ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(2) (2023), 232-235**
(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

FIVE SERIES EQUATIONS INVOLVING GENERALIZED BATEMAN *k*-FUNCTIONS

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(Received: November 06, 2023; In format: November 13, 2023; Revised: November 23, 2023; Accepted: November 26, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53227

Abstract

In this paper, the solution of five series equations involving generalized Bateman k-functions is obtained by reducing them to Fredholm integral equation of the second kind. The solution presented in this paper is obtained by employing the techniques of Narain and Lal [12] involving generalized Bateman k-functions by reducing them to the solution of a Fredholm inregral equation of second kind with different bounday conditions. Thus we have seen that Bateman k-functions are having interesting properties to solve double, triple, quadruple and five series equations as special functions. These solutions are very useful in Mathematical and Quantum Physics, Aero and Fluid Dynamics and Thermodynamics.

2020 Mathematical Sciences Classification: 45XX, 45B05, 45F10,33C45.
Keywords and Phrases: Five Series equations, generalized Bateman k-functions, Fredholm integral equation.

1 Introduction

Chakraborty [3] discussed on generalization of Bateman functions. Erdélyi [7] has given tables of Integrals Transforms. Noble [13] presented formal solution of dual series equations involving Jacobi polynomials. Srivastava [15,16] and Srivastav [17] obtained solutions on dual series relations involving series of generalized Bateman k-functions and triple series equations involving series of Jacobi polynomials. Later on Lowndes [9, 10], Dwivedi and Trivedi [6] gave the solution of triple and quadruple series equations involving Jacobi polynomials. Chandel [2] solved a problem on Heat Conduction over the surface of a sphere by making an appeal to dual series equations involving Legendre polynomials by employing Mehler Dirichlet integrals [19, (2.6.20), (2.6.21)] and Fredholm integral equation. Srivastava [14] obtained solutions of a pair of dual series equations involving generalized Bateman k-functions. Narain, Singh and Lal [11] obtained solution of triple series equations involving generalized Bateman k-functions. Narain and Lal [12] gave a method for the solution of five Series equations by reducing them to Fredholm integral equations of the second kind. Dwivedi and Singh [7] gave the solution of some five series equations involving generalized Bateman k-functions by reducing them to simultaneous Fredholm integral equations. Recently Tripathi and Dixit [20] have obtained formal solution of four series equations involving generalized Bateman k-functions. Apelblat, Consiglia and Mainardi [1] in a recent survey expressed that Havlock(1925) and Bateman(1931) has introduced new functions as solutions of fluid dynamics problems. Recently, Shrivastava, Narain and Shrivastava [17] obtained solution of triple series equations involving generalized Laguerre polynomials. In this paper, we discuss the problem to obtain the solution of five series equations involving generalized Bateman K-functions, employing the technique due to Narain and Lal [12].

We shall obtain the solution of the following five series equations :

(1.1) $$\sum_{n=0}^{\infty} D_n \Gamma(l+1+n) k_{2n}^{2l}\left(\frac{x}{2}\right) = 0; 0 < x < a,$$

(1.2) $$\sum_{n=0}^{\infty} D_n \Gamma(l+m+n) k_{2n}^{2l}\left(\frac{x}{2}\right) = g(x); a < x < b,$$

$$(1.3)\qquad \sum_{n=0}^{\infty} D_n\Gamma(l+1+n)k_{2n}^{2l}\left(\frac{x}{2}\right) = 0; b<x<c,$$

$$(1.4)\qquad \sum_{n=0}^{\infty} D_n\Gamma(l+m+n)k_{2n}^{2l}\left(\frac{x}{2}\right) = h(x); c<x<d,$$

$$(1.5)\qquad \sum_{n=0}^{\infty} D_n\Gamma(l+1+n)k_{2n}^{2l}\left(\frac{x}{2}\right) = 0; d<x<\infty,$$

where $l>-m, 0<m<1,\ k_{2n}^{2l}(x)$ is the generalized Bateman k-function as given by Chakrabarty ([2], 6), $g(x)$ and $h(x)$ are known functions. Solution is obtained by reducing them to Fredholm integral equation of the second kind.

2 Some Useful Results

From the orthogonality relation of Srivastava ([15],p.589, eq. (2.6)]), we have

$$(2.1)\qquad \int_0^{\infty} x^{-2l-1}k_{2n}^{2l}(x)k_{2m}^{2l}(x)dx = 2^{2l}\frac{\Gamma(n-l)}{\Gamma(n+l+1)}\delta_{m,n}$$

where $\delta_{m,n}$ is Kronecker delta.

