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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

ON COMPLEX VALUED FUZZY b - METRIC SPACE

Gopal Yadav[1], Rajesh Kumar Sharma[2] and Gend Lal Prajapati[3]
[1]Department of Engineering Science & Humanities, Indore Institute of Science & Technology, Rau-Pithampur Road, Indore, Madhya Pradesh, India-453331
[2]Department of Mathematics, Jawaharlal Nehru Smriti PG College, Shujalpur, Madhya Pradesh, India-465333
[3]Department of Computer Engineering, Institute of Engineering & Technology, Devi Ahilya University, Khandwa Road, Indore, Madhya Pradesh, India-452017
Email: yadavgopal0902@gmail.com, raj_rma@yahoo.co.in, glprajapati@ietdavv.edu.in
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Abstract

In this paper, the notion of complex-valued fuzzy b-metric space is introduced. In this newly developed structure, we have established a sufficient condition for a sequence to be Cauchy. Moreover, under suitable conditions of contractive type, the existence and uniqueness of fixed points of self-maps are established in this structure. To demonstrate the validity of the hypothesis and the degree of generality of our results, some examples are also furnished.
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1 Introduction

In 1965, Zadeh [17] introduced the concept of fuzzy sets. Due to the widespread use of this concept in various fields, numerious authors have expansively developed the theory of fuzzy sets and its applications in variety of domain. Using the concept of fuzziness, Kramosil and Mechalek [9] introduced the notion of fuzzy metric space by generalizing the concept of probabilistic metric space. Grabiec [7] extended the well-known fixed point theorem of Banach[4] in complete fuzzy metric space in the sense of Kramosil and Michalek. In a paper, George and Vermani [6] modified the concept of fuzzy metric space and defined Hausdroff topology on fuzzy metric space. By observing weaker conditions of the triangle inequality, Bakhtin [2] and Czerwik [5] introduced the structure of b-metric space and generalized the Banach contraction principle. In this sequence, a relation between b-metric and fuzzy metric spaces has been studied by Hassanzadeh et al.[8]. On the other hand Sedghi et al.[15] introduced the notion of b-fuzzy metric spaces by weakening the triangle inequality. The concept of fuzzy b-metric space was first developed by Nadaban [11]. Recently, Mehmood et al. introduced the concept of extended fuzzy b-metric space [10].

In a paper, Buckley [3] introduced the fuzzy complex numbers and fuzzy complex analysis. After that many authors initiated work in fuzzy complex number by acknowledging the Buckleys work. In this series Ramot el al.[12] established the innovative concept of complex fuzzy sets. In this context, the range of membership function of complex fuzzy set is not limited to $[0, 1]$ as the membership function of traditional fuzzy set but, it extended to the unit circle in the complex plane. Then, here we see that the range of membership function of crisp set $\{0, 1\}$ is extended to the range of membership function of fuzzy set $[0, 1]$ and the range of membership function of fuzzy set $[0, 1]$ is extended the range of membership function of complex fuzzy set to the unit circle in complex plane.

In 2011, Azam et al.[1] defined a partial order $\precsim$ on set of complex numbers $\mathbb{C}$ for comparing the two complex numbers and introduced the concept of complex valued metric spaces. Also they obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions.

Recognizing the notion of complex valued fuzzy set of Ramot et al.[12], Sigh et al.[14] developed the structure of complex valued fuzzy metric spaces. They also established the complex valued fuzzy version of Banach contraction principle.

In a paper, Rao et al. [13] introduced the complex valued b-metric space and gave a common fixed point theorem for four maps in this structure. In this paper, we establish the structure of complex-valued fuzzy b-metric space along with its properties. Moreover, we establish a theorem of existence and uniqueness for a fixed point of self-map defined on this newly developed structure.

