ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(2) (2023), 265-272**
(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

SOME RESULTS OF THE GROWTH PROPERTIES OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES ON THE BASIS OF THEIR (p,q)th-RELATIVE GOL'DBERG ORDER AND (p,q)th-RELATIVE GOL'DBERG TYPE

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(Received: January 22, 2023; In format: August 27, 2023; Revised: December 15, 2023; Accepted: December 16, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53233

Abstract

Biswas [2] introduced the idea of (p,q)th-relative Gol'dberg order and (p,q)th-relative Gol'dberg type of an entire function of several complex variables. In this paper we want to establish some results of the growth analysis of entire function of several complex variables on the basis of their (p,q)-ψ relative Gol'dberg order and (p,q)-ψ relative Gol'dberg type of an entire function of several complex variables.

2020 Mathematical Sciences Classification: 32A15, 30D35

Keywords and Phrases: (p,q)-ψ relative Gol'dberg order, (p,q)-ψ relative Gol'dberg type, growth, entire function of several complex variables, (p,q)-ψ relative Gol'dberg weak type.

1 Introduction

Let $\mathbb{C}^n$ and $\mathbb{R}^n$ respectively denotes the complex and real n-spaces. Also, let us indicate the point $(z_1, z_2, \ldots, z_n)$, $(m_1, m_2, \ldots, m_n)$ of $\mathbb{C}^n$ or $\mathbb{I}^n$ by their corresponding unsuffixed symbols z, m respectively where I denotes the set of non negative integers. The modulus of z, denoted by $|z|$, is defined as $|z| = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}$. If the coordinates of the vector m are non-negative integers, then z^n will denote $z_1^{m_1}, z_2^{m_2}, \ldots, z_n^{m_n}$ and $\|m\| = m_1 + m_2 + \cdots + m_n$. If $D \subseteq \mathbb{C}^n$ be an arbitrary bounded complex n-circular domain with center at the origin of coordinates, then for any entire function $f(z)$ on n-complex variables and $R > 0$, $M_{f,D}(R)$ may be defined as $M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$, where a point $z \in D_R$ iff $\frac{z}{R} \in D$. If $f(z)$ is non-constant, then $M_{f,D}(R)$ is strictly increasing and its inverse $M_{f,D}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists such that $\lim_{R\to\infty} M_{f,D}^{-1}(R) = \infty$. For $k \in \mathbb{N}$, we define $\exp^{[k]} R = \exp(\exp^{[k-1]} R)$ and $\log^{[k]} R = \log(\log^{[k-1]} R)$, where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} R = R$, $\log^{[-1]} R = \exp R$, $\exp^{[0]} R = R$ and $\exp^{[-1]} R = \log R$, where p and q always denote positive integers. Maji and Datta [9] introduced the definitions of (p,q)th-Gol'dberg order and (p,q)th-Gol'dberg lower order of an entire function $f(z)$ of n-complex variables, where $p \geq q$ in the following ways;

$$\rho_D^{(p,q)}(f) = \limsup_{R\to\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} R}, \tag{1.1}$$

and

$$\lambda_D^{(p,q)}(f) = \liminf_{R\to\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} R}. \tag{1.2}$$

For $p = 2$ and $q = 1$ the symbols $\rho_D^{(p,q)}(f)$ and $\lambda_D^{(p,q)}(f)$ are respectively denoted by $\rho_D(f)$ and $\lambda_D(f)$ which are actually classical growth indicators [7, 8]. However in the line of Gol'dberg [7, 8], it may be easily established that $\rho^{(p,q)}(f)$ and $\lambda^{(p,q)}(f)$ instead of $\rho_D^{(p,q)}(f)$ and $\lambda_D^{(p,q)}(f)$ respectively.

2 Definitions

Biswas [5] introduced the defintions of (p,q)-ψ order and (p,q)-ψ lower order of an entire function of n-complex variables.

