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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

ON WEIGHTED PÁL TYPE (0,2) – INTERPOLATION ON THE UNIT CIRCLE

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Abstract

In this paper, we study the explicit representation of weighted Pál - type (0,2) - interpolation on two pairwise disjoint sets of nodes on the unit circle, which are obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ and $P_n^{''}(x)$ respectively, where $P_n(x)$ stands for n^{th} Legendre polynomial.

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1 Introduction

In 1979, Turán [12] studied the (0,2) – Interpolation for getting an approximate solution of differential equation $y'' + fy = 0$. Balázs [8] introduced the weighted (0, 2)– Interpolation on the zeros of Ultraspherical polynomial $P_n^{(\alpha)}(x), \alpha > -1$. In 1960, Kiš [10] initiated the Lacunary interpolation on the unit circle. He considered (0,2)- Interpolation on the unit circle and established the convergence theorem. After that several mathematician have considered (0,2) – Interpolation viz. on the unit circle, infinite interval and on the real line. In 1996, Xie [13] considered $(0,1,3)^*$- interpolation on the vertically projected nodes onto the unit circle. He claimed the regularity, explicit representation and convergence of (0,1,3)* - Interpolation. In 2003, Dikshit [9] considered the Pál – type Interpolation on non uniformly distributed nodes on the unit circle. After that author and Mathur [1] considered the weighted (0,2)* – Interpolation on the set of nodes obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ on the unit circle and established a convergence theorem for that interpolatory polynomial. In 2012, she [2,3] considered weighted (0;0,2) and (0,2;0) – Interpolation on projected nodes onto the unit circle, obtained the regularity, fundamental polynomial and established a convergence theorem. In 2017, authors [4] considerd the regularity and explicit forms of weighted (0,2;0)- interpolation on the unit circle on two pairwise disjoint sets of nodes obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ and $P_n^{''}(x)$ respectively onto the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial. After that the auhors [5] also established convergence for the above said interpolatory polynomials. Recently, authors [6] considered weighted Lacunary interpolation on the nodes, which are obtained by projecting vertically the zeros of the $(1-x^2)P_n^{'}(x)$ onto the unit circle and established a convergence theorem for the same. Recently, author with Iqram [7] considered generalized Hermite-Fejér interpolation on the nodes, which are obtained by vertically projected zeros of the $(1+x)P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial established the convergence theorem. These have motivated us to consider (0;0,2) interpolation on two pairwise disjoint sets of nodes on the unit circle. Let

$$Z_n = \begin{cases} z_k = cos\theta_k + i\ sin\theta_k \\ z_{n+k} = \overline{z_k}\ , \quad k = 1\,(1)\,n, \end{cases} \tag{1.1}$$

$$T_n = \begin{cases} t_k = cos\varphi_k + i\ sin\varphi_k, \\ t_{(n-2)+k} = \overline{t_k}\ , \quad k = 1\,(1)\,n-2, \end{cases} \tag{1.2}$$

be two set of nodes. In which the Lagrange data is prescribed on the first set of nodes whereas Lacunary data on the other one.We obtained regularity, explicit forms and established a convergence theorem of the interpolatory polynomials. In Section 2, we give some preliminaries, in Section 3, we describe the problem and regularity, in Section 4 and Section 5, we present the explicit forms and convergence of weighted Pál – type (0,2) – interpolation on the unit circle respectively.

2 Preliminaries

The differential equation satisfied by $P_n(x)$ is

$$(1-x^2)\,P_n''(x) - 2x\;P_n'(x) + n\,(n+1)\,P_n(x) = 0. \tag{2.1}$$

$$W(z) = \prod_{k=1}^{2n}(z-z_k) = K_n\;P_n\left(\frac{1+z^2}{2z}\right)z^n, \tag{2.2}$$

$$R(z) = \left(z^2-1\right)W(z), \tag{2.3}$$

$$H(z) = \prod_{k=1}^{2n-4}(z-t_k) = K_n^{**}\;\;P_n^{''}\left(\frac{1+z^2}{2z}\right)z^{n-2}. \tag{2.4}$$