2 Preliminaries

Definition 2.1 ([14])**.** *A complex Fuzzy set S, defined on a universe of discourse U, is characterized by a membership function $\mu_s(x)$ that assigns every element $x \in U$, a complex valued grade of membership in S. The values $\mu_s(x)$ lie within the unit circle in the complex plane, and thus of the form*

$$\mu_s(x) = r_s(x).e^{i\omega_s(x)} \qquad (i = \sqrt{-1})$$

where $r_s(x)$ and $\omega_s(x)$ both are real valued, with $r_s(x) \in [0,1]$. The complex fuzzy set S, may be represented as the set of ordered pairs , given by

$$S = \{(x, \mu_s(x) \mid x \in U\}.$$

Clearly, each complex grade of membership is defined by an amplitude term $r_s(x)$ and a phase term $\omega_s(x)$. Notice that it is possible to represent any ordinary fuzzy set in terms of a complex fuzzy set. If any ordinary fuzzy set S is characterized by the real valued membership function $\lambda_s(x)$ where $x \in U$, then S can be transformed into complex fuzzy set by setting the amplitude terms $r_s(x)$ equal to $\lambda_s(x)$ and the phase term $\omega_s(x)$ equal to zero for all $x \in U$. Thus one can say that without a phase term, the complex fuzzy set effectively reduces to conventional fuzzy set.

Definition 2.2 ([1])**.** *Let $\mathbb{C}$ be the set of complex numbers and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as: $\alpha_1 \precsim \alpha_2 \Leftrightarrow Re(\alpha_1) \le Re(\alpha_2)$, $Im(\alpha_1) \le Im(\alpha_2)$. It follows that $\alpha_1 \precsim \alpha_2$ if one of the following conditions hold:*

(i) $Re(\alpha_1) = Re(\alpha_2)$ *and* $Im(\alpha_1) = Im(\alpha_2)$,
(ii) $Re(\alpha_1) < Re(\alpha_2)$ *and* $Im(\alpha_1) = Im(\alpha_2)$,
(iii) $Re(\alpha_1) = Re(\alpha_2)$ *and* $Im(\alpha_1) < Im(\alpha_2)$,
(iv) $Re(\alpha_1) < Re(\alpha_2)$ *and* $Im(\alpha_1) < Im(\alpha_2)$.

We write $\alpha_1 \precnsim \alpha_2$ if $\alpha_1 \neq \alpha_2$ and one of (ii), (iii) and (iv) is satisfied and we write $\alpha_1 \prec \alpha_2$ if only (iv) is satisfied.

Here we note the following condition trivially hold:

(i) If $0 \precsim \alpha_1 \precsim \alpha_2$ then $|\alpha_1| \le |\alpha_2|$,
(ii) If $0 \precsim \alpha_1 \precnsim \alpha_2$ then $|\alpha_1| < |\alpha_2|$,
(iii) If $\alpha_1 \prec \alpha_2$ and $\alpha_2 \prec \alpha_3$ then $\alpha_1 \prec \alpha_3$,
(iv) If $a, b \in \mathbb{R}$ and $a \le b$ then $a\alpha \precsim b\alpha$ for all $\alpha \in \mathbb{C}$,
(v) If $a, b \in \mathbb{R}$ and $0 \le a \le b$ then $\alpha_1 \precsim \alpha_2$ implies $a\alpha_1 \precsim b\alpha_2$.

Utilizing the concept due to Azam et al.[1] and the definition of max function by Verma et al.[16], Singh at al.[14] gave the similar definition of min function as follows

Definition 2.3 ([14])**.** *Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ and the partial order relation $\precsim$ is defined on $\mathbb{C}$. Then, min functions for complex numbers with partial order relations is defined as:*

(1) $\min\{\alpha_1, \alpha_2\} = \alpha_1 \Leftrightarrow \alpha_1 \precsim \alpha_2$,
(2) $\min\{\alpha_1, \alpha_2\} \precsim \alpha_3 \Rightarrow \alpha_1 \precsim \alpha_3$ *or* $\alpha_2 \precsim \alpha_3$.

Note 2.1. Throughout this paper the symbol $\le$ or $\ge$ used in sense of real numbers while symbol $\precsim$ or $\succsim$ used in sense of complex numbers.