Definition 2.1 ([5]). *Let $\psi(R) : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function. Then the (p, q)-ψ Gol'dberg order $\rho_D^{(p,q)}(f, \psi)$ and (p, q)-ψ Gol'dberg lower order $\lambda_D^{(p,q)}(f, \psi)$ of an entire function $f(z)$ of n-complex variables are defined as,*

$$\rho_D^{(p,q)}(f, \psi) = \limsup_{R\to\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} \psi(R)}, \tag{2.1}$$

and

$$\lambda_D^{(p,q)}(f, \psi) = \liminf_{R\to\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} \psi(R)}. \tag{2.2}$$

Definition 2.1 avoids the restriction for $p \geq q$. However, an entire function $f(z)$ for which $\rho_D^{(p,q)}(f, \psi)$ and $\lambda_D^{(p,\ q)}(f, \psi)$ are called regular (p, q)-ψ Gol'dberg growth. Otherwise, $f(z)$ is said to be irregular (p, q)-ψ Gol'dberg growth. For any non-decreasing unbounded function $\psi(R) : [0, +\infty) \to (0, +\infty)$, if it is assumed that $\lim_{R\to+\infty} \frac{\log^{[q]} \psi(\alpha R)}{\log^{[q]} \psi(R)} = 1$, for all $\alpha > 0$, then one can easily verify that $\rho_D^{(p,q)}(f, \psi)$ and $\lambda_D^{(p,q)}(f, \psi)$ are independent of the choice of the domain D and use the symbols $\rho^{(p,q)}(f, \psi)$ and $\lambda^{(p,q)}(f, \psi)$ instead of $\rho_D^{(p,q)}(f, \psi)$ and $\lambda_D^{(p,q)}(f, \psi)$ respectively. Now for any two entire functions $f(z)$ and $g(z)$ of n-complex variables, Mondal and Roy [11] introduced the concept of relative Gol'dberg order of $f(z)$ with respect to $g(z)$ and relative Gol'berg lower order of $f(z)$ with respect to $g(z)$. For the (p, q)-ψ relative Goldberg order introduced by Biswas and Biswas [5] in the following definitions:

Definition 2.2 ([5]). *Let $\psi(R) : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function. Also, let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables. The (p, q)-ψ relative Gol'dberg order and the (p, q)-ψ relative Gol'dberg lower order of $f(z)$ with respect to $g(z)$ are defined as*

$$\rho_{g,D}^{(p,q)}(f, \psi) = \limsup_{R\to\infty} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} \psi(R)}, \tag{2.3}$$

and

$$\lambda_{g,D}^{(p,q)}(f, \psi) = \liminf_{R\to\infty} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} \psi(R)}. \tag{2.4}$$

Further an entire function $f(z)$ of n-complex variables for which $\rho_{g,D}^{(p,q)}(f, \psi)$ and $\lambda_{g,D}^{(p,q)}(f, \psi)$ are same, is called a function of (p, q)-ψ relative Gol'dberg growth with respect to an entire function $g(z)$ of n-complex variables. Otherwise, $f(z)$ is said to be irregular (p, q)-ψ relative Gol'dberg growth with respect to $g(z)$.

Definition 2.3 ([6]). *Let $f(z)$ and $g(z)$ be two entire functions of n-complex variables with the index-pair (m, q) and (m, p) respectively, where p, q, m are the integers such that $m \geq q + 1 \geq 1$ and $m \geq p + 1 \geq 1$, if $b < \rho_{g,D}^{(p,q)}(f, \psi) < +\infty$ and $\rho_{g,D}^{(p-1,q-1)}(f, \psi)$ is not a non-zero finite number, where $b = 1$, if $p = q$, and $b = 0$ for otherwise. Moreover, if $0 < \rho_{g,D}^{(p,q)}(f, \psi) < \infty$,*

$$\begin{cases} \rho_{g,D}^{(p-n,q)}(f, \psi) = \infty & , \quad for \quad n < p; \\ \rho_{g,D}^{(p,q-n)}(f, \psi) = 0 & , \quad for \quad n < q; \\ \rho_{g,D}^{(p+n,q+n)}(f, \psi) = 1 & , \quad for \quad n = 1, 2, \ldots. \end{cases} \tag{2.5}$$

Similarly for $0 < \lambda_{g,D}^{(p,q)}(f, \psi) < \infty$, then

$$\begin{cases} \lambda_{g,D}^{(p-n,q)}(f, \psi) = \infty & , \quad for \quad n < p; \\ \lambda_{g,D}^{(p,q-n)}(f, \psi) = 0 & , \quad for \quad n < q; \\ \lambda_{g,D}^{(p+n,q+n)}(f, \psi) = 1 & , \quad for \quad n = 1, 2, \ldots. \end{cases} \tag{2.6}$$

If $\psi(R) = R$ and $p \geq q$, then Definition 2.2 coincides with the definition of (p, q)-ψ relative Gol'dberg order and (p, q)-ψ relative Gol'dberg lower order introduced by T. Biswas and R. Biswas [6]. Consequently for $\psi(R) = R$ and $p \geq q$, Definition 2.3 reduces to the definition of index-pair (p, q) of an entire function with respect to another entire function of n-complex variables [3].

