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*(Dedicated to Professor V. P. Saxena on His $80^{th}$ Birth Anniversary Celebrations)*

# IDENTITIES RELATED TO GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS

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**Abstract**

The famous Fibonacci and Lucas polynomials possess various astonishing properties and identities. The Fibonacci polynomial has been generalized in many ways by preserving the recurrence relation and others by preserving the initial condition. In this paper, we define generalized Fibonacci and Lucas polynomials and proved some famous identities in our settings.

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## 1 Introduction

Belgian mathematician Eugene Charles Catalan and the German Mathematician E. Jacobsthal [7] were first studied Fibonacci polynomials in 1883. Fibonacci polynomials are of great importance in Mathematics. The Fibonacci and Lucas polynomials are extensively explored by many mathematicians like, Basin [2] , Horadam and Mahon [6], and Lucas [11] (for details see Koshy [7]) and connected to various branches of mathematics. Recently, many new identities of Generalized Fibonacci and Lucas polynomials are studied by Agrawal et al. [1].

A set of Fibonacci polynomials generated by the $Q$ matrix, satisfying the following recurrence relation, was proved by Basin [2].

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x), n \geq 2 \text{ with } f_0(x) = 0, f_1(x) = 1. \tag{1.1}$$

The initial terms of the Fibonacci polynomials are

$$f_2(x) = x, f_3(x) = x^2 + 1, f_4(x) = x^3 + 2x, f_5(x) = x^4 + 3x^2 + 1 \text{ and so on }. \tag{1.2}$$

Jacobsthal polynomials are given by (for more details see Koshy [7])

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x), n \geq 3 \text{ with } J_1(x) = 1 = J_2(x). \tag{1.3}$$

Pell polynomials due to Horadam and Mahon [6] are defined by

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), n \geq 2 \text{ with } P_0(x) = 0, P_1(x) = 1. \tag{1.4}$$

The generating function of Fibonacci and Lucas polynomials due to Doman and Williams [4] is given by

$$\sum_{n=0}^{\infty} f_n(x)t^n = t(1 - xt - t^2)^{-1}, \sum_{n=0}^{\infty} L_n(x)t^n = (2 - xt)(1 - xt - t^2)^{-1}. \tag{1.5}$$

For Fibonacci and Lucas polynomials, the explicit sum formula due to Horadam and Mahon [6] and Koshy [7] is given by

$$f_n(x) = \sum_{n=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} x^{n-1-2k}, L_n(x) = \sum_{n=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}. \tag{1.6}$$

where $\binom{n-k}{k}$ is a binomial coefficient and $[x]$ is defined as the greatest integer less than or equal to $x$.

Many more interesting properties for Fibonacci and Lucas polynomials have been studied by Doman and Williams [4], Koshy [7], and Lucas [11].

Many famous identities which we have proved for our polynomial have been studied for generalized Fibonacci sequence in [9].

In this paper, we derive famous identities such as Catalan's, d'Ocagne's, and many other for our generalized Lucas polynomials which is derived for the generalized Fibonacci polynomials by Rathore et al. [8]. Also, we proved some identities for our generalized Fibonacci polynomials with the help of generating function and Binet's formula.

## 2 Preliminaries

In this section, we give some basic definitions which are useful throughout the paper.

**Definition 2.1.** ***Fibonacci Polynomials:*** *A polynomial sequence that can be considered as a generalization of Fibonacci numbers are Fibonacci polynomials (for more details see Lucas* [11]*). The Fibonacci polynomial due to Koshy* [7] *is defined by the following recurrence relation,*

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x), n \geq 3 \textit{ with } f_1(x) = 1, f_2(x) = x.$$

**Definition 2.2.** ***Lucas Polynomials:*** *The Lucas Polynomials due to Bicknell* [3] *and Lucas* [8] *are defined by the recurrence relation,*

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), n \geq 2 \textit{ with } L_0(x) = 2, L_1(x) = x.$$