Definition 2.4 ([14])**.** *A binary operation $*$: $r_s e^{i\theta} \times r_s e^{i\theta} \to r_s e^{i\theta}$, where $r_s \in [0,1]$ and a fix $\theta \in [0, \frac{\pi}{2}]$, is called complex valued continuous t-norm if it satisfies the following conditions:*

(1) $*$ *is associative and commutative,*
(2) $*$ *is continuous,*
(3) $a * e^{i\theta} = a, \forall a \in r_s e^{i\theta}$ *where* $r_s \in [0,1]$,
(4) $a * b \precsim c * d$ *whenever* $a \precsim c$ *and* $b \precsim d$, *for all* $a, b, c, d \in r_s e^{i\theta}, r_s \in [0,1]$.

Definition 2.5. *Let $*$ be a complex valued continuous t-norm, let $*_n : r_s e^{i\theta} \to r_s e^{i\theta}$ where $n \in \mathbb{N}, r_s \in [0,1]$ be defined in the following way*

$$*_1(x) = x * x, \qquad *_{n+1}(x) = (*_n(x) * x) \qquad n \in \mathbb{N}, x \in r_s e^{i\theta}.$$

Each complex valued t-norm $*$ can be extended by associativity in a unique way to an n-aray operation taking for $(x_1, x_2, \dots x_n) \in [r_s]^n e^{i\theta}$ where $r_s \in [0,1]$ the values

$$*_{i=1}^{1} x_i = x_1, *_{i=1}^{n} x_i = ((*_{i=1}^{n-1} x_i) * x_n) = (x_1 * x_2 * \cdots * x_n).$$

A complex valued t-norm $*$ can be extended to a countable infinite operation taking for any sequence $(x_n)_{n\in\mathbb{N}}$ from $r_s \in [0,1]$ the value

$$*_{i=1}^{\infty} x_i = \lim_{n\to\infty} *_{i=1}^{n} x_i.$$

The sequence $(*_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is nonincreasing and bounded from below, and hence the limit $*_{i=1}^{\infty} x_i$ exists.

Definition 2.6 ([14])**.** *The triplet $(X, M, *)$ is said to be complex valued fuzzy metric space if X is a non-empty set, $*$ is a complex valued t-norm and $M : X \times X \times (0,\infty) \to r_s e^{i\theta}$ is a complex valued fuzzy set, where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$, satisfying the following conditions:*

$(CF1)$ $\quad M(x,y,t) \succ 0,$

$(CF2)$ $\quad M(x,y,t) = e^{i\theta}$ *for all* $t > 0 \Leftrightarrow x = y,$

$(CF3)$ $\quad M(x,y,t) = M(y,x,t),$

$(CF4)$ $\quad M(x,z,t+s) \succsim M(x,y,t) * M(y,z,s),$

$(CF5)$ $\quad M(x,y,.) : (0,\infty) \to r_s e^{i\theta}$ *is continuous,*

for all $x, y, z \in X, s, t > 0, r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$. Also $(M, *)$ is called a complex valued fuzzy metric.

Remark 2.1. *If we take $\theta = 0$ then complex valued fuzzy metric simply goes to real valued fuzzy metric.*

Now, in this paper we introduce the complex valued fuzzy b-metric space as.

Definition 2.7. *Let X be a non-empty set, $b \geq 1$ be a given real number, $*$ is a complex valued t-norm and $M : X \times X \times (0,\infty) \to r_s e^{i\theta}$ is a complex valued fuzzy set, where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$, satisfying the following conditions:*

(CF_bM1) $\quad M(x,y,t) \succ 0,$

(CF_bM2) $\quad M(x,y,t) = e^{i\theta}$ *for all* $t > 0 \Leftrightarrow x = y,$

(CF_bM3) $\quad M(x,y,t) = M(y,x,t),$

(CF_bM4) $\quad M(x,z,t+s) \succsim M(x,y,\frac{t}{b}) * M(y,z,\frac{s}{b}),$

(CF_bM5) $\quad M(x,y,.) : (0,\infty) \to r_s e^{i\theta}$ *is continuous and* $\lim_{t\to\infty} M(x,y,t) = r_s e^{i\theta},$

for all $x, y, z \in X, s, t > 0, r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$. A quadruple $(X, M, *, b)$ is said to be complex valued fuzzy b-metric space.

