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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

ON THE DEGREE OF APPROXIMATION OF FUNCTION $f \in W(L_p, \xi(t))$ CLASS BY $(C,2)(e,c)$ MEANS OF ITS FOURIER SERIES

H. L. Rathore
Department of Mathematics, Government College Pendra, Bilaspur, Chhattisgarh, India-495119.
Email: hemlalrathore@gmail.com

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Abstract

We study on degree of approximation of function belonging to weighted $(L_p, \xi(t))$ class by $(C,1)(e,c)$ mean and weighted $(L_p, \xi(t))$ class by $(C,2)(E,q)$ has been discussed by Rathore and Shrivastava. Since (e,c) includes (E,q) method, so for obtaining more generalized result we replace (E,q) by (e,c) mean. Which is a regular method of summation for $c > 0$. In this paper we obtain the degree of approximation of the function belonging to weighted $(L_p, \xi(t))$ class by $(C,2)(e,c)$ product means of its Fourier series has been proved.

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Keywords and Phrases: Degree of approximation, $W(L_p, \xi(t))$ class of function, $(C,2)$ summability, (e,c) summability, $(C,2)(e,c)$ product summability, Fourier series, Lebesgue integral.

1 Introduction

The (e,c) summability method was introduced by Hardy and Littlewood [6], which is a regular for $c > 0$ including the method of summability for Borel, (E,q) etc. We study on approximation of f belonging to many classes. Also $W(L_p, (\xi(t))$ by Cesăro mean, Nörlund mean, has been discussed by several researchers like Alexits [1], Khan [6], Chandra [3], Sahney and Goel [17], Quereshi [12], Shrivastava and Verma [19], Mishra et al.[10] etc. Rathore and Shrivastava [13] extended the result on degree of approximation of a function belonging to $W(L_r, \xi(t))$ class by $(C,1)(e,c)$ means of Fourier series. Further Rathore and Shrivastava [14] studied about product summability on approximation of a function belonging to $W(L_r, \xi(t))$ class by $(C, 2)$ (E,q) means. In this direction several researchers like Lal and Singh [9], Lal and Kushwaha [8], Nigam [11], Albayrak, Koklu and Bayramov [2], Rathore, Shrivastava and Mishra ([15], [16]) etc. Recently Kushwaha [7] has determined on approximation of function by $(C,2)(E,1)$ product summability method of Fourier series, but till now no work done to extend the result on approximation of function $f \epsilon W(L_p, \xi(t))$ class by $(C,2)(e,c)$ mean has been seen.

2 Definition and Notations

Let $f(x)$ be periodic with period -2π and integrable in the sense of Lebesgue. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

with n^{th} partial sum $S_n(f;x)$.

A series $\sum_{n=0}^{\infty} u_n$ with the sequence of partial sum $\{S_n\}$ is said to be summable (e,c), $(c > 0)$ to sum S.

Let $\left\{t_n^{(e,c)}\right\}$ denotes the sequence of (e,c) mean of the sequence $\{S_n\}$. If the (e,c) transform of S_n defined as

$$\lim_{n\to\infty} t_n^{(e,c)} = \lim_{n\to\infty} \sqrt{\frac{c}{\pi n}} \sum_{r=-\infty}^{\infty} \exp\left(-\frac{cr^2}{k}\right) S_{k+r} \tag{2.2}$$

exists, where $S_{k+r} = 0$, when $k + r < 0$.

We write

$$\left\| t_n^{(e,c)} - f \right\| = \sup_{-\pi \le x \le \pi} \left| t_n^{(e,c)}(f:x) - f(x) \right|, \tag{2.3}$$

where $t_n^{(e,c)}(f:x)$ is n^{th} (e,c) means of the Fourier series f at x. Thus if

$$t_n^{(e,c)}(f;x)-f(x)=\frac{1}{\pi}\sqrt{\frac{c}{\pi n}}\int_0^{\pi}\frac{\Phi_x(t)}{\sin t/2}\left[\sum_{r=-k}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t\right]dt. \tag{2.4}$$

The series $\sum_{n=0}^{\infty}u_n$ with the partial sum S_n is said to be summable (e,c) to the definite number S, (see Hardy [4]).