We shall require the following fundamental polynomials of Lagrange interpolation based on the zeros of $W(z)$ and $R(z)$, are respectively defined as

$$L_{1k}(z) = \frac{W(z)}{(z-z_k)\,W'(z_k)}\;,\quad k = 1\,(1)\,2n, \tag{2.5}$$

$$L_k(z) = \frac{R(z)}{(z-z_k)\,R'(z_k)}\;,\quad k = 0\,(1)\,2n+1, \tag{2.6}$$

$$l_{2k}(z) = \frac{H(z)}{(z-t_k)\;\;H^{'}(t_k)}\;,\quad k = 1\,(1)\,2n-4, \tag{2.7}$$

$$J_k(z) = \int_0^z t\;\;l_{2k}(t)\;dt, \tag{2.8}$$

$$J(z) = \int_0^z\;H(t)\;dt, \tag{2.9}$$

which satisfies

$$J(-z) \;=\; -J(z). \tag{2.10}$$

We shall also use the following results in our investigations :

$$W'(z_k) = \frac{K_n}{2}\left(z_k^{2}-1\right)P_n'(x_k)\,z_k^{n-2},\quad k = 1(1)2n-2, \tag{2.11}$$

$$W^{''}(z_k) = \;K_n\;\left[(n-1)\left(z_k^2\;-1\right)-1\right]z_k^{n-3}P_n^{'}(x_k)\,,\;\; k = 1\,(1)\,2n, \tag{2.12}$$

$$H^{'}(t_k) = \frac{K_n^{**}}{2}\left(t_k^2\;-1\right)t_k^{n-4}\;\;P_n^{'''}(x_k^*)\,,\;\; k = 1(1)2n-4, \tag{2.13}$$

$$W^{'}(t_k) = \;\;K_n\frac{n\left\{(n+3)\left(t_k^2\;-1\right)+4\right\}}{2\left(t_k^2\;+1\right)}t_k^{n-1}P_n(x_k^*)\,, \tag{2.14}$$

$$W^{''}(t_k) = \;K_n\frac{n\,(n-1)\left\{(n-1)\left(t_k^2\;-1\right)-1\right\}}{2\left(t_k^2\;+1\right)}t_k^{n-2}P_n(x_k^*)\,, \tag{2.15}$$

$$R^{'}(t_k) = \left(\mathrm{z_k^2}\;-1\right)\mathrm{W}^{'}(\mathrm{z_k})\,, \tag{2.16}$$

$$R^{''}(z_k) = 4z_k\;W^{'}(z_k) + \left(z_k^2\;-1\right)W^{''}(z_k)\,, \tag{2.17}$$

$$R^{'}(t_k) = \left(t_k^2\;-1\right)W^{'}(t_k)\,, \tag{2.18}$$

$$R^{''}(t_k) = 4t_k W^{'}(t_k) + \left(t_k^2 \; - 1\right) W^{''}(t_k) + 2W(t_k), \tag{2.19}$$

$$H^{''}(t_k) = K_n^* \left\{(n-5)\left(t_k^2 \; - 1\right) - 5\right\} t_k^{n-5} P_n^{'''}(x_k^*). \tag{2.20}$$

We shall also use the following well known inequalities:
For $-1 < x < 1$

$$|P_n(x)| \le 1, \tag{2.21}$$

$$\left(1 - x^2\right)^{1/4} |P_n(x)| \le \sqrt{\frac{2}{\pi}} n^{-1/2}, \tag{2.22}$$

$$\left(1 - x^2\right)^{3/4} |P_n'(x)| \le \sqrt{2}\, n^{1/2}, \tag{2.23}$$

$$\left(1 - x^2\right) \left|P_n^{''}(x)\right| \sim n^2 \; . \tag{2.24}$$

Let $x_k = cos\theta_k, \quad k = 1(1)n$ are the zeros of n^{th} Legendre polynomial $P_n(x)$, with $1 > x_1 > x_2 > \cdots > x_n > -1$, then

$$\left(1 - x_k^2\right)^{-1} \sim \left(\frac{k}{n}\right)^{-2}, \tag{2.25}$$

$$\left|P_n^{(s)}(x_k)\right| \sim \; k^{-s-\frac{1}{2}} \;\; n^{2s} \;, \quad s = 0, 1, 2, 3 \; . \tag{2.26}$$

For more details one can refer to [11].

