Abstract

In this paper, we study the explicit representation of weighted Pál-type (0,2) - interpolation on two pairwise disjoint sets of nodes on the unit circle, which are obtained by projecting vertically the zeros of $(1 - x^2)P_n(x)$ and $P_n'(x)$ respectively, where $P_n(x)$ stands for the $n^{th}$ Legendre polynomial.

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1 Introduction

In 1979, Turán [12] studied the (0,2) - Interpolation for getting an approximate solution of differential equation $y'' + fy = 0$. Balázs [8] introduced the weighted (0,2) - Interpolation on the zeros of Ultraspherical polynomial $P_n^{(\alpha)}(x)$, $\alpha > -1$. In 1960, Kiš [10] initiated the Lacunary interpolation on the unit circle. He considered (0,2)- Interpolation on the unit circle and established the convergence theorem. After that several mathematician have considered (0,2) – Interpolation viz. on the unit circle, infinite interval and on the real line. In 1996, Xie [13] considered (0,1,3)* - interpolation on the vertically projected nodes onto the unit circle. He claimed the regularity, explicit representation and convergence of (0,1,3)* - Interpolation. In 2003, Dikshit [9] considered the Pál – type Interpolation on non uniformly distributed nodes on the unit circle. After that author and Mathur [1] considered the weighted (0,2)* – Interpolation on the set of nodes obtained by projecting vertically the zeros of $(1 - x^2)P_n(x)$ on the unit circle and established a convergence theorem for that interpolatory polynomial. In 2012, she [2,3] considered weighted $(0;0,2)$ and $(0,2;0)$ – Interpolation on projected nodes onto the unit circle, obtained the regularity, fundamental polynomial and established a convergence theorem. In 2017, authors [4] considered the regularity and explicit forms of weighted (0,2;0) - interpolation on the unit circle on two pairwise disjoint sets of nodes obtained by projecting vertically the zeros of $(1 - x^2)P_n(x)$ and $P_n'(x)$ respectively onto the unit circle, where $P_n'(x)$ stands for the $n^{th}$ Legendre polynomial. After that the authors [5] also established convergence for the above said interpolatory polynomials. Recently, authors [6] considered weighted Lacunary interpolation on the nodes, which are obtained by projecting vertically the zeros of the $(1 - x^2)P_n'(x)$ onto the unit circle and established a convergence theorem for the same. Recently, author with Iqram [7] considered generalized Hermite-Fejér interpolation on the nodes, which are obtained by vertically projected zeros of the $(1 + x)P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial established the convergence theorem. These have motivated us to consider $(0;0,2)$ interpolation on two pairwise disjoint sets of nodes on the unit circle. Let

\begin{align}
Z_n = \left\{ z_k = \cos \theta_k + i \sin \theta_k \\ z_{n+k} = \frac{z_k}{2}, \quad k = 1 \right\} n,
\end{align}

\begin{align}
T_n = \left\{ t_k = \cos \varphi_k + i \sin \varphi_k, \\ t_{(n-2)+k} = \frac{t_k}{2}, \quad k = 1 \right\} n - 2,
\end{align}

be two set of nodes. In which the Lagrange data is prescribed on the first set of nodes whereas Lacunary data on the other one. We obtained regularity, explicit forms and established a convergence theorem of the interpolatory polynomials. In Section 2, we give some preliminaries, in Section 3, we describe the problem and regularity, in Section 4 and Section 5, we present the explicit forms and convergence of weighted Pál – type (0,2) – interpolation on the unit circle respectively.
2 Preliminaries

The differential equation satisfied by \( P_n(x) \) is

\[
(1 - x^2) P''_n(x) - 2x P'_n(x) + n(n + 1) P_n(x) = 0.
\]

(2.1)

\[
W(z) = \prod_{k=1}^{2n} (z - z_k) = K_n P_n \left( \frac{1 + z^2}{2z} \right) z^n;
\]

(2.2)

\[
R(z) = (z^2 - 1) W(z),
\]

