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ON ARITHMETIC FUNCTIONS, THEIR EXTENDED COEFFICIENTS: VARIOUS RESULTS AND RELATIONS

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Abstract

In this article, we represent a recurrence relation of the arithmetic function connected with an ascending factorial function, Lah and Stirling numbers. We then obtain a relation of harmonic numbers and again extend the coefficients of these arithmetic functions involving Bell polynomials through introducing the sequence of Hankel type integrals. On the other hand, making some of the extensions of these arithmetic functions, we derive some more results and the summation formulae in terms of Riemann Zeta function.

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1 Introduction

In this representation, we consider the following recurrence relation recently studied by Pathan et al. [17] as

$$f_k(n) = \frac{k}{n}\sum_{j=1}^{n} g_j f_k(n-j), \quad f_k(0)=1, \quad n,k \geq 1, \tag{1.1}$$

which is satisfied by interesting arithmetic functions [2, 11]. From (1.1), it is clear that $f_k(n)$ is a polynomial of degree n in k

$$f_k(n) = a(n,1)k + a(n,2)k^2 + \cdots + a(n,n-1)k^{n-1} + a(n,n)k^n, \quad n \geq 1, \tag{1.2}$$

where the coefficients $a(n,m)$ are in terms of the quantities g_j, in fact due to [17], we have

$$a(n,n) = \frac{1}{n!}\left(g_1\right)^n, \quad n \geq 1. \tag{1.3}$$

Here in (1.2) the coefficients are

$$a(n,m) = \frac{1}{m!(n-m)!}\sum_{j=1}^{n-m}\left(g_1\right)^{m-j}\binom{m}{j} j! B_{n-m,j}\left(\frac{1!}{2}g_2, \frac{2!}{3}g_3, \ldots, \frac{(n-m-j+1)!}{n-m-j+2}g_{n-m-j+2}\right) \quad \forall n \geq m+1, \tag{1.4}$$

that involving the incomplete exponential Bell polynomials [9, 16, 17, 18].

The relations (1.3) and (1.4) are in harmony with the expressions of Jakimczuk [12, Eqns. (8)-(11)].

Further, we also show that due to Stirling numbers the relations (1.2) and (1.4) imply an important property as given by [17]

$$a(n,1) = \frac{1}{n}g_n = -\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k} f_k(n), \quad n \geq 1, \tag{1.5}$$

and we realize that applications of the results (1.1)-(1.5) for the cases $g_j = 1$ and $g_r = r$. We discuss these conditions in the next section on an application of Stirling numbers [6, 8], Lah numbers [1, 14] and then describe the harmonic numbers [26] and again derive the Truesdell's polynomials [3-5] and their discussions on generalizations.

2 Various properties of (1.5) and applications

In this section, we derive various results due to the formula (1.5) through following theorem:

Theorem 2.1. *Due to Stirling numbers, the results* (1.2) *and* (1.4) *imply the property given by*

$$a(n,1)=\frac{1}{n}g_n=-\sum_{k=1}^{n}\frac{(-1)^k}{k}\begin{pmatrix} n \\ k \end{pmatrix}f_k(n),\quad n\geq 1. \tag{2.1}$$

Proof. Considering the results (1.2) and (1.4) we find that

$$\begin{aligned}\sum_{k=1}^{n}\frac{(-1)^k}{k}\begin{pmatrix} n \\ k \end{pmatrix}f_k(n) &= \sum_{k=1}^{n}\frac{(-1)^k}{k}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{j=1}^{n}a(n,j)k^j,\\ &=\sum_{j=1}^{n}a(n,j)\sum_{k=1}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}k^{j-1},\\ &=a(n,1)\sum_{k=1}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}+\sum_{t=1}^{n-1}a(n,t+1)\sum_{k=1}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}k^t,\\ &=-a(n,1)+(-1)^n n!\sum_{t=1}^{n-1}a(n,t+1)S_t^{[n]},\end{aligned} \tag{2.2}$$

but (1.4) gives $a(n,1)=\frac{1}{n}g_n$, and for the Stirling numbers of the second kind [1,10,15,21,23], we have that $S_t^{[n]}=0$, because $t<n$, therefore (2.2) implies (2.1) q.e.d. □