Remark 2.2. *The class of complex valued fuzzy b-metric spaces is effectively larger than that of complex valued fuzzy metric spaces[4], since a complex valued fuzzy b-metric is a complex valued fuzzy metric when $b = 1$.*

Example 2.1. Let $M(x,y,t) = e^{i\theta} e^{\frac{-d(x,y)}{t}}$, where d is a b-metric on X and $a * c = a.c$ for all $a, c \in r_s e^{i\theta}$. Then it is easy to show that $(X, M, *)$ is a complex valued fuzzy b- metric space. Obviously conditions from $(CF_bM1 - CF_bM3)$ of Definition 2.7 are satisfied. For each $x, y, z \in X$ we obtain

$$\begin{aligned} M(x,y,t+s) &= e^{i\theta} e^{\frac{-d(x,y)}{t+s}} \\ &\succsim e^{i\theta} e^{\frac{-b[d(x,z)+d(z,y)]}{t+s}} \\ &\succsim e^{i\theta} e^{-\frac{d(x,z)}{\frac{t}{b}}} . e^{-\frac{d(z,y)}{\frac{s}{b}}} \\ &= M\left(x,z,\frac{t}{b}\right) * M\left(z,y,\frac{s}{b}\right). \end{aligned}$$

So condition (CF_bM4) of Definition 2.7 holds and $(X, M, *)$ is a complex valued fuzzy b-metric space.

Example 2.2. Let $X = \mathbb{R}$. We define $a * c = a.c, \forall a, c \in r_s e^{i\theta}$, where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$. Furthermore for all $x, y \in X$ and $t \in (0,\infty)$, we define

$$M(x,y,t) = e^{i\theta} e^{\frac{-|x-y|^p}{t}},$$

where $p > 1$ is a real number. Then, $(X, M, *)$ is a complex valued fuzzy b-metric space with $b = 2^{p-1}$.

Example 2.3. Let $M(x,y,t) = e^{i\theta}\frac{t}{t+d(x,y)}$ where d is a b-metric on X and $a * c = a.c, \forall a, c \in r_s e^{i\theta}$. Then $M(x,y,*)$ is a complex valued fuzzy b-metric space.

Definition 2.8. *Let $b \geq 1$ be a given real number. A function $f \colon \mathbb{R} \to \mathbb{C}$ will be called b- non-decreasing if $t < s$ we will have $f(t) \precsim f(bs)$.*

Lemma 2.1. *The mapping $M(x,y,.) \colon [0,\infty) \to r_s e^{i\theta}$ is b-non decreasing for all $x, y \in X$.*

Proof. For some $0 < t < s$, we have

$$M(x,y,bs) \succsim M(x,y,t) * M(y,y,s-t) = M(x,y,t) * e^{i\theta} = M(x,y,t).$$

Therefore for all $x, y \in X, M(x,y,.)$ is b-non-decreasing.

Definition 2.9. *Let $(X, M, *)$ be a complex valued fuzzy b-metric space. We define an open ball $B(x,r,t)$ with centre $x \in X$ and radius $r \in \mathbb{C}, 0 \prec r \prec e^{i\theta}, t > 0$ as*

$$B(x,r,t) = \{y \in X \colon M(x,y,t) \succ e^{i\theta} - r\}, \text{ where } \theta \in \left[0, \frac{\pi}{2}\right].$$

Definition 2.10. *Let $(X, M, *)$ be a complex valued fuzzy b-metric space then*

(a) A sequence $\{x_n\}$ in X is said to be convergent to x in X, if and only if $\lim_{n\to\infty} M(x_n, x, t) = e^{i\theta}$ for any $n > 0$ and for all $t > 0$.

(b) A sequence $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n\to\infty} M(x_n, x_{n+m}, t) = e^{i\theta}$ for any $m > 0$ and for all $t > 0$.

(c) The complex valued fuzzy b-metric space $(X, M, *)$ is called complete if every Cauchy sequence is convergent.

3 Main Results

Lemma 3.1. *Let $(X, M, *)$ be a complex valued fuzzy b-metric space such that $\lim_{t\to\infty} M(x_n, x_{n+m}, t) = e^{i\theta}$ for all $x, y \in X$ if*

$$M(x,y,t) \succsim M\left(x, y, \frac{t}{\lambda}\right)$$

for all $x, y \in X, 0 < \lambda < 1, t \in (0,\infty)$ then $x = y$.