T. Biswas and C. Biswas [4] introduced the definition of (p, q)-ψ relative Gol'dberg type $\triangle_{g,D}^{(p,q)}(f, \psi)$ and (p, q)-ψ relative Gol'dberg lower type $\nabla_{g,D}^{(p,q)}(f, \psi)$, (p, q)-ψ relative Gol'dberg weak type $\overline{\triangle}_{g,D}^{(p,q)}(f, \psi)$ and the growth indicator $\overline{\nabla}_{g,D}^{(p,q)}(f, \psi)$ in the following ways;

Definition 2.4 ([4])**.** *Let $\psi(R):[0,+\infty)\to(0,+\infty)$ be a non-decreasing unbounded function. Also, let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables. The (p,q)-ψ relative Gol'dberg type and (p,q)-ψ relative Gol'dberg lower type of $f(z)$ with respect to $g(z)$ are defined as,*

$$\triangle_{g,D}^{(p,q)}(f,\psi)=\limsup_{R\to\infty}\frac{\log^{[p-1]}M_{g,D}^{-1}(M_{f,D}(R))}{\left(\log^{[q-1]}\psi(R)\right)^{\rho_{g,D}^{(p,q)}(f,\psi)}},\quad 0<\rho_{g,D}^{(p,q)}(f,\psi)<+\infty, \tag{2.7}$$

and

$$\nabla_{g,D}^{(p,q)}(f,\psi)=\liminf_{R\to\infty}\frac{\log^{[p-1]}M_{g,D}^{-1}(M_{f,D}(R))}{\left(\log^{[q-1]}\psi(R)\right)^{\rho_{g,D}^{(p,q)}(f,\psi)}},\quad 0<\rho_{g,D}^{(p,q)}(f,\psi)<+\infty. \tag{2.8}$$

Definition 2.5 ([4])**.** *Let $\psi(R):[0,+\infty)\to(0,+\infty)$ be a non-decreasing unbounded function. Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables. The relative (p,q)-ψ Gol'dberg weak type and the growth indicator of $f(z)$ with respect to $g(z)$ are defined as,*

$$\overline{\triangle}_{g,D}^{(p,q)}(f,\psi)=\liminf_{R\to\infty}\frac{\log^{[p-1]}M_{g,D}^{-1}(M_{f,D}(R))}{\left(\log^{[q-1]}\psi(R)\right)^{\lambda_{g,D}^{(p,q)}(f,\psi)}},\quad 0<\lambda_{g,D}^{(p,q)}(f,\psi)<+\infty, \tag{2.9}$$

and

$$\overline{\nabla}_{g,D}^{(p,q)}(f,\psi)=\limsup_{R\to\infty}\frac{\log^{[p-1]}M_{g,D}^{-1}(M_{f,D}(R))}{\left(\log^{[q-1]}\psi(R)\right)^{\lambda_{g,D}^{(p,q)}(f,\psi)}},\quad 0<\lambda_{g,D}^{(p,q)}(f,\psi)<+\infty. \tag{2.10}$$

During the past decades, the several authors [1, 2, 3, 5, 6, 10] made closed investigation on the growth properties of entire functions of n-complex variables using different growth indicator such as (p,q)-ψ relative order, (p,q)-ψ relative lower order etc.

