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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

COEFFICIENT INEQUALITY FOR A CLASS OF ANALYTIC FUNCTIONS APPROACHING TO CLASS OF CONVEX FUNCTIONS IN THE LIMIT FORM AND CLASS OF STARLIKE FUNCTIONS DIRECTLY

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Abstract

We introduce a class of analytic functions and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to this class with special character that it tends to the class of convex functions as $\alpha \to \frac{\pi}{2}$.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disc $\mathbb{E} = \{z : |z| < 1\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach [1, 2] proved that $|a_2| \leq 2$ for the functions $f(z)\mathcal{S}$. In 1923, Löwner [10] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$.

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was expected to try to find some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$**,** Fekete and Szegö [4] [8]used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$\left|a_3 - \mu a_2^2\right| = \begin{cases} 3 - 4\mu & if \mu \leq 0 \\ 1 + 2\exp\left(\dfrac{-2\mu}{1-\mu}\right) & if \ 0 \ \leq \mu \leq 1 \\ 4\mu - 3 & if \mu \geq 1 \end{cases}. \tag{1.2}$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ [3, 9].

Let us define some subclasses of $\mathcal{S}$.

We denote by $\mathcal{S}^*$, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \ \in \ \mathcal{A}$$

and satisfying the condition

$$Re \left(\frac{zg\left(z\right)}{g\left(z\right)}\right) > 0, \ z \in \mathbb{E}. \tag{1.3}$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = \ z + \sum_{n=2}^{\infty} c_n z^n \ \in \ \mathcal{A}$$

and satisfying the condition

$$Re\left(\frac{(zh'(z)}{h'(z)}\right) > 0,\ z \in \mathbb{E}. \tag{1.4}$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in \mathcal{S}^*$ such that

$$Re\ \left(\frac{zf'(z)}{g(z)}\right) > 0,\ z \in \mathbb{E}. \tag{1.5}$$

The class of close to convex functions is denoted by C and was introduced by Kaplan [7] and it was shown by him that all close to convex functions are univalent.

$$S^*\ (A, B) = \left\{f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} \prec \ \frac{1 + Az}{1 + Bz},\ -1 \leq B < A \leq 1, z \in \mathbb{E}\right\}, \tag{1.6}$$

$$\mathcal{K}(A, B) = \left\{f(z) \in \mathcal{A}; \frac{(zf'(z))'}{f'(z)} \prec \ \frac{1 + Az}{1 + Bz},\ -1 \leq B < A \leq 1, z \in \mathbb{E}\right\}. \tag{1.7}$$

It is obvious that $S^*(A, B)$ is a subclass of S^* and $\mathcal{K}\ (A, B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclass as

$$\left\{f(z) \in \mathcal{A}; \left(\frac{zf'(z)}{f(z)}\right)^{\beta} + \tan\alpha \left(\frac{(zf'(z))'}{f'(z)}\right)^{1-\beta} \prec \left\{\frac{1 + w(z)}{1 - w(z)}\right\}^{\gamma}; z \in \mathbb{E}\right\}$$

and we shall denote this class as $\mathcal{K}S^*\ (\alpha, \beta\}$.

We shall deal with two subclasses of $S^*\ (f, f', \alpha, \beta)$ defined as follows in our next paper:

$$\mathcal{KS}^*\ (\alpha, \beta, A, B) = \left\{f(z) \in \mathcal{A}; \left(\frac{zf'(z)}{f(z)}\right)^{\beta} + \tan\alpha \left(\frac{(zf'(z))'}{f'(z)}\right)^{1-\beta} \prec \frac{1 + Az}{1 + Bz}; z \in \mathbb{E}\right\}, \tag{1.8}$$

$$\mathcal{KS}^*\ (A, B, \alpha, \beta, \gamma) = \left\{f(z) \in \mathcal{A}; \left(\frac{zf'(z)}{f(z)}\right)^{\beta} + \tan\alpha \left(\frac{(zf'(z))'}{f'(z)}\right)^{1-\beta} \prec \left\{\frac{1 + Az}{1 + Bz}\right\}^{\gamma}; z \in \mathbb{E}\right\}. \tag{1.9}$$

Several researchers established new subclasses using these classes and gave amazing results about coefficient inequality. [12], [15].