3 The Problem and Regularity

Let $Z_n \cup \{-1, 1\}$ and T_n be two disjoint set of nodes obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ and $P_n^{''}(x)$ onto the unit circle respectively, where $P_n(x)$ stands for n^{th} Legendre polynomial, Z_n and T_n are defined in (1.1) and (1.2), we take here $z_0 = 1, z_{2n+1} = -1$.
Here we are interested to determine the following polynomial $Q_{6n-7}(z)$ of degree $\le 6n - 7$ satisfying the conditions:

$$\begin{cases} Q_{6n-7}(z_k) & = \alpha_k, & k = 0\,(1)\,2n+1 \\ Q_{6n-7}(t_k) & = \beta_k, & k = 1\,(1)\,2n-4 \\ \left[p(z)\, Q_{6n-7}(z)\right]^{''}_{z=t_k} & = \gamma_k, & k = 1\,(1)\,2n-4, \end{cases} \tag{3.1}$$

where $\alpha_k' s$, $\beta_k' s$ and $\gamma_k' s$ are arbitrary complex constants and

$$p(z) = z^{n(n-3)/2}\left(z^2 - 1\right)^{7/2}\left(z^2 + 1\right)^{-n(n+1)/2}$$

is a weight function.

Theorem 3.1. *$Q_{6n-7}(z)$ is regular on $Z_n \cup \{-1, 1\}$ and T_n.*

Proof. It is sufficient, if we show that the unique solution of (3.1) is

$$Q_{6n-7}(z) \equiv 0,$$

when all data $\alpha_k = \beta_k = \gamma_k = 0$.
In this case, we have

$$Q_{6n-7}(z) = W(z)\, H(z)\, q(z)\,,$$

where $q(z)$ is a polynomial of degree $\le 2n - 3$, $W(z)$ and $H(z)$ are defined in (2.2) and (2.4) respectively. Obviously

$$Q_{6n-7}(z_k) = 0, \qquad k = 1\,(1)\,2n,$$

$$Q_{6n-7}(t_k) = 0, \qquad k = 1\,(1)\,2n-4.$$

From

$$[p(z)\,Q_{6n-7}(z)]''_{z=t_k}=0,$$

using (2.13) - (2.15) and (2.20), we get

(3.2) $$q'(t_k)=0.$$

Therefore, we have

(3.3) $$q'(z)=\ a\ H(z),$$

where a is an arbitrary constant.
Thus, we get

(3.4) $$q(z)=a\ J(z)+b,$$

where

(3.5) $$J(z)=\int_0^z H(t)\ dt.$$

For $q(\pm 1)=0$, we have

(3.6) $$\begin{cases} a\ \ J(1)+b=0 \\ a\ J(-1)+b=0. \end{cases}$$

Since

(3.7) $$J(-z)=-J(z),$$

therefore, using (3.7) in (3.6), we get $a=b=0$.
Hence the theorem follows. □

4 Explicit Representation of Interpolatory Polynomials

We shall write $Q_{6n-7}(z)$ satisfying (3.1) as

(4.1) $$Q_{6n-7}(z)=\sum_{k=0}^{2n+1}\alpha_k B^*_{0k}(z)\ +\sum_{k=1}^{2n-4}\beta_k B_{0k}(z)+\sum_{k=1}^{2n-4}\gamma_k B_{2k}(z),$$

where B^*_{0k}, B_{0k} and B_{2k} are unique polynomials, each of degree at most $6n-7$ satisfying the conditions:

For $k=0\,(1)\,2n+1$

(4.2) $$\begin{cases} B^*_{0k}(z_j) & =\ \delta_{jk}, & j=0\,(1)\,2n+1 \\ B^*_{0k}(t_j) & =\ 0, & j=1\,(1)\,2n-4 \\ [p(z)\,B^*_{0k}(z)]''_{z=t_j} & =\ 0, & j=1\,(1)\,2n-4. \end{cases}$$

For $k=1\,(1)\,2n-4$

(4.3) $$\begin{cases} B_{0k}(z_j) & =0, & j=0\,(1)\,2n+1 \\ B_{0k}(t_j) & =\delta_{jk}, & j=1\,(1)\,2n-4 \\ [p(z)\,B_{0k}(z)]''_{z=t_j} & =0, & j=1\,(1)\,2n-4. \end{cases}$$

For $k=1\,(1)\,2n-4$

(4.4) $$\begin{cases} B_{2k}(z_j) & =0, & j=0\,(1)\,2n+1 \\ B_{2k}(t_j) & =0, & j=1\,(1)\,2n-4 \\ [p(z)\,B_{2k}(z)]''_{z=t_j} & =\delta_{jk}, & j=1\,(1)\,2n-4. \end{cases}$$