Proof. Suppose $\lambda \in (0,1)$ such that

$$M(x,y,\lambda t) \succsim M(x,y,t) \forall x, y \in X, t \in (0,\infty)$$

so that

$$M(x,y,t) \succsim M\left(x, y, \frac{t}{\lambda}\right).$$

On repeated application, we have

$$M(x,y,t) \succsim M\left(x, y, \frac{t}{\lambda^n}\right) \text{ for some positive integer } n.$$

On making $n \to \infty$, reduces to $M(x,y,t) \succsim e^{i\theta}$. This implies $M(x,y,t) = e^{i\theta}$. Thus by (CF_bM2), we have $x = y$.

Lemma 3.2. *Let $\{x_n\}$ be a sequence in a complex valued fuzzy b-metric space $(X, M, *)$. suppose that there exists $\lambda \in (0, \frac{1}{b})$ such that*

$$M(x_n, x_{n+1}, t) \succsim M\left(x_{n-1}, x_n, \frac{t}{\lambda}\right), n \in \mathbb{N}, t > 0 \tag{3.1}$$

and there exist $x_0, x_1 \in X$ and $v \in (0,1)$ such that

$$\lim_{n\to\infty} *_{i=n}^{\infty} M\left(x_0, x_1, \frac{t}{v^i}\right) = e^{i\theta} t > 0. \tag{3.2}$$

Then $\{x_n\}$ is a Cauchys sequence.

Proof. Let $\sigma \in (\lambda b, 1)$. Then the sum $\sum_{i=1}^{\infty} \sigma^i$ is convergent, and there exists $n_0 \in \mathbb{N}$ such that $\sum_{i=1}^{\infty} \sigma^i < 1$ for every $n > n_0$. Let $n > m > n_0$. Since M is b non-decreasing, by (CF_bM4) every $t > 0$

$$\begin{aligned} M(x_n, x_{n+m}, t) &\succsim M(x_n, x_{n+m}, \frac{t\sum_{i=n}^{n+m-1} \sigma^i}{b}) \\ &\succsim \left(M\Big(x_n, x_{n+1}, \frac{t\sigma^n}{b^2}\Big) * M\Big(x_{n+1}, x_{n+m}, \frac{t\sum_{i=n+1}^{n+m-1} \sigma^i}{b^2}\Big)\right) \\ &\succsim M\Big(x_n, x_{n+1}, \frac{t\sigma^n}{b^2}\Big) \\ &\quad * \left(M\Big(x_{n+1}, x_{n+2}, \frac{t\sigma^{n+1}}{b^3}\Big) * \cdots * M\Big(x_{n+m-1}, x_{n+m}, \frac{t\sigma^{n+m-1}}{b^m}\Big)\right). \end{aligned}$$

By (3.1) it follows that

$$M(x_n, x_{n+1}, t) \succsim M\Big(x_0, x_1, \frac{t}{\lambda^n}\Big), n \in \mathbb{N}, t > 0$$

and since $n > m$ and $b > 1$, we have

$$\begin{aligned} M(x_n, x_{n+m}, t) &\succsim M\Big(x_0, x_1, \frac{t\sigma^n}{b^2\lambda^n}\Big) \\ &\quad * \left(M\Big(x_0, x_1, \frac{t\sigma^{n+1}}{b^3\lambda^{n+1}}\Big) * \cdots * M\Big(x_0, x_1, \frac{t\sigma^{n+m-1}}{b^{m+1}\lambda^{n+m-1}}\Big)\right) \\ &\succsim *_{i=m}^{n+m-1} M\Big(x_0, x_1, \frac{t\sigma^i}{b^{i-n+2}\lambda^i}\Big) \\ &\succsim *_{i=m}^{n+m-1} M\Big(x_0, x_1, \frac{t\sigma^i}{b^i\lambda^i}\Big) \\ &\succsim *_{i=m}^{n+m-1} M\Big(x_0, x_1, \frac{t}{v^i}\Big) \text{ where } v = \frac{b\lambda}{\sigma} \end{aligned}$$

Since, $v \in (0, 1)$. By(3.2) it follows that $\{x_n\}$ is a Cauchys sequence.