3 Mains Results

Theorem 3.1. *Let f, g, h and k be any four entire functions of n-complex variables such that $0<\lambda_{h,D}^{(p,m)}(f,\psi)<\rho_{h,D}^{(p,m)}(f,\psi)<+\infty$, and $0<\lambda_{k,D}^{(q,m)}(g,\psi)<\rho_{k,D}^{(q,m)}(g,\psi)<+\infty$, where p, q, m, are all positive integers.*
Then

$$\frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)}\le\liminf_{R\to\infty}\frac{\log^{[p]}M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]}M_{k,D}^{-1}(M_{g,D}(R))}\le\frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}$$
$$\le\limsup_{R\to\infty}\frac{\log^{[p]}M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]}M_{k,D}^{-1}(M_{g,D}(R))}\le\frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}.$$

Proof. From the definition of $\lambda_{h,D}^{(p,m)}(f,\psi)$ and $\rho_{k,D}^{(q,m)}(g,\psi)$, we get for arbitrary positive $\epsilon>0$ for all large values of R,

$$\log^{[p]}M_{h,D}^{-1}(M_{f,D}(R))\ge(\lambda_{h,D}^{(p,m)}(f,\psi)-\epsilon)\log^{[m]}\psi(R), \tag{3.1}$$

and

$$\log^{[q]}M_{k,D}^{-1}(M_{g,D}(R))\le(\rho_{k,D}^{(q,m)}(g,\psi)+\epsilon)\log^{[m]}\psi(R). \tag{3.2}$$

Now from (3.1) and (3.2), it follows that for all sufficiently large values of R

$$\frac{\log^{[p]}M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]}M_{k,D}^{-1}(M_{g,D}(R))}\ge\frac{\lambda_{h,D}^{(p,m)}(f,\psi)-\epsilon}{\rho_{k,D}^{(q,m)}(g,\psi)+\epsilon}.$$

As $\epsilon>0$ is arbitrary,

$$\liminf_{R\to\infty}\frac{\log^{[p]}M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]}M_{k,D}^{-1}(M_{g,D}(R))}\ge\frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)}. \tag{3.3}$$

Again for a sequence of value of R tending to infinity,

$$\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R)) \leq (\lambda_{h,D}^{(p,m)}(f,\psi)+\epsilon)\log^{[m]}\psi(R), \tag{3.4}$$

and for all sufficiently large values of R,

$$\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R)) \geq (\lambda_{k,D}^{(q,m)}(g,\psi)-\epsilon)\log^{[m]}\psi(R). \tag{3.5}$$

Now from (3.4) and (3.5), we obtain for a sequence of values of R tending to infinity

$$\frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\lambda_{h,D}^{(p,m)}(f,\psi)+\epsilon}{\lambda_{k,D}^{(q,m)}(g,\psi)-\epsilon}.$$

As $\epsilon > 0$ is arbitrary,

$$\liminf_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}. \tag{3.6}$$

Also for all sufficient values of R,

$$\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R)) \leq (\lambda_{k,D}^{(q,m)}(g,\psi)+\epsilon)\log^{[m]}\psi(R). \tag{3.7}$$

Combining (3.1) and (3.7), we obtain for a sequence of values of R tending to infinity,

$$\frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{\lambda_{h,D}^{(p,m)}(f,\psi)-\epsilon}{\lambda_{k,D}^{(q,m)}(g,\psi)+\epsilon}.$$

As $\epsilon > 0$ is arbitrary,

$$\limsup_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}. \tag{3.8}$$

Also for all sufficiently large values of R,

$$\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R)) \leq (\rho_{h,D}^{(p,m)}(f,\psi)+\epsilon)\log^{[m]}\psi(R). \tag{3.9}$$

Now combining (3.5) and (3.9), for all sufficiently large values of R,

$$\frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\rho_{h,D}^{(p,m)}(f,\psi)+\epsilon}{\lambda_{k,D}^{(q,m)}(g,\psi)-\epsilon}.$$

Since ϵ is arbitrary

$$\limsup_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}. \tag{3.10}$$

Thus the theorem follows from (3.3), (3.6), (3.8) and (3.10). □

Theorem 3.2. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \rho_{h,D}^{(p,m)}(f,\psi) < +\infty$, and $0 < \rho_{k,D}^{(q,m)}(g,\psi) < +\infty$, where p, q, m, are all positive integers. Then*

$$\liminf_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)} \leq \limsup_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))}.$$

Proof. From the definition of $\rho_{k,D}^{(q,m)}(g,\psi)$, we get for a sequence of values of R tending to infinity,

$$\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R)) \geq (\rho_{k,D}^{(q,m)}(g,\psi)-\epsilon)\log^{[m]}\psi(R). \tag{3.11}$$