Theorem 4.1. *For $k=1\,(1)\,2n-4$, we have*

(4.5) $$B_{2k}(z)=W(z)\,H(z)\left\{c_k\ J_k(z)+c^*_k\ J(z)+c^{**}_k\right\},$$

where $J_k(z)$ is defined in (2.8)

(4.6) $$c_k\ =\ \frac{1}{2t_k\ p(t_k)\,W(t_k)\,H'(t_k)}\ ,$$

(4.7) $$c^*_k\ =\ -\ c_k\ \frac{\{J_k(1)-J_k(-1)\}}{2\ J(1)}\ ,$$

(4.8) $$c^{**}_k=-\ c_k\frac{\{J_k(1)+J_k(-1)\}}{2},$$

and $J(z)$ is defined in (2.9) .

From (4.5), we have

$$B_{2k}(z_j) = 0, \qquad j = 1\,(1)\,2n,$$

$$B_{2k}(t_j) = 0, \qquad j = 1\,(1)\,2n-4.$$

For $j = 1\,(1)\,2n-4$, we get

$$[p(z)\,B_{2k}(z)]''_{z=t_j} = 0, \quad \text{for} \quad j \neq k\ .$$

For $j = k$, we get (4.6).
From $B_{2k}(z_j) = 0$, for $j = 0$ and $2n+1$, we get (4.7) –(4.8).

Theorem 4.2. *For $k = 1\,(1)\,2n-4$, we have*

$$B_{0k}(z) = \frac{\left(z^2-1\right)W(z)}{(t_k^2-1)\,W(t_k)}\; l_{2k}^2(z) + \frac{W(z)\,H(z)}{(t_k^2-1)\,W(t_k)\,H'(t_k)}\left\{S_k(z) + b_k^*\ J(z) + b_k^{**}\right\} + b_k B_{2k}(z) \tag{4.9}$$

where

$$S_k(z) = -\int_0^z \left(t^2-1\right)\frac{[l'_{2k}(t) - l'_{2k}(t_k)\,l_{2k}(t)]}{(t-t_k)}\,dt, \tag{4.10}$$

$$b_k = -4\{l'_{2k}(t_k)\}^2 p(t_k) - \frac{\left\{p(z)\left(z^2-1\right)W(z)\right\}''_{z=t_k}}{(t_k^2-1)\,W(t_k)} - 4\ l'_{2k}(t_k)\,\frac{\left\{p(z)\left(z^2-1\right)W(z)\right\}'_{z=t_k}}{(t_k^2-1)\,W(t_k)} \tag{4.11}$$

$$b_k^* = -\frac{\{S_k(1) - S_k(-1)\}}{2\ \ J(1)}, \tag{4.12}$$

$$b_k^{**} = -\frac{\{S_k(1) + S_k(-1)\}}{2}\ . \tag{4.13}$$

From (4.9) one can see

$$B_{0k}(z_j) = 0, \qquad j = 1\,(1)\,2n\ .$$

$$B_{0k}(t_j) = \delta_{jk}\ , \qquad j = 1\,(1)\,2n-4.$$

Now from

$$[p(z)\,B_{0k}(z)]''_{z=t_j} = 0, \qquad \text{for} \qquad j \neq k,$$

we get

$$S'_k(t_j) = -\frac{\left(t_j^2-1\right)}{\left(t_j-t_k\right)}\,l'_{2k}(t_j)\,.$$

Owing to third condition of (4.3), we derive

$$S'_k(z) = \left(z^2-1\right)\frac{\left[l'_{2k}(z) - l'_{2k}(t_k)\ \ l_{2k}(z)\right]}{(z-t_k)}.$$

On solving it we obtain (4.10).
From

$$[p(z)\,B_{0k}(z)]''_{z=t_j} = 0, \qquad \text{for} \qquad j = k,$$

we establish (4.11).
From (4.9), for

$$B_{0k}(z_j) = 0, \qquad j = 0 \text{ and } 2n+1,$$

we derive (4.12) - (4.13).