Theorem 3.1. *Let $(X, M, *)$ be a complete complex valued fuzzy b-metric space and let $f\colon X \to X$. Suppose there exist $\lambda \in (0, \frac{1}{b})$ such that*

$$M(fx, fy, t) \succsim M\Big(x, y, \frac{t}{\lambda}\Big), \qquad x, y \in X, \quad t > 0 \tag{3.3}$$

and there exists $x_0 \in X$ and $v \in (0, 1)$ such that

$$\lim_{n\to\infty} *_{i=n}^{\infty} M\Big(x_0, x_1, \frac{t}{v^i}\Big) = e^{i\theta} \qquad t > 0. \tag{3.4}$$

Then f has a unique fixed point in X.

Proof. Let $x_0 \in X$ and $x_{n+1} = fx_n, n \in \mathbb{N}$. If we take $x = x_n$ and $y = x_{n-1}$ in (3.3) then we have

$$M(x_n, x_{n+1}, t) \succsim M\Big(x_{n-1}, x_n, \frac{t}{\lambda}\Big) \qquad n \in \mathbb{N}, \quad t > 0$$

By Lemma 3.2 it follows that $\{x_n\}$ is a Cauchys sequence. Since $(X, M, *)$ is complete there exist $x \in X$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} M(x_n, x, t) = e^{i\theta} \qquad t > 0. \tag{3.5}$$

Using condition (3.3) and (CF_bM4) we show that x is a fixed point of f.

$$\begin{aligned} M(fx, x, t) &\succsim \left(M\Big(fx, x_n, \frac{t}{2b}\Big) * M\Big(x_n, x, \frac{t}{2b}\Big)\right) \\ &\succsim \left(M\Big(fx_n, x_n, \frac{t}{2b}\Big) * M\Big(x_n, x, \frac{t}{2b}\Big)\right) \\ &\succsim \left(M\Big(x_{n+1}, x_n, \frac{t}{2b}\Big) * M\Big(x_n, x, \frac{t}{2b}\Big)\right) \\ &\succsim \left(M\Big(x_{n-1}, x_n, \frac{t}{2b\lambda}\Big) * M\Big(x_n, x, \frac{t}{2b}\Big)\right), \end{aligned}$$

for all $t > 0$, by (3.5) as $n \to \infty$, we get
$$M(fx, x, t) \succsim (e^{i\theta} * e^{i\theta}) = e^{i\theta}.$$
Suppose that x and y are fixed point for f. By (3.3)
$$M(x, y, t) = M(fx, fy, t) \succsim M\Big(x, y, \frac{t}{\lambda}\Big), \qquad t > 0.$$
Lemma 3.1 implies $x = y$.

Example 3.1. Let $X = [0, 1]$ and $M(x, y, t) = e^{i\theta} e^{\frac{-|x-y|^2}{t}}$ be a complex valued fuzzy b-metric space with $b = 2$ and $\theta \in [0, \frac{\pi}{2}]$. Let $f(x) = kx$, where $k = \frac{1}{\sqrt{2}}$ and $x \in X$. Then

$$M(fx, fy, t) = e^{i\theta} e^{\frac{-k^2|x-y|^2}{t}} \succsim e^{i\theta} e^{\frac{-\lambda|x-y|^2}{t}} = M\Big(x, y, \frac{t}{\lambda}\Big), x, y \in X, t > 0. \tag{3.6}$$

For $k^2 < \lambda < \frac{1}{b}$. So, the condition of the Theorem 3.1 fulfilled and f has a unique fixed point in X.

Theorem 3.2. *Let $(X, M, *)$ be a complex valued complete fuzzy b-metric space and let $f\colon X \to X$. Suppose that there exist $\lambda \in (0, \frac{1}{b})$ such that*

$$M(fx, fy, t) \succsim \min\Big\{M\Big(x, y, \frac{t}{\lambda}\Big), M\Big(fx, x, \frac{t}{\lambda}\Big), M\Big(fy, y, \frac{t}{\lambda}\Big)\Big\} \tag{3.7}$$

for all $x, y \in X, t > 0$ and there exist $x_0 \in X$ and $v \in (0, 1)$ such that

$$\lim_{n\to\infty} *_{i=n}^{\infty} M\Big(x_0, fx_0, \frac{t}{v^i}\Big) = e^{i\theta} \tag{3.8}$$

for all $t > 0$. Then f has a fixed point in X.