(2.3)

\[
H(z) = \prod_{k=1}^{2n-4} (z - t_k) = K_{n}^{\ast \ast} P^n_\ast \left( \frac{1 + z^2}{2z} \right) z^{n-2}.
\]

(2.4)

We shall require the following fundamental polynomials of Lagrange interpolation based on the zeros of \( W(z) \) and \( R(z) \), are respectively defined as

\[
L_{1k}(z) = \frac{W(z)}{(z - z_k) W'(z_k)}, \quad k = 1 \text{ or } 2n,
\]

(2.5)

\[
L_k(z) = \frac{R(z)}{(z - z_k) R'(z_k)}, \quad k = 0 \text{ or } 2n + 1,
\]

(2.6)

\[
l_{2k}(z) = \frac{H(z)}{(z - t_k) H'(t_k)}, \quad k = 1 \text{ or } 2n - 4,
\]

(2.7)

\[
J_k(z) = \int_0^z t \ l_{2k}(t) \ dt,
\]

(2.8)

\[
J(z) = \int_0^z H(t) \ dt,
\]

(2.9)

which satisfies

\[
J(-z) = -J(z).
\]

(2.10)

We shall also use the following results in our investigations:

\[
W'(z_k) = \frac{K_n}{2} \left( z_k^2 - 1 \right) P'_n(x_k) z_k^{n-2}, \quad k = 1 \text{ or } 2n - 2,
\]

(2.11)

\[
W''(z_k) = K_n \left[ (n - 1)(z_k^2 - 1) - 1 \right] z_k^{n-3} P'_n(x_k), \quad k = 1 \text{ or } 2n,
\]

(2.12)

\[
H'(t_k) = \frac{K_{n}^{\ast \ast}}{2} \left( t_k^2 - 1 \right) t_k^{n-4} P''_n(x_k), \quad k = 1 \text{ or } 2n - 4,
\]

(2.13)

\[
W'(t_k) = K_n \frac{n \left( (n + 3) \left( t_k^2 - 1 \right) + 4 \right)}{2 \left( t_k^2 + 1 \right)} t_k^{n-1} P_n(x_k^*),
\]

(2.14)

\[
W''(t_k) = K_n \frac{n(n - 1) \left( (n - 1) \left( t_k^2 - 1 \right) - 1 \right)}{2 \left( t_k^2 + 1 \right)} t_k^{n-2} P_n(x_k^*),
\]

(2.15)

\[
R'(t_k) = (x_k^2 - 1) W'(z_k),
\]

(2.16)

\[
R''(z_k) = 4x_k \ W'(z_k) + (z_k^2 - 1) \ W''(z_k),
\]

(2.17)

\[
R'(t_k) = (t_k^2 - 1) W'(t_k),
\]

(2.18)
Proof. It is sufficient, if we show that the unique solution of (3.1) is

(2.20) \[ H''(t_k) = K^*_n \left\{ (n - 5) (t_k^2 - 1) - 5 \right\} t_k^{n-5} P_n''(x_k^1) \] .

We shall also use the following well known inequalities:
For \(-1 < x < 1\)
(2.21) \[ |P_n(x)| \leq 1, \]
(2.22) \[ (1 - x^2)^{1/4} |P_n(x)| \leq \sqrt{\frac{2}{\pi}} n^{-1/2}, \]
(2.23) \[ (1 - x^2)^{3/4} |P'_n(x)| \leq \sqrt{2} n^{1/2}, \]
(2.24) \[ (1 - x^2) |P''_n(x)| \sim n^2. \]

Let \(x_k = \cos \theta_k, \ k = 1(1)n\) are the zeros of \(n^{th}\) Legendre polynomial \(P_n(x)\), with \(1 > x_1 > x_2 > \cdots > x_n > -1\), then
(2.25) \[ (1 - x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}, \]
(2.26) \[ |P^{(s)}_n(x_k)| \sim k^{-s - \frac{1}{2}} n^{2s}, \ s = 0, 1, 2, 3. \]

For more details one can refer to [11].

