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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

ON HARMONIOUS CHROMATIC NUMBER OF $C(T_{m,n})$, $M(T_{m,n})$, $C(L(T_{m,n}))$ AND $M(L(T_{m,n}))$

Akhlak Mansuri[1] and R. S. Chandel[2]
Department of Mathematics
[1]Government Girls College, Mandsaur, Madhya Pradesh, India-458001
[2]Government Geetanjali Girls College, Bhopal, Madhya Pradesh, India-462001
Email: [1]akhlaakmansuri@gmail.com, [2]rs_chandel2009@yahoo.co.in
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Abstract

In this paper, we discuss the harmonious coloring and investigate the harmonious chromatic number of central and middle graph of tadpole graph and harmonious chromatic number of central and middle graph of line graph of tadpole graph, denoted by $\chi_H(C(T_{m,n}))$, $\chi_H(M(T_{m,n}))$ and $\chi_H(C(L(T_{m,n})))$, $\chi_H(M(L(T_{m,n})))$ respectively.

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1 Introduction

A proper vertex coloring of a graph G is a function $c : V(G) \longrightarrow \{1, 2, , k\}$ in which $c(u)$ and $c(v)$ are different for the adjacent vertices u and v and smallest number of colors are needed to color a graph G is called its chromatic number, and is often denoted $\chi(G)$. The Harmonious coloring [5, 6, 7, 9] of a simple graph G is proper vertex coloring in which no any two edges share the same color and minimum number of colors are to be used for harmonious coloring is known as the harmonious chromatic number, denoted by $\chi_H(G)$. For a graph $G = (V, E)$, subdividing each edge of the given graph G exactly once and joining all the non-adjacent vertices of it is the Central graph [3, 7] $C(G)$ of G and the middle graph $M(G)$ [8] is defined in such a way that the vertex set of $M(G)$ is $V(G) \cup E(G)$ and two vertices x, y of $M(G)$ are adjacent in $M(G)$) when one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G and the line Graph [4] of a simple graph G, denoted by $L(G)$ and defined in such a way that there exactly one vertex $v(e)$ in $L(G)$ for each edge e in G and for any two edges e and e' in G, $L(G)$ has an edge between $v(e)$ and $v(e')$, if and only if e and e' are incident with the same vertex in G. The (m, n)-tadpole graph [1, 2, 4] denoted by $T_{m,n}$ $(m \geq 3, n \geq 2)$ is obtained by joining cycle C_m and path P_n, with a bridge that consists $m + n$ vertices and $m + n$ edges.

2 Harmonious Chromatic Number of Tadpole Graph

Theorem 2.1. *For central graph of tadpole graph $T_{m,n}$, the harmonious chromatic number, $\chi_H(C(T_{m,n}) = 2m + n$.*

Proof. Let $T_{m,n}$ be a tadpole graph consisting $m + n$ vertices and $m + n$ edges. $V(T_{m,n}) = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ and $E(T_{m,n}) = \{u_1u_2, u_2u_3, ..., u_mu_1\} \cup \{u_1v_1, v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$. To get the central graph subdivide each edge of $T_{m,n}$ by the vertices u'_i and v'_j $(1 \leq i \leq m)(1 \leq j \leq n)$. $V(C(T_{m,n})) = \{u_i : 1 \leq i \leq m\} \cup \{u'_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\} \cup \{v'_j : 1 \leq j \leq n\}$. Coloring the vertices as follows; define coloring $c : V(C(T_{m,n})) \longrightarrow \{1, 2, 3, ..., (2m + n)\}$ by $c(u_i) = i$ $(1 \leq i \leq m)$, $c(u'_i) = m + i$ $(1 \leq i \leq m)$, $c(v_j) = 2m + j$ $(1 \leq j \leq n)$, $c(v'_j) = m + 1 + j$ $(1 \leq j \leq n)$.

Claim 2.1: c is proper; from above each $c(u_i)$ and $c(v_i)$ and its neighbors are assigned by different colors *i.e.* $c(u_i) \neq c(v_i)$, although $c(u'_i) = c(v_j)$, but these vertices are at least at a distance 2, which leads to proper coloring.