**Definition 2.3.** ***Generalized Fibonacci Polynomials:*** *The generalized Fibonacci polynomials are defined by*

$$f_n(x) = \begin{cases} s, & \textit{if } n = 0; \\ sx, & \textit{if } n = 1; \\ xf_{n-1}(x) + f_{n-2}(x), & \textit{if } n \geq 2. \end{cases} \tag{2.1}$$

**Definition 2.4.** ***Generalized Lucas Polynomials:*** *The generalized Lucas polynomials are defined by*

$$l_n(x) = \begin{cases} 2s, & \textit{if } n = 0; \\ sx, & \textit{if } n = 1; \\ xl_{n-1}(x) + l_{n-2}(x), & \textit{if } n \geq 2. \end{cases} \tag{2.2}$$

**Definition 2.5.** ***Generating Function:*** *Let $a_0, a_1, a_2,$ be a sequence of real numbers. Then the function (2.5) $g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ is called a generating function for the sequence $\{a_n\}$. Generating functions provides a powerful tool for solving linear homogeneous recurrence relations with constant coefficients (for more details see Lucas* [11]*).*

## 3 Generalized Fibonacci Polynomials

The generalization of Fibonacci polynomials can be done in many ways by changing the initial condition and others by changing the recurrence relation. Rathore et al. [9] defined the generalized Fibonacci polynomials $w_n(x)$ by recurrence relation $w_n = xw_{n-1} + w_{n-2}, n \geq 2$, with $w_0(x) = 2b, w_1(x) = a + b$ where $a$ and $b$ are integer. Sikhwal et al. [9] defined the generalized Fibonacci polynomials $u_n(x)$ by recurrence relation with $u_n = xu_{n-1} + u_{n-2}, n \geq 2$, with $u_0(x) = a, u_1(x) = 2a + 1$ where $a$ is an integer. In this paper, we define generalized Fibonacci polynomials $g_n(x)$ by the recurrence relation

$$g_n^{(x)} = xg_{n-1}^{(x)} + g_{n-2}^{(x)}, n \geq 2 \text{ with } g_0(x) = a + b, g_1(x) = 2a + 1 \tag{3.1}$$

where $a$ and $b$ are integers.

The starting few terms of a generalized Fibonacci polynomials are given by

$$g_0(x) = a + b, g_1(x) = 2a + 1, g_2(x) = x(2a + 1) + a + b, g_3(x) = x^2(2a + 1) + x(a + b) + 2a + 1.$$

For $x = 1, a = 0, b = 0$, we obtain the classical Fibonacci sequence.

Binet's Formula for generalized Fibonacci polynomial is given by $g_n(x) = (A\alpha^n + B\beta^n)$, where

$$A = \frac{(2a + 1) - (a + b)\beta}{\alpha - \beta}, B = \frac{(a + b)\alpha - (2a + 1)}{\alpha - \beta}.$$

Also, Note that $\alpha\beta = -1, \alpha + \beta = x, \alpha + \beta = \sqrt{4 + x^2}$ where $\alpha$ and $\beta =$ are the roots of the quadratic multline given by $\lambda^2 - x\lambda - 1 = 0$ (Koshy [7]).

**Lemma 3.1.** *The generating function for generalized Fibonacci polynomials defined in equation* (3.1) *is given by*

$$\sum_{n=0}^{\infty} g_n(x)t^n = \frac{(a+b)(1-xt)+(2a+1)t}{1-xt-t^2}.$$

*Proof.* Replace $n$ by $n+1$ in (3.1), we have

$$g_{n+1}(x) = xg_n(x) + g_{n-1}(x); n \geq 1. \tag{3.2}$$

Let

$$F(t) = \sum_{n=0}^{\infty} g_n(x)t^n. \tag{3.3}$$

From equation (3.2), we have

$$\sum_{n\geq 1} g_{n+1}(x)t^n = x\sum_{n\geq 1} g_n(x)t^n + \sum_{n\geq 1} g_{n-1}(x)t^n. \tag{3.4}$$