Let $\left\{t_n^{(C,2)}\right\}$ denote the sequence of $(C,2)$ mean of the sequence $\{S_n\}$. If the $(C,2)$ transform of S_n is defined as

$$t_n^{(C,2)}(f:x)=\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)S_k\to S \quad \text{as } n\to\infty \tag{2.5}$$

then the series $\sum_{n=0}^{\infty}u_n$ is said to be summable to the number S by $(C,2)$ method. Thus if

$$t_n^{(C,2)(e,c)}(f:x)=\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)t_k^{(e,c)}\to S \text{ as } n\to\infty \tag{2.6}$$

where $t_n^{(C,2)(e,c)}$ denotes the sequence of $(C,2)(e,c)$ product mean of the sequence S_n, the series $\sum_{\mathrm{n=0}}^{\infty}\mathrm{u_n}$ is said to be summable to the number S by $(C,2)(e,c)$ method.

We observe that $(C,2)(e,c)$ method is regular if $c>0$.

A function $f\in W(L_p,\xi(t))$ class, if

$$\left(\int_0^{2\pi}\left(\left|[f(x+t)-f(x)]\sin^{\beta}x\right|^p dx\right)\right)^{1/p}=O(\xi(t)),(\beta\geq 0). \tag{2.7}$$

Given a positive increasing function $\xi(t)$ and an integer $p\geq 1$, we observe that

$$W(L_p,\xi(t))\overset{\beta=0}{\longrightarrow}L(\xi(t),p)\overset{\xi(t)=t^{\alpha}}{\longrightarrow}L(\alpha,p)\overset{p\to\infty}{\longrightarrow}\text{Lip }\alpha.$$

That is

$$\text{Lip}\,\alpha\subseteq\text{Lip}(\alpha,p)\subseteq\text{Lip}(\xi(t),p)\subseteq W(L_p,\xi(t)),\ \text{for } 0<\alpha\leq 1, p\geq 1.$$

Now we define norm by

$$\|f\|_p=\left(\int_0^{2\pi}|f(x)|^p dx\right)^{1/p}, p\geq 1. \tag{2.8}$$

The degree of approximation $E_n(f)$ be given by

$$E_n(f)=\min\|T_n-f\|_p, \tag{2.9}$$

where $T_n(x)$ is a trigonometric polynomial of degree n by (see Zygmund [21]).

We shall use following notation:

$$\phi(t)=f(x+t)+f(x-t)-2f(x). \tag{2.10}$$

3 Inequalities

We use the following inequalities in our further investigations

$$\sum_{r=k+1}^{\infty}r\exp\left(-\frac{cr^2}{k}\right)\leq\frac{k}{2c}\exp(-ck) \tag{3.1}$$

$$\left|\sum_{r=k+1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t\right|\leq\frac{kt}{2c}\exp(-ck), \tag{3.2}$$

$$\sum_{r=k+1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos(rt)=O\left\{\frac{\exp(-ck)}{t}\right\}, \tag{3.3}$$

$$1+2\sum_{r=1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos(rt)=\sqrt{\frac{\pi k}{c}}\left\{\exp\left(\frac{-kt^2)}{4c}\right)+O\left(\exp\left(\frac{-k\pi}{4c}\right)\right)\right\}. \tag{3.4}$$

The inequality (3.2) follows from (3.1). (3.3) may be obtained by using Able's Lemma and (3.4) may be obtained by classical formula for theta function by Siddiqui [18] and (3.1) is due to Shrivastava & Verma [19].

4 Main Theorem

We prove the following theorem

Theorem 4.1. *If $f : R \to R$ is 2π-periodic function, Lebesgue integrable on $[0, 2\pi]$ and belonging to the $W(L_p, \xi(t))$ class then the degree of approximation of f by the $(C, 2)(e, c)$ product summability means of Fourier series satisfies*

$$\left\| t_n^{(C,2)(e,c)} - f(x) \right\|_p = O\left[(n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right)\right] \tag{4.1}$$

provided $\xi(t)$ satisfies the following condition :

$$\left\{\frac{\xi(t)}{t}\right\} \text{ be a decreasing sequence,} \tag{4.2}$$

$$\left\{\int_0^{1/n+1} \left(\frac{t|\phi(t)|}{\xi(t)}\right)^p \sin^{\beta p} t dt\right\}^{1/p} = O\left(\frac{1}{n+1}\right), \tag{4.3}$$

$$\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}\phi(t)}{\xi(t)}\right)^p \sin^{\beta p} t dt\right\}^{1/p} = O\left((n+1)^{\delta}\right), \tag{4.4}$$

where δ is an arbitrary number such that $q(1-\delta) - 1 > 0, \frac{1}{p} + \frac{1}{q} = 1$. conditions (4.3) and (4.4) hold uniformly in x and $t_n^{(C,2)(e,c)}$ is (C, 2) (e, c) mean of the Fourier series (2.1).