Now from (3.9) and (3.11), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\rho_{h,D}^{(p,m)}(f,\psi)+\epsilon}{\rho_{k,D}^{(q,m)}(g,\psi)-\epsilon}.$$

As $\epsilon > 0$ is arbitrary,

$$\liminf_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)}. \tag{3.12}$$

Also for all sufficiently large values of R ,

$$\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R)) \geq (\rho_{h,D}^{(q,m)}(f,\psi) - \epsilon) \log^{[m]} \psi(R). \tag{3.13}$$

Now from (3.2) and (3.13), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{\rho_{h,D}^{(p,m)}(f,\psi) - \epsilon}{\rho_{k,D}^{(q,m)}(g,\psi) + \epsilon}.$$

As $\epsilon > 0$ is arbitrary,

$$\limsup_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)}. \tag{3.14}$$

Thus the theorem follows from (3.12) and (3.14). □

Theorem 3.3. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \lambda_{h,D}^{(p,m)}(f,\psi) < \rho_{h,D}^{(p,m)}(f,\psi) < +\infty$ and $0 < \lambda_{k,D}^{(q,m)}(g,\psi) < \rho_{k,D}^{(q,m)}(g,\psi) < +\infty$, where p, q, m, are all positive integers.*
Then

$$\frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)} \leq \liminf_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \min\left\{\frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}, \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)}\right\}$$
$$\leq \max\left\{\frac{\lambda_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}, \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\rho_{k,D}^{(q,m)}(g,\psi)}\right\} \leq \limsup_{R\to\infty} \frac{\log^{[p]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\rho_{h,D}^{(p,m)}(f,\psi)}{\lambda_{k,D}^{(q,m)}(g,\psi)}.$$

Theorem 3.4. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \nabla_{h,D}^{(p,m)}(f,\psi) < \triangle_{h,D}^{(p,m)}(f,\psi) < +\infty$ and $0 < \nabla_{k,D}^{(q,m)}(g,\psi) < \triangle_{k,D}^{(q,m)}(g,\psi) < +\infty$ and $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$, where p, q, m are all positive integers then*

$$\frac{\nabla_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)} \leq \liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\nabla_{h,D}^{(p,m)}(f,\psi)}{\nabla_{k,D}^{(q,m)}(g,\psi)}$$
$$\leq \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\nabla_{k,D}^{(q,m)}(g,\psi)}.$$

Proof. From the definition of $\triangle_{k,D}^{(q,m)}(g,\psi)$ and $\nabla_{h,D}^{(p,m)}(f,\psi)$, we have for arbitrary $\epsilon > 0$ and for all sufficient large values of R,

$$\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R)) \geq (\nabla_{h,D}^{(p,m)}(f,\psi) - \epsilon)\{\log^{m-1} \psi(R)\}^{\rho_{h,D}^{(p,m)}(f,\psi)}, \tag{3.15}$$

and

$$\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R)) \leq (\triangle_{k,D}^{(q,m)}(g,\psi) + \epsilon)\{\log^{m-1} \psi(R)\}^{\rho_{k,D}^{(q,m)}(g,\psi)}. \tag{3.16}$$

Now using the condition $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$, combining (3.15) and (3.16), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{(\nabla_{h,D}^{(p,m)}(f,\psi) - \epsilon)}{(\triangle_{k,D}^{(q,m)}(g,\psi) + \epsilon)}.$$

As $\epsilon > 0$ is arbitrary,

$$\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{f,D}(R))} \geq \frac{\nabla_{h,D}^{(p,m)}(g,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)}. \tag{3.17}$$

Also, for all sufficient large values of R,

$$\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R)) \leq (\nabla_{h,D}^{(p,m)}(f,\psi) + \epsilon)\{\log^{m-1} \psi(R)\}^{\rho_{h,D}^{(p,m)}(f,\psi)}, \tag{3.18}$$

and for all sufficiently large values of R,

$$\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R)) \geq (\nabla_{k,D}^{(q,m)}(f,\psi) - \epsilon)\{\log^{m-1} \psi(R)\}^{\rho_{k,D}^{(q,m)}(g,\psi)}. \tag{3.19}$$

Using the condition $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$, combining (3.18) and (3.19), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{(\nabla_{h,D}^{(p,m)}(f,\psi) + \epsilon)}{(\nabla_{k,D}^{(q,m)}(g,\psi) - \epsilon)}.$$