Now,

$$\sum_{n\geq 1} g_n(x)t^n = \sum_{n\geq 1} g_n(x)t^n + g_0(x) - g_0(x) = F(t) - (a+b). \tag{3.5}$$

and

$$\sum_{n\geq 1} g_{n-1}(x)t^n = tF(t). \tag{3.6}$$

Therefore, R.H.S of (3.4) becomes

$$\sum_{n\geq 1} g_{n+1}(x)t^n = x[F(t) - (a+b)] + tF(t). \tag{3.7}$$

Now,

$$\sum_{n\geq 1} g_{n+1}(x)t^n = \sum_{n\geq 1} g_n(x)t^n + g_0(x) - g_0(x) + g_1(x) - g_1(x) = \frac{1}{t}[F(t) - (a+b) - t(2a+1)] \tag{3.8}$$

Therefore, (3.7) becomes

$$\frac{1}{t}[F(t) - (a+b) - t(2a+1)] = x[F(t) - (a+b)] + tF(t)$$

i.e.

$$F(t)(1-xt-t^2) = [(a+b)(1-xt) + (2a+1)t].$$

Thus,

$$\sum_{n=0}^{\infty} g_n(x)t^n = \frac{(a+b)(1-xt)+(2a+1)t}{1-xt-t^2}. \tag{3.9}$$

□

## 4 Generalized Lucas Polynomials

We define generalized Lucas polynomials $k_n(x)$ by the recurrence relation

$$k_n(x) = xk_{n-1}(x) + k_{n-2}(x); n \geq 2 \text{ with } k_0(x) = a, k_1(x) = x, \tag{4.1}$$

where $a$ is an integer. The first few terms of generalized Lucas polynomials are given by

$$k_0(x) = a, k_1(x) = x, k_2(x) = ax + a = a(x+1), k_3(x) = x(a+1) + a.$$

For $x = 1, a = 2$, we obtain Lucas sequence.

Following the same idea as in proof of Lemma 3.1, we can derive a generating function for generalized Lucas polynomials (defined as above), is given by

$$\sum_{n=0}^{\infty} k_n(x)t^n = \frac{a(1-xt)+xt}{1-xt-t^2}.$$

Binet's Formula for generalized Lucas polynomials is given by

$$k_n(x) = A(\alpha^n + \beta^n), \text{ where } A = \frac{a}{2} \text{ (Koshy [7])}.$$

## 5 Some Identities of generalized Fibonacci polynomials

In this section, we investigate some of the identities of our generalized Fibonacci polynomials with the help of a generating function and Binet's formula.

**Theorem 5.1.** *If the nth term of a generalized Fibonacci polynomial is $g_n(x)$ and $g'_n(x)$ denotes the derivative of $g_n(x)$ with respect to $x$, then*

$$g'_n(x) = xg'_{n-1}(x) + g'_{n-2}(x) + g_{n-1}(x), n \geq 2. \tag{5.1}$$

*Proof.* The generating function of generalized Fibonacci polynomials is given by

$$\sum_{n=0}^{\infty} g_n(x)t^n = [(a+b)(1-xt) + (2a+1)t](1-xt-t^2)^{-1}.$$

Differentiating both sides with respect to $x$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} g'_n(x)t^n =& [(a+b)(1-xt) + (2a+1)t](-t)(-1)(1-xt-t^2)^{-2} \\ &+[-t(a+b)](1-xt-t^2)^{-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}(1-xt-t^2)\sum_{n=0}^{\infty} g'_n(x)t^n &= t[(a+b)(1-xt) + (2a+1)t](1-xt-t^2)^{-1} - t(a+b) \\ &= t\sum_{n=0}^{\infty} g_n(x)t^n - t(a+b).\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} g'_n(x)t^n - x\sum_{n=0}^{\infty} g'_n(x)t^{n+1} - \sum_{n=0}^{\infty} g'_n(x)t^{n+2} = \sum_{n=0}^{\infty} g_n(x)t^{n+1} - t(a+b).$$

Equating the coefficients of $t^n$ on both sides, we have

$$g'_n(x) = xg'_{n-1}(x) + g'_{n-2}(x) + g_{n-1}(x),$$

which proves the Theorem 5.1.