5 Lemmas

We shall use the following Lemmas

Lemma 5.1. *Let* $M_n(t) = \frac{1}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (n-k+1)\left[\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin t/2}\right]$.

Then $|M_n(t)| = O(n+1)$, for $0 < t < \frac{\pi}{(n+1)}$.

Proof. Applying $\sin nt \leq n \sin t$, for $0 < t < \frac{\pi}{(n+1)}$, we have

$$\begin{aligned} |M_n(t)| &\leq \frac{1}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (n-k+1)\left[\frac{(2k+1)\sin t/2}{\sin t/2}\right] \\ &\leq \frac{1}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (n-k+1)(2k+1) \\ &= \frac{(n+1)}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (2k+1) - \frac{1}{(n+1)(n+2)\pi} \sum_{k=0}^{n} k(2k+1) \\ &= \frac{(n+1)^2}{(n+2)\pi} - \frac{1}{(n+1)(n+2)\pi}\left[\sum_{k=0}^{n} \left(2k^2 + k\right)\right] \\ &= \frac{(n+1)^2}{(n+2)\pi} - \frac{1}{(n+1)(n+2)\pi} \cdot \frac{n(n+1)(2n+1)}{3} - \frac{1}{(n+1)(n+2)\pi} \frac{n(n+1)}{2} \\ &= O(n+1). \end{aligned} \tag{5.1}$$

□

Lemma 5.2. *Let $|M_n(t)| = O\left(\frac{1}{t}\right)$, for $\frac{\pi}{(n+1)} \leq t \leq \pi$.*

Proof. Applying Jordon's Lemma $\sin\left(\frac{t}{2}\right) \geq t/\pi$ and $\sin kt \leq 1$ for $\frac{\pi}{(n+1)} \leq t \leq \pi$

$$\begin{aligned} |M_n(t)| &\leq \frac{1}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (n-k+1)\left[\frac{1}{t/\pi}\right] \\ &= \frac{(n+1)\pi}{(n+1)(n+2)t\pi} - \frac{\pi}{(n+1)(n+2)t\pi} \sum_{k=0}^{n} k \end{aligned} \tag{5.2}$$

$$= \frac{1}{(n+2)t} - \frac{n(n+1)}{2(n+1)(n+2)t}$$
$$= O\left(\frac{1}{t}\right).$$

□

6 Proof of the Main Theorem

Using (Titchmarsh [20]) and Riemann - Lebesgue theorem, the partial sum $S_n(f;x)$ of the series (2.1) is given by

$$S_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}} \sin\left(n+\frac{1}{2}\right) t dt. \tag{6.1}$$

If $t_n^{(e,c)}$ denotes (e,c) transform of $S_n(f;x)$ then

$$t_n^{(e,c)}(f;x) - f(x) = \frac{1}{2\pi}\sqrt{\frac{c}{\pi n}}\int_0^\pi \frac{\emptyset_x(t)}{\sin t/2}\left[\sum_{r=-k}^{\infty} \exp\left(-\frac{cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t\right] dt$$
$$= \frac{1}{2\pi}\sqrt{\frac{c}{\pi n}}\int_0^\pi \frac{\emptyset_x(t)}{\sin t/2}\left[\left\{1+2\sum_{r=1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos rt\right\}\sin\left(k+\frac{1}{2}\right)t\right.$$
$$\left. + \sum_{r=k+1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t\right] dt$$
$$= \frac{1}{\pi}\sqrt{\frac{c}{\pi n}}\int_0^\pi \frac{\phi_x(\mathrm{t})}{\sin t/2}\left[\left\{1+2\sum_{r=1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos rt\right\}\sin\left(k+\frac{1}{2}\right)t\right] dt$$
$$-2\sum_{r=k+1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos rt \sin\left(k+\frac{1}{2}\right)t + \sum_{r=1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right) dt.$$

For $(C,2)(e,c)$ transform, $t_n^{(C,2)(e,c)}(f;x)$ of $S_n(f;x)$, we write

$$t_n^{(C,2)(e,c)}(f;x) - f(x) = \frac{2}{2\pi(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\sqrt{\frac{c}{\pi k}}\int_0^\pi \frac{\emptyset_x(t)}{\sin t/2}$$
$$\cdot\left[\begin{array}{c} \left\{1+2\sum_{r=1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos rt\right\}\sin\left(k+\frac{1}{2}\right)t \\ -2\sum_{r=k+1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos rt\sin\left(k+\frac{1}{2}\right)t \\ +\sum_{r=k+1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t \end{array}\right] dt,$$
$$I = I_1 + I_2 + I_3 \text{ (say) }. \tag{6.2}$$

