psi(\hbar_n, \hbar_{n+1}) \eta^n \eth_{\mathbb{C}}(\hbar_0, \hbar_1) + \sum_{i=n+1}^{m-1} \prod_{j=0}^{i} \zeta(\hbar_j, \hbar_m) \psi(\hbar_i, \hbar_{i+1}) \eta^i \eth_{\mathbb{C}}(\hbar_0, \hbar_1)$$

Let,

$$\mho_\iota = \sum_{i=0}^{\iota} \prod_{j=0}^{\iota} \psi(\hbar_j, \hbar_m) \psi(\hbar_i, \hbar_{i+1}) \eta^i.$$

$$\eth_{\mathbb{C}}(\hbar_n, \hbar_m) \precsim \eth_{\mathbb{C}}(\hbar_0, \hbar_1)[\eta^n \psi(\hbar_n, \hbar_{n+1}) + (\mho_{m-1}, \mho_n)]. \tag{3.14}$$

By using ratio test and (3.12), $\lim_{m,n\to\infty} \mho_n$ exists which implies sequence $\{\mho_n\}$ is Cauchy. Applying $\lim_{m,n\to\infty}$ to (3.14), we get

$$\lim_{m,n\to\infty} \eth_{\mathbb{C}}(\hbar_n, \hbar_m) = 0. \tag{3.15}$$

As we know $\{\hbar_n\}$ is Cauchy in complete $\eth_{\mathbb{C}}$ -metric space, then we say that $\{\hbar_n\}$ is converges to a point $\hbar^* \in \nabla$. Now next part $\hbar^*$ is fixed point of $\tilde{\beth}$. We use definition of continuity of $\tilde{\beth}$,

$$\hbar^* = \lim_{n\to\infty} \hbar_{n+1} = \lim_{n\to\infty} \tilde{\beth} \hbar_n = \tilde{\beth}(\lim_{n\to\infty} \hbar_n) = \tilde{\beth} \hbar^*$$

and finally remaining part is uniqueness of fixed point. On contrary we assume $\tilde{\beth}$ has two fixed point say $F$ and $F^*$,

$$\eth_{\mathbb{C}}(F, F^*) = \eth_{\mathbb{C}}(\tilde{\beth} F, \tilde{\beth} F^*) \precsim \psi \eth_{\mathbb{C}}(F, F^*),$$

which holds only when $\eth_{\mathbb{C}}(F, F^*) = 0$ and Hence it finally gives uniqueness of fixed point. □

If we assume $\tilde{\beth}_1$ & $\tilde{\beth}_2$ are equal and which is equal to $\tilde{\beth}$ along with we avoid map $\tilde{\beth} : \nabla \to \nabla$ is continuous; $\Re, \flat = 1$ & $\lambda, \mu = 0$ then Theorem 3.1 reduces to following result:

**Theorem 3.3.** *Consider* $(\nabla, \eth_{\mathbb{C}})$ *be a complete* $\eth_{\mathbb{C}}$ *metric space. Suppose*

$$\frac{1}{\eta} > \sup_{1 \leq m} \lim_{i\to\infty} \frac{\psi(\hbar_{i+1}, \hbar_{i+2})}{\psi(\hbar_i, \hbar_{i+1})} \zeta(\hbar_{i+1}, \hbar_m). \tag{3.16}$$

*For every* $\hbar, \ell \in \nabla$ & $0 \prec \eth_{\mathbb{C}}(\hbar, \ell)$*, we use* $\eta$ *non negative real numbers with* $0 < \eta < 1$*, we choose* $\hbar_n = \tilde{\beth}^n \hbar_0 \in \nabla$ *for all* $\hbar_0 \in \nabla$ *then the map* $\tilde{\beth} : \nabla \to \nabla$ *be a mapping such that,*

$$\eth_{\mathbb{C}}(\tilde{\beth} \hbar, \tilde{\beth} \ell) \precsim \eta \{\eth_{\mathbb{C}}(\hbar, \ell)\}, \tag{3.17}$$

*afterward assume that,* $\lim_{n\to\infty} \zeta(\hbar_n, \hbar)$ *,* $\lim_{n\to\infty} \psi(\hbar, \hbar_n)$ *both exist and finite, then* $\tilde{\beth}$ *admits unique common fixed point.*

*Proof.* If we follow similar steps like Theorem 3.2 we can easily get the Cauchy sequence $\{\hbar_n\}$ under $\eth_{\mathbb{C}}$-metric space $(\nabla, \eth_{\mathbb{C}})$. Subsequently we say $\{\hbar_n\}$ converges to $\hbar^* \in \nabla$. We shall prove $\tilde{\beth}$ admits $\hbar^*$ as a fixed point, we consider the triangle inequality of complex valued controlled metric space,

$$\eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) \precsim \psi(\hbar^*, \hbar_n) \eth_{\mathbb{C}}(\hbar^*, \hbar_n) + \psi(\hbar_n, \hbar_{n+1}) \eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}).$$

with the help of Statement (b) of Theorem 3.3, we write

$$\lim_{n\to\infty} \eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) = 0. \tag{3.18}$$

Again by (3.17) and triangle inequality, we get

$$\eth_{\mathbb{C}}(\hbar^*, \tilde{\beth} \hbar^*) \precsim \psi(\hbar^*, \hbar_{n+1}) \eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) + \psi(\hbar_{n+1}, \tilde{\beth} \hbar^*) \eth_{\mathbb{C}}(\hbar_{n+1}, \tilde{\beth} \hbar^*)$$

$$\precsim \psi(\hbar^*, \hbar_{n+1}) \eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) + \eta \psi(\hbar_{n+1}, \tilde{\beth} \hbar^*) \eth_{\mathbb{C}}(\hbar_n, \tilde{\beth} \hbar^*).$$

Letting $\lim_{n\to\infty}$ and Statement of Theorem 3.3, we get $\eth_{\mathbb{C}}(\hbar^*, \tilde{\beth} \hbar^*) = 0$, Hence proved. □

We use following example to verify above results:

**Example 3.1.** Let $\eth_{\mathbb{C}} : \nabla \times \nabla \to \mathbb{C}$ be a symmetric metric. Suppose $\nabla = \{1, 2, 0\}$ and $\eth_{\mathbb{C}}(1, 2) = \eth_{\mathbb{C}}(0, 1) = 1 + i$ & $\eth_{\mathbb{C}}(0, 2) = 4 + 4i$ again function $\psi : \nabla \times \nabla \to [1, \infty)$ is symmetric and

$$\psi(1,1) = \tfrac{4}{3}, \psi(2,2) = \tfrac{6}{5}, \psi(1,2) = \tfrac{5}{4}.$$
$$\psi(0,2) = \tfrac{4}{3}, \psi(0,1) = \tfrac{3}{2}, \psi(0,0) = 2.$$

It's easy to verify $\eth_{\mathbb{C}}$ is a metric space, Suppose self map $\tilde{\daleth}$ follows $\tilde{\daleth}(2) = \tilde{\daleth}(1) = \tilde{\daleth}(0) = 0$ & use $\eta = \frac{2}{5}$ and we clearly see that (3.17) holds for $\hbar_0 \in \nabla$ then condition (3.16) is satisfied. We follow the following cases to verify hypothesis of Theorem 3.3,

***Case I.*** If $\hbar = 1$, $\ell = 2$ then,
$\eth_{\mathbb{C}}(\tilde{\daleth}\hbar, \tilde{\daleth}\ell) = \eth_{\mathbb{C}}(\tilde{\daleth}1, \tilde{\daleth}2) = \eth_{\mathbb{C}}(2, 2) = 0 \precsim \frac{2}{5}(1 + i) = \eta\eth_{\mathbb{C}}(1, 2) = \eta\eth_{\mathbb{C}}(\hbar, \ell)$.

