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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

COMMON FIXED POINT THEOREMS IN COMPLEX VALUED METRIC SPACES SATISFYING E.A. PROPERTY AND INTIMATE MAPPING

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Abstract

In this paper we prove the common fixed point theorems in complex valued metric spaces satisfying E.A. property and intimate mapping. Our result generalizes some recent results in the literature due to Azam et al.(2011) . Also we improve the results of Rajput & Singh(2014) satisfying E.A. property and Meena(2015) regarding intimate mapping. Some concepts have been taken from the results obtained by Choi et al.(2017) and Jebril et al.(2019) to improve our results. Also some examples are given to illustrate our obtained results.

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1 Introduction, Definitions and Notations

The concept of fixed point theorem first introduced by Poincare & Miranda[15] in 1883. After that Brouwer[3] published his famous fixed point theorem in 1912. The theorem states that "If B is a closed unit ball in R^n and if $T : B \to B$ is continuous then T has a fixed point in B". In 1992 Banach[4] proved his famous fixed point theorem in which contraction principle is the main tools. Banach's fixed point theorem plays a major role in fixed point theory. It has applications in many branches of mathematics. Because of its usefulness, a lot of articles have been dedicated to the improvement and generalization of that result. Most of these generalizations have been made by considering different contractive type conditions in different spaces$\{cf.[5] - [21]\}$. In 2011, Azam et al.[2] made a generalization by introducing a complex valued metric space using some contractive type conditions.Very recently, Rajput & Singh [18] generalized this result by replacing the constants of contraction by some control functions. The purpose of this work is to obtain a common fixed point result for three self mappings in complex valued metric spaces which generalizes the results of [1] and improve the results of Rajput & Singh[18] satisfying E.A. property and Meena[14] regarding intimate mapping.

We write regular complex number as $z = x + iy$ where x and y are real numbers and $i^2 = -1$. Let $\mathbb{C}_1$ be the set of complex numbers and z_1 and $z_2 \in \mathbb{C}_1$. Define a partial order relation $\precsim$ on $\mathbb{C}_1$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the following conditions is satisfied:

$(i)Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$, $(ii)Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$, $(iii)Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$, $(iv)Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of (ii), (iii) and (iv) is satiesfied and we write $z_1 \prec z_2$ if only (iv) is satisfied.

Taking this into account some fundamental properties of the partial order $\precsim$ on $\mathbb{C}_1$ as follows:

(1) If $0 \precsim z_1 \precsim z_2$ then $|z_1| < |z_2|$;
(2) If $z_1 \precsim z_2, z_2 \precsim z_3$ then $z_1 \precsim z_3$ and
(3) If $z_1 \precsim z_2$ and $0 < \lambda < 1$ is a real number then $\lambda z_1 \precsim z_2$.

Azam et al. defined the complex valued valued metric space in the following way:

Definition 1.1. *Let X be a nonempty set where as $\mathbb{C}_1$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}_1$ satisfies the following conditions:*

$(d_1) : 0 \precsim d(x,y)$,*for all* $x, y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$;

$(d_2) : d(x,y) = d(y,x)$ *for all* $x, y \in X$;

$(d_3) : d(x,y) \precsim d(x,z) + d(z,y)$,*for all* $x, y, z \in X$.

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

Definition 1.2 ([2]). *Let $\{x_n\}$ be sequence in X and $x \in X$. If for every $c \in \mathbb{C}_1$ with $0 \prec c$, there is an $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec c$ for all $n > n_0$ then x is called the limit of $\{x_n\}$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.*

Definition 1.3 ([2]). *If every Cauchy sequence is convergent in $\mathbb{C}_1$ then the space is called a complete complex valued metric space.*

Definition 1.4 ([6]). *Let f and g be two self-maps defined on a set X. Then f and g are said to be weakly compatible if they commute at their coincidence points.*

Definition 1.5 ([22]). *Let $T, S : X \to X$ be two self mappings of a bicomplex valued metric space (X,d). The pair (T,S) are said to satisfy E. A. property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.*

Definition 1.6 ([21]). *The self mappings T, $S : X \to X$ are said to satisfy the common limit in the range of S property ($CLRs$ property) if $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Sx$ for some $x \in X$.*

Definition 1.7 ([14]). *Let S and T be self maps on a bicomplex valued metric space (X,d). Then the pair $\{S,T\}$ is said to be $T-$intimate if and only if $\alpha d(TSz_n, Tz_n) \precsim \alpha d(SSz_n, Sz_n)$, where $\alpha = \limsup\{z_n\}$ or $\liminf\{z_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Tz_n = t$ for some t in X.*

Some common fixed point results are established by Rajput & Singh [18] for rational type contraction mapping in $\mathbb{C}_1$ on which they have proved the following theorem.