Claim 2.2; c is harmonious; it is clear that no two edges share the same color pair and we assign different colors on the vertices in such a way that they are at least at a distance 3. Therefore it is harmonious. *Claim* 2.3; c is minimum; all the vertices are colored by $2m+n$ colors, if we repeat (assign) any color on any vertex from these assigned colors, color pairs will be repeated which contradicts the harmonious coloring, therefore it is minimum. Hence the theorem. Figure 2.1 shows the central graph of $T_{4,3}$ for with coloring.

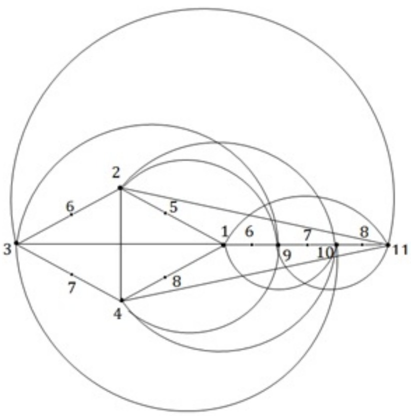

***Figure* 2.1:** $C(T_{4,3})$ with coloring, $\chi_H C(T_{4,3}) = 11$.

$\square$

Theorem 2.2. *For middle graph of tadpole graph, $T_{m,n}$, the harmonious chromatic number, $\chi_H(M(T_{m,n})) = 2m+n$.*

Proof. Let $T_{m,n}$ be a tadpole graph consisting $m+n$ vertices and $m+n$ edges. $V(T_{m,n}) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\}$ and $E(T_{m,n}) = \{u_1u_2, u_2u_3, ..., u_mu_1\} \cup \{u_1v_1, v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$. For getting middle graph, let subdivide each edge of $T_{m,n}$ by the vertices u'_i and v'_j $(1 \le i \le m)(1 \le j \le n)$. $V(M(T_{m,n})) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{u'_i : 1 \le i \le m\} \cup \{v'_j : 1 \le j \le n\}$. Coloring the vertices as follows; define coloring $c : V(M(T_{m,n})) \longrightarrow \{1, 2, 3, ..., (2m+n)\}$ by $c(u_i) = i$ $(1 \le i \le m)$, $c(u'_i) = m+i$ $(1 \le i \le m)$, $c(v_j) = j+1$ $(1 \le j \le n)$, $c(v'_j) = 2m+j$ $(1 \le j \le n)$. For further proof follow Theorem 2.1. Figure 2.2 shows the middle graph of $T_{3,4}$ with coloring.

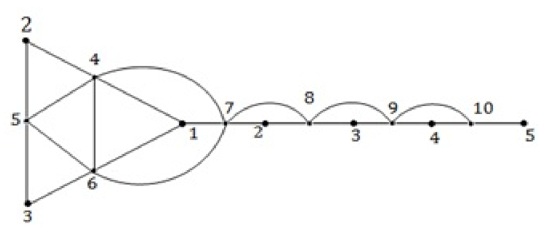

***Figure* 2.2:** $T_{3,4}$ with coloring, $\chi_H(M(T_{3,4})) = 10$.

$\square$

3 Harmonious Chromatic Number of Line Graph of Tadpole Graph

Theorem 3.1. *For central graph of line graph of tadpole graph $L(T_{m,n})$, the harmonious chromatic number, $\chi_H(C(L(T_{m,n}))) = 2m+n+2$.*

Proof. Let $L(T_{m,n})$ be a line graph of tadpole graph consisting $m+n$ vertices and $m+n+1$ edges. $V(L(T_{m,n})) = \{x_i : 1 \le i \le m\} \cup \{y_j : 1 \le j \le n\}$ and $E(L(T_{m,n})) = \{x_1x_2, x_2x_3, ..., x_mx_1\} \cup \{x_1y_1, x_my_1, y_1y_2, y_2y_3, ..., y_{n-1}y_n\}$. Now to get the central graph subdivide each edge of $L(T_{m,n})$ by the vertices z_1, z_2, x'_i and y'_j $(1 \le i \le m)(1 \le j \le n-1)$. $V(C(L(T_{m,n}))) = \{z_1, z_2\} \cup \{x_i : 1 \le i \le m\} \cup \{x'_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{y'_j : 1 \le j \le n-1\}$. Coloring the vertices as follows; define coloring