***Case II.*** If $\hbar = 0$, $\ell = 1$ then,
$\eth_{\mathbb{C}}(\tilde{\daleth}\hbar, \tilde{\daleth}\ell) = \eth_{\mathbb{C}}(\tilde{\daleth}0, \tilde{\daleth}1) = \eth_{\mathbb{C}}(2, 2) = 0 \precsim \frac{2}{5}(1 + i) = \eta\eth_{\mathbb{C}}(0, 1) = \eta\eth_{\mathbb{C}}(\hbar, \ell)$.

***Case III.***If $\hbar = 0$, $\ell = 2$ then,
$\eth_{\mathbb{C}}(\tilde{\daleth}\hbar, \tilde{\daleth}\ell) = \eth_{\mathbb{C}}(\tilde{\daleth}0, \tilde{\daleth}2) = \eth_{\mathbb{C}}(2, 2) = 0 \precsim \frac{2}{5}(4 + 4i) = \eta\eth_{\mathbb{C}}(0, 2) = \eta\eth_{\mathbb{C}}(\hbar, \ell)$

***Case IV.*** If $\hbar = 0$, $\ell = 0$ ; $\hbar = 1$, $\ell = 1$ ; $\hbar = 2$, $\ell = 2$ then, the results hold good. Then we say that $\tilde{\daleth}$ admits a unique fixed point as $\hbar^* = 0$.

If we assume $\tilde{\daleth}_1$ and $\tilde{\daleth}_2$ are equal and which is equal to $\tilde{\daleth}$; $\Re, \flat = 1$ & $\lambda = 0$ then Theorem 3.1 reduces to following result,

**Theorem 3.4.** *Consider* $(\nabla, \eth_{\mathbb{C}})$ *be a Complete* $\eth_{\mathbb{C}}$ *metric space. Suppose* $\aleph = \frac{\eta}{(1-\mu)} < 1$ *and*

$$\frac{1}{\aleph} > \sup_{1 \leq m} \lim_{i \to \infty} \frac{\psi(\hbar_{i+1}, \hbar_{i+2})}{\psi(\hbar_i, \hbar_{i+1})} \zeta(\hbar_{i+1}, \hbar_m). \tag{3.19}$$

*For every* $\hbar, \ell \in \nabla$ & $0 \prec \eth_{\mathbb{C}}(\hbar, \ell)$, *we use* $\mu, \eta$ *are non negative real numbers with* $0 \leq \eta < 1$, $0 \leq \mu < 1$ *we choose* $\hbar_n = \tilde{\daleth}^n \hbar_0 \in \nabla$ *for all* $\hbar_0 \in \nabla$ *then the map* $\tilde{\daleth} : \nabla \to \nabla$ *be a Continuous map satisfying,*

$$\eth_{\mathbb{C}}(\tilde{\daleth}\hbar, \tilde{\daleth}\ell) \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar, \tilde{\daleth}\hbar)\eth_{\mathbb{C}}(\ell, \tilde{\daleth}\ell)}{1 + \eth_{\mathbb{C}}(\hbar, \ell)}\} + \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\} \tag{3.20}$$

*afterward Assume that,* $\lim_{n\to\infty} \zeta(\hbar_n, \hbar)$ , $\lim_{n\to\infty} \psi(\hbar, \hbar_n)$ *both are exist and finite, then* $\tilde{\daleth}$ *admits unique common fixed point.*

*Proof.* The proof of the above result is similar to Theorem 3.1 therefore we omit it. □

**Proposition 3.1.** *Above results gives generalization of* [10] *D.Lateef rational functions result, Fisher type, under Complex valued double controlled metric space* [11].

Suppose that $\tilde{\daleth}_1$ & $\tilde{\daleth}_2$ are equal and which is equal to $\tilde{\daleth}$ along with we map $\tilde{\daleth} : \nabla \to \nabla$ is not continuous; $\Re, \flat = 1$ & $\lambda = 0$ then Theorem 3.1 reduces to following result,

**Theorem 3.5.** *Consider* $(\nabla, \eth_{\mathbb{C}})$ *be a Complete* $\eth_{\mathbb{C}}$ *metric space. Suppose* $\aleph = \frac{\eta}{(1-\mu)} < 1$ *and*

$$\frac{1}{\aleph} > \sup_{1 \leq m} \lim_{i \to \infty} \frac{\psi(\hbar_{i+1}, \hbar_{i+2})}{\psi(\hbar_i, \hbar_{i+1})} \zeta(\hbar_{i+1}, \hbar_m). \tag{3.21}$$

*For every* $\hbar, \ell \in \nabla$ *and* $0 \prec \eth_{\mathbb{C}}(\hbar, \ell)$, *we use* $\mu, \eta$ *are non negative real numbers with* $0 \leq \eta < 1$, $0 \leq \mu < 1$ *we choose* $\hbar_n = \tilde{\daleth}^n \hbar_0 \in \nabla$ *for all* $\hbar_0 \in \nabla$ *then the map* $\tilde{\daleth} : \nabla \to \nabla$ *be a mapping such that,*

$$\eth_{\mathbb{C}}(\tilde{\daleth}\hbar, \tilde{\daleth}\ell) \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar, \tilde{\daleth}\hbar)\eth_{\mathbb{C}}(\ell, \tilde{\daleth}\ell)}{1 + \eth_{\mathbb{C}}(\hbar, \ell)}\} + \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\}. \tag{3.22}$$

*afterward assume that,* $\lim_{n\to\infty} \zeta(\hbar_n, \hbar)$ , $\lim_{n\to\infty} \psi(\hbar, \hbar_n)$ *both are exist and finite, then* $\tilde{\daleth}$ *admits unique common fixed point.*

*Proof.* If we follow similar steps like Theorem 3.1 we can easily get the Cauchy sequence $\{\hbar_n\}$ under $\eth_{\mathbb{C}}$-metric space $(\nabla, \eth_{\mathbb{C}})$. Subsequently we say $\{\hbar_n\}$ converges to $\hbar^* \in \nabla$. We shall prove $\tilde{\daleth}$ admits $\hbar^*$ as a fixed point. Lets consider the triangle inequality of complex valued controlled metric space,

$$\eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) \precsim \psi(\hbar^*, \hbar_n)\eth_{\mathbb{C}}(\hbar^*, \hbar_n) + \psi(\hbar_n, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1}).$$

with the help of Statement of Theorem 3.5, we write

$$\lim_{n\to\infty} \eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) = 0. \tag{3.23}$$