Corollary 2.1. *Applying the results* (1.1) *and* (1.5) *in the Theorem* 2.1 *and choosing $g_j=j\geq 1$, following relation holds true*

$$f_k(n)=\sum_{l=1}^{n}\frac{1}{l!}\begin{pmatrix} n-1 \\ l-1 \end{pmatrix}k^l,\quad n\geq 1. \tag{2.3}$$

Proof. In (1.1), choosing $g_j=j\geq 1$, we have

$$f_k(n)=\frac{k}{n}\sum_{j=1}^{n}jf_k(n-j),\quad f_k(0)=1, \tag{2.4}$$

$$a(n,m)=\frac{1}{m!}\sum_{j=1}^{n-m}\begin{pmatrix} m \\ j \end{pmatrix}\begin{pmatrix} n-m-1 \\ j-1 \end{pmatrix}\overset{[17]}{=}\frac{1}{m!}\begin{pmatrix} n-1 \\ m-1 \end{pmatrix}, \tag{2.5}$$

where in (2.5), we applied the following relation in terms of the Lah numbers $[1,14,15,23]$ as

$$B_{n-m,j}(1!,2!,\ldots,(n-m-j+1)!)=L_{n-m}^{[j]}=\frac{(n-m)!}{j!}\begin{pmatrix} n-m-1 \\ j-1 \end{pmatrix}. \tag{2.6}$$

Hence making an appeal to the results (2.4)-(2.6), we find the relation

$$f_k(n)=\sum_{l=1}^{n}\frac{1}{l!}\begin{pmatrix} n-1 \\ l-1 \end{pmatrix}k^l,\quad n\geq 1, \tag{2.7}$$

which verifies the relation (2.3). □

Corollary 2.2. *Applying the results* (1.1) *and* (1.5) *in the Theorem* 2.1, *for all $j,g_j=1$, following relations hold true*

$$f_k(n)=\frac{(-1)^n}{n!}\sum_{j=0}^{n}(-1)^jS_n^{(j)}k^j=\frac{1}{n!}(k)_n,\quad n\geq 1, \tag{2.8}$$

where $S_n^{(j)}$ are the Stirling numbers $\forall j=1,2,3,\ldots,n$.

Proof. In the results (1.1) and (1.4), setting $g_j = 1 \quad \forall j$, we have

$$f_k(n) = \frac{k}{n}\sum_{j=1}^{n} f_k(n-j), \quad f_k(0) = 1 \tag{2.9}$$

and

$$a(n,m) = \frac{(-1)^n}{m!}\sum_{l=1}^{n-m} \frac{(-1)^l l!}{(n-m+l)!} S^{(l)}_{n-m+l}\delta_{lm} = \frac{(-1)^{n-m}}{n!} S^{(m)}_n, \tag{2.10}$$

where the following identity [20] in terms of the Stirling numbers of the first kind [1, 20, 21, 23] was employed as

$$B_{n-m,j}\left(\frac{1!}{2}, \frac{2!}{3}, \ldots, \frac{(n-m-j+1)!}{n-m-j+2}\right) = (-1)^{n-m-j}(n-m)!\sum_{l=0}^{j} \frac{(-1)^l}{(j-l)!(n-m+l)!} S^{(l)}_{n-m+l}. \tag{2.11}$$

Hence, by the results (2.9)-(2.11), we obtain following identities

$$f_k(n) = \frac{(-1)^n}{n!}\sum_{j=0}^{n} (-1)^j S^{(j)}_n k^j \overset{[21]}{=} \frac{1}{n!}(k)_n, \quad n \geq 1, \tag{2.12}$$

such that $(k)_n = k(k+1)\cdots(k+n-1)$.

Finally, the identities in (2.12) give us the relations (2.8). □

Thus the Corollary 2.2 implies an interesting recurrence relation for the ascending factorial function

$$(k)_n = (n-1)!k\sum_{j=1}^{n} \frac{1}{(n-j)!}(k)_{n-j}, \quad n, k \geq 1. \tag{2.13}$$

If we remember that $(n)_n = \frac{\Gamma(2n)}{\Gamma(n)} = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma\left(n+\frac{1}{2}\right)$, here given that $\Gamma(n+1) = n!, n \geq 1$.