Theorem 1.1 ([18]). *Let (X,d) be a complex valued metric space and $A, B, S, T : X \to X$ be four self mappings satisfying the following conditions*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(ii) *For all* $x, y \in X$ *and* $0 < \alpha < 1$,

$$\begin{aligned} d(Ax, By) &\precsim \alpha \frac{[d(Ax,Sx)(dAx,Ty) + d(By,Ty)\,d(By,Sx)]}{d(Ax,Ty) + d(By,Sx)} \\ &\quad + \beta \frac{\left[\{d(Ax,Ty)\}^2 + \{d(By,Sx)\}^2\right]}{d(Ax,Ty) + d(By,Sx)}; \end{aligned}$$

(iii) *The pairs (A,S) and (B,T) are weakly compatible and*

(iv) *The pair (A,S) or (B,T) satisfies E. A. property if the range of mappings $S(X)$ or $T(X)$ is closed subspace of X then A, B, S and T have a unique common fixed point in X.*

Meena[14] investigated a common fixed point for intimate mappings in $\mathbb{C}_1$ as follows:

Theorem 1.2. *Let A, B, S and T be the four mappings from a complex valued metric space (X,d) into itself, such that*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(ii) *For all* $x, y \in X$,

$$d(Ax, By) \precsim \alpha d(Sx, Ty) + \beta \frac{d(Ax,Sx)\,.d(By,Ty)}{d(Ax,Ty) + d(Sx,By) + d(Sx,Ty)}$$

and $d(Ax,Ty) + d(Sx,By) + d(Sx,Ty) \neq 0$, *where* α, β *are non-negative real numbers with* $\alpha + \beta < 1$;

(iii) (A,S) *is* $S-$*intimate and* (B,T) *is* $T-$*intimate and*

(iv) $S(X)$ *is complete.*

Then A, B, S and T have a unique common fixed piont in X.

2 Main Results

In this section we prove some theorems and give some examples.

Theorem 2.1. *Let (X,d) be a complex valued metric space and $A,B,S,T : X \to X$ four self-mappings satisfying the conditions:*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(ii) for all $x,y \in X$,

$$d(Ax,By) \precsim \alpha d(Sx,Ty) + \beta \frac{d(Ax,Sx).d(By,Ty)}{[1+d(Sx,Ty)]} + \gamma \frac{d(Ty,By)[1+d(Sx,Ax)]}{[1+d(Sx,Ty)]}; \tag{2.1}$$

(iii) the pair (A,S) and (B,T) are weakly compatible;

(iv) one of the pair (A,S) or (B,T) satiesfies E.A. property.

If the range of one of the mapping $S(X)$ or $T(X)$ is a closed subspace of X then the mapping A,B,S and T have a unique common fixed point in X.

Proof. First we suppose that the pair (B,T) satisfies $E.A.$ property. Then by difinition there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Further since $B(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence, $\lim_{n\to\infty} Sy_n = t$. We claim that $\lim_{n\to\infty} Ay_n = t$. Let $\lim_{n\to\infty} Ay_n = t_1 \neq t$, then putting $x = y_n$, $y = x_n$ in condition (ii), we have

$$\begin{aligned} & d(Ay_n, Bx_n) \\ \precsim\ & \alpha d(Sy_n,Tx_n) + \beta \frac{d(Ay_n,Sy_n).d(Bx_n,Tx_n)}{[1+d(Sy_n,Tx_n)]} \\ & + \gamma \frac{d(Tx_n,Bx_n)[1+d(Sy_n,Ay_n)]}{[1+d(Sy_n,Tx_n)]}. \end{aligned}$$

or,

$$d(t_1,t) \precsim \alpha d(t,t) + \beta \frac{d(t_1,t).d(t,t)}{[1+d(t,t)]} + \gamma \frac{d(t,t)[1+d(t,t_1)]}{1+d(t,t)}.$$

Then $|d(t_1,t)| \leq 0$.