For $l>-\frac{1}{2}, 0<m<1$, it is easily shown by Erdélyi ([8], p.401 eq. (1); p.405 eq. (20)) that

$$(2.2)\qquad \int_0^{y}(y-x)^{m-1}e^{+x}k_{2n}^{2l}(x)dx = \frac{\Gamma(m)}{2^m}e^{y}k_{2n+m}^{2l+m}(y),$$

$$(2.3)\qquad \int_y^{\infty}(x-y)^{-m}x^{-l-1}e^{-x}k_{2n}^{2l}(x)dx = \frac{\Gamma(1-m)\Gamma(l+m+n)}{2^{\frac{(1-m)}{2}}\Gamma(l+1+n)}y^{(-l+\frac{m}{2}+\frac{1}{2})}e^{-y}\cdot k_{2n+m-1}^{2l+m-1}(y).$$

The following summation result can be easily established by using (2.1), (2.2) and (2.3):

$$(2.4)\qquad S(x,u) = \sum_{n=0}^{\infty}\frac{\Gamma(l+m+n)}{2^{2l}\Gamma(n-l)}k_{2n}^{2l}(x)k_{2n}^{2l}(u),$$

$$(2.5)\qquad \begin{aligned} S(x,u) &= \frac{e^{-x}x'\cdot 2^{\frac{3(1-m)}{2}}}{\{\Gamma(1-m)\}^2}\int_0^{r}E(y)(x-y)^{-m}(u-y)^{-m}dy,\\ &= \frac{e^{-x}\cdot x'\cdot 2^{\frac{3(1-m)}{2}}}{\{\Gamma(1-m)\}^2}S_r(x,u),\end{aligned}$$

where, $E(y)=e^{2y}\cdot y^{l+\frac{m}{2}+\frac{1}{2}}, r=\min(x,u)$.

3 Solution of Five Series Equations

To solve eqns. (1.1) to (1.5), we assume $\frac{x}{2}=X$ and

$$(3.1)\qquad \begin{aligned}\sum_{n=0}^{\infty} D_n\Gamma(l+1+n)k_{2n}^{2l}(x) &= p(x), a<x<b\\ &= q(x), c<x<d.\end{aligned}$$

Using the orthogonality relation (2.1), with an appeal to (1.1), (1.3) and (3.1), we obtain

$$(3.2)\qquad D_n = \frac{2^{-21}}{\Gamma(n-l)}\left\{\int_a^b p(u)+\int_c^d q(u)\right\}u^{-2l-1}\cdot k_{2n}^{2l}(u)du.$$

After substituting the value of D_n in eqns. (1.2) and (1.4), and then interchanging the order of summation and integration, we find the equation

$$(3.3)\qquad \begin{aligned}\int_a^b u^{-2l-1}p(u)S_u(X,u)du + \int_x^b u^{-2l-1}p(u)S_x(X,u)du + \int_c^d u^{-2l-1}q(u)S_x(X,u)du\\ = \frac{\{\Gamma(1-m)\}^2}{2^{\frac{3(1-m)}{2}}}e^{x}\cdot X^{-l}g(X)\ (a<x<b),\end{aligned}$$

$$(3.4)\quad \int_a^b u^{-2l-1}p(u)S_u(X,u)du+\int_c^x u^{-2l-1}q(u)S_x(X,u)du+\int_x^d u^{-2l-1}q(u)S_x(x,u)du$$
$$=\frac{\{\Gamma(1-m)\}^2}{2^{\frac{3(1-m)}{2}}}e^x, X^{-l}h(x),\ (c<x<d).$$

Inverting the order of integration in the above equations, we derive

$$(3.5)\qquad \int_a^x \frac{E(y)}{(x-y)^m}\left\{p_1(y)+\int_c^d \frac{u^{-2l-1}q(u)}{(u-y)^m}dudy=\frac{\{\Gamma(1-m)\}^2}{2^{\frac{3(1-m)}{2}}}e^x\cdot x^{l-1}g(x), a<x<b\right. ,$$

(3.6)
$$\int_c^x \frac{E(y)}{(x-y)^m}q_1(y)dy=\frac{\{\Gamma(1-m)\}^2}{2^{\frac{3(1-m)}{2}}}\cdot e^x\cdot x^{-1}h(x)-\int_a^b\frac{E(y)}{(x-y)^m}p_1(y)dy-\int_a^c\frac{E(y)dy}{(x-y)^m}\int_c^d\frac{u^{2l-1}q(u)du}{(u-y)^m},$$
$(c<x<d)$, where,

$$(3.7)\qquad \left\{\begin{array}{l}\text{(i) } p_1(y)=\int_y^b \frac{u^{-2l-1}p(u)}{(u-y)^m}du\\ \text{(ii) } q_1(y)=\int_y^d \frac{u^{-2l-1}q(u)}{(u-y)^m}du\end{array}\right. .$$