Again by inequalities (3.22) and triangle inequality, we get

$$\eth_{\mathbb{C}}(\hbar^*, \tilde{\beth}\hbar^*) \precsim \psi(\hbar^*, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) + \psi(\hbar_{n+1}, \tilde{\beth}\hbar^*)\eth_{\mathbb{C}}(\hbar_{n+1}, \tilde{\beth}\hbar^*)$$

$$\precsim \psi(\hbar^*, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) + \eta\psi(\hbar_{n+1}, \tilde{\beth}\hbar^*)\mu\{\frac{\eth_{\mathbb{C}}(\hbar_n, \tilde{\beth}\hbar_n)\eth_{\mathbb{C}}(\hbar^*, \tilde{\beth}\hbar^*)}{1+\eth_{\mathbb{C}}(\hbar_n, \hbar^*)}\} + \eta\eth_{\mathbb{C}}(\hbar_n, \hbar^*)$$

$$\precsim \psi(\hbar^*, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar^*, \hbar_{n+1}) + \eta\psi(\hbar_{n+1}, \tilde{\beth}\hbar^*)\mu\{\frac{\eth_{\mathbb{C}}(\hbar_n, \hbar_{n+1})\eth_{\mathbb{C}}(\hbar^*, \tilde{\beth}\hbar^*)}{1+\eth_{\mathbb{C}}(\hbar_n, \hbar^*)}\} + \eta\eth_{\mathbb{C}}(\hbar_n, \hbar^*).$$

Letting $\lim_{n\to\infty}$ and Statement of Theorem 3.5, we get $\eth_{\mathbb{C}}(\hbar^*, \tilde{\beth}\hbar^*) = 0$, Hence proved. □

Lets verify above result through the following example.

**Example 3.2.** Let $\eth_{\mathbb{C}} : \nabla \times \nabla \to \mathbb{C}$ be a symmetric metric. Suppose $\nabla = \{1, 2, 0\}$ and $\eth_{\mathbb{C}}(1, 2) = \eth_{\mathbb{C}}(0, 1) = 1 + i$ & $\eth_{\mathbb{C}}(0, 2) = 4 + 4i$ again function $\psi : \nabla \times \nabla \to [1, \infty)$ is symmetric and

$$\psi(1,1) = \tfrac{7}{3}, \psi(2,2) = \tfrac{9}{5}, \psi(1,2) = 2;\ \psi(0,2) = \tfrac{7}{3}, \psi(0,1) = 3, \psi(0,0) = 5$$

It's easy to verify $\eth_{\mathbb{C}}$ is a metric space, Suppose self map $\tilde{\beth}$ follows $\tilde{\beth}(2) = \tilde{\beth}(1) = \tilde{\beth}(0) = 1$ & use $\mu, \eta = \frac{2}{5}$ and we clearly see that (3.20) holds for $\hbar_0 \in \nabla$ then condition (3.19) is satisfied. We follow the following cases to verify hypothesis of Theorem 3.5,

***Case I.)*** If $\hbar = 1$, $\ell = 2$ then,
$\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) = 0 \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar,\tilde{\beth}\hbar)\eth_{\mathbb{C}}(\ell,\tilde{\beth}\ell)}{1+\eth_{\mathbb{C}}(\hbar,\ell)}\} + \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\}$.

***Case II.*** If $\hbar = 0$, $\ell = 1$ then,
$\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) = 0 \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar,\tilde{\beth}\hbar)\eth_{\mathbb{C}}(\ell,\tilde{\beth}\ell)}{1+\eth_{\mathbb{C}}(\hbar,\ell)}\} + \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\}$.

***Case III.***If $\hbar = 0$, $\ell = 2$ then,
$\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) = 0 \precsim \mu\{\frac{\eth_{\mathbb{C}}(\hbar,\tilde{\beth}\hbar)\eth_{\mathbb{C}}(\ell,\tilde{\beth}\ell)}{1+\eth_{\mathbb{C}}(\hbar,\ell)}\} + \eta\{\eth_{\mathbb{C}}(\hbar, \ell)\}$.

***Case IV.*** If $\hbar = 0$, $\ell = 0$ ; $\hbar = 1$, $\ell = 1$ ; $\hbar = 2$, $\ell = 2$ then $\eth_{\mathbb{C}}(\tilde{\beth}\hbar, \tilde{\beth}\ell) = 0$, results hold good. Then we say that $\tilde{\beth}$ admits a unique fixed point as $\hbar^* = 1$.

## 4 Application of the Main theorem

We divide application part of main Theorem in to two following folds,

### 4.1 Application Part I

In this part we would like to introduce the notion of Existence and unique fixed point solution in the context of fractional $FVI_dE$. By applying fractional Adams Bashforth method to the (1.1) $FVI_dE$,

$$\tilde{\beth}_0 = \tilde{\beth}(0;\ell) \text{ and } {}_0^{ABC}D_{\hbar}^{\omega}\tilde{\beth}(\hbar;\ell) = \aleph(\hbar) + \Re(\hbar).\tilde{\beth}(\hbar,\ell) + \int_0^{\hbar} \mho_1(\hbar,\xi).\chi_1(\tilde{\beth}(\xi,\ell))d\xi + \int_0^1 \mho_2(\hbar,\xi)\chi_2(\tilde{\beth}(\xi,\ell))d\xi$$

in the setting of complex valued controlled metric space we prove following application part.

#### 4.1.1 Application to fractional Fredholm Volterra integro differential equation.

We consider the following hypothesis,

1. $\Re$ and $\aleph$ both function are continuous,
2. $$| \eth(\tilde{\beth}_1(\hbar;\ell), \tilde{\beth}_2(\hbar;\ell)) | .\alpha_1 \geq | (\eth(\chi_1(\tilde{\beth}_1(\hbar;\ell))), (\chi_1(\tilde{\beth}_2(\hbar;\ell)))) |, \forall \tilde{\beth}_1, \tilde{\beth}_2 \in C^z(\nabla), \alpha_1, \alpha_2 > 0,$$
   $$| \eth(\tilde{\beth}_1(\hbar;\ell), \tilde{\beth}_2(\hbar;\ell)) | .\alpha_2 \geq | (\eth(\chi_2(\tilde{\beth}_1(\hbar;\ell))), (\chi_2(\tilde{\beth}_2(\hbar;\ell)))) |, \forall \tilde{\beth}_1, \tilde{\beth}_2 \in C^z(\nabla), \alpha_1, \alpha_2 > 0. \tag{4.1}$$
3. For the function $\mho_1^*$ and $\mho_2^*$,
   $$\mho_1^* < \infty \Rightarrow \mho_1^* = \sup_{\hbar\in\nabla} \int_0^{\hbar} | \mho_1(\hbar,\xi) | d\xi \text{ and } \mho_2^* < \infty \Rightarrow \mho_2^* = \sup_{\hbar\in\nabla} \int_0^{\hbar} | \mho_2(\hbar,\xi) | d\xi, \tag{4.2}$$

$C^z(\nabla, \mathbb{R})$ be the space of all continuous functions $\beth : \nabla \to \mathbb{R}$ which has $\|\beth\|_\infty = max\{| \beth(\rho) | : \forall \rho \in \nabla\}$ then $(C^z(\nabla, \mathbb{R}), \|.\|_\infty)$ is banach space.

