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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

A NEW CLASS OF PÁL TYPE INTERPOLATION IN COMPLEX PLANE-II

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Abstract

In this paper, the author considered a new class for Pál type interpolation problems. They termed Pál type interpolation problems as $PTIP$. This new class for $PTIP$ is defined by omitting a non-zero complex node from the set of value nodes and simultaneously adding another complex node to the set of derivative nodes.

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1 Introduction

Lacunary Polynomial Interpolation is an extension of Hermite Interpolation. It comprises the matching of values and derivatives at certain points but does not insist that these points be consecutive. The author termed Lacunary Polynomial Interpolation as LPI. LPI problems are not always regular due to matching at non-consecutive derivatives.

The study on LPI started with the evolution of Birkhoff interpolation. It is a finely honed theory on real nodes [9, 20]. LPI problems on non-uniformly distributed nodes received attention after the investigations of Brueck [1]. He studied non-uniformly distributed nodes on the unit disk, obtained by applying Mbius transform to the set of zeros of roots of unity. He defined the following polynomials;

$$v_n^\alpha(z) = (z+\alpha)^n - (1+\alpha z)^n, \tag{1.1}$$

$$w_n^\alpha(z) = (z+\alpha)^n + (1+\alpha z)^n. \tag{1.2}$$

where $0 < \alpha < 1$.

A revolution in the theory of LPI at special nodes was due to Pál [12]. He introduced a new type of interpolation on zeros of two different polynomials, referred as Pál type interpolation.

Let $A(z) \in \pi_m$ and $B(z) \in \pi_n$, where π_n be the set of polynomials of degree less than or equal to n with complex coefficients. For a given positive integer r the problem of $(0, r)$ Pál type interpolation i.e. $(0, r)$-$PTIP$ consists finding a polynomial $P(z) \in \pi_{m+n-1}$, that has prescribed values at m pairwise distinct nodes and prescribed value for r^{th} derivative at n pairwise distinct nodes. These m nodes are called value nodes, and n nodes are called derivative nodes.

The $(0, r)$-$PTIP$ on the pair $\{A(z), B(z)\}$ is regular if and only if any $P(z) \in \pi_{m+n-1}$ with the following sets of interpolation conditions:

$P(y_i) = 0$; where $A(y_i) = 0$; $i = 1, 2, \ldots, m$,

$P^{(r)}(z_j) = 0$; where $B(z_j) = 0$; $j = 1, 2, \ldots, n$.

implies that $P(z) \equiv 0$. Here the zeros of $A(z)$, $B(z)$ are assumed to be simple.

De Bruin [2, 4, 5], De Bruin et al. [3], De Bruin and Dikshit [6], Bokari et al. [7], Dikshit [8], Pathak [13], Mandloi and Pathak [10], Modi et al. [11], studied regularity of Pál type interpolation problems with some additional nodes.

De Bruin [2] evaluated regularity of incomplete Pál type interpolation on the zeros of polynomials given by (1.1) and (1.2). He omitted one or two real nodes from zeros of $w_n^\alpha(z)$ and/or $v_n^\alpha(z)$.

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The author [14, 15] investigated the regularity of Birkhoff interpolation in certain dimensions. They revisited $PTIP$ for the sets consisting of the zeros of polynomials with complex coefficients with some additional nodes. Also, they assessed the maximum number of nodes that can be added at value nodes to get regular $PTIP$ [16]. They studied the regularity of 'incomplete' $PTIP$ for several pairs, where they omitted real as well as complex nodes from zeros of certain polynomials [17, 18]. The author [19] defined a new class of Pál type interpolation obtained by adding a real node to one set of interpolation points and omitting a real node from another set of interpolation points.
In section 2, we consider $(0,1)$-$PTIP$, where we omit a non-zero complex node ζ from $v_{2n}^{\alpha}(z)$ and add $-\zeta$ to $w_n^{\alpha}(z)$ or $v_n^{\alpha}(z)$.
In section 3, We consider the polynomials $a_m(z) \in \pi_m$ and $b_n(z) \in \pi_n$ $(m \geq n)$ with simple zeros and take $A_m(z)$ and $B_n(z)$ as the sets of the zeros of these polynomials respectively with $B_n(z) \subseteq A_m(z)$. We assess the regularity of $(0,1)$-$PTIP$ and $(0,2)$- $PTIP$ by omitting a non-zero complex node ζ from $a_m(z)$ and adding $-\zeta$ to $b_n(z)$.