Hence $t_1 = t$ and this implies that $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = t$.

Now suppose that $S(X)$ is a closed subspace of X, then $t = Su$ for some $u \in X$. Subsequntly, we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = t = Su$.

We claim that $Au = Su$. For this we put $x = u$ and $y = x_n$ in contractive condition (ii). Then we have

$$\begin{aligned} d(Au,Bx_n) \precsim\ & \alpha d(Su,Tx_n) + \beta \frac{d(Au,Su).d(Bx_n,Tx_n)}{[1+d(Su,Tx_n)]} \\ & + \gamma \frac{d(Tx_n,Bx_n)[1+d(Su,Au)]}{[1+d(Su,Tx_n)]}. \end{aligned}$$

Taking $n \to \infty$, we get

$$d(Au,t) \precsim \alpha d(Su,t) + \beta \frac{d(Au,Su).d(t,t)}{[1+d(Su,t)]} + \gamma \frac{d(t,t)[1+d(Su,Au)]}{[1+d(Su,t)]}.$$

Then $|d(Au,t)| \leq 0$, which is a contradiction. Hence u is a coincident point of (A,S).

Now the weak compatibility of the pair (A,S) implies that $ASu = SAu$ or $At = St$.

On the other hand since $A(X) \subset T(X)$, there exists a v in X such that $Au = Tv$. Thus $Au = Su = Tv = t$. Now we show that v is a coincidence point of (B,T), i.e., $Bv = Tv = t$.

Putting $x = u$, $y = v$ in contractive condition (ii), we have

$$\begin{aligned} d(Au,Bv) \precsim\ & \alpha d(Su,Tv) + \beta \frac{d(Au,Su).d(Bv,Tv)}{[1+d(Su,Tv)]} \\ & + \gamma \frac{d(Tv,Bv)[1+d(Su,Au)]}{[1+d(Su,Tv)]}. \end{aligned}$$

or,

$$d(t,Bv) \precsim \alpha d(t,t) + \beta \frac{d(t,t).d(Bv,t)}{[1+d(t,t)]} + \gamma \frac{d(t,Bv)[1+d(t,t)]}{[1+d(t,t)]}.$$

or,

$$d(t,Bv) \precsim \gamma d(t,Bv).$$

This implies that $|d(t,Bv)| \leq 0$, which is a contradiction. Thus $Bv = t$.
Hence $Bv = Tv = t$ and v is a coincident point of B and T.
Further, the weak compatibility of the pair (B,T) implies that $BTv = TBv$, i.e., $Bt = Tt$. Therefore t is a common coincidence point of A, B, S and T.
Now we show that t is a common coincident point of A, B, S and T.
Putting $x = u$ and $y = t$ in contractive condition (ii), we get

$$\begin{aligned} d(t,Bt) &= d(Au,Bt) \precsim \alpha d(Su,Tt) + \beta \frac{d(Au,Su).d(Bt,Tt)}{[1+d(Su,Tt)]} \\ &\quad + \gamma \frac{d(Tt,Bt)[1+d(Su,Au)]}{[1+d(Su,Tt)]}. \end{aligned}$$

or,

$$|d(t,Bt)| \leq \alpha |d(t,Bt)| \, [\text{as } Bt = Tt \text{ and } Au = Su = Tv = t]$$

$$\text{or, } |d(t,Bt)| \leq 0,$$

which is a contradiction. Thus $Bt = t$.
Therefore, $At = Bt = St = Tt = t$.