**Theorem 4.1.** *Suppose* (4.1),(4.2) *and* (1) *are satisfied. If*

$$\Psi_1 = [\frac{\zeta\omega.[\Xi(\omega+1).(1-\omega)+\omega]}{\zeta^2(\omega).\Xi(\omega+1)}]_{\|\Re\|_\infty<1}. \tag{4.3}$$

*Then above problem* (1.1) *FVI$_d$E admits at least one solution* $\tilde{\beth}(\hbar,\ell)$.

Before starting our proof we go through following result;

$$0<\omega\leq 1 \text{ and } \tilde{\beth}(\hbar,\ell)-\tilde{\beth}_0 = \frac{1}{\zeta(\omega)}[(1-\omega)\tilde{\aleph}(\hbar,\ell)+\frac{\omega}{\Xi(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}\tilde{\aleph}(\xi,\ell)d\xi], \tag{4.4}$$

which is the solution of, $\tilde{\beth}_0=\tilde{\beth}(0;\ell)$ and $\tilde{\aleph}(\hbar,\ell)={}_0^{ABC}D_\hbar^\omega\,\tilde{\beth}(\hbar;\ell)$. Applying the operator $({}_0^{AB}\,B_\hbar^\omega)$ to above equation, $({}_0^{AB}\,B_\hbar^\omega)\,\tilde{\aleph}(\hbar,\ell)=({}_0^{AB}\,B_\hbar^\omega)\,{}_0^{ABC}D_\hbar^\omega\,\tilde{\beth}(\hbar;\ell)$. Hence we write (4.4) as,

$$\tilde{\beth}(\hbar,\ell)-\tilde{\beth}(0;\ell) = \frac{1}{\zeta(\omega)}[(1-\omega)\tilde{\aleph}(\hbar,\ell)+\frac{\omega}{\Xi(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}\tilde{\aleph}(\xi,\ell)d\xi].$$

*Proof.* As we know that,

$$\tilde{\beth}(\hbar,\ell)-\tilde{\beth}(0;\ell) = \frac{1}{\zeta(\omega)}[(1-\omega)\tilde{\aleph}(\hbar,\ell)+\frac{\omega}{\Xi(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}\tilde{\aleph}(\xi,\ell)d\xi]. \tag{4.5}$$

We write main equation (1.1) as,

$$\tilde{\aleph}(\hbar,\ell)-\aleph(\hbar)=\Re(\hbar).\tilde{\beth}(\hbar,\ell)+\int_0^\hbar \mho_1(\hbar,\xi).\chi_1(\tilde{\beth}(\xi,\ell))d\xi+\int_0^1 \mho_2(\hbar,\xi).\chi_2(\tilde{\beth}(\xi,\ell))d\xi.$$

Similarly, we write

$$\tilde{\aleph}(\xi,\ell)-\aleph(\xi)=\Re(\xi).\tilde{\beth}(\xi,\ell)+\int_0^\xi \mho_1(\xi,\digamma).\chi_1(\tilde{\beth}(\digamma,\ell))d\digamma+\int_0^1 \mho_2(\xi,\digamma).\chi_2(\tilde{\beth}(\digamma,\ell))d\digamma.$$

Applying above two equations in (4.5), we get

$$\tilde{\beth}(\hbar,\ell)-\tilde{\beth}(0;\ell)=\frac{(1-\omega)}{\zeta(\omega)}[\aleph(\hbar)+\Re(\hbar).\tilde{\beth}(\hbar,\ell)+\int_0^\hbar \mho_1(\hbar,\xi).\chi_1(\tilde{\beth}(\xi,\ell))d\xi+\int_0^1 \mho_2(\hbar,\xi)\chi_2(\tilde{\beth}(\xi,\ell))d\xi]$$
$$+\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}[\aleph(\xi)+\Re(\xi)\tilde{\beth}(\xi,\ell)+\int_0^\xi \mho_1(\xi,\digamma)\chi_1(\tilde{\beth}(\digamma,\ell))d\digamma+\int_0^1 \mho_2(\xi,\digamma)\chi_2(\tilde{\beth}(\digamma,\ell))d\digamma]d\xi.$$

Now lets use operator $\Upsilon$ in above equation,

$$\Upsilon\tilde{\beth}(\hbar,\ell)-\tilde{\beth}(0;\ell)=\frac{(1-\omega)}{\zeta(\omega)}[\aleph(\hbar)+\Re(\hbar).\tilde{\beth}(\hbar,\ell)+\int_0^\hbar \mho_1(\hbar,\xi).\chi_1(\tilde{\beth}(\xi,\ell))d\xi+\int_0^1 \mho_2(\hbar,\xi)\chi_2(\tilde{\beth}(\xi,\ell))d\xi]$$
$$+\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}[\aleph(\xi)+\Re(\xi).\tilde{\beth}(\xi,\ell)+\int_0^\xi \mho_1(\xi,\digamma)\chi_1(\tilde{\beth}(\digamma,\ell))d\digamma+\int_0^1 \mho_2(\xi,\digamma)\chi_2(\tilde{\beth}(\digamma,\ell))d\digamma]d\xi.$$

Here we claim, operator $\Upsilon$ admits fixed point and we defined it as,

$$\Upsilon : L^z(\nabla,\mathbb{R})\cap C^z(\nabla,\mathbb{R})\to L^z(\nabla,\mathbb{R})\cap C^z(\nabla,\mathbb{R})$$

So, we divide our proof into following folds, **Firstly**, we show $\chi_1,\chi_2$ continuous which finally gives $\Upsilon$ is continuous. Suppose $\{\tilde{\beth}_n\}$ be a sequence such that $\tilde{\beth}_n\to\tilde{\beth}$ in $C(\nabla,\mathbb{R}^z)$. Then $\hbar\in\nabla$ we get,

$$|\,\eth(\Upsilon\tilde{\beth}_n(\hbar,\ell),\Upsilon\tilde{\beth}(\hbar,\ell))\,|\leq\tilde{\beth}_n(0;\ell)+\frac{(1-\omega)}{\zeta(\omega)}[\aleph(\hbar)+\Re(\hbar)\tilde{\beth}_n(\hbar,\ell)+\int_0^\hbar \mho_1(\hbar,\xi)\chi_1(\tilde{\beth}_n(\xi,\ell))d\xi+\int_0^1 \mho_2(\hbar,\xi)$$
$$\chi_2(\tilde{\beth}_n(\xi,\ell))d\xi]+\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}[\aleph(\xi)+\Re(\xi)\tilde{\beth}_n(\xi,\ell)+\int_0^\xi \mho_1(\xi,\digamma)\chi_1(\tilde{\beth}_n(\digamma,\ell))d\digamma+\int_0^1 \mho_2(\xi,\digamma)$$
$$\chi_2(\tilde{\beth}_n(\digamma,\ell))d\digamma]d\xi-[\tilde{\beth}(0;\ell)+\frac{(1-\omega)}{\zeta(\omega)}[\aleph(\hbar)+\Re(\hbar)\tilde{\beth}(\hbar,\ell)+\int_0^\hbar \mho_1(\hbar,\xi)\chi_1(\tilde{\beth}(\xi,\ell))d\xi+\int_0^1 \mho_2(\hbar,\xi)\chi_2(\tilde{\beth}(\xi,\ell))d\xi]$$
$$+\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}[\aleph(\xi)+\Re(\xi)\tilde{\beth}(\xi,\ell)+\int_0^\xi \mho_1(\xi,\digamma)\chi_1(\tilde{\beth}(\digamma,\ell))d\digamma+\int_0^1 \mho_2(\xi,\digamma)\chi_2(\tilde{\beth}(\digamma,\ell))d\digamma]d\xi].$$