Theorem 4.3. *For $k = 1\,(1)\,2n$, we have*

$$B_{0k}^{*}\,(z) = \frac{\left(z^{2}-1\right) H^{2}\,(z)}{(z_k^{2}-1)\,H^{2}\,(z_k)}\;L_{1k}\,(z) + \frac{W\,(z)\,H(z)}{(z_k^{2}-1)W'\,(z_k)\,H^{3}\,(z_k)}\left\{M_k\,(z) + \;a_k^{*}\;J\,(z) + a_k^{**}\right\}, \tag{4.14}$$

where

$$M_k\,(z) = -\int_0^z \frac{\left[\left(t^{2}-1\right) H'\,(t)\,H\,(z_k) - \left(z_k^{2}-1\right) H'\,(z_k)\,H\,(t)\right]}{(t-z_k)}\;dt, \tag{4.15}$$

$$a_k^{*} = -\frac{\{M_k\,(1) - M_k\,(-1)\}}{2\;J\,(1)} \tag{4.16}$$

$$a_k^{**} = -\frac{\{M_k\,(1) + M_k\,(-1)\}}{2}. \tag{4.17}$$

For $k = 0$ and $2n+1$, we have

$$B_{0k}^{*}\,(z) = W\,(z)\,H\,(z)\left\{a_{1k}^{*}\;J\,(z) + \;a_{2k}^{*}\right\}, \tag{4.18}$$

where

$$a_{1k}^{*} \;=\; \frac{1}{2\;W\,(z_k)\,H\,(z_k)\,J\,(z_k)}, \tag{4.19}$$

$$a_{2k}^{*} \;=\; \frac{1}{2\;W\,(z_k)\,H\,(z_k)}. \tag{4.20}$$

From (4.14)

$$B_{0k}^{*}\,(z_j) \;= \delta_{jk}, \qquad j = 1\,(1)\,2n,$$
$$B_{0k}^{*}\,(t_j) \;= 0, \qquad j = 1\,(1)\,2n-4.$$

From

$$\left[p\,(z)\,B_{0k}^{*}\,(z)\right]_{z=t_j}^{''} \;=\; 0, \qquad j = 1\,(1)\,2n-4,$$

we derive

$$M_k^{'}\,(t_j) = \;-H\,(z_k)\,\frac{\left(t_j^{2}-1\right) H^{'}\,(t_j)}{(t_j - z_k)}\quad,$$

Employing to third condition of (4.2), we establish

$$M_k^{'}\,(z) = \;-\frac{\left[\left(z^{2}-1\right) H^{'}\,(z)\,H\,(z_k) - \left(z_k^{2}-1\right) H^{'}\,(z_k)\,H\,(z)\right]}{(z-z_k)}\quad.$$

On solving it we get (4.15).
From (4.14), for

$$B_{0k}^{*}\,(z_j) \;= 0, \qquad j = 0 \text{ and } \;2n+1,$$

we derive (4.16) and (4.17).
For $k = 0$ and $2n+1$, from (4.18), we have

$$B_{0k}^{*}\,(z_j) \;= 0, \qquad j = 1\,(1)\,2n,$$
$$B_{0k}^{*}\,(t_j) \;= 0, \qquad j = 1\,(1)\,2n-4.$$
$$\left[p\,(z)\,B_{0k}^{*}\,(z)\right]_{z=t_j}^{''} \;= 0, \qquad j = 1\,(1)\,2n-4.$$

For

$$B_{0k}^{*}\,(z_j) \;= \delta_{jk}\;, \quad j = 0 \text{ and } 2n+1,$$

we get (4.19) and (4.20).

5 Estimation of Fundamental Polynomials

Lemma 5.1. *For $z = e^{i\theta}$, $(0 \ \le \theta < 2\pi)$, we have*

$$\sum_{k=1}^{2n-4} |p(z) \ B_{2k}(z)| \le c \ \log n \ , \tag{5.1}$$

where $B_{2k}(z)$ be defined in Theorem 4.1 *and c is a constant independent of n and z.*

Lemma 5.2. *For $z = e^{i\theta}$, $(0 \ \le \theta < 2\pi)$, we have*

$$\left|p(z) \, B^*_{0,0}(z)\right| \le c, \qquad \left|p(z) \, B^*_{0,2n+1}(z)\right| \le c, \tag{5.2}$$

and

$$\sum_{k=1}^{2n} |p(z) \, B^*_{0k}(z)| \le cn^2 \log n \ , \tag{5.3}$$

Lemma 5.3. *For $z = e^{i\theta}$, $(0 \ \le \theta < 2\pi)$, we have*

$$\sum_{k=1}^{2n-4} |p(z) \, B_{0k}(z)| \le cn^2 \log n \ , \tag{5.4}$$

where $B_{0k}(z)$ be defined in Theorem 4.2 and c is a constant independent of n and z.