3 The Problem and Regularity
Let \(Z_n \cup \{-1, 1\}\) and \(T_n\) be two disjoint set of nodes obtained by projecting vertically the zeros of \((1-x^2)P_n(x)\) and \(P_n''(x)\) onto the unit circle respectively, where \(P_n(x)\) stands for \(n^{th}\) Legendre polynomial, \(Z_n\) and \(T_n\) are defined in (1.1) and (1.2), we take here \(z_0 = 1, z_{2n+1} = -1\).

Here we are interested to determine the following polynomial \(Q_{6n-7}(z)\) of degree \(\leq 6n - 7\) satisfying the conditions:

(3.1) \[ \begin{align*}
Q_{6n-7}(z_k) &= \alpha_k, & k = 0 \ (2n + 1) \\
Q_{6n-7}(t_k) &= \beta_k, & k = 1 \ (2n - 4) \\
[p(z)Q_{6n-7}'(z)]_{z=t_k}'' &= \gamma_k, & k = 1 \ (2n - 4),
\end{align*} \]

where \(\alpha_k, \beta_k\) and \(\gamma_k\) are arbitrary complex constants and

\[ p(z) = z^{n(n-3)/2} (z^2 - 1)^{7/2} (z^2 + 1)^{-n(n+1)/2} \]

is a weight function.

Theorem 3.1. \(Q_{6n-7}(z)\) is regular on \(Z_n \cup \{-1, 1\}\) and \(T_n\).

Proof. It is sufficient, if we show that the unique solution of (3.1) is

\[ Q_{6n-7}(z) \equiv 0, \]

when all data \(\alpha_k = \beta_k = \gamma_k = 0.\)

In this case, we have

\[ Q_{6n-7}(z) = W(z) H(z) q(z), \]

where \(q(z)\) is a polynomial of degree \(\leq 2n - 3\), \(W(z)\) and \(H(z)\) are defined in (2.2) and (2.4) respectively. Obviously

\[ \begin{align*}
Q_{6n-7}(z_k) &= 0, & k = 1 \ (2n), \\
Q_{6n-7}(t_k) &= 0, & k = 1 \ (2n - 4).
\end{align*} \]
From
\[ p(z)Q_{6n-7}(z) = 0, \]
using (2.13) - (2.15) and (2.20), we get
\[ q'(t_k) = 0. \]
Therefore, we have
\[ q'(z) = a H(z), \]
where \( a \) is an arbitrary constant.
Thus, we get
\[ q(z) = a J(z) + b, \]
where
\[ J(z) = \int_0^z H(t) \, dt. \]
For \( q(\pm 1) = 0 \), we have
\[ \begin{cases} a J(1) + b = 0 \\ a J(-1) + b = 0. \end{cases} \]
Since
\[ J(-z) = -J(z), \]
therefore, using (3.7) in (3.6), we get \( a = b = 0 \).
Hence the theorem follows.

4 Explicit Representation of Interpolatory Polynomials

We shall write \( Q_{6n-7}(z) \) satisfying (3.1) as
\[ Q_{6n-7}(z) = \sum_{k=0}^{2n+1} \alpha_k B^*_0(z) + \sum_{k=1}^{2n-4} \beta_k B_0(z) + \sum_{k=1}^{2n-4} \gamma_k B_2(z), \]
where \( B^*_0, B_0 \) and \( B_2 \) are unique polynomials, each of degree at most \( 6n - 7 \) satisfying the conditions:
For \( k = 0 \) \( \left( 1 \right) 2n + 1 \)
\[ \begin{cases} B^*_0(z_j) = \delta_{jk}, & j = 0 \left( 1 \right) 2n + 1 \\ B^*_0(t_j) = 0, & j = 1 \left( 1 \right) 2n - 4 \\ [p(z)B^*_0(z)]'' = 0, & j = 1 \left( 1 \right) 2n - 4. \end{cases} \]

For \( k = 1 \) \( \left( 1 \right) 2n - 4 \)
\[ \begin{cases} B_0(z_j) = 0, & j = 0 \left( 1 \right) 2n + 1 \\ B_0(t_j) = \delta_{jk}, & j = 1 \left( 1 \right) 2n - 4 \\ [p(z)B_0(z)]'' = 0, & j = 1 \left( 1 \right) 2n - 4. \end{cases} \]