$$\leq \frac{(1-\omega)}{\zeta(\omega)}[\mid \Re(\hbar) \mid\mid \tilde{\beth}_n(\hbar,\ell) - \tilde{\beth}(\hbar,\ell) \mid + \int_0^{\hbar} \mid \mho_1(\hbar,\xi) \mid\mid \chi_1(\tilde{\beth}_n(\xi,\ell)) - \chi_1(\tilde{\beth}(\xi,\ell)) \mid d\xi + \int_0^1 \mid \mho_2(\xi,F) \mid\mid \chi_2(\tilde{\beth}_n(F,\ell)) - \chi_2$$

$$(\tilde{\beth}(F,\ell)) \mid dF] + \frac{\omega}{\Xi(\omega)\zeta(\omega)} \int_0^{\hbar} \frac{(\hbar-\xi)^{\omega}}{(\hbar-\xi)}[\mid \Re(\xi) \mid\mid \tilde{\beth}_n(\xi,\ell) - \tilde{\beth}(\xi,\ell) \mid + \int_0^{\xi} \mid \mho_1(\xi,F) \mid\mid \chi_1(\tilde{\beth}_n(F,\ell)) - \chi_1(\tilde{\beth}(F,\ell)) \mid dF]d\xi$$

Apply supremum then,

$$\|\eth(\Upsilon\tilde{\beth}_n(\hbar,\ell) - \Upsilon\tilde{\beth}(\hbar,\ell))\|_\infty \leq \frac{(1-\omega)}{\zeta(\omega)}[\|\aleph\|_\infty + \|\tilde{\beth}_n - \tilde{\beth}\|_\infty + \mho_1^*\|\chi_1(\tilde{\beth}_n) - \chi_1(\tilde{\beth})\|_\infty + \mho_2^*\|\chi_2(\tilde{\beth}_n) - \chi_2(\tilde{\beth})\|_\infty]$$

$$+ \frac{\omega\hbar^{\omega}}{\Xi(\omega+1)\zeta(\omega)}[\|\aleph\|_\infty + \|\tilde{\beth}_n - \tilde{\beth}\|_\infty + \mho_1^*\|\chi_1(\tilde{\beth}_n) - \chi_1(\tilde{\beth})\|_\infty + \mho_2^*\|\chi_2(\tilde{\beth}_n) - \chi_2(\tilde{\beth})\|_\infty].$$

$$\leq \frac{(1-\omega)}{\zeta(\omega)} + \frac{\omega\hbar^{\omega}}{\Xi(\omega+1)\zeta(\omega)}[\|\aleph\|_\infty + \|\tilde{\beth}_n - \tilde{\beth}\|_\infty + \mho_1^*\|\chi_1(\tilde{\beth}_n) - \chi_1(\tilde{\beth})\|_\infty + \mho_2^*\|\chi_2(\tilde{\beth}_n) - \chi_2(\tilde{\beth})\|_\infty]$$

$$\|\eth(\Upsilon\tilde{\beth}_n(\hbar,\ell) - \Upsilon\tilde{\beth}(\hbar,\ell))\|_\infty \to 0, when \tilde{\beth}_n \to \tilde{\beth},$$

which finally gives that $\Upsilon$ is continuous. **Secondly** we work on compactness property for $\Upsilon$ and then completely continuous. Let $\mathfrak{C}_R = \{\tilde{\beth} \in C(\nabla, \mathbb{R}^z) : \|\tilde{\beth}\|_\infty \leq R\}$ be a convex, closed and bounded set with, $\frac{\Psi_2}{1-\Psi_1} \leq R$ and we define $\lambda_j = \sup_{\tilde{\beth}\in\nabla\times[0,R]} \chi_j(\tilde{\beth}(\xi,\ell)+1), j = 1, 2, ..$

$$\Psi_2 = \mid \tilde{\beth}_0 \mid + [\frac{\zeta\omega.[\Xi(\omega+1).(1-\omega)+\omega]}{\zeta^2(\omega).\Xi(\omega+1)}]_{[\|\aleph\|_\infty + \mho_1^*\lambda_1 + \mho_2^*\lambda_2]}, \tag{4.6}$$

$$\tilde{\beth} \in \mathfrak{C}_R, \mid \Upsilon\tilde{\beth}(\hbar,\ell) \mid - \mid \tilde{\beth}(0;\ell) \mid \leq \frac{(1-\omega)}{\zeta(\omega)}[\mid \aleph(\hbar) \mid + \mid \Re(\hbar) \mid\mid \tilde{\beth}(\hbar,\ell) \mid + \int_0^{\hbar} \mid \mho_1(\hbar,\xi) \mid\mid \chi_1(\tilde{\beth}(\xi,\ell)) \mid d\xi +$$

$$\int_0^1 \mid \mho_2(\hbar,\xi) \mid\mid \chi_2(\tilde{\beth}(\xi,\ell)) \mid d\xi] + \frac{\omega}{\Xi(\omega)\zeta(\omega)} \int_0^{\hbar} \frac{(\hbar-\xi)^{\omega}}{(\hbar-\xi)}[\mid \aleph(\xi) \mid + \mid \Re(\xi) \mid\mid \tilde{\beth}(\xi,\ell) \mid + \int_0^{\xi} \mid \mho_1(\xi,F) \mid$$

$$\mid \chi_1(\tilde{\beth}(F,\ell)) \mid dF + \int_0^1 \mid \mho_2(\xi,F) \mid\mid \chi_2(\tilde{\beth}(F,\ell)) \mid dF]d\xi,$$

$$\mid \Upsilon\tilde{\beth}(\hbar,\ell) \mid - \mid \tilde{\beth}(0;\ell) \mid \leq \frac{(1-\omega)}{\zeta(\omega)}[\mid \aleph(\hbar) \mid + \mid \Re(\hbar) \mid\mid \tilde{\beth}(\hbar,\ell) \mid + \mho_1^*\lambda_1 + \mho_2^*\lambda_2] + \frac{\omega\hbar^{\omega}}{\Xi(\omega+1)\zeta(\omega)}$$

$$[\mid \aleph(\xi) \mid + \mid \Re(\xi) \mid\mid \tilde{\beth}(\xi,\ell) \mid + \mho_1^*\lambda_1 + \mho_2^*\lambda_2].$$