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination. *Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z\ \mathbb{E}$ and we write $f(z)\ \prec\ F(z)$.*[11]

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$$w(z) = \ \sum_{n=1}^{\infty} d_n z^n,\ w(0) = 0,\ |w(z)| < 1. \tag{1.10}$$

It is known that

$$|d_1| \leq 1,\ |d_2| \leq \ 1 - |d_1|^2. \tag{1.11}$$

2 Preliminary Lemmas.

For $0\ <\ c\ <\ 1$, we write $w(z) = \left(\dfrac{c + z}{1 + cz}\right)$ so that

$$\frac{1 + w(z)}{1 - w(z)} \ = \ 1 + 2cz + 2z^2 + \cdots . \tag{2.1}$$

3 Main Results

Theorem 3.1. Let $f(z) \in \mathcal{KS}^*(\alpha,\beta,\gamma)$
$|a_3 - \mu a_2^2| \leq$

$$\begin{cases} \dfrac{1}{\{\beta+2(1-\beta)\tan\alpha\}^2}\left[\dfrac{\{4(1-\beta)(\beta+2)\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}}{\{\beta+3(1-\beta)\tan\alpha\}}-4\gamma^2\mu\right] \\ if\ \ \mu \leq \dfrac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}-\gamma\{(1-\alpha)\beta+2\alpha(1-\beta)\}}{4\{\beta+3(1-\beta)\tan\alpha\}} & (3.1) \\ \\ \dfrac{1}{3\alpha+\beta-4\alpha\beta} \\ if\ \ \dfrac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}-\gamma\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4\{\beta+3(1-\beta)\tan\alpha\}} \leq \\ \mu \leq \dfrac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}+\gamma\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4\gamma^2\{\beta+3(1-\beta)\tan\alpha\}} & (3.2) \\ \\ \dfrac{1}{\{\beta+2(1-\beta)\tan\alpha\}^2}\left[4\gamma^2\mu-\dfrac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}}{\{\beta+3(1-\beta)\tan\alpha\}}\right] \\ if\ \ \mu \geq \dfrac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}+\gamma\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4\gamma^2\{\beta+3(1-\beta)\tan\alpha\}} & (3.3) \end{cases}$$

The results are sharp.

Proof. By definition of $\mathcal{KS}^*(\alpha,\beta,\gamma)$, we have

$$f(z) \in \mathcal{A};\left(\frac{zf'(z)}{f(z)}\right)^{\beta}+\tan\alpha\left(\frac{(zf'(z))'}{f'(z)}\right)^{1-\beta}=\left\{\frac{1+w(z)}{1-w(z)}\right\}^{\gamma};w(z)\in\mathcal{U} \tag{3.4}$$

Expanding the series (2.1), we get

(3.5)
$$\left\{1+\beta a_2 z+(2\beta a_3+\frac{\beta(\beta-3)}{2}a_2^2)z^2+\ldots\right\}+\tan\alpha\ \{1+2(1-\beta)\,a_2 z+2(1-\beta)(3a_3-(\beta+2)a_2^2)z^2+\ldots\}$$
$$=\left(1+2\gamma c_1 z+2\gamma\left(c_2+\gamma {c_1}^2\right)z^2+\ldots\right).$$

Identifying terms in (3.5), we get

$$a_2=\frac{2\gamma}{\beta+2(1-\beta)\tan\alpha}\ c_1. \tag{3.6}$$

$$a_3=\frac{\gamma}{\beta+3(1-\beta)\tan\alpha}\ c_2+\ \frac{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)}{\{\beta+3(1-\beta)\tan\alpha\}\{\beta+2(1-\beta)\tan\alpha\}}\gamma^2\ {c_1}^2. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$\begin{aligned} a_3-\mu a_2^2 =&\frac{\gamma c_2}{\beta+3(1-\beta)\tan\alpha} \\ &+\left[\frac{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)}{\{\beta+3(1-\beta)\tan\alpha\}\{\beta+2(1-\beta)\tan\alpha\}}-\frac{4\gamma^2\mu}{\{\beta+2(1-\beta)\tan\alpha\}^2}\right]c_1^2. \end{aligned} \tag{3.8}$$

Taking absolute value and using Triangular inequality, (3.8) can be rewritten as

$$
(3.9) \quad \begin{aligned} |a_3-\mu a_2^2| &\le \frac{\gamma|c_2|}{\beta+3(1-\beta)\tan\alpha} \\ &+\frac{1}{\{\beta+2(1-\beta)\tan\alpha\}^2}\left|\frac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}}{\{\beta+3(1-\beta)\tan\alpha\}}-4\gamma^2\mu\right||c_1^2| \end{aligned}
$$

Using (1.9) in (3.6), simple calculations yield

$$
(3.10) \qquad |a_3-\mu a_2^2| \le \frac{\gamma}{\beta+3(1-\beta)\tan\alpha}+\frac{1}{\{\beta+2(1-\beta)\tan\alpha\}^2}
$$

$$
\left[\left|\frac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}}{\{\beta+3(1-\beta)\tan\alpha\}}-4\gamma^2\mu\right|-\frac{\gamma\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{\beta+3(1-\beta)\tan\alpha}\right]|c_1|^2
$$

***Case* I.** $\mu \le \dfrac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}}{4\gamma^2\{\beta+3(1-\beta)\tan\alpha\}}$. In this case, (3.10) can be rewritten as

$$
(3.11) \qquad |a_3-\mu a_2^2| \le \frac{\gamma}{\beta+3(1-\beta)\tan\alpha}+\frac{1}{\{\beta+2(1-\beta)\tan\alpha\}^2}
$$

$$
\left[\frac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}-\gamma\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{\{\beta+3(1-\beta)\tan\alpha\}}-4\mu\right]|c_1|^2.
$$

***Subcase* I (a).** $\mu \le \dfrac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}-\gamma\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4\{\beta+3(1-\beta)\tan\alpha\}}$.