For \( k = 1 \) \( \left( 1 \right) 2n - 4 \)
\[ \begin{cases} B_2(z_j) = 0, & j = 0 \left( 1 \right) 2n + 1 \\ B_2(t_j) = 0, & j = 1 \left( 1 \right) 2n - 4 \\ [p(z)B_2(z)]'' = \delta_{jk}, & j = 1 \left( 1 \right) 2n - 4. \end{cases} \]

Theorem 4.1. For \( k = 1 \) \( \left( 1 \right) 2n - 4 \), we have
\[ B_2(z) = W(z)H(z) \{ c_k J_k(z) + c^*_k J(z) + c^{**}_k \}, \]
where \( J_k(z) \) is defined in (2.8)
\[ c_k = \frac{1}{2t_k p(t_k) W(t_k) H'(t_k)}, \]
\[ c^*_k = -c_k \frac{J_k(1) - J_k(-1)}{2J(1)}, \]
\[ c^{**}_k = -c_k \frac{J_k(1) + J_k(-1)}{2}, \]
and \( J(z) \) is defined in (2.9) .
From (4.5), we have

\[ B_{2k} (z_j) = 0, \quad j = 1 \text{ or } 2n, \]
\[ B_{2k} (t_j) = 0, \quad j = 1 \text{ or } 2n - 4. \]

For \( j = 1 \text{ or } 2n - 4 \), we get

\[ [p(z) B_{2k}(z)]''_{z=t_j} = 0, \quad \text{for } j \neq k. \]

For \( j = k \), we get (4.6).

From \( B_{2k}(z_j) = 0 \), for \( j = 0 \) and \( 2n + 1 \), we get (4.7) – (4.8).

**Theorem 4.2.** For \( k = 1 \text{ or } 2n - 4 \), we have

\[ B_{0k} (z) = \left( \frac{z^2 - 1}{(t^2 - 1)} W(z) \right) \frac{l_k^2(z)}{W(t_k)} + \frac{W(z) H(z)}{(t^2 - 1) W(t_k) H'(t_k)} \{ S_k(z) + b_k^* J(z) + b_k^{**} \} + b_k B_{2k} (z) \]

where

\[ S_k(z) = -\int_0^z \left( \frac{(t^2 - 1)}{(t - t_k)} \right) \left[ \frac{l_k^2(t) - l_k^2(t_k)}{l_k^2(t_k)} \right] dt, \]

\[ b_k = -4 \left( \frac{l_k^2(t_k)}{l_k^2(t_k)} \right)^2 p(t_k) - \frac{\{ p(z) \left( \frac{z^2 - 1}{(t^2 - 1)} W(z) \right) \}''_{z=t_k}}{(t^2 - 1) W(t_k)} - 4 \left( \frac{l_k^2(t_k)}{l_k^2(t_k)} \right) \frac{\{ p(z) \left( \frac{z^2 - 1}{(t^2 - 1)} W(z) \right) \}'}{(t^2 - 1) W(t_k)} \]

\[ b_k^* = -\frac{\{ S_k(1) - S_k(-1) \}}{2 J(1)}, \]

\[ b_k^{**} = -\frac{\{ S_k(1) + S_k(-1) \}}{2}. \]

From (4.9) one can see

\[ B_{0k} (z_j) = 0, \quad j = 1 \text{ or } 2n. \]
\[ B_{0k} (t_j) = \delta_{jk}, \quad j = 1 \text{ or } 2n - 4. \]

Now from

\[ [p(z) B_{0k}(z)]''_{z=t_j} = 0, \quad \text{for } j \neq k, \]

we get

\[ S_k'(t_j) = -\frac{(t_j^2 - 1)}{(t_j - t_k)} l_k^2(t_j). \]

Owing to third condition of (4.3), we derive

\[ S_k'(z) = (z^2 - 1) \left( \frac{l_k^2(z) - l_k^2(t_k)}{l_k^2(t_k)} \right) \frac{l_k^2(z)}{(z - t_k)}. \]

On solving it we obtain (4.10).