Take Supremum on both side,

$$\|\Upsilon\tilde{\beth}\|_\infty - \mid \tilde{\beth}(0;\ell) \mid \leq \frac{(1-\omega)}{\zeta(\omega)} + \frac{\omega}{\Xi(\omega+1)\zeta(\omega)}[\|\aleph\|_\infty + \|\Re\|_\infty R + \mho_1^*\lambda_1 + \mho_2^*\lambda_2], \tag{4.7}$$

$$\leq [\frac{\zeta\omega[\Xi(\omega+1)(1-\omega)+\omega]}{\zeta^2(\omega)\Xi(\omega+1)}][\|\aleph\|_\infty + \mho_1^*\lambda_1 + \mho_2^*\lambda_2] + [\frac{\zeta\omega[\Xi(\omega+1)(1-\omega)+\omega]}{\zeta^2(\omega)\Xi(\omega+1)}]\|\Re\|_\infty R$$

$$\leq \Psi_1 R + \Psi_2 \leq R.$$

It gives that $\Upsilon$ is uniformly bounded. Now our next claim is that $\Upsilon$ is equicontinuous. Let $\hbar_1 < \hbar_2$,

$$\mid \eth(\Upsilon\tilde{\beth}(\hbar_2,\ell), \Upsilon\tilde{\beth}(\hbar_1,\ell)) \mid = \mid \frac{(1-\omega)}{\zeta(\omega)}[\aleph(\hbar_2) + \Re(\hbar_2).\tilde{\beth}(\hbar_2,\ell) + \int_0^{\hbar_2} \mho_1(\hbar_2,\xi)\chi_1(\tilde{\beth}(\xi,\ell))d\xi + \int_0^1 \mho_2(\hbar_2,\xi)$$

$$\chi_2(\tilde{\beth}(\xi,\ell))d\xi] + \frac{\omega}{\Xi(\omega)\zeta(\omega)} \int_0^{\hbar_2} \frac{(\hbar_2-\xi)^{\omega}}{(\hbar_2-\xi)}[\aleph(\xi) + \Re(\xi)\tilde{\beth}(\xi,\ell) + \int_0^{\xi} \mho_1(\xi,F)\chi_1(\tilde{\beth}(F,\ell))dF + \int_0^1 \mho_2(\xi,F)$$

$$\chi_2(\tilde{\beth}(F,\ell))dF]d\xi - [\frac{(1-\omega)}{\zeta(\omega)}[\aleph(\hbar_1) + \Re(\hbar_1)\tilde{\beth}(\hbar_1,\ell) + \int_0^{\hbar_1} \mho_1(\hbar_1,\xi).\chi_1(\tilde{\beth}(\xi,\ell))d\xi + \int_0^1 \mho_2(\hbar_1,\xi)\chi_2(\tilde{\beth}(\xi,\ell))d\xi]$$

$$+ \frac{\omega}{\Xi(\omega)\zeta(\omega)} \int_0^{\hbar_1} \frac{(\hbar_1-\xi)^{\omega}}{(\hbar_1-\xi)}[\aleph(\xi) + \Re(\xi).\tilde{\beth}(\xi,\ell) + \int_0^{\xi} \mho_1(\xi,F)\chi_1(\tilde{\beth}(F,\ell))dF + \int_0^1 \mho_2(\xi,F).\chi_2(\tilde{\beth}(F,\ell))dF]d\xi] \mid$$

$$\leq \frac{(1-\omega)}{\zeta(\omega)}(\mid \aleph(\hbar_2) - \aleph(\hbar_1) \mid + \mid \Re(\hbar_2).\tilde{\beth}(\hbar_2,\ell) - \Re(\hbar_1).\tilde{\beth}(\hbar_1,\ell) \mid + \int_0^{\hbar_1} (\mho_1(\hbar_2,\xi) - \mho_1(\hbar_1,\xi))\chi_1(\tilde{\beth}(\xi,\ell))d\xi$$

$$+\int_{\hbar_1}^{\hbar_2}\mho_1(\hbar_2,\xi)\chi_1(\tilde{\beth}(\xi,\ell))d\xi+\int_0^1(\mho_2(\hbar_2,\xi)-\mho_2(\hbar_2,\xi))\chi_2(\tilde{\beth}(\xi,\ell))d\xi+\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^{\hbar_1}(\frac{(\hbar_2-\xi)^\omega}{(\hbar_2-\xi)}-\frac{(\hbar_1-\xi)^\omega}{(\hbar_1-\xi)})$$

$$[\Re(\xi).\tilde{\beth}(\xi,\ell)+\int_0^\xi\mho_1(\xi,F)\chi_1(\tilde{\beth}(F,\ell))dF+\int_0^1\mho_2(\xi,F).\chi_2(\tilde{\beth}(F,\ell))dF]d\xi\int_{\hbar_1}^{\hbar_2}\frac{(\hbar_2-\xi)^\omega}{(\hbar_2-\xi)}$$

$$[\Re(\xi).\tilde{\beth}(\xi,\ell)+\int_0^\xi\mho_1(\xi,F)\chi_1(\tilde{\beth}(F,\ell))dF+\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^1\mho_2(\xi,F).\chi_2(\tilde{\beth}(F,\ell))dF]d\xi$$

$=\mathfrak{S}+\mathfrak{T}+\mathfrak{U}$, where

$$\mathfrak{S}=\frac{(1-\omega)}{\zeta(\omega)}(\mid\aleph(\hbar_2)-\aleph(\hbar_1)\mid+\mid\Re(\hbar_2).\tilde{\beth}(\hbar_2,\ell)-\Re(\hbar_1).\tilde{\beth}(\hbar_1,\ell)\mid+\int_0^{\hbar_1}(\mho_1(\hbar_2,\xi)-\mho_1(\hbar_1,\xi)) \tag{4.8}$$

$$\chi_1(\tilde{\beth}(\xi,\ell))d\xi+\int_{\hbar_1}^{\hbar_2}\mho_1(\hbar_2,\xi)\chi_1(\tilde{\beth}(\xi,\ell))d\xi+\int_0^1(\mho_2(\hbar_2,\xi)-\mho_2(\hbar_2,\xi))\chi_2(\tilde{\beth}(\xi,\ell))d\xi.$$

If we use $\hbar_2\to\hbar_1$ then $\mathfrak{S}\to 0$. Again for $\mathfrak{T}$ we write ,

$$\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^{\hbar_1}(\frac{(\hbar_2-\xi)^\omega}{(\hbar_2-\xi)}-\frac{(\hbar_1-\xi)^\omega}{(\hbar_1-\xi)})[\Re(\xi)\tilde{\beth}(\xi,\ell)+\int_0^\xi\mho_1(\xi,F)\chi_1(\tilde{\beth}(F,\ell))dF+\int_0^1\mho_2(\xi,F)\chi_2(\tilde{\beth}(F,\ell))dF]d\xi$$

$$\leq(\|\Re\|_\infty R+\mho_1^*\lambda_1+\mho_2^*\lambda_2)\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^{\hbar_1}(\frac{(\hbar_2-\xi)^\omega}{(\hbar_2-\xi)}-\frac{(\hbar_1-\xi)^\omega}{(\hbar_1-\xi)})d\xi$$

$$\mathfrak{T}\leq((\hbar_2-\hbar_1)^\omega-(\hbar_2)^\omega+(\hbar_1)^\omega)\frac{(\|\Re\|_\infty R+\mho_1^*\lambda_1+\mho_2^*\lambda_2)\omega}{\Xi(\omega+1)\zeta(\omega)}. \tag{4.9}$$