Then due to the formula (2.13), we find the results

$$\sum_{j=1}^{n} \frac{1}{(n-j)!}(n)_{n-j} = \frac{1}{2}\frac{(2n)!}{(n!)^2} = \frac{1}{2}\begin{pmatrix} 2n \\ n \end{pmatrix}, \quad n \geq 1. \tag{2.14}$$

Furthermore, if we accept that in (2.8) the symbol k is a continuous variable, then we apply $\frac{d}{dk}$ to (2.13) and then we make $k = 1$ to deduce the following identity [26] involving harmonic numbers $[1, 10, 21, 23]$

$$\sum_{j=1}^{n} H_j = (n+1)H_n - n, \quad n \geq 1, \tag{2.15}$$

where in (2.15), we applied the expression

$$\left[\frac{d}{dx}(x)_m\right]_{x=1} = m!H_m. \tag{2.16}$$

Remark 2.1. *If F is the generating function of $f_k(n)$, then following convolution holds true*

$$\sum_{n=0}^{\infty} f_{k_3}(n)q^n = F^{k_3} = F^{k_1+k_2} = F^{k_1}F^{k_2} = \left(\sum_{j=0}^{\infty} f_{k_1}(j)q^j\right)\left(\sum_{l=0}^{\infty} f_{k_2}(l)q^l\right),$$

that is there exists

$$f_{k_3}(n) = \sum_{j=0}^{n} f_{k_1}(j)f_{k_2}(n-j), \quad k_3 = k_1 + k_2, \quad k_1, k_2 \geq 1, \tag{2.17}$$

which means that f_{k_3} is the Cauchy convolution of f_{k_1} with f_{k_2}.

3 Identities due to the formula (2.14)

The formula (2.14) has a great importance when we multiply it by a Beta function. Then we evaluate some of its identities and relations by employing the Beta function given by

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \forall m,n > 0, \tag{3.1}$$

and the gamma function defined by [25, p.19]

$$\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} dt \quad \forall m > 0, \tag{3.2}$$

Theorem 3.1. *$\forall n \geq 1$, by the formula (2.14) following identities hold*

$$\frac{\sqrt{\pi}\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \sum_{j=1}^{n} \frac{1}{(n-j)!}(n)_{n-j} = \frac{4^{\mathrm{n}}}{(2n+1)} = B\left(\frac{1}{2}, n+1\right) \sum_{j=1}^{n} \frac{1}{(n-j)!}(n)_{n-j} \tag{3.3}$$

and

$$\frac{\sqrt{\pi}\mathrm{n}!}{2^{2\mathrm{n}+1}\left(n+\frac{1}{2}\right)!} \sum_{j=1}^{n} \frac{1}{(n-j)!}(n)_{n-j} = \frac{1}{2(2n+1)} = \left\{\sum_{j=1}^{n} \frac{1}{(n-j)!}(n)_{n-j}\right\} \int_0^1 x^n(1-x)^n dx. \tag{3.4}$$

Proof. Considering the formula (2.14) we find that

$$\left\{\sum_{j=1}^{n} \frac{1}{(n-j)!}(n)_{n-j}\right\} \int_0^1 x^n(1-x)^n dx = \frac{1}{2}\frac{(2n)!}{(n!)^2}\frac{\Gamma(n+1)\Gamma(n+1)}{(2n+1)\Gamma(2n+1)} = \frac{1}{2(2n+1)}. \tag{3.5}$$

Now, on making an appeal to well known Legendre duplication formula in the middle of the Eqn. (3.5), we obtain the identity (3.4).

Finally, by the Eqns. (3.1) and (3.4), we derive the identities in the Eqn. (3.3). □

4 Some of the extensions of the arithmetic function (1.1), their results and relations

In this section we introduce some extensions of the arithmetic function (1.1) and the identity (3.5). Then make their applications to derive some more other results connected to Bell polynomials [16, 17, 18] and the Riemann Zeta functions [13, 19, 24].

For the $g_j \forall j \geq 1$, given in (1.5), one of the extensions of (1.1) is taken by

$$f_k(n,t) = \frac{k}{n}\sum_{j=1}^{n} e^{g_j t} f_k(n-j), \quad f_k(0) = 1, \quad n,k \geq 1. \tag{4.1}$$

Clearly, from (4.1) we have a relation with (1.1) as found by

$$\left.\frac{d}{dt} f_k(n,t)\right|_{t=0} = f_k(n), f_k(0) = 1, n,k \geq 1. \tag{4.2}$$

Theorem 4.1. *Due to the extension (4.1), a formula exists as*

$$\left.\frac{d^n}{dt^n} f_k(n,t)\right|_{t=0} = \frac{k}{n}\sum_{j=1}^{n} a(j,1) f_k(n-j) j^n, \quad f_k(0) = 1, \quad n,k \geq 1. \tag{4.3}$$