For $0<m<1$, we can solve Abel-type integral eqns. (3.5), (3.6) and (3.7) to obtain the equations

$$(3.8)\qquad E(y)p_1(y)=G(y)-E(y)\int_c^d\frac{u^{-2l-1}q(u)}{(u-y)^m}du,$$

$$(3.9)\quad E(y)q_1(y)=H(y)-\frac{\sin m\pi}{\pi(y-c)^{1-m}}\int_a^b\frac{(c-t)^{1-m}}{(y-t)}E(t)p_1(t)dt-\frac{\sin m\pi}{\pi(y-c)^{1-m}}\cdot$$
$$\times\int_a^c\frac{(c-t)^{1-m}}{(y-t)}E(t)dt\int_c^d\frac{u^{-2l-1}q(u)}{(u-t)^m}du,$$

$$(3.10)\qquad u^{-2l-1}p(u)=-\frac{\sin m\pi}{\pi}\frac{d}{du}\int_u^b\frac{p_1(y)du}{(y-u)^{1-m}}, a<u<b,$$

$$(3.11)\qquad u^{-2l-1}q(u)=-\frac{\sin m\pi}{\pi}\frac{d}{du}\int_u^d\frac{p_1(y)du}{(y-u)^{1-m}}, a<u<d,$$

where $G(y)$ and $H(y)$ are known functions given by

$$(3.12)\qquad \left\{\begin{array}{l} G(y)=\frac{\Gamma(1-m)}{2^{\frac{3(1-m)}{2}}\Gamma(m)}\frac{d}{dy}\int_a^y\frac{e^x\cdot x^{-1}h(x)dx}{(y-x)^{1-m}}, a<y<b\\ H(y)=\frac{\Gamma(1-m)}{2^{\frac{3(1-m)}{2}}\Gamma(m)}\frac{d}{dy}\int_c^y\frac{e^x\cdot x^{-1}h(x)dx}{(y-x)^{1-m}}, c<y<d\end{array}\right. .$$

From eqns. (3.8) and (3.10), we see that the functions $p(u)$ and $q(u)$ are related by the equation

$$(3.13)\qquad u^{-2l-1}p(u)=-\frac{\sin m\pi}{\pi}\frac{d}{du}\int_u^b\frac{G(y)dy}{E(y)(y-u)^{1-m}}+\frac{\sin m\pi}{\pi(b-u)^{1-m}}\int_c^d\frac{t^{-2l-1}(t-b)^{1-m}q(t)}{(u-t)}dt$$

where $a<u<b$.

Now

$$(3.14)\qquad \int_c^d\frac{u^{-2l-1}q(u)}{(u-y)^m}du=\frac{\sin m\pi}{\pi(c-y)^{m-1}}\int_c^d\frac{(t-c)^{m-1}q_1(t)dt}{(t-y)}.$$

Using this result together with eqn. (3.8), we see that eqn. (3.9) can be written in the form

$$(3.15)\qquad E(y)q_1(y)+\int_c^d q_1(x)T(x,y)dX=H(y)-\frac{\mathrm{sinm}\,\pi}{\pi(y-c)^{1-m}}\cdot\int_a^b(c-t)^{1-m}G(t)dt$$

where

$$(3.16)\qquad T(x,y)=\frac{\sin^2 m\pi}{\pi^2[(x-c)(y-c)]^{1-m}}\cdot\int_b^c\frac{E(t)(c-t)^{2(1-m)}}{(x-t)(y-t)}dt, c<y<d$$

is a symmetric kernel.

Eqn. (3.15) is a Fredholm integral equation of the second kind which determines $q_1(y).q(u)$ can be found from eqn. (3.11) and $p(u)$ from eqn. (3.13). Finally the coefficients D_n for $l>-\frac{1}{2}, 0<m<1$ are given by the eqn. (3.2).

If we replace X by $x/2$, we get the solution of equations (1.1) to (1.5) by eqn. (3.2).

In Particular if $a=0, b=a, c=b$ and $d\to\infty$ in equations (1.1) to (1.5), we get the solution of triple series equations considered by Dwivedi [4].

4 Conclusion

The generalized Bateman k-functions have been applied to solve the problems of different integral and series equations by many scholars like Srivastava [14], Srivastava[15], Dwivedi [5], Dwivedi and Trivedi [6], Narain, Singh and Lal [11], Narain and Lal [12] Dwivedi and Singh [7], Tripathi and Dixit [19] to solve pair of dual series, triple series, quadruple series, and some five series equations. The solution presented in this paper is obtained by employing the techniques of Narain and Lal involving generalized Bateman k-functions by reducing them to the solution of a Fredholm inregral equation of second kind with different bounday conditions. Thus we have seen that Bateman k-functions are having interesting properties to solve double, triple, quadruple and five series equations as special functions. These solutions are very useful in Mathematical and Quantum Physics, aero and Fluid Dynamics and Thermodynamics.

Acknowledgement. The authors express their sincere gratitude to the editors and referees for carefully reading the manuscript and for their valuable comments and suggestions which greatly improved this paper.

Conflict of interest: We declare that authors have no conflict of interest.

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