As $\epsilon > 0$ is arbitrary,

$$\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\nabla_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)}. \tag{3.20}$$

Also for all sufficiently large values of R,

$$\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R)) \leq (\nabla_{k,D}^{(q,m)}(g,\psi) + \epsilon)\{\log^{m-1} \psi(R)\}^{\rho_{k,D}^{(q,m)}(g,\psi)}. \tag{3.21}$$

Using the condition $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$, combining (3.15) and (3.21), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{(\nabla_{h,D}^{(p,m)}(f,\psi) - \epsilon)}{(\nabla_{k,D}^{(q,m)}(g,\psi) + \epsilon)}.$$

As $\epsilon > 0$ is arbitrary,

$$\limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{f,D}(R))} \geq \frac{\nabla_{h,D}^{(p,m)}(f,\psi)}{\nabla_{k,D}^{(q,m)}(f,\psi)}. \tag{3.22}$$

Also for all sufficiently large values of R,

$$\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R)) \leq (\triangle_{h,D}^{(p,m)}(f,\psi) + \epsilon)\{\log^{m-1} \psi(R)\}^{\rho_{h,D}^{(p,m)}(f,\psi)}. \tag{3.23}$$

Using the condition $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$ and combining (3.19) and (3.23), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{f,D}(R))} \leq \frac{(\triangle_{h,D}^{(p,m)}(g,\psi) + \epsilon)}{(\nabla_{k,D}^{(q,m)}(g,\psi) - \epsilon)}.$$

As $\epsilon > 0$ is arbitrary,

$$\limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\nabla_{k,D}^{(q,m)}(g,\psi)}. \tag{3.24}$$

Thus the theorem follows from (3.17), (3.20), (3.22) and (3.24). □

Theorem 3.5. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \triangle_{h,D}^{(p,m)}(f,\psi) < +\infty$, and $0 < \triangle_{k,D}^{(q,m)}(g,\psi) < +\infty$ and $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$ where p, q, m are all positive integers.*
Then

$$\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)} \leq \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))}.$$

Proof. From the definition of $\triangle_{k,D}^{(q,m)}(g,\psi)$, we have for arbitrary ϵ and for all sufficient large values of R,

$$\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R)) \geq (\triangle_{k,D}^{(q,m)}(g,\psi) - \epsilon)\{\log^{m-1} \psi(R)\}^{\rho_{h,D}^{(q,m)}(g,\psi)}, \tag{3.25}$$

Using condition $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$ and combining (3.23) and (3.25), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\triangle_{h,D}^{(p,m)}(f,\psi) + \epsilon}{\triangle_{k,D}^{(q,m)}(g,\psi) - \epsilon}.$$

As $\epsilon > 0$ is arbitrary,

$$\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)}. \tag{3.26}$$

Also for all sufficiently large values of R,

$$\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R)) \geq (\triangle_{h,D}^{(p,m)}(f,\psi) - \epsilon)\{\log^{m-1}\psi(R)\}^{\rho_{h,D}^{(p,m)}(f,\psi)}. \tag{3.27}$$

Using condition $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$ and combining (3.16) and (3.27), we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{\triangle_{h,D}^{(p,m)}(f,\psi) - \epsilon}{\triangle_{k,D}^{(q,m)}(g,\psi) + \epsilon}.$$

As $\epsilon > 0$ is arbitrary,

$$\limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \geq \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)}. \tag{3.28}$$

Thus, the theorem follows from (3.26) and (3.28). □

Theorem 3.6. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \nabla_{h,D}^{(p,m)}(f,\psi) < \triangle_{h,D}^{(p,m)}(f,\psi) < +\infty$, and $0 < \nabla_{k,D}^{(q,m)}(g,\psi) < \triangle_{k,D}^{(q,m)}(g,\psi) < +\infty$ and $\rho_{h,D}^{(p,m)}(f,\psi) = \rho_{k,D}^{(q,m)}(g,\psi)$ where p, q, m are all positive integers.*
Then

$$\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \min\left\{\frac{\nabla_{h,D}^{(p,m)}(f,\psi)}{\nabla_{k,D}^{(q,m)}(g,\psi)}, \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)}\right\}$$
$$\leq \max\left\{\frac{\nabla_{h,D}^{(p,m)}(f,\psi)}{\nabla_{k,D}^{(q,m)}(g,\psi)}, \frac{\triangle_{h,D}^{(p,m)}(f,\psi)}{\triangle_{k,D}^{(q,m)}(g,\psi)}\right\} \leq \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))}.$$