From

\[ [p(z) B_{0k}(z)]''_{z=t_j} = 0, \quad \text{for } j = k, \]

we establish (4.11).

From (4.9), for

\[ B_{0k} (z_j) = 0, \quad j = 0 \text{ and } 2n + 1, \]

we derive (4.12) - (4.13).
Theorem 4.3. For $k = 1 \ldots 2n$, we have

\begin{equation}
B^*_0 k(z) = \left( \frac{z^2 - 1}{z_k^2 - 1} \right) H^2(z) \left( \frac{L_{1k}(z) + \frac{W(z)H(z)}{(z_k^2 - 1)H^2(z_k)}}{(z_k^2 - 1)H^2(z_k)} \right) M_k(z) + a_k^* J(z) + a_{k^*}^*,
\end{equation}

where

\begin{equation}
M_k(z) = -\int_0^z \frac{[(t^2 - 1) H'(t) H(z_k) - (z_k^2 - 1) H'(z_k) H(t)] dt}{(t - z_k)},
\end{equation}

\begin{equation}
a_k^* = -\frac{\{M_k(1) - M_k(-1)\}}{2 J(1)}
\end{equation}

\begin{equation}
a_{k^*}^* = -\frac{\{M_k(1) + M_k(-1)\}}{2}.
\end{equation}

For $k = 0$ and $2n + 1$, we have

\begin{equation}
B^*_0 k(z) = W(z) H(z) \{a_{1k}^* J(z) + a_{2k}^*\},
\end{equation}

where

\begin{equation}
a_{1k}^* = \frac{1}{2 W(z_k) H(z_k) J(z_k)},
\end{equation}

\begin{equation}
a_{2k}^* = \frac{1}{2 W(z_k) H(z_k)}.
\end{equation}

From (4.14)

\begin{align*}
B^*_0 k(z_j) & = \delta_{jk}, & j & = 1 \ldots 2n, \\
B^*_0 k(t_j) & = 0, & j & = 1 \ldots 2n - 4.
\end{align*}

From

\begin{equation}
[p(z) B^*_0 k(z)]''_{z = t_j} = 0, & j = 1 \ldots 2n - 4,
\end{equation}

we derive

\begin{equation}
M'_k(t_j) = -H(z_k) \frac{(t_j^2 - 1) H'(t_j)}{(t_j - z_k)},
\end{equation}

Employing to third condition of (4.2), we establish

\begin{equation}
M'_k(z) = -\frac{[(z^2 - 1) H'(z) H(z_k) - (z_k^2 - 1) H'(z_k) H(z)]}{(z - z_k)}.
\end{equation}

On solving it we get (4.15).

From (4.14), for

\begin{equation}
B^*_0 k(z_j) = 0, & j = 0 \text{ and } 2n + 1,
\end{equation}

we derive (4.16) and (4.17).

For $k = 0$ and $2n + 1$, from (4.18), we have

\begin{align*}
B^*_0 k(z_j) & = 0, & j & = 1 \ldots 2n, \\
B^*_0 k(t_j) & = 0, & j & = 1 \ldots 2n - 4.
\end{align*}

From

\begin{equation}
[p(z) B^*_0 k(z)]''_{z = t_j} = 0, & j = 1 \ldots 2n - 4.
\end{equation}

For

\begin{equation}
B^*_0 k(z_j) = \delta_{jk}, & j = 0 \text{ and } 2n + 1,
\end{equation}

we get (4.19) and (4.20).
5 Estimation of Fundamental Polynomials

Lemma 5.1. For $z = e^{i\theta}$, $(0 \leq \theta < 2\pi)$, we have

$$\sum_{k=1}^{2n-4} |p(z) B_{2k}(z)| \leq c \log n,$$

where $B_{2k}(z)$ be defined in Theorem 4.1 and $c$ is a constant independent of $n$ and $z$.