2 A new class of $PTIP$ on non-uniformly distributed nodes

Theorem 2.1. *Let $0 < \alpha < 1$, $n \geq 2$ then $(0,1)$-$PTIP$ on $\left\{\frac{v_{2n}^{\alpha}(z)}{(z-\zeta)}, (z+\zeta)w_n^{\alpha}(z)\right\}$ is regular, for $\pm\zeta \in v_{2n}^{\alpha}(z)$, $\pm\zeta \notin w_n^{\alpha}(z)$.*

Proof. Here, we have total $3n$ interpolation points.
The problem is to find a polynomial $P(z) \in \pi_{3n-1}$ with
$P(y_i) = 0$; y_i is zero of $\frac{v_{2n}^{\alpha}(z)}{(z-\zeta)}$; $i = 1, 2, \ldots, (2n-1)$,
$P'(-\zeta) = 0$,
$P'(z_j) = 0$; z_j is zero of $w_n^{\alpha}(z)$; $j = 1, 2, \ldots, n$.
Let $P(z) = \frac{v_{2n}^{\alpha}(z)}{(z-\zeta)}Q(z)$; where $Q(z) \in \pi_n$,
then $P(z) \in \pi_{3n-1}$.
The problem will be regular if $P(z) \equiv 0$.
As $P'(z_j) = 0$, we get

$$\frac{v_{2n}^{\alpha}(z_j)}{(z_j-\zeta)}Q'(z_j) + \left\{\frac{\{v_{2n}^{\alpha}(z_j)\}'}{(z_j-\zeta)} - \frac{v_{2n}^{\alpha}(z_j)}{(z_j-\zeta)^2}\right\}Q(z_j) = 0.$$

Also, $z_j \in w_n^{\alpha}(z) \subseteq v_{2n}^{\alpha}(z)$, thus we have

$$\frac{\{v_{2n}^{\alpha}(z_j)\}'}{(z_j-\zeta)}Q(z_j) = 0.$$

Since,

$$\{v_{2n}^{\alpha}(z_j)\}' = \frac{2n(1-\alpha^2)(z_j+\alpha)^{n-1}}{(1+\alpha z_j)} \neq 0,$$

Therefore,

$$Q(z_j) = 0.$$

Since z_j has n zeros, therefore

$$Q(z) = Cq_n(z). \tag{2.1}$$

Since,

$$P'(-\zeta) = 0,$$

Therefore,

$$\frac{\{v_{2n}^{\alpha}(-\zeta)\}'}{(2\zeta)}Q(-\zeta) = 0.$$

As,

$$\frac{\{v_{2n}^{\alpha}(-\zeta)\}'}{2\zeta} \neq 0,$$

we have

$$Q(-\zeta) = 0. \tag{2.2}$$

Equations (2.1), (2.2) and interpolatory conditions, give

$$C = 0.$$

Hence,

$$Q(z) \equiv 0.$$

□

Remark 2.1. *One of the essential conditions for the above result is* $w_n^\alpha(z) \subseteq V_{2n}^\alpha(z)$ *. Since one more similar condition satisfies for the polynomials given by equations* (1.1) *and* (1.2) *viz.* $v_n^\alpha(z) \subseteq V_{2n}^\alpha(z)$*. Therefore, the following result must hold.*

Theorem 2.2. *Let* $0 < \alpha < 1$, $n \geq 2$ *then* $(0,1)$*-*$PTIP$ *on* $\left\{\frac{v_{2n}^\alpha(z)}{(z+\zeta)}, (z-\zeta)v_n^\alpha(z)\right\}$ *is regular, for* $\pm\zeta \in v_{2n}^\alpha(z)$, $\pm\zeta \notin v_n^\alpha(z)$.

3 A new class of $PTIP$ on the zeros of the polynomials with complex coefficients

Theorem 3.1. $(0,1)$*-*$PTIP$ *on* $\left\{\frac{a_m(z)}{(z-\zeta)}, (z+\zeta)b_n(z)\right\}$, $m > n \geq 1$ *for* $\pm\zeta \in A_m(z), \pm\zeta \notin B_n(z)$ *is regular.*

Proof. Here we have total $m+n$ interpolation points.
The problem is to find a polynomial $P(z) \in \pi_{m+n-1}$ with
$P(y_i) = 0$; where y_i is a zero of $\frac{a_m(z)}{(z-\zeta)}$; $i = 1, 2, \ldots, (m-1)$,
$P'(-\zeta) = 0$,
$P'(z_j) = 0$; where z_j is a zero of $b_n(z)$; $j = 1, 2, \ldots, n$.
Let $P(z) = \frac{a_m(z)}{(z-\zeta)}Q(z)$; where $Q(z) \in \pi_n$,
then $P(z) \in \pi_{m+n-1}$.
The problem will be regular if $P(z) \equiv 0$.
Now,
$P'(z_j) = 0$, we have

$$\frac{a_m(z_j)}{(z_j-\zeta)}Q'(z_j) + \left\{\frac{a'_m(z_j)}{(z_j-\zeta)} - \frac{a_m(z_j)}{(z_j-\zeta)^2}\right\}Q(z_j) = 0.$$