Theorem 3.7. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \overline{\triangle}_{h,D}^{(p,m)}(f,\psi) < \overline{\nabla}_{h,D}^{(p,m)}(f,\psi) < +\infty$ and $0 < \overline{\triangle}_{k,D}^{(q,m)}(g,\psi) < \overline{\nabla}_{k,D}^{(q,m)}(g,\psi) < +\infty$ and $\lambda_{h,D}^{(p,m)}(f,\psi) = \lambda_{k,D}^{(q,m)}(g,\psi)$ where p, q, m are all positive integers.*
Then

$$\frac{\overline{\triangle}_{h,D}^{(p,m)}(f,\psi)}{\overline{\nabla}_{k,D}^{(q,m)}(g,\psi)} \leq \liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\overline{\triangle}_{h,D}^{(p,m)}(f,\psi)}{\overline{\triangle}_{k,D}^{(q,m)}(g,\psi)}$$
$$\leq \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\overline{\nabla}_{h,D}^{(p,m)}(f,\psi)}{\overline{\triangle}_{k,D}^{(q,m)}(g,\psi)}.$$

Similarly, in line with Theorem 3.8 and Theorem 3.9 and with help of Theorems 3.5 and 3.6, one may easily prove the following two theorems, and therefore their proofs are omitted.

Theorem 3.8. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \overline{\nabla}_{h,D}^{(p,m)}(f,\psi) < +\infty$, $0 < \overline{\nabla}_{k,D}^{(q,m)}(g,\psi) < +\infty$ and $\lambda_{h,D}^{(p,m)}(f,\psi) = \lambda_{k,D}^{(q,m)}(g,\psi)$ where p, q, m are all positive integers,*

$$\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \frac{\overline{\nabla}_{h,D}^{(p,m)}(f,\psi)}{\overline{\nabla}_{k,D}^{(q,m)}(g,\psi)} \leq \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))}.$$

Theorem 3.9. *Let f, g, h and k be any four entire functions of n-complex variables such that $0 < \overline{\triangle}_{h,D}^{(p,m)}(f,\psi) < \overline{\nabla}_{h,D}^{(p,m)}(f,\psi) < +\infty$, $0 < \overline{\triangle}_{k,D}^{(q,m)}(g,\psi) < \overline{\nabla}_{k,D}^{(q,m)}(g,\psi) < +\infty$ and $\lambda_{h,D}^{(p,m)}(f,\psi) = \lambda_{k,D}^{(q,m)}(g,\psi)$ where p, q, m are all positive integers*

$$\liminf_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))} \leq \min\left\{\frac{\overline{\triangle}_{h,D}^{(p,m)}(f,\psi)}{\overline{\triangle}_{k,D}^{(q,m)}(g,\psi)}, \frac{\overline{\nabla}_{h,D}^{(p,m)}(f,\psi)}{\overline{\nabla}_{k,D}^{(q,m)}(g,\psi)}\right\}$$
$$\leq \max\left\{\frac{\overline{\triangle}_{h,D}^{(p,m)}(f,\psi)}{\overline{\triangle}_{k,D}^{(q,m)}(g,\psi)}, \frac{\overline{\nabla}_{h,D}^{(p,m)}(f,\psi)}{\overline{\nabla}_{k,D}^{(q,m)}(g,\psi)}\right\} \leq \limsup_{R\to\infty} \frac{\log^{[p-1]} M_{h,D}^{-1}(M_{f,D}(R))}{\log^{[q-1]} M_{k,D}^{-1}(M_{g,D}(R))}.$$

4 Conclusion

In this paper, we want to establish some growth properties of entire function of n-complex variables on the basis of their of (p,q)-ψ relative Gol'dberg order, (p,q)-ψ relative Gol'dberg type, (p,q)-ψ relative Gol'dberg weak type and growth indicator where p, q are any positive integer.

Acknowledgement. We are very grateful to the Editor and Referees for their useful suggestions in bringing the paper to its present form.

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