Replacing $n$ by $n+1$, we also derive,

$$g'_{n+1}(x) = xg'_n(x) + g'_{n-1}(x) + g_n(x).$$

□

**Theorem 5.2.** *Let $g_n(x)$ be the $n^{th}$ term of a generalized Fibonacci polynomial, then*

$$ng_n(x) - x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x). = xg_n(x) + (2-x^2)g_{n-1}(x) - 3xg_{n-2}(x) - 2g_{n-3}(x); n \geq 3. \tag{5.2}$$

*Proof.* The generating function of a generalized Fibonacci polynomials is given by

$$\sum_{n=0}^{\infty} g_n(x)t^n = [(a+b)(1-xt) + (2a+1)t](1-xt-t^2)^{-1}. \tag{i}$$

Differentiate it both sides partially with respect to $t$, we get

$$\sum_{n=0}^{\infty} ng_n(x)t^{n-1} = [(a+b)(1-xt) + (2a+1)t](x+2t)(1-xt-t^2)^{-2} + [-x(a+b) + (2a+1)](1-xt-t^2)^{-1}. \tag{ii}$$

Differentiating (i) both sides partially with respect to $x$, we have

$$\sum_{n=0}^{\infty} g'_n(x)t^n = [(a+b)(1-xt) + (2a+1)t](t)(1-xt-t^2)^{-2} + [-t(a+b)](1-xt-t^2)^{-1}. \tag{iii}$$

On dividing both sides by $t$, we derive

$$\sum_{n=0}^{\infty} g'_n(x)t^{n-1} = [(a+b)(1-xt) + (2a+1)t](1-xt-t^2)^{-2} + [-(a+b)](1-xt-t^2)^{-1}.$$

Hence,

(iv) $$\sum_{n=0}^{\infty} g'_n(x)t^(n-1) + (a+b)(1-xt-t^2)^{-1} = [(a+b)(1-xt)+(2a+1)t](1-xt-t^2)^{-2}.$$

On substituting the value of R.H.S of equation (iv) in equation (ii), we have

$$\sum_{n=0}^{\infty} ng_n(x)t^{n-1} = (x+2t)\sum_{n=0}^{\infty} g'_n(x)t^{n-1} + (a+b)(1-xt-t^2)^{-1}$$
$$+[-x(a+b)+(2a+1)](1-xt-t^2)^{-1}$$
$$=x\sum_{n=0}^{\infty} g'_n(x)t^{n-1} + 2\sum_{n=0}^{\infty} g'_n(x)t^{n-1} + (x+2t)(a+b)(1-xt-t^2)^{-1}$$
$$+[-x(a+b)+(2a+1)](1-xt-t^2)^{-1}$$
$$=x\sum_{n=0}^{\infty} g'_n(x)t^{n-1} + 2\sum_{n=0}^{\infty} g'_n(x)t^{n} + (2t)(a+b)+(2a+1)(1-xt-t^2)^{-1}.$$

Therefore,

$$(1-xt-t^2)\sum_{n=0}^{\infty} ng_n(x)t^{n-1}$$
$$= x(1-xt-t^2)\sum_{n=0}^{\infty} g'_n(x)t^{n-1} + 2(1-xt-t^2)\sum_{n=0}^{\infty} g'_n(x)t^{n} + (2t)(a+b)+(2a+1).$$