If we use $\hbar_2\to\hbar_1$ then $\mathfrak{T}\to 0$. Again similar for $\mathfrak{U}$,

$$\mathfrak{U}=\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_{\hbar_1}^{\hbar_2}\frac{(\hbar_2-\xi)^\omega}{(\hbar_2-\xi)}[\Re(\xi).\tilde{\beth}(\xi,\ell)+\int_0^\xi\mho_1(\xi,F)\chi_1(\tilde{\beth}(F,\ell))dF+\int_0^1\mho_2(\xi,F)\chi_2(\tilde{\beth}(F,\ell))dF]d\xi$$

$$\leq(\|\Re\|_\infty R+\mho_1^*\lambda_1+\mho_2^*\lambda_2)\frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_{\hbar_1}^{\hbar_2}(\frac{(\hbar_2-\xi)^\omega}{(\hbar_2-\xi)})d\xi$$

$$\mathfrak{U}\leq((\hbar_2-\hbar_1)^\omega)\frac{(\|\Re\|_\infty R+\mho_1^*\lambda_1+\mho_2^*\lambda_2)\omega}{\Xi(\omega+1)\zeta(\omega)}. \tag{4.10}$$

and hence, If we use $\hbar_2\to\hbar_1$ then $\mathfrak{U}\to 0$. By using above condition of $\mathfrak{U}$ of (4.10) , $\mathfrak{S}$ of (4.9) and $\mathfrak{T}$ of (4.9),

$$\|\eth(\Upsilon\tilde{\beth}(\hbar_2,\ell),\Upsilon\tilde{\beth}(\hbar_1,\ell))\|_\infty\to 0,$$

as $\hbar_2\to\hbar_1$, which implies $\Upsilon$ is equicontinuous. With the help of Arzelà-Ascoli theorem, we say $\Upsilon$ is completely continuous, as we have $\Upsilon$ is compact in $C(\nabla,\mathbb{R}^z)$.

**Thirdly** we deduce $\Upsilon$ admits at least one fixed point in $\nabla$. Suppose, $\tilde{\beth}(\hbar,\ell)\in ß$. Then $\tilde{\beth}(\hbar,\ell)=\flat\Upsilon\tilde{\beth}(\hbar,\ell)$ Set $ß=\{\tilde{\beth}(\hbar,\ell)\in C(\nabla,\mathbb{R}^z):\tilde{\beth}(\hbar,\ell)=\flat\Upsilon\tilde{\beth}(\hbar,\ell),0<\flat<1\}$ bounded. Now for $0\leq\hbar\leq 1$,

$$\mid\tilde{\beth}(\hbar,\ell)\mid=\mid\flat\Upsilon\tilde{\beth}(\hbar,\ell)\mid\leq\mid\Upsilon\tilde{\beth}(\hbar,\ell)\mid\leq\mid\tilde{\beth}(0;\ell)\mid+(\frac{(1-\omega)}{\zeta(\omega)}+\frac{\omega}{\Xi(\omega+1)\zeta(\omega)})[\|\aleph\|_\infty+\|\Re\|_\infty R+\mho_1^*\lambda_1+\mho_2^*\lambda_2]$$

$$\leq\mid\tilde{\beth}(0;\ell)\mid+[\frac{\zeta\omega[\Xi(\omega+1)(1-\omega)+\omega]}{\zeta^2(\omega)\Xi(\omega+1)}][\|\aleph\|_\infty+\mho_1^*\lambda_1+\mho_2^*\lambda_2]+[\frac{\zeta\omega[\Xi(\omega+1)(1-\omega)+\omega]}{\zeta^2(\omega)\Xi(\omega+1)}].$$

$\|\Re\|_\infty R\leq\Psi_2R+\Psi_2$. By equation (4.3), we get $\Psi_2R+\Psi_2\leq R$. Hence we say ß is bounded, as we given in our main Theorem 3.2 we conclude $\Upsilon$ admits at least one fixed point. Hence, we deduce that Problem (1.1) admits at least one fixed point solution in $\nabla$. **Lastly** We work on the uniqueness of solution for our FV$I_d$E (1.1). We have to show here $\Upsilon$ admits unique solution. For that we consider $\tilde{\beth}_1(\hbar,\ell)$,$\tilde{\beth}_2(\hbar,\ell)\in C(\nabla,\mathbb{R}^z)$,

$$\curlywedge=(\frac{\zeta\omega.[\Xi(\omega+1).(1-\omega)+\omega\hbar^\omega]}{\zeta^2(\omega).\Xi(\omega+1)})[\|\Re\|_\infty+\mho_1^*c_1+\mho_2^*c_2]<1, \tag{4.11}$$

$$\mid\eth(\Upsilon\tilde{\beth}_1(\hbar,\ell),\Upsilon\tilde{\beth}_2(\hbar,\ell))\mid\leq\frac{(1-\omega)}{\zeta(\omega)}[\mid\Re(\hbar)\mid\mid\tilde{\beth}_1(\hbar,\ell)-\tilde{\beth}_2(\hbar,\ell)\mid+\int_0^\hbar\mid\mho_1(\hbar,\xi)\mid\mid\chi_1(\tilde{\beth}_1(\xi,\ell))-\chi_1(\tilde{\beth}_2(\xi,\ell))\mid d\xi$$

$$+\int_0^1 \mid \mho_2(\hbar,\xi) \mid\mid \chi_2(\tilde{\beth}_1(\xi,\ell)) - \chi_2(\tilde{\beth}_2(\xi,\ell)) \mid d\xi] + \frac{\omega}{\Xi(\omega)\zeta(\omega)}\int_0^\hbar \frac{(\hbar-\xi)^\omega}{(\hbar-\xi)}[\mid \Re(\xi) \mid\mid \tilde{\beth}_1(\xi,\ell) - \tilde{\beth}_2(\xi,\ell) \mid +$$

$$\int_0^\xi \mid \mho_1(\xi,F) \mid\mid \chi_1(\tilde{\beth}_1(F,\ell)) - \chi_1(\tilde{\beth}_2(F,\ell)) \mid dF + \int_0^1 \mid \mho_2(\xi,F) \mid\mid \chi_2(\tilde{\beth}_1(F,\ell)) - \chi_2(\tilde{\beth}_2(F,\ell)) \mid dF]d\xi.$$

Apply supremum both sides, we get

$$\|\eth(\Upsilon\tilde{\beth}_1(\hbar,\ell),\Upsilon\tilde{\beth}_2(\hbar,\ell))\|_\infty \le \frac{(1-\omega)}{\zeta(\omega)}[\|\Re\|_\infty + \mho_1^* c_1 + \mho_2^* c_2] + \|\tilde{\beth}_1 - \tilde{\beth}_2\|_\infty + \frac{\omega\hbar^\omega}{\Xi(\omega+1)\zeta(\omega)}[\|\Re\|_\infty +$$

$$\mho_1^* c_1 + \mho_2^* c_2] + \|\tilde{\beth}_1 - \tilde{\beth}_2\|_\infty = (\frac{(1-\omega)}{\zeta(\omega)} + \frac{\omega\hbar^\omega}{\Xi(\omega+1)\zeta(\omega)})[\|\Re\|_\infty + \mho_1^* c_1 + \mho_2^* c_2] + \|\tilde{\beth}_1 - \tilde{\beth}_2\|_\infty$$

$$= (\frac{\zeta\omega.[\Xi(\omega+1).(1-\omega)+\omega\hbar^\omega]}{\zeta^2(\omega).\Xi(\omega+1)})[\|\Re\|_\infty + \mho_1^* c_1 + \mho_2^* c_2] + \|\tilde{\beth}_1 - \tilde{\beth}_2\|_\infty.$$

So, By equation (4.11), we write $\|\eth(\Upsilon\tilde{\beth}_1(\hbar,\ell),\Upsilon\tilde{\beth}_2(\hbar,\ell))\|_\infty \le \curlywedge\|\tilde{\beth}_1 - \tilde{\beth}_2\|_\infty$, which shows that $\Upsilon$ is a Contraction map. Thus using Theorem 3.2, $\Upsilon$ admits a unique fixed point solution and hence we say system (1.1) admits a unique solution $\tilde{\beth}(\hbar,\ell)$. □

### 4.2 Application Part II

We consider the integral type of equation (1.2), which has two bounded continuous function namely $\curlywedge(\hbar) : [0,1] \to \mathbb{R}$ and $\aleph(\hbar,\Re_1(\hbar)) : [0,1]\times\mathbb{R} \to \mathbb{R}$. The function $\chi : [0,1)\times[0,1) \to [0,\infty)$ with $\chi(\hbar,.) \in L^1[0,1]$ and $0 \le \hbar \le 1$. Here we present Theorem 4.2 for existence and common solution to the equation (1.2).