Proof. Let $x_0 \in X$ and $x_{n+1} = fx_n, n \in \mathbb{N}$. By 3.6 with $x = x_n$ and $y = x_{n-1}$ for every $n \in \mathbb{N}$ and every $t > 0$, we have

$$M(fx_n, fx_{n-1}, t) \succsim \min\Big\{M\Big(x_n, x_{n-1}, \frac{t}{\lambda}\Big), M\Big(fx_n, x_n, \frac{t}{\lambda}\Big), M\Big(fx_{n-1}, x_{n-1}, \frac{t}{\lambda}\Big)\Big\}$$
$$M(x_{n+1}, x_n, t) \succsim \min\Big\{M\Big(x_n, x_{n-1}, \frac{t}{\lambda}\Big), M\Big(x_{n+1}, x_n, \frac{t}{\lambda}\Big), M\Big(x_n, x_{n-1}, \frac{t}{\lambda}\Big)\Big\}$$
$$M(x_{n+1}, x_n, t) \succsim \min\Big\{M\Big(x_n, x_{n-1}, \frac{t}{\lambda}\Big), M\Big(x_{n+1}, x_n, \frac{t}{\lambda}\Big)\Big\}.$$

If $M(x_{n+1}, x_n, t) \succsim M\Big(x_{n+1}, x_n, \frac{t}{\lambda}\Big) n \in \mathbb{N}, t > 0$. By Lemma3.1 it follows that $x_n = x_{n+1}, n \in \mathbb{N}$. So,

$$M(x_{n+1}, x_n, t) \succsim M\Big(x_n, x_{n-1}, \frac{t}{\lambda}\Big), \qquad n \in \mathbb{N}, t > 0.$$

By Lemma 3.2 we have that $\{x_n\}$ is a Cauchys sequence. Hence there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} M(x, x_n, t) = e^{i\theta}, t > 0. \tag{3.9}$$

Now we prove that x is a fixed point of f. Let $\sigma_1 \in (\lambda b, 1)$ and $\sigma_2 = 1 - \sigma_1$. By (3.6)

$$\begin{aligned} M(fx, x, t) &\succsim \Big(M\Big(fx, fx_n, \frac{t\sigma_1}{b}\Big) * M\Big(fx_n, x, \frac{t\sigma_2}{b}\Big)\Big) \\ &\succsim \Big(\min\Big\{M\Big(x, x_n, \frac{t\sigma_1}{b\lambda}\Big), M\Big(x, fx, \frac{t\sigma_1}{b\lambda}\Big), M\Big(x_n, x_{n+1}, \frac{t\sigma_1}{b\lambda}\Big)\Big\} * M\Big(fx_n, x, \frac{t\sigma_2}{b}\Big)\Big) \end{aligned}$$

Taking $n \to \infty$ and using(3.8) we have

$$\begin{aligned} M(fx, x, t) &\succsim \min\Big\{e^{i\theta}, M\Big(x, fx, \frac{t\sigma_1}{b\lambda}\Big), e^{i\theta}\Big\} * e^{i\theta} \\ &\succsim \Big\{M\Big(x, fx, \frac{t\sigma_1}{b\lambda}\Big) * e^{i\theta}\Big\} = M\Big(x, fx, \frac{t}{v}\Big), t > 0, \end{aligned}$$

where $v = \frac{b\lambda}{\sigma_1} \in (0, 1)$. Therefore,

$$M(fx, x, t) \succsim M\Big(x, fx, \frac{t}{v}\Big), t > 0,$$

By Lemma 3.1 it follows that $fx = x$. Suppose x and y are the fixed point for f, that is $fx = x$ and $fy = y$. By condition (3.6), we get

$$\begin{aligned} M(fx, fy, t) &\succsim \min\Big\{M\Big(x, y, \frac{t}{\lambda}\Big), M\Big(x, fx, \frac{t}{\lambda}\Big), M\Big(y, fy, \frac{t}{\lambda}\Big)\Big\} \\ &= \min\Big\{M\Big(x, y, \frac{t}{\lambda}\Big), e^{i\theta}, e^{i\theta}\Big\} = M\Big(x, y, \frac{t}{\lambda}\Big). \end{aligned}$$

For $t > 0$, by Lemma 3.1 it follows that $fx = fy$, that is $x = y$.