Now,

$$I_1 \le \frac{1}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\sqrt{\frac{c}{\pi k}}\int_0^\pi \frac{\emptyset_x(t)}{\sin t/2}$$
$$\cdot\left[\left\{1+2\sum_{r=1}^{\infty}\exp\left(-\frac{cr^2}{k}\right)\cos rt\right\}\sin\left(k+\frac{1}{2}\right)t\right] dt$$
$$= \frac{1}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\sqrt{\frac{c}{\pi k}}\int_0^\pi \frac{\emptyset_x(t)}{\sin t/2}$$
$$\cdot\sqrt{\frac{\pi k}{c}}\left[\left\{\exp\left(\frac{-kt^2}{4c}\right) + O\left(\exp\left(\frac{-k\pi}{4c}\right)\right)\right\}\sin\left(k+\frac{1}{2}\right)t\right] dt$$
$$= \frac{1}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\int_0^\pi \frac{\emptyset_x(t)}{\sin t/2}\sin\left(k+\frac{1}{2}\right)t\exp\left(\frac{-kt^2}{4c}\right) dt$$

$$+\frac{1}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\left[\left\{\int_0^{\pi}\frac{\emptyset_x(t)}{\sin t/2}\sin\left(k+\frac{1}{2}\right)t\cdot O\left(\exp\left(\frac{-k\pi}{4c}\right)\right)\right\}dt\right],$$

$$I_1 = I_{1.1} + I_{1.2} \quad \text{(say)}. \tag{6.3}$$

Now,

$$I_{1.1} = \frac{1}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\int_0^{\pi}\frac{\emptyset_x(t)}{\sin t/2}\left[\left\{\sin\left(k+\frac{1}{2}\right)t\exp\left(\frac{-kt^2)}{4c}\right)\right\}\right]dt$$

$$= O\left(\exp\left(\frac{-nt^2)}{4c}\right)\right)\int_0^{\pi}\emptyset_x(t)M_n(t)dt, \quad \text{using Lemma 5.1}$$

Then

$$|I_{1.1}| \le O(1)\left[\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}.\right]\emptyset_x(t)M_n(t)dt$$

$$I_{1.1} = I_{1.11} + I_{1.12} \tag{6.4}$$

Now

$$|I_{1.11}| \le \int_0^{\pi/n+1}|\varnothing_x(t)|\,|M_n(t)|\,dt$$

We have

$$|\emptyset_x(x+t) - \emptyset_x(x)| \le |f(v+x+t) - f(v+x)| + |f(v-x-t) - f(v-x)|.$$

Hence by Minkowiski inequality

$$\left[\int_0^{2\pi}|\{|\,\emptyset_x(x+t)\ \ -\emptyset_x(x)\}\sin^{\beta}x|^p\,dx\right]^{1/p}$$

$$\le\left[\int_0^{2\pi}\left|\{f(v+x+t)-f(v+x)\}\sin^{\beta}x\right|^p dx\right]^{1/p}$$

$$+\left[\int_0^{2\pi}\left|\{f(v-x-t)-f(v-x)\}\sin^{\beta}x\right|^p dx\right]^{1/p}$$

$$=O(\xi(t)).$$

Then $f \in W(L_p, \xi(t)) \Rightarrow \emptyset_x(t) \in W(L_p, \xi(t))$,
Applying Hölder's inequality and second mean value theorem for integral

$$|I_{1.11}| \le \left[\int_0^{\pi/n+1}\left\{\frac{t\,|\emptyset_x(t)|\sin^{\beta}t}{\xi(t)}\right\}^p dt\right]^{1/p}\left[\int_0^{\pi/n+1}\left\{\frac{\xi(t)\,|M_n(t)|}{t\sin^{\beta}t}\right\}^q dt\right]^{1/q} \tag{6.5}$$

$$= O\left(\frac{\pi}{(n+1)}\right)\left[\int_0^{\pi/n+1}\left\{\frac{\xi(t)(n+1)}{t^{1+\beta}}\right\}^q dt\right]^{1/q}$$

$$= O\left\{\xi\left(\frac{\pi}{(n+1)}\right)\right\}\left[\left(t^{-(1+\beta)q++1}\right)^{1/q}\right]_0^{\pi/n+1}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\}.$$