#### 4.2.1 Application to the integral type equation

**Theorem 4.2.** *Suppose,*

I) *The continuous function,* $\curlywedge(\hbar) : [0,1] \to \mathbb{R}$ *and* $\aleph(\hbar,\Re_1(\hbar)) : [0,1]\times\mathbb{R}\to\mathbb{R}$. *Let* $\tilde{\beth} : \nabla\times\nabla$ *be an operator of,*

$$\nabla\Re_1(\hbar) - \curlywedge(\hbar) = \int_0^\hbar \chi(\hbar,\ell)\aleph(\ell,\Re_1\ell)d\ell \tag{4.12}$$

II) $\mid \aleph(\hbar,\Re_1(\hbar)) - \aleph(\hbar,\Re_2(\hbar)) \mid \le \frac{1}{F e^{iF\hbar}} \mid \Re_1(\hbar) - \Re_2(\hbar) \mid$ *for* $\forall\ \Re_1,\Re_2 \in \nabla$ & $1 < F \le \frac{1}{\eta}$; $0<\eta<1$.

III) *The function* $\chi : [0,1)\times[0,1) \to [0,\infty)$ *with* $\chi(\hbar,.) \in L^1[0,1]$ *and* $0 \le \hbar \le 1$*;*

$$1 \ge \|\int_0^\hbar \chi(\hbar,\ell)d\ell\|, \tag{4.13}$$

*where,* $\nabla = C([0,1],\mathbb{R})$ *be real valued continuous function on* $[0,1]$ *and* $\Re_1(\hbar) \in \nabla$ *then* (1.2) *admits unique solution.*

*Proof.* Let the mapping, $\psi(\hbar) : \nabla\times\nabla \to [1,\infty)$ defined as,

$$\psi(\hbar) = \begin{cases} F + \max\{\Re_1(\hbar),\Re_2(\hbar)\}, & \text{Otherwise} \\ \\ 1, & \text{if } \Re_1,\Re_2 \in [0,1]. \end{cases}$$

Assume $\eth_\mathbb{C} : \nabla\times\nabla \to \mathbb{C}$ be a complex valued $\eth_\mathbb{C}$ metric space,

$$\eth_\mathbb{C}(\Re_1,\Re_2) = \|\Re_1\|_\infty = \sup_{0\le\hbar\le1} \mid \Re_1(\hbar) \mid e^{-iF\hbar},$$

where, $\nabla = C([0,1],\mathbb{R})$, $1 < F \le \frac{1}{\eta}; 0 < \eta < 1$ and $(i)^2 = -1$. Here its easy to say $(\nabla,\eth_\mathbb{C})$ is complete complex valued $\eth_\mathbb{C}$ metric space. Main integral type equation (1.2) can be again resumed to find the element $\hbar^* \in \nabla$ which gives fixed point for $\tilde{\beth}$, Now

$$\mid \tilde{\beth}\Re_1(\hbar) - \tilde{\beth}\Re_2(\hbar) \mid \le \mid \int_0^\hbar [\chi(\hbar,\ell)\aleph(\ell,\Re_1\ell) - \chi(\hbar,\ell)\aleph(\ell,\Re_2\ell)] \mid d\ell \le \mid \int_0^\hbar \mid \chi(\hbar,\ell)[\aleph(\ell,\Re_1\ell) - \aleph(\ell,\Re_2\ell)] \mid d\ell$$

$$\le (\int_0^\hbar \chi(\hbar,\ell)d\ell)\int_0^\hbar \mid [\aleph(\ell,\Re_1\ell) - \aleph(\ell,\Re_2\ell)] \mid d\ell \le \frac{1}{\beta e^{iF\hbar}}(\int_0^\hbar \chi(\hbar,\ell)d\ell)\int_0^\hbar \mid [\Re_1\ell - \Re_2\ell] \mid d\ell$$

$$= \frac{e^{iF\hbar}}{\beta e^{iF\hbar}e^{-iF\hbar}}(\int_0^\hbar \chi(\hbar,\ell)e^{-iF\hbar}d\ell)\int_0^\hbar \mid [\Re_1\ell - \Re_2\ell] \mid e^{-iF\hbar}d\ell.$$

Apply Supremum to both side, we get

$$[\sup_{0\leq\hbar\leq 1} \mid \tilde{\beth}\Re_1(\hbar) - \tilde{\beth}\Re_2(\hbar) \mid e^{-iF\hbar}] \leq \frac{1}{F}(\int_0^{\hbar} \sup_{0\leq\hbar\leq 1} \chi(\hbar,\ell)e^{-iF\hbar}d\ell)[\sup_{0\leq\hbar\leq 1} \mid [\Re_1\hbar - \Re_2\hbar] \mid e^{-iF\hbar}d\ell].$$

with the help of (4.2) and II, we get

$$\eth_{\mathbb{C}}(\tilde{\beth}\Re_1, \tilde{\beth}\Re_2) = \|\tilde{\beth}\Re_1 - \tilde{\beth}\Re_2\|_\infty \leq \frac{1}{F}\|\Re_1 - \Re_2\|_\infty = \frac{1}{F}\eth_{\mathbb{C}}(\Re_1, \Re_2).$$

We can check easily both cases of $\psi(\Re_1, \Re_2)$ when $0 \leq \Re_1 \leq 1$ ; $0 \leq \Re_2 \leq 1$ or else (3.13) true. Hence for $0 < \frac{1}{F} < 1$, all hypothesis of Theorem 3.2 hold true, which finally gives that (1.2) admits unique solution. □

## 5 Conclusion

To study and contribute to worldly problems we consider the concept of controlled, double controlled metric in the setting of Extended complex valued metric space. Afterwards, we present our paper in three folds as, **Firstly**, we introduce fixed point theorem which is the extended version of famous results from literature, namely Fisher and Banach [16] contraction type results along with some examples to sustain our results. **Secondly** with the help of $ABC$ fractional derivative (1.1), we introduced common fixed point Theorem 4.1 for $FVI_dE$ and its unique fixed point solution. **Thirdly** we introduced a fixed point solution to the integral type equation (1.2) in $\eth_{\mathbb{C}}$ metric as the application part of main results.

**Acknowledgment**

The author is very much thankful to the Editor and Referee for their valuable suggestions to bring the manuscript in its present form.

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