Proof. Using the conditions from (2.21) – (2.26), we get the result. □

6 Convergence

Theorem 6.1. *Let $f(z)$ be continuous for $|z| \le 1$ and analytic for $|z| < 1$. Let the arbitrary numbers $\gamma_k' s$ be such that*

$$|\gamma_k| = O\left(n^2 \ \omega_3\left(f, \frac{1}{n}\right)\right), \qquad k = 1\,(1)\,2n-4. \tag{6.1}$$

Then $\{Q_{6n-7}(z)\}$ defined by

$$Q_{6n-7}(z) = \sum_{k=0}^{2n+1} f(z_k) \, B^*_{0k}(z) \ + \sum_{k=1}^{2n-4} f(t_k) \, B_{0k}(z) + \sum_{k=1}^{2n-4} \gamma_k \ B_{2k}(z), \tag{6.2}$$

satisfies the relation

$$\left|p(z) \left\{ \ Q_{6n-7}(z) - f(z) \right\}\right| = O\left(\omega_3\left(f, n^{-1}\right) \log n \ \right), \tag{6.3}$$

where $\omega_3\left(f, n^{-1}\right)$ be the third modulus of continuity of $f(z)$.

To prove the Theorem 6.1, we shall need the followings:

Remark 6.1. Let $f(z)$ be continuous for $|z| \le 1$ and analytic for $|z| < 1$ and $f'' \in Lip \ \alpha, \ \alpha > 0$, then the sequence $\{Q_{6n-7}(z)\}$ converges uniformly to $f(z)$ in $|z| \le 1$, which follows from (6.3) provided

$$\omega_3\left(f, n^{-1}\right) = O\left(n^{-2-\alpha}\right). \tag{6.4}$$

There exists a polynomial $F_n(z)$ of degree $\le 6n-7$, satisfying Jackson's inequality

$$|f(z) - F_n(z)| \le c \ \omega_3\left(f, n^{-1}\right), \quad z = e^{i\theta}\,(0 \le \theta < 2\pi), \tag{6.5}$$

and the inequality due to Kiš [10],

$$\left|F_n^{(m)}(z)\right| \le c \ n^m \ \omega_3\left(f, n^{-1}\right) \ , \quad m \ \in I^+. \tag{6.6}$$

Proof. Since $Q_{6n-7}(z)$ be a uniquely determined polynomial of degree $\le 6n-7$ and the polynomial $F_n(z)$ of degree $\le 6n-7$ satisfying (6.5) and (6.6) can be expressed as

$$F_n(z) = \sum_{k=0}^{2n+1} \ F_n(z_k) \ \ B^*_{0k}(z) + \sum_{k=1}^{2n-4} \ F_n(t_k) \, B_{0k}(z) + \sum_{k=1}^{2n-4} F_n''(t_k) \, B_{2k}(z).$$

Then

$$|p(z) \{Q_{6n-7}(z) - f(z)\}| \le |p(z) \{Q_{6n-7}(z) - F_n(z)\}| + |p(z) \{F_n(z) - f(z)\}|$$

$$
\begin{aligned}
&\leq \sum_{k=0}^{2n+1} \left| f\left(z_k\right) - \quad F_n\left(z_k\right)\right| \left| p\left(z\right) \;\; B_{0k}^{*}\left(z\right)\right| \\
&+ \sum_{k=1}^{2n-4} \left| f\left(t_k\right) - \quad F_n\left(t_k\right)\right| \left| p\left(z\right) \;\; B_{0k}\left(z\right)\right| \\
&+ \sum_{k=1}^{2n-4} \left\{ \left|\gamma_k\right| + \left| F_n^{''}\left(t_k\right)\right| \right\} \left| p\left(z\right) \;\; B_{2k}\left(z\right)\right| \\
&\quad + \; \left|p(z)\right| \left|F_n\left(z\right) - f\left(z\right)\right|.
\end{aligned}
$$

Using (6.1), (6.2), (6.4) – (6.6) and Lemmas 5.1 – 5.3, we get (6.3) . □

7 Conclusion

In this paper, we defined the weighted Pal - type (0,2) - interpolation on two pairwise disjoint sets of nodes on the unit circle, which converges uniformly to $f^{''} \in Lip\ \alpha,\ \alpha > 0$.

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