Proof. Operate (4.1) by the operator $\frac{d^n}{dt^n}$ to find that

$$\frac{d^n}{dt^n} f_k(n,t) = \frac{k}{n}\sum_{j=1}^{n} e^{g_j t}\left(\frac{g_j}{j}\right)^n f_k(n-j) j^n, \text{ provided that } f_k(0) = 1, n,k \geq 1. \tag{4.4}$$

Then in (4.4) apply the formula (1.5), to find that

$$\frac{d^n}{dt^n} f_k(n,t) = \frac{k}{n}\sum_{j=1}^{n} e^{g_j t} a(j,1) f_k(n-j) j^n, \text{ provided that } f_k(0) = 1, n,k \geq 1. \tag{4.5}$$

Finally, making an appeal to the result (4.5), we derive the result of (4.3). □

Theorem 4.2. *Due to the Theorem* 4.1, *there also exists another generating function*

$$f_k(n,t) = \frac{k}{n}\sum_{j=1}^{n} a(j,1) f_k(n-j)\frac{(jt)^n}{n!}, \quad f_k(0)=1, \quad n,k \geq 1. \tag{4.6}$$

Proof. Consider the Maclaurin series

$$f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f^{(n)}(0), f^{(n)}(0) = \left.\frac{d^n}{dt^n} f(t)\right|_{t=0}, \tag{4.7}$$

where $f(t)$ possesses continuous derivative of all orders in the interval $[0,t]$. Then make an appeal to the formula (4.3) of the Theorem 4.1 to find the function (4.7). $\square$

Theorem 4.3. *For the generalized Riemann Zeta function defined and studied by* [13, 19, 24]

$$\zeta(a,s) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \forall a.s \in \mathbb{C} \textit{ and } \Re(a) > 0, \Re(s) > 1, \tag{4.8}$$

and

$$\zeta(a,s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{e^{-at}}{(1-e^{-t})} dt \forall a.s \in \mathbb{C} \textit{ and } \Re(a) > 0, \Re(s) > 0, \tag{4.9}$$

there exists following summation formulae

$$\sum_{n=0}^{\infty} \frac{\sqrt{\pi}\mathrm{n}!\left(n-\frac{1}{2}\right)!}{2^{2\mathrm{n}-1}\left\{\left(n+\frac{1}{2}\right)!\right\}} \sum_{j=1}^{n} \frac{1}{(n-j)!}(n)_{n-j} = \sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^2} = \zeta\left(\frac{1}{2},2\right), \tag{4.10}$$

$$\sum_{n=0}^{\infty} \frac{\sqrt{\pi}\mathrm{n}!\left(n-\frac{1}{2}\right)!}{2^{2\mathrm{n}-1}\left\{\left(n+\frac{1}{2}\right)!\right\}^2} \sum_{j=1}^{n} \frac{1}{(n-j)!}(n)_{n-j} = \int_0^{\infty} \left(\frac{t}{1-e^{-t}}\right) e^{-\frac{1}{2}t} dt. \tag{4.11}$$

Proof. Considering the results (3.3) and (3.4) and Making an appeal to the formulae of generalized Riemann Zeta function (4.8) and (4.9), we derive the formulae (4.10) and (4.11), respectively. $\square$

5 Extensions in the coefficients $a(n,m)$ defined in (2.10) via sequence of Hankel type integral operators, to find different polynomials

The Hankel's contour integral is defined by [25, p. 219]

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^u u^{-z} du, \sigma > 0, \Re(z) > 0, i = \sqrt{(-1)}. \tag{5.1}$$

Therefore to make extensions in the coefficients $a(n,m)$ given in the Eqn. (2.10), we introduce a sequence of Hankel type integral operators due to (5.1) and again apply the formula of the generating function for the Stirling numbers due to Riordan [22] (see also in (Chandel [6], Chandel and Yadava [8]) of first kind which is given by

$$S_n^{(k)} = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n. \tag{5.2}$$

Now from (2.10) considering the coefficients $a(n,m) \forall n,m \in \mathbb{N} \cup \{0\}$ as $a(n,m) = \frac{(-1)^{n-m}}{n!} S_n^{(m)}$ and in it applying (5.2), we find the formula of $a(n,m)$ consisting of sequence of Hankel's type contour integrals (5.1) in the form

$$a(n,m) = \frac{(-1)^n}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{j^n}{n!} = \frac{(-1)^n}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left\{ \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{ju} u^{-(n+1)} du \right\}, \sigma > 0. \tag{5.3}$$