Hence

$$\sum_{n=0}^{\infty} ng_n(x)t^{n-1} - x\sum_{n=0}^{\infty} ng_n(x)t^{n} - \sum_{n=0}^{\infty} ng_n(x)t^{n+1}$$
$$= x\sum_{n=0}^{\infty} g'_n(x)t^{n-1} - x^2\sum_{n=0}^{\infty} g'_n(x)t^{n} - x\sum_{n=0}^{\infty} g'_n(x)t^{n+1}$$
$$+2\sum_{n=0}^{\infty} g'_n(x)t^{n} - 2x\sum_{n=0}^{\infty} g'_n(x)t^{n+1} - 2\sum_{n=0}^{\infty} g'_n(x)t^{n+2} + (2t)(a+b)+(2a+1).$$

By equating the coefficients of $t^{n-1}$ on both the sides, we finally derive (5.2). □

**Theorem 5.3.** *For the generalized Fibonacci polynomials $g_n(x)$, we derive the following identities*

*(i)* $g'_{n+1}(x) - g'_{n-1}(x) = xg'_n(x) + g_n(x)$,

*(ii)* $g'_{n+1}(x) - (1-x^2)g'_{n-1}(x)$
$= (x+1)g_n(x) - x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x) + 3xg'_{n-2}(x) + 2g'_{n-3}(x); n \geq 3$,

*(iii)* $(2-x^2)g'_{n-1}(x)$
$= xg_n(x) - x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x) - xg'_n(x) + 3xg'_{n-2}(x) + 2g'_{n-3}(x); n \geq 3$,

*(iv)* $(2-x^2)g'_{n-1}(x) = x(1-x^2)g'_n(x) + 3xg'_{n-2}(x) + 2g'_{n-3}(x) + (n+2-x^2)g_n(x)$
$- x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x); n \geq 3.$

*Proof.* Differentiating (3.2) both sides with respect to $x$, we obtain

(i) $$g'_{n+1}(x) - g'_{n-1}(x) = xg'_n(x) + g_n(x).$$

Using Theorem 5.2 in (i), we derive

(ii) $$g'_{n+1}(x) + (1-x^2)g'_{n-1}(x) = (n+1)g_n(x) - x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x) + 3xg'_{n-2}(x) + 2g'_{n-3}(x).$$

On subtracting (i) from (ii), we prove

(iii) $$(2-x^2)g_(n-1)'(x) = ng_n(x) - x(n-1)g_{n-1}(x) - (n-2)g_{n-2}(x) - xg'_n(x) + 3xg'_{n-2}(x) + 2g'_{n-3}(x).$$

On multiplying (i) by $(1-x^2)$ and adding it to (ii), we establish

(iv) $$(2-x^2)g'_{n+1}(x) = x(1-x^2)g'_n(x) + 3xg'_{n-2}(x) + 2g'_{n-3}(x) + (n+2-x^2)g_n(x).$$

□

**Theorem 5.4.** *Let $g_n(x)$ be the nth term of a generalized Fibonacci polynomial, then*

$$ng'_{n+1}(x)-(n+2-x^2)g'_{n-1}(x)-(n+1)xg'_n(x)+2g'_{n-3}(x)+3xg'_{n-2}(x) = x(n-1)g_{n-1}(x)+(n-2)g_{n-2}(x); n\geq 3. \tag{5.3}$$

*Proof.* From Theorem 5.3(i), we have

$$g'_{n+1}(x)-g'_{n-1}(x)-xg'_n(x)=g_n(x). \tag{I}$$

and from Theorem 5.3(ii), we have

$$g'_{n+1}(x)+(1-x^2)g'_{n-1}(x)=(n+1)g_n(x)-x(n-1)g_{n-1}(x)-(n-2)g_{n-2}(x)+3xg'_{n-2}(x)+2g'_{n-3}(x). \tag{II}$$