Uniqueness:

For uniquness suppose t^* be another common fixed point of A, B, S and T.
Then we have $At^* = Bt^* = St^* = Tt^* = t^*$.
Therefore using (2.1) we get

$$\begin{aligned} d(At,Bt^*) &\precsim \alpha d(St,Tt^*) + \beta \frac{d(At,St).d(Bt^*,Tt^*)}{[1+d(St,Tt^*)]} + \gamma \frac{d(Tt^*,Bt^*)[1+d(St,At)]}{[1+d(St,Tt^*)]} \\ \text{or, } d(t,t^*) &\precsim \alpha d(t,t^*) + \beta \frac{d(t,t).d(t^*,t^*)}{[1+d(t,t^*)]} + \gamma \frac{d(t^*,t^*)[1+d(t,t)]}{[1+d(t,t^*)]} \\ \text{or, } d(t,t^*) &\precsim \alpha d(t,t^*). \end{aligned}$$

Hence,

$$|d(t,t^*)| \precsim \alpha |d(t,t^*)|,$$

which implies that $|d(t,t^*)| \leq 0$ i.e., $t = t^*$.
Therefore t is the unique common fixed point of A, B, S and T. □

Example 2.1. Let $X = [0,1]$. We define the mapping $d : X \times X \to \mathbb{C}_1$ as follows

$$d(x,y) = (1+i)|x-y|, \quad x,y \in X$$

Then (X,d) is a complex valued metric space.
Let $A, B, S, T : X \to X$ be defined by

$$Ax = \tfrac{x}{4},\ 0 \leq x \leq 1;\ Bx = \begin{cases} 0, & x \neq \frac{1}{2} \\ \frac{1}{8}, & x = \frac{1}{2}; \end{cases}$$

$$Tx = x,\ 0 \leq x < 1;\ Sx = \begin{cases} \frac{x}{4}, & 0 \leq x < 1 \\ \frac{1}{8}, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Clearly, $S(X)$ is closed and $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. We consider the sequence $\left\{x_n :\ x_n = \frac{1}{2} + \frac{1}{n+2}\right\}$ in X. Then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{8}$. So that the pair (A,S) satisfies the E.A. property. Thus all the conditions of Theorem 2.1 are satiefied and 0 is the unique common fixed point of A, B, S and T.

Theorem 2.2. *Let A, B, S and T be the four mapping from a complex valued metric space (X, d) into itself such that*

(i) $A(X) \subset T(X)$ and $B(X) \subset S(X)$;

(ii) $d(Ax, By) \precsim \alpha d(Sx, Ty) + \beta \frac{d(Ax,Sx).d(By,Ty)}{1+d(Sx,Ty)} + \gamma \frac{d(Ty,By).[1+d(Sx,Ax)]}{1+d(Sx,Ty)}$,
where α, β, γ are non negative real numbers with $\alpha + \beta + 2\gamma < 1$;

(iii) (A, S) is $S-$intimate and (B, T) is $T-$intimate;

(iv) $S(X)$ is complete.

Then A, B, S and T have a unique common fixed point in X.

Proof. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, therefore there exists a sequence $\{y_{2n}\}$ in X such that
$y_{2n} = Ax_{2n} = Tx_{2n+1}$.
$y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$.
Then using these conditions in contractive condition (ii), we get

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Ax_{2n}, Bx_{2n+1}) \\
&\precsim \alpha d(Sx_{2n}, Tx_{2n+1}) + \beta \frac{d(Ax_{2n}, Sx_{2n}).d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Sx_{2n}, Tx_{2n+1})} \\
&\quad + \gamma \frac{d(Tx_{2n+1}, Bx_{2n+1}).[1 + d(Sx_{2n}Ax_{2n})]}{1 + d(Sx_{2n}, Tx_{2n+1})} \\
&= \alpha d(y_{2n-1}, y_{2n}) + \beta \frac{d(y_{2n}, y_{2n-1}).d(y_{2n+1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} \\
&\quad + \gamma \frac{d(y_{2n}, y_{2n+1}).[1 + d(y_{2n-1}, y_{2n})]}{1 + d(y_{2n-1}, y_{2n})} \\
&\precsim \alpha d(y_{2n-1}, y_{2n}) + \beta d(y_{2n}, y_{2n+1}) + \gamma d(y_{2n}, y_{2n+1})
\end{aligned}$$

$$(1 - \beta - \gamma) d(y_{2n}, y_{2n+1}) \precsim \alpha d(y_{2n}, y_{2n-1})$$

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &\precsim \frac{\alpha}{(1 - \beta - \gamma)} d(y_{2n}, y_{2n-1}) \\
&\precsim \frac{\alpha + \gamma}{1 - \beta - \gamma} d(y_{2n}, y_{2n-1})
\end{aligned}$$

$$\text{or, } d(y_{2n}, y_{2n+1}) \precsim h d(y_{2n}, y_{2n-1}),$$

where $h = \frac{\alpha+\gamma}{1-\beta-\gamma} < 1$, as $\alpha + \beta + 2\gamma < 1$.