Example 3.2. Let $X = (0, 2)$ with a b-metric d defined by

$$d(x, y) = |x - y|^2, \forall x, y \in X.$$

For all $x, y \in X$ and $t \in (0, \infty)$, we define

$$M(x, y, t) = e^{i\theta} e^{\frac{-(d(x,y))}{t}}.$$

Clearly $M(x, y, *)$ is complex valued complete fuzzy b-metric space with t-norm $*$ defined as $a * b = a.b$ where $a, b \in r_s e^{i\theta}$ for a fixed $\theta \in [0, \frac{\pi}{2}]$ and $r_s \in [0, 1]$. Here $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$ for all $x, y \in X$. Then $M(x, y, t)$ is a complex valued fuzzy b-metric space with b=2. Define the map $f : X \to X$

$$f(x) = \begin{cases} 2 - x, & x \in (0, 1), \\ 1, & x \in [1, 2). \end{cases}$$

Case 3.1. If $x, y \in [1, 2)$ then, $M(fx, fy, t) = e^{i\theta}, t > 0$ and condition 3.6 are trivially satisfied.

Case 3.2. If $x \in [1, 2)$ and $y \in (0, 1)$, then for $\lambda(\frac{1}{4}, \frac{1}{2})$, we have

$$M(fx, fy, t) = e^{i\theta} e^{-\frac{|x-y|^2}{t}} = e^{i\theta} e^{-\frac{|1-y|^2}{t}} \succsim e^{i\theta} e^{-\frac{4\lambda|1-y|^2}{t}} = M\Big(fy, y, \frac{t}{\lambda}\Big).$$

Case 3.3. If $x \in (0, 1)$ and $y \in [1, 2)$, then for $\lambda \in (\frac{1}{4}, \frac{1}{2})$, we have

$$M(fx, fy, t) = e^{i\theta} e^{-\frac{|x-y|^2}{t}} = e^{i\theta} e^{-\frac{|1-x|^2}{t}} \succsim e^{i\theta} e^{-\frac{4\lambda|1-x|^2}{t}} = M\Big(fx, x, \frac{t}{\lambda}\Big).$$

Case 3.4. If $x, y \in (0, 1)$, then for $\lambda \in (\frac{1}{4}, \frac{1}{2})$, we have

$$M(fx, fy, t) = e^{i\theta} e^{-\frac{|x-y|^2}{t}} = e^{i\theta} e^{-\frac{|1-y|^2}{t}} \succsim e^{i\theta} e^{-\frac{4\lambda|1-y|^2}{t}} = M\big(fy, y, \frac{t}{\lambda}\big), x > y, t > 0$$

and $M(fx, fy, t) \succsim M\Big(fx, x, \frac{t}{\lambda}\Big), x < y, t > 0$. So conditions (3.6) are satisfied for all $x, y \in X, t > 0$, and by Theorem 3.2 it follows that $x = 1$ is a unique fixed point for f.

4 Conclusion

In the present study, we defined a new concept of complex-valued fuzzy b-metric space. We also established the condition of being Cauchy and convergence in this newly developed space. Several allied aspects of complex-valued fuzzy b-metric space are also defined, which fortify the concept. In our main result, we obtained the Banach contraction principle in the "complex valued fuzzy b-metric space". For the sustainability of our result, we also furnished an example that satisfied our main result.

5 Open problem

It is the introduction of a phase term that makes the complex fuzzy set a distinctive and novel concept. Quantum mechanics allows an object to exhibit a wave-like nature associated with a phase term. Therefore, making use of this concept of complex valued fuzzy b metric space to show the existence of a fixed quantum state associated with quantum operations is an open problem.

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