Now

$$I_{1.12} \le \int_{\pi/n+1}^{\pi}\emptyset_x(t)M_n(t)dt$$

Using Hölder's inequality $|\sin t| < 1, |\sin t| \ge \left(\frac{2t}{\pi}\right)$, Lemma 5.2 and second mean value theorem

$$|I_{1.12}| \le \left[\int_{\pi/n+1}^{\pi}\left\{\frac{t^{-\delta}\,|\emptyset_x(t)|\sin^{\beta}t}{\xi(t)}\right\}^p dt\right]^{1/p}\left[\int_{\pi/n+1}^{\pi}\left\{\frac{\xi(t)\,|M_n(t)|}{t^{-\delta}\sin\beta}\right\}^q dt\right]^{1/q} \tag{6.6}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{\pi/n+1}^{\pi}\left\{\frac{\xi(t)}{t^{1+\beta-\delta}}\right\}^{q}dt\right]^{1/q}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{1}^{n+1}\left\{\frac{\xi(\pi/y)}{\left(\left(1/y\right)^{(1+\beta-\delta)q}\right.}\right\}^{q}\frac{dy}{y^{2}}\right]^{1/q} \quad \left.\text{Lput } t=(\pi/y)\right]$$

$$= O\left\{(n+1)^{\delta}\right\}\xi\left(\frac{\pi}{n+1}\right)\left[\int_{1}^{n+1}\frac{dy}{y^{-(1+\beta-\delta)q+2}}\right]^{1/q}$$

$$= O\left\{(n+1)^{\delta}\right\}\xi\left(\frac{\pi}{n+1}\right)\left[y^{(1+\beta-\delta)-1/q}\right]_{1}^{n+1}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\}.$$

Now,

$$I_{1.2} = \frac{1}{(n+1)(n+2)\pi}\sum_{k=0}^{n}(n-k+1)\left[\left\{\int_{0}^{\pi}\frac{\emptyset_x(t)}{\sin t/2}\sin\left(k+\frac{1}{2}\right)tO\left(\exp\left(\frac{-k\pi}{4c}\right)\right)\right\}dt\right]$$

$$= O\left(\exp\left(\frac{-n\pi}{4c}\right)\right)\int_{0}^{\pi}\emptyset_x(t)M_n(t)dt \quad \text{(using Lemma 5.1).}$$

Then,

$$I_{1.2} = \left[\int_{0}^{\pi/n+1} + \int_{\pi/n+1}^{\pi}\cdot\right]\emptyset_x(t)M_n(t)dt, \tag{6.7}$$

$$I_{1.2} = I_{1.21} + I_{1.22}.$$

Now,

$$I_{1.21} = \int_{0}^{\pi/n+1}\emptyset_x(t)M_n(t)dt$$

Using Hölder's inequality and second mean value theorem

$$|I_{1.21}| \leq \left[\int_{0}^{\pi/n+1}\left\{\frac{t\,|\emptyset_x(t)|\sin^{\beta}t}{\xi(t)}\right\}^{p}dt\right]^{1/p}\left[\int_{0}^{\pi/n+1}\left\{\frac{\xi(t)\,|M_n(t)|}{t\sin^{\beta}t}\right\}^{q}dt\right]^{1/q} \tag{6.8}$$

$$= O\left(\frac{\pi}{n+1}\right)\left[\int_{0}^{\pi/n+1}\left\{\frac{\xi(t)(n+1)}{t^{\beta+1}}\right\}^{q}dt\right]^{1/q}$$

$$= O\left\{(n+1)^{\beta+1/p}\xi\left(\frac{1}{n+1}\right)\right\},\ \text{since } \frac{1}{p}+\frac{1}{q}=1.$$

Now,

$$I_{1.22} = \int_{\pi/n+1}^{\pi}\emptyset_x(t)M_n(t)dt$$

Using Hölder inequality and $|\sin t| < 1, |\sin t| \geq \left(\frac{2t}{\pi}\right)$

$$|I_{1.22}| \leq \left[\int_{\pi/n+1}^{\pi}\left\{\frac{t^{-\delta}\,|\emptyset_x(t)|\sin^{\beta}t}{\xi(t)}\right\}^{p}dt\right]^{1/p}\left[\int_{\pi/n+1}^{\pi}\left\{\frac{\xi(t)M_n(t)}{t^{-\delta}\sin\beta}\right\}^{q}dt\right]^{1/q} \tag{6.9}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{\pi/n+1}^{\pi}\left\{\frac{\xi(t)}{t^{1+\beta-\delta}}\right\}^{q}dt\right]^{1/q}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{1}^{n+1}\left\{\frac{\xi(\pi/y)}{(\pi/y)^{(1+\beta-\delta)q}}\right\}^{q}\frac{dy}{y^{2}}\right]^{1/q} \quad \text{put } t=(\pi/y)$$