Using (1.9), (3.8) becomes

(3.12)

$$
\left|a_3-\mu a_2^2\right| \le \frac{1}{\{\beta+2(1-\beta)\tan\alpha\}^2}\left[\frac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}}{\{\beta+3(1-\beta)\tan\alpha\}}-4\gamma^2\mu\right].
$$

***Subcase* I (b).** $\mu \ge \dfrac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}-\gamma\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4\{\beta+3(1-\beta)\tan\alpha\}}$.

We obtain from (3.8)

$$
(3.13) \qquad \left|a_3-\mu a_2^2\right| \le \frac{\gamma}{\beta+3(1-\beta)\tan\alpha}.
$$

***Case* II.** $\mu \ge \dfrac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}}{4\gamma^2\{\beta+3(1-\beta)\tan\alpha\}}$

Preceding as in case I, we get

$$
(3.14) \qquad \left|a_3-\mu a_2^2\right| \le \frac{1}{3\alpha+\beta-4\alpha\beta}+\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}
$$

$$
\left[4\gamma^2\mu-\frac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}+\gamma\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{\{\beta+3(1-\beta)\tan\alpha\}}\right]|c_1|^2.
$$

***Subcase* II (a).**$\mu \le \dfrac{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan\alpha\}+\gamma\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{4\gamma^2\{\beta+3(1-\beta)\tan\alpha\}}$

(3) takes the form

$$
(3.15) \qquad \left|a_3-\mu a_2^2\right| \le \frac{\gamma}{\beta+3(1-\beta)\tan\alpha}
$$

Combining subcase I (b) and subcase II (a), we obtain

(3.16) $$\left|a_3-\mu a_2^2\right| \leq \frac{\gamma}{\beta+3\left(1-\beta\right)\tan\alpha} \text{ if}$$

$$\frac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}\}-\gamma\{\left(1-\alpha\right)\beta+2\alpha(1-\beta)\}^2}{4\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}} \leq$$

$$\mu \leq \frac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}\}+\gamma\{\left(1-\alpha\right)\beta+2\alpha(1-\beta)\}^2}{4\gamma^2\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}}$$

***Subcase* II (b).** $\mu\ \geq \dfrac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}\}+\gamma\{\left(1-\alpha\right)\beta+2\alpha(1-\beta)\}^2}{4\gamma^2\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}}$

Preceding as in subcase I (a), we get

(3.17)

$$\left|a_3-\mu a_2^2\right| \leq \frac{1}{\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}^2}\left[4\gamma^2\mu-\frac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}\}}{\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}}\right].$$

Combining (3.9), (3.13) and (3.14), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$$f_1\left(z\right)=\left(1+az\right)^b,$$

where
$$a=\frac{2\gamma\left\{\beta+3(1-\beta)\tan a\right\}}{\{4(1-\beta)(\beta+2)\tan\alpha-\beta(\beta-3)\}\{\beta+2(1-\beta)\tan a\}^3-2\gamma}$$

and
$$b=\frac{\left\{4\left(1-\beta\right)\left(\beta+2\right)\tan\alpha\ -\beta(\beta-3)\right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}\}^3-2\gamma}{\left\{\beta+3\left(1-\beta\right)\tan\alpha\ \right\}\left\{\beta+2\left(1-\beta\right)\tan\alpha\ \right\}\}}.$$

Extremal function for (3.2) is defined by $f_2\left(z\right)=z(1+cz^2)^d$,
where $c=\dfrac{\tan\alpha}{\beta+3\left(1-\beta\right)\tan\alpha}$ and $d=\dfrac{\gamma}{\tan\alpha}$.

Corollary 3.1. *Putting $\gamma=1,\beta=0$ and applying limit as $\alpha\to\frac{\pi}{2}$ in the theorem, we get*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} 1-\mu,\ if\ \mu\leq\ 1; \\ \dfrac{1}{3}\ \ if\ \ 1\leq\mu\leq\dfrac{4}{3}; \\ \mu-1,\ if\ \mu\geq\ \dfrac{4}{3}. \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.

Corollary 3.2. *Putting $\alpha=0,\beta=1,\ \gamma=0$ in the theorem, we get*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} 3-4\mu\ if\ \mu\leq\ \dfrac{1}{2}; \\ 1\ \ if\ \ \dfrac{1}{2}\leq\mu\leq 1; \\ 4\mu-3,\ if\ \mu\geq\ 1. \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent starlike functions.

Conclusion : A subclass of analytic functions which take a broad view of some well-known subclasses of analytic and univalent functions was demarcated. The better estimates for the Fekete-Szeg functional for the defined class were obtained along with extremal functions. The study combines existing results and attains new outcomes in geometric function theory. Forthcoming researches can be done to acquire the geometric properties.

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