Lemma 5.2. For $z = e^{i\theta}$, $(0 \leq \theta < 2\pi)$, we have

$$|p(z) B_{*0,0}^*(z)| \leq c, \quad |p(z) B_{*0,2n+1}^*(z)| \leq c,$$

and

$$\sum_{k=1}^{2n} |p(z) B_{0k}^*(z)| \leq cn^2 \log n,$$

Lemma 5.3. For $z = e^{i\theta}$, $(0 \leq \theta < 2\pi)$, we have

$$\sum_{k=1}^{2n-4} |p(z) B_{0k}(z)| \leq cn^2 \log n,$$

where $B_{0k}(z)$ be defined in Theorem 4.2 and $c$ is a constant independent of $n$ and $z$.

Proof. Using the conditions from (2.21) – (2.26), we get the result.

6 Convergence

Theorem 6.1. Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Let the arbitrary numbers $\gamma_k$ be such that

$$|\gamma_k| = O \left( n^2 \omega_3 \left( f \frac{1}{n} \right) \right), \quad k = 1 (1) 2n-4.$$

Then $\{Q_{6n-7}(z)\}$ defined by

$$Q_{6n-7}(z) = \sum_{k=0}^{2n+1} f(z_k) B_{0k}^*(z) + \sum_{k=1}^{2n-4} f(t_k) B_{0k}(z) + \sum_{k=1}^{2n-4} \gamma_k B_{2k}(z),$$

satisfies the relation

$$|p(z) \{ Q_{6n-7}(z) - f(z) \} | = O \left( \omega_3 \left( f, n^{-1} \right) \log n \right),$$

where $\omega_3 (f, n^{-1})$ be the third modulus of continuity of $f(z)$.

To prove the Theorem 6.1, we shall need the followings:

Remark 6.1. Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$ and $f'' \in \text{Lip} \alpha$, $\alpha > 0$, then the sequence $\{Q_{6n-7}(z)\}$ converges uniformly to $f(z)$ in $|z| \leq 1$, which follows from (6.3) provided

$$\omega_3 \left( f, n^{-1} \right) = O \left( n^{-2-\alpha} \right).$$

There exists a polynomial $F_n(z)$ of degree $\leq 6n - 7$, satisfying Jackson’s inequality

$$|f(z) - F_n(z)| \leq c \omega_3 \left( f, n^{-1} \right), \quad z = e^{i\theta} \left( 0 \leq \theta < 2\pi \right),$$

and the inequality due to Kiš [10],

$$|F_n^{(m)}(z)| \leq c n^m \omega_3 \left( f, n^{-1} \right), \quad m \in I^+.$$

Proof. Since $Q_{6n-7}(z)$ be a uniquely determined polynomial of degree $\leq 6n - 7$ and the polynomial $F_n(z)$ of degree $\leq 6n - 7$ satisfying (6.5) and (6.6) can be expressed as

$$F_n(z) = \sum_{k=0}^{2n+1} F_n(z_k) B_{0k}^*(z) + \sum_{k=1}^{2n-4} F_n(t_k) B_{0k}(z) + \sum_{k=1}^{2n-4} F_n''(t_k) B_{2k}(z).$$

Then

$$|p(z) \{ Q_{6n-7}(z) - f(z) \} | \leq |p(z) \{ Q_{6n-7}(z) - F_n(z) \} | + |p(z) \{ F_n(z) - f(z) \} |$$
\[ \leq \sum_{k=0}^{2n+1} |f(z_k) - F_n(z_k)| |p(z)| B_{0k}^*(z) \]
\[ + \sum_{k=1}^{2n-4} |f(t_k) - F_n(t_k)| |p(z)| B_{0k}(z) \]
\[ + \sum_{k=1}^{2n-4} |\gamma_k| + \left| F_n''(t_k) \right| |p(z)| B_{2k}(z) \]
\[ + |p(z)| |F_n(z) - f(z)|. \]

Using (6.1), (6.2), (6.4) – (6.6) and Lemmas 5.1 – 5.3, we get (6.3).

7 Conclusion
In this paper, we defined the weighted Pal - type (0,2) - interpolation on two pairwise disjoint sets of nodes on the unit circle, which converges uniformly to \( f'' \in \text{Lip } \alpha, \alpha > 0 \).

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References