$$= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\}.$$

Now,

$$I_2 = -\frac{1}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (n-k+1)\sqrt{\frac{c}{\pi k}} \int_0^{\pi} \frac{\emptyset_x(t)}{\sin t/2}$$
$$\cdot \left[2 \sum_{r=k+1}^{\infty} \exp\left(-\frac{cr^2}{k}\right) \cos rt \sin\left(k+\frac{1}{2}\right) t \right] dt$$
$$I_2 \le \frac{-2}{(n+1)(n+2)\pi} \sum_{k=0}^{n} (n-k+1)\sqrt{\frac{c}{\pi k}} \int_0^{\pi} \frac{\emptyset_x(t)}{\sin t/2} \left[O\left(\frac{\exp(-ck)}{t}\right) \sin\left(k+\frac{1}{2}\right) t \right] dt$$
$$= O\left(n^{-1/2}\exp(-cn)\right) \int_0^{\pi} \emptyset_x(t) \frac{M_n(t)}{t} dt, \text{ (using inequality (3.3))}$$

Then

$$I_2 = \left[\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi} \cdot \right] \emptyset_x(t) \frac{M_n(t)}{t} dt, \quad (6.10)$$
$$= I_{2.1} + I_{2.2}.$$

Now, using Hölder's inequality

$$I_{2.1} \le \left[\int_0^{\pi/n+1} \left\{ \frac{t\,|\emptyset_x(t)| \sin^{\beta} t}{\xi(t)} \right\}^p dt \right]^{1/p} \left[\int_0^{\pi/n+1} \left\{ \frac{\xi(t)\,|M_n(t)|}{t^2 \sin\beta} \right\}^q dt \right]^{1/q} \quad (6.11)$$
$$= O\left(\frac{1}{n+1}\right) \left[\int_0^{\pi/n+1} \left\{ \frac{\xi(t)(n+1)}{t^{\beta+2}} \right\}^q dt \right]^{1/q}$$
$$= O\left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\} \quad \left(\text{since } \frac{1}{p}+\frac{1}{q}=1 \right).$$

Now

$$I_{2.2} = \int_{\pi/n+1}^{\pi} \emptyset_x(t) \frac{M_n(t)}{t} dt.$$

Using Hölder's inequality and similarly

$$I_{2.2} = O\left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\} \quad (\text{ from } I_{1.22}). \quad (6.12)$$

Similarly,

$$I_3 = O\left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\}. \quad (6.13)$$

Now combining (6.2) to (6.13), we set

$$\left| t_n^{(C,2)(e,c)}(f;x) - f(x) \right| = O\left[(n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right],$$
$$\left\| t_n^{(C,2)(e,c)}(f;x) - \text{f(x)} \right\|_p = \left\{ \int_0^{2\pi} \left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\}^p dx \right\}^{1/p}$$
$$= 0\left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\} \left[\left\{ \int_0^{2\pi} dx \right\}^{1/p} \right]$$
$$= 0\left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\}.$$

This completes the proof of the theorem.

7 Corollaries

Following corollaries can be derived from main theorem:

Corollary 7.1. *If $\beta = 0$ and $\xi(t) = t^\alpha$ then the degree of approximation of a function $f \in \mathrm{Lip}(\alpha, p)$, $0 < \alpha \leq 1$ is given by*

$$\left\| t_n^{(C,2)(e,c)}(f; x) - \mathrm{f(x)} \right\|_p = \mathrm{O}\left\{ \frac{1}{(n+1)^{\alpha - 1/p}} \right\}$$

Corollary 7.2. *If $p \to \infty$ in corollary (7.1), and for $0 < \alpha < 1$.*

$$\left\| t_n^{(C,2)(e,c)}(f; x) - \mathrm{f(x)} \right\|_\infty = 0\left\{ \frac{1}{(n+1)^{\alpha}} \right\}.$$

8 Conclusion

The summability method (e, c) includes method of summability like Borel, $(E, 1), (E, q), F(a, q)$ and $[F, d_n]$ then by using the result of main theorem we can derive more generalizing result and also the result of Kushwaha [7] can be derived directly.

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