This implies that

$$|d(y_{2n}, y_{2n+1})| \precsim h |d(y_{2n}, y_{2n-1})|. \tag{2.2}$$

Similarly,

$$\begin{aligned}
d(y_{2n+2}, y_{2n+1}) &= d(Ax_{2n}, Bx_{2n+1}) \\
&\precsim \alpha d(Sx_{2n+2}, Tx_{2n+1}) + \beta \frac{d(Ax_{2n+2}, Sx_{2n+2}).d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Sx_{2n+2}, Tx_{2n+1})} \\
&\quad + \gamma \frac{d(Tx_{2n+1}, Bx_{2n+1}).[1 + d(Sx_{2n+2}Ax_{2n+2})]}{1 + d(Sx_{2n+2}, Tx_{2n+1})} \\
&= \alpha d(y_{2n+1}, y_{2n}) + \beta \frac{d(y_{2n+2}, y_{2n+1}).d(y_{2n+1}, y_{2n})}{1 + d(y_{2n+1}, y_{2n})} \\
&\quad + \gamma \frac{d(y_{2n+2}, y_{2n+1}).[1 + d(y_{2n+1}y_{2n})]}{1 + d(y_{2n+1}, y_{2n})} \\
&= \alpha d(y_{2n+1}, y_{2n}) + \beta \frac{d(y_{2n+2}, y_{2n+1}).d(y_{2n+1}, y_{2n})}{1 + d(y_{2n+1}, y_{2n})} \\
&\quad + \gamma \frac{d(y_{2n+2}, y_{2n+1}).}{1 + d(y_{2n+1}, y_{2n})} + \gamma \frac{[1 + d(y_{2n+1}y_{2n})].}{1 + d(y_{2n+1}, y_{2n})} \\
&\precsim \alpha d(y_{2n+1}, y_{2n}) + (\beta + \gamma) d(y_{2n+2}, y_{2n+1}) + \gamma d(y_{2n+1}, y_{2n})
\end{aligned}$$

$$\text{or, } d(y_{2n+2}, y_{2n+1}) \precsim \frac{\alpha+\gamma}{1-\beta-\gamma} d(y_{2n+1}, y_{2n})$$

$$\text{or, } d(y_{2n+2}, y_{2n+1}) \precsim h d(y_{2n+1}, y_{2n}),$$

where $h = \frac{\alpha+\gamma}{1-\beta-\gamma}$.

This implies that

(2.3) $$|d(y_{2n+2}, y_{2n+1})| \leq h |d(y_{2n+1}, y_{2n})|.$$

Thus from (2.2) and (2.3) we can write

$$|d(y_{2n+2}, y_{2n+1})| \leq h |d(y_{2n+1}, y_{2n})| \leq ... \leq h^{n+1} |d(y_0, y_1)|.$$

So that for any $m > n$

$$\begin{aligned} |d(y_n, y_m)| &\leq |d(y_n, y_{n+1})| + |d(y_{n+1}, y_{n+2})| + ... + |d(y_{m-1}, y_m)| \\ &\leq h^n |d(y_0, y_1)| + h^{n-1} |d(y_0, y_1)| + ... + h^{m-1} |d(y_0, y_1)| \end{aligned}$$

$$\text{i.e., } |d(y_n, y_m)| \leq \frac{h^n}{1-h} |d(y_0, y_1)|,$$

which accounts to say that $\{y_n\}$ is a Cauchy sequence. i.e., $\{Sx_{2n}\}$ is Cauchy in $S(X)$, also $S(X)$ is complete, then $\{y_n\}$ converges to a point $p = Su$ for some $u \in X$.

Thus Ax_{2n}, Sx_{2n}, Bx_{2n+1}, $Tx_{2n+1} \to p$.