$c : V(C(L(T_{m,n}))) \longrightarrow \{1, 2, 3, ..., (2m + n + 2)\}$ by $c(x_i) = i$ $(1 \le i \le m)$, $c(x'_i) = m + i$ $(1 \le i \le m)$, $c(z_1) = 2m + 1$, $c(z_2) = 2m + 2$, $c(y_j) = 2m + 2 + j$ $(1 \le j \le n)$, $c(y'_j) = m + j$ $(1 \le j \le n - 1)$. Figure 3.1 shows the central graph of $L(T_{3,4})$ with coloring. Now we proceed as done in Theorem 2.1.

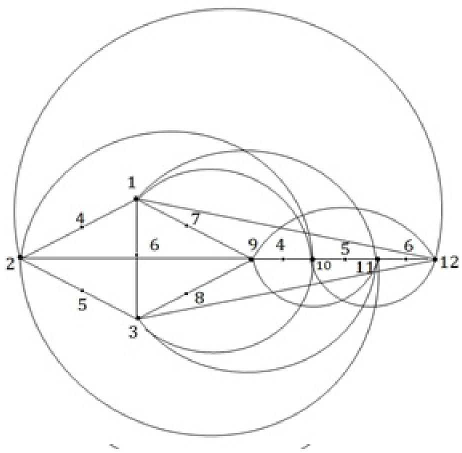

***Figure* 3.1:** $C(L(T_{3,4}))$ with coloring, $\chi_H C(L(T_{3,4})) = 12$.

□

Theorem 3.2. *For middle graph of line graph of tadpole graph $L(T_{m,n})$, the harmonious chromatic number, $\chi_H(M(L(T_{m,n}))) = 2m + n + 2$.*

Proof. Let $L(T_{m,n})$ be a line graph of tadpole graph consisting $m + n$ vertices and $m + n + 1$ edges. $V(L(T_{m,n})) = \{x_i : 1 \le i \le m\} \cup \{y_j : 1 \le j \le n\}$ and $E(L(T_{m,n})) = \{x_1x_2, x_2x_3, ..., x_mx_1\} \cup \{x_1y_1, x_my_1, y_1y_2, y_2y_3, ..., y_{n-1}y_n\}$. Now to get the middle graph subdivide each edge of $L(T_{m,n})$ by the vertices z_1, z_2, x'_i and y'_j $(1 \le i \le m)(1 \le j \le n - 1)$. $V(C(L(T_{m,n}))) = \{z_1, z_2\} \cup \{x_i : 1 \le i \le m\} \cup \{x'_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{y'_j : 1 \le j \le n - 1\}$. For further proof follow Theorem 2.1. Figure 3.2 shows the middle graph of $L(T_{3,4})$ with coloring.

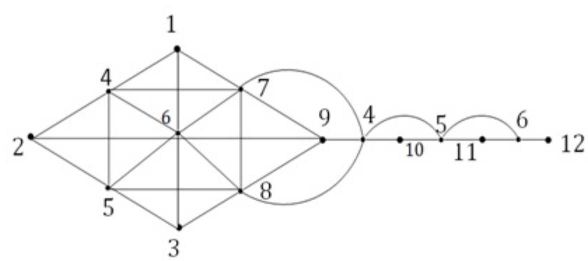

***Figure* 3.2:** $M(L(T_{3,4}))$ with coloring, $\chi_H M(L(T_{3,4})) = 12$.

□

4 Conclusion

In this paper, we investigate the harmonious chromatic number of central graph, middle graph and line graph of tadpole graph and we find that the harmonious chromatic number of central graph of line graph of tadpole graph is same as the harmonious chromatic number of middle graph of line graph of tadpole graph *i.e.* $\chi_H(C(L(T_{m,n}))) = \chi_H(M(L(T_{m,n})))$.

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