As $P'(-\zeta) = 0, -\zeta \in A_m(z)$ and $a_m(z)$ has simple zeros, the polynomial and its derivative cant vanish simultaneously

$$Q(-\zeta) = 0. \tag{3.1}$$

Also, $z_j \in B_n(z) \subseteq A_m(z)$, we have

$$\frac{a'_m(z_j)}{(z_j-\zeta)}Q(z_j) = 0.$$

Since $\zeta \notin B_n(z)$ and $a'_m(z_j) \neq 0$, therefore we get

$$Q(z_j) = 0.$$

As z_j has n zeros, thus we have

$$Q(z) = Cq_n(z). \tag{3.2}$$

Equations (3.1), (3.2) and interpolatory conditions, give

$$C = 0.$$

Hence,

$$Q(z) \equiv 0.$$

□

Theorem 3.2. *The* $(0,2)$*-*$PTIP$ *on* $\left\{\frac{a_m(z)}{(z-\zeta)}, (z+\zeta)b_n(z)\right\}$, $m > n \geq 1$ *for* $\pm\zeta \in A_m(z), \pm\zeta \notin B_n(z)$ *is regular.*

Proof. Here we have total $m+n$ interpolation points.
The problem is to find a polynomial $P(z) \in \pi_{m+n-1}$ with
$P(y_i) = 0$; where y_i is a zero of $\frac{a_m(z)}{(z-\zeta)}$; $i = 1, 2, \ldots, (m-1)$,
$P''(-\zeta) = 0$,
$P''(z_j) = 0$; where z_j is a zero of $b_n(z)$; $j = 1, 2, \ldots, n$.
Let $P(z) = \frac{a_m(z)}{(z-\zeta)} Q(z)$; where $Q(z) \in \pi_n$,
then $P(z) \in \pi_{m+n-1}$.
The problem will be regular if $P(z) \equiv 0$.
Now,
$P''(z_j) = 0$,
Therefore,

$$\frac{a_m(z_j)}{(z_j-\zeta)} Q''(z_j) + 2\left\{\frac{a'_m(z_j)}{(z_j-\zeta)} - \frac{a_m(z_j)}{(z_j-\zeta)^2}\right\} Q'(z_j) + \left\{\frac{a''_m(z_j)}{(z_j-\zeta)} - 2\frac{a'_m(z_j)}{(z_j-\zeta)^2} + 2\frac{a_m(z_j)}{(z_j-\zeta)^3}\right\} Q(z_j) = 0.$$

Also $z_j \in B_n(z) \subseteq A_m(z)$ and $a_m(z)$ has simple zeros, the polynomial and its derivative cant vanish simultaneously, therefore we get

$$2(z_j-\zeta)a'_m(z_j)Q'(z_j) + \left\{(z_j-\zeta)a''_m(z_j) - 2a'_m(z_j)\right\} Q(z_j) = 0.$$

Since z_j has n zeros and $Q(z) \in \pi_n$, therefore the differential equation is given by

$$2(z-\zeta)a'_m(z)Q'(z) + \left\{(z-\zeta)a''_m(z) - 2a'_m(z)\right\} Q(z) = C(z+\zeta)b_n(z). \tag{3.3}$$

The integrating factor of the differential equation (3.3) is given by

$$\Phi(z) = \frac{\left\{a'_m(z)\right\}^{1/2}}{(z-\zeta)}.$$

The solution of the differential equation (3.3) is given by

$$\frac{\left\{a'_m(z)\right\}^{1/2}}{(z-\zeta)} Q(z) = C \int \frac{b_n(t)(t+\zeta)}{\left\{a'_m(z)\right\}^{1/2}(t-\zeta)^2} dt.$$

$$C \left. \frac{b_n(t)(t+\zeta)}{\left\{a'_m(z)\right\}^{1/2}(t-\zeta)^2} \right|_{t=\zeta} = 0 \Rightarrow C = 0.$$

Hence,

$$Q(z) \equiv 0$$

. ∎

4 Conclusion

The posed problems of $(0,1)$-$PTIP$ and $(0,2)$-$PTIP$ obtained by adding and omitting a non-zero complex node simultaneously are found to be regular on considered sets of value nodes and derivative nodes.

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