Now,

$$\begin{aligned} d(Au, Bx_{2n+1}) &\precsim \alpha d(Su, Tx_{2n+1}) + \beta \frac{d(Au, Su) . d(Bx_{2n+1} Tx_{2n+1})}{1 + d(Su, Tx_{2n+1})} \\ &\quad + \gamma \frac{d(Tx_{2n+1}, Bx_{2n+1}) . [1 + d(Su, Au)]}{1 + d(Su, Tx_{2n+1})}. \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$|d(Au, p)| \leq \alpha |d(Su, p)|.$$

Thus $|d(Au, p)| \leq 0$, [as $p = Su$] i.e., $Au = p = Su$.

Again $A(X) \subset T(X)$, therefore there exists $v \in X$ such that $Au = Tv = p$.

Now we cosider

$$d(p, Bv) = d(Au, Bv)$$

$$\precsim \alpha d(Su, Tv) + \beta \frac{d(Au, Su) . d(Bv, Tv)}{1 + d(Su, Tv)} + \gamma \frac{d(Tv, Bv) . [1 + d(Su, Au)]}{1 + d(Su, Tv)}$$

$$\text{or, } d(p, Bv) \leq \gamma d(p, Bv)$$

$$\text{or, } d(p, Bv) = 0,$$

which implies that $p = Bv = Tv = Au = Su$.

Now since $Au = Su = p$ and (A, S) is $S-$intimate,

Therefore we have

(2.4) $$|d(Sp, p)| \leq |d(Ap, p)|.$$

Also

$$d(Ap, p) = d(Ap, Bv)$$

$$\precsim \alpha d(Sp, Tv) + \beta \frac{d(Ap, Sp) . d(Bv, Tv)}{1 + d(Sp, Tv)} + \gamma \frac{d(Tv, Bv) . [1 + d(Sp, Ap)]}{1 + d(Sp, Tv)}$$

$$\text{or, } |d(Ap, p)| \leq \alpha |d(Sp, p)| \leq \alpha |d(Ap, p)| . [\text{using } (2.4)]$$

Thus $|d(Ap, p)| = 0$, implies that $Ap = p$ and $Sp = p$. Similarly $Bp = Tp = p$.

Uniqueness:

Let us suppose that q be another common fixed point of A, B, S and T such that $p \neq q$.

Then

$$d(p, q) = d(Ap, Bq)$$

$$\precsim \alpha d(Sp,Tq)+\beta\frac{d(Ap,Sp).d(Bq,Tq)}{1+d(Sp,Tq)}+\gamma\frac{d(Tq,Bq).[1+d(Sp,Ap)]}{1+d(Sp,Tq)}$$
$$\precsim \alpha d(p,q),$$

which implies that $|d(p,q)|\le\alpha|d(p,q)|\Rightarrow|d(p,q)|=0$, i.e., $p=q$.
This proves that the mappings A,B,S and T have a unique common fixed point. □

Example 2.2. :Let $X=\mathbb{C}_1$ be the set of complex numbers. Define $d:X\times X\to\mathbb{C}_1$ by $d(z_1,z_2)=(1+i)|z_1-z_2|$ where $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ .Then (X,d) is a complete complex valued metric space. Define $A,B,S,T:X\to X$ as $Az=0, Bz=0$, $Sz=z$ and $Tz=\frac{z}{2}$. Clearly $A(X)\subset T(X)$ and $B(X)\subset S(X)$. Now consider the sequence $\{z_n=\frac{1}{n}, n\in\mathbb{N}\}$ in $\mathbb{C}_1$, then $\lim\limits_{n\longrightarrow\infty}Az_n=\lim\limits_{n\longrightarrow\infty}Sz_n=0$. Also we have $\lim\limits_{n\longrightarrow\infty}d(SAz_n,Sz_n)\precsim\lim\limits_{n\longrightarrow\infty}d(AAz_n,Az_n)$. Thus the pair (A,S) is $S-$intimate. Again $\lim\limits_{n\longrightarrow\infty}d(TBz_n,Tz_n)\precsim\lim\limits_{n\longrightarrow\infty}d(BBz_n,Bz_n)$ implies that the pair (B,T) is $T-$intimate. Therefore the mappings satisfies all the conditions of Theorem 2.2. Hence A,B,S and T have a unique common fixed point in X.

3 Future Prospect

In the line of the works as carried out in the paper one may think of the deduction of fixed point theorems using fuzzy metric, quasi metric, partial metric and other different types of metrics under the flavour of bicomplex analysis. This may be an active area of research to the future workers in this branch.

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