Substituting the value of $g_n(x)$ from (I) in (II), we finally derive

$$g'_{n+1}(x)+(1-x^2)g_(n-1)'(x)$$
$$=(n+1)g'_{n+1}(x)-g'_{n-1}(x)-xg'_n(x)-x(n-1)g_{n-1}(x)-(n-2)g_{n-2}(x)+3xg'_{n-2}(x)+2g'_{n-3}(x),$$

which is (5.3). □

**Theorem 5.5** (**Explicit Summation formula**). *For generalized Fibonacci polynomials*

$$g_n(x) = (a+b)\left\{\sum_{k=0}^{[n/2]}\binom{n-k}{k}x^{n-2k}-\sum_{k=0}^{[n/2]}\binom{n-k-1}{k}x^{n-2k-1}\right\} +(2a+1)\sum_{k=0}^{[n/2]}\binom{n-k-1}{k}x^{n-2k-1}.$$

*Proof.* The generating function for generalized Fibonacci polynomials is given by

$$\sum_{n=0}^{\infty}g_n(x)t^n=[(a+b)(1-xt)+(2a+1)t](1-xt-t^2)^{-1}$$
$$=[(a+b)(1-xt)+(2a+1)t]\sum_{n=0}^{\infty}(x+t)^nt^n$$
$$=[(a+b)(1-xt)+(2a+1)t]\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}\binom{n}{k}(nk)x^{n-k}t^k$$
$$=[(a+b)(1-xt)+(2a+1)t]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}x^{n-k}t^{n+k}$$
$$=[(a+b)(1-xt)+(2a+1)t]\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+k)!}{k!(n)!}x^nt^{n+2k}.$$

On equating the coefficients of $t^n$ on both sides, we prove

$$g_n(x) = (a+b)\left\{\sum_{k=0}^{[n/2]}\binom{n-k}{k}x^{n-2k}-\sum_{k=0}^{[n/2]}\binom{n-k-1}{k}x^{n-2k-1}\right\} +(2a+1)\sum_{k=0}^{[n/2]}\binom{n-k-1}{k}x^{n-2k-1}.$$

□

**Theorem 5.6.** ***A Variant Property:*** *For generalized Fibonacci polynomials*

$$g_{n-2}(x)g_{n+1}(x)-g_{n-1}(x)g_n(x)=(-1)^{n-2}x[(2a+1)(a+b)x+(2a+1)^2+(a+b)^2].$$

*Proof.* We know that the Binet's formula for generalized Fibonacci polynomials is given by

$$g_n(x)=(A\alpha^n+B\beta^n).$$

Therefore,

$$g_{n-2}(x)g_{n+1}(x)-g_{n-1}(x)g_n(x)$$

$$
\begin{aligned}
&= (A\alpha^{n-2} + B\beta^{n-2})(A\alpha^{n+1} + B\beta^{n+1}) - (A\alpha^{n-1} + B\beta^{n-1})(A\alpha^{n} + B\beta^{n}) \\
&= (A^2\alpha^{2n-1} + AB\alpha^{n-2}\beta^{n+1} + AB\alpha^{n+1}\beta^{n-2} + B^2\beta^{2n-1}) \\
&-(A^2\alpha^{2n-1} + AB\alpha^{n-1}\beta^{n} + AB\alpha^{n}\beta^{n-1} + B^2\beta^{2n-1}) \\
&= AB(\alpha^{n-2}\beta^{n+1} + \alpha^{n+1}\beta^{n-2} + \alpha^{n-1}\beta^{n} + \alpha^{n}\beta^{n-1}) \\
&= AB(\alpha\beta)[(\alpha+\beta)(\beta^2 - \alpha\beta + \alpha^2) - \alpha\beta(\alpha+\beta)] \\
&= AB(\alpha\beta)^{n-2}(\alpha+\beta)(\alpha-\beta)^2 \\
&= (-1)^{n-2}x[(2a+1)(a+b)x - (2a+1)^2 + (a+b)^2].
\end{aligned}
$$

For $a = 0, b = 0, x = 1$, the above identity reduces to the identity for classical Fibonacci sequence. □

## 6 Some Identities of generalized Lucas polynomial

Next, we explore the Lucas counterparts of Catalan's identity which have been stated for Fibonacci due to Sikhwal et al. [9].