Due to (5.3), for exploring new ideas in the field of arithmetic functions and further extensions in these results, we define a sequence of Hankel type integral operators (5.1) in the form

$$K(j,n;\sigma)\{f\} = \frac{\Gamma(n+1)}{j^n} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{ju} u^{-(n+1)} f(u) du, \ \sigma > 0, \ f(0) = 1. \tag{5.4}$$

It is clear that when $f(u) \equiv 1$, then for $\sigma > 0$ the formulae (5.3) and (5.4) give us the relations with Bell coefficients

$$\frac{(-1)^n}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{j^n}{n!}\left\{K(j,n;\sigma)\{1\}\right\} = a(n,\ m) = a(n,\ m,1), (\text{let})\,. \tag{5.5}$$

Therefore in the formula (5.4) when we set $f^{\alpha,r}(u) = e^{(\alpha+(r-1)j)u}$ and $\sigma > 0$, we find

$$K(j,n,\alpha;\sigma)\{f^{\alpha,r}\} = \frac{\Gamma(n+1)}{j^n}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} u^{-(n+1)}e^{(\alpha+rj)u}du = \frac{\{(\alpha+rj)\}^{n+1}}{j^{n+1}}. \tag{5.6}$$

Further for $\sigma > 0$, making an application of the formulae (5.4) and (5.6), we get the coefficients of Bell polynomials in following generalized form

$$\begin{aligned} a\left(n,\ m, f^{\alpha,r}\right) &= \frac{(-1)^n}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{j^n}{n!}\left\{jK(j,n,\alpha;\sigma)\{f^{\alpha,r}\}\right\} \\ &= \frac{(-1)^n}{n!}\frac{1}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\{\alpha+rj\}^{n+1} \\ &= \frac{(-1)^{n-m}}{n!}S^{\alpha}(n+1,m,r), \end{aligned} \tag{5.7}$$

where $f^{\alpha,r}(u) = e^{(\alpha+(r-1)j)u}$.

Here in (5.7), the generalized Stirling formula is given by Chandel and Yadava [8]

$$S^{\alpha}(n,m,r) = \frac{(-1)^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\{(\alpha+rj)\}^n\,. \tag{5.8}$$

Now making an appeal to (5.7), we obtain a generating function equivalent to the generating function due to Chandel and Yadava [8 , Eqn. (2.6)] as given by

$$\begin{aligned} (-1)^m\sum_{n=0}^{\infty}(-t)^n a\left(n-1,\ m, f^{\alpha,r}\right) &= \frac{(-1)^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+rj)^n \\ &= e^{\alpha t}\frac{(-1)^m}{m!}\sum_{j=0}^{m}(-m)_j\frac{\left(e^{rt}\right)^j}{j!} = e^{\alpha t}\frac{(-1)^m}{m!}\,{}_1F_0\left(-m;-;e^{rt}\right). \end{aligned} \tag{5.9}$$

Again considering the formula (5.7) we get Truesdell polynomials due to Chandel [3, 4, 5]

$$\begin{aligned} &(-1)^{m+n}n!\sum_{m=0}^{n}(-1)^m a\left(n-1,\ m, f^{\alpha,r}\right)p^r x^{rm} \\ &= \sum_{m=0}^{n}\frac{(-1)^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(\alpha+rj)^n p^r x^{rm} = T_n^{\alpha}(x,r,-p). \end{aligned} \tag{5.10}$$

6 Concluding remarks

In this article**,** a recurrence relation of the arithmetic function is considered to obtain the coefficients of Bell polynomials. Then we derive various results and relations connected with an ascending factorial function, Lah and Stirling numbers and to find a relation of harmonic numbers. To exploring of this work in multidisciplinary aspect, we make some of the extensions of the coefficients of Bell polynomials in terms of sequence of the Hankel type integral operators to derive generalized Stirling numbers and Truesdell's polynomials. We also derive the summation formulae in terms of Riemann Zeta function.

On the other hand, making an appeal to [7] in (5.7), we may introduce the coefficients of Bell polynomials into multivariable Truesdell's polynomials and then we may apply the techniques due to [7] to derive various results and generating functions.

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