**Theorem 6.1.** *Let $k_n(x)$ be the nth term of generalized Lucas polynomial, then*

$$k_n^2(x) - k_{n+r}(x)k_{n-r}(x) = (-1)^{n-r}\left[\frac{a^2(-1)^r}{2} - \frac{ak_{2r}(x)}{2}\right].$$

*Proof.* Binet's formula for Lucas polynomial is given by

$$k_n(x) = A(\alpha^n + \beta^n).$$

Therefore,

$$
\begin{aligned}
&k_n^2(x) - k_{n+r}(x)k_{n-r}(x) = [A(\alpha^n + \beta^n)]^2 - A(\alpha^{n+r} + \beta^{n+r})A(\alpha^{n-r} + \beta^{n-r}) \\
&= [A(\alpha^2 n + \beta^2 n + 2\alpha^n\beta^n)]^2 - A^2(\alpha^2 n + \alpha^(n+r)\beta^{n-r} + \alpha^{n-r}\beta^{n+}) + \beta^2 n) \\
&= 2A^2(\alpha\beta)^n - A^2(\alpha\beta)^{n-r}(\alpha^2 r + \beta^2 r) \\
&= 2A^2(-1)^n - A(-1)^{n-r}k_{2r}(x) \\
&= (-1)^n 2A^2 - A(-1)^{-r}k_{2r}(x) \\
&= (-1)^{n-r}\left\{\frac{a^2(-1)^r}{2} - \frac{ak_{2r}(x)}{2}\right\}.
\end{aligned}
$$

□

The following theorem gives the identity for Lucas polynomial which is already derived for generalized Fibonacci polynomials known as d'Ocagne's identity in Sikhwal et al. [10].

**Theorem 6.2.** *If the nth term of generalized Lucas polynomial is $k_n(x)$, then*

$$k_m(x)k_{n+1}(x) - k_{m+1}(x)k_n(x) = \frac{a}{2}\{(-1)^{n+1}k_{m-n-1}(x) - (-1)^{m+1}k_{n-m-1}(x)\}.$$

*Proof.* Binet's formula for Lucas polynomials is given by

$$k_n(x) = A(\alpha^n + \beta^n).$$

Therefore,

$$
\begin{aligned}
&k_m(x)k_{n+1}(x) - k_{m+1}(x)k_n(x) \\
&= A(\alpha^m + \beta^m)A(\alpha^{n+1} + \beta^{n+1}) - A(\alpha^{m+1} + \beta^{m+1})A(\alpha^n + \beta^n) \\
&= A^2(\alpha^m\beta^{n+1} + \alpha^{n+1}\beta^m - \alpha^{m+1}\beta^n - \alpha^n\beta^{m+1}) \\
&= A^2\{(\alpha\beta)^{n+1}(\alpha^{m-n-1} + \beta^{m-n-1}) - (\alpha\beta)^{m+1}(\alpha^{n-m-1} + \beta^{n-m-1})\} \\
&= A\{(-1)^{n+1}k_{m-n-1}(x) - (-1)^{m+1}k_{n-m-1}(x)\} \\
&= \frac{a}{2}\{(-1)^{n+1}k_{m-n-1}(x) - (-1)^{m+1}k_{n-m-1}(x)\}.
\end{aligned}
$$

□

The next theorem gives the relevant results to Theorems 6.1 and 6.2 for our Lucas polynomials.

**Theorem 6.3.** *Let $k_n(x)$ be the nth term of generalized Lucas polynomial, then*

*(i)* $k_n^2(x) + k_{n+r}(x)k_{n-r}(x) = ak_2n(x) + \frac{a^2}{2}(-1)^n + \frac{a}{2}(-1)^{n-r}k_{2r}(x)$,
*(ii)* $k_m(x)k_{n+1}(x) + k_{m+1}(x)k_n(x)$
$= \frac{a}{2}\{2k_{m+n+1}(x) + (-1)^{n+1}k_{m-n-1}(x) + (-1)^{m+1}k_{n-m-1}(x)\}$.

*Proof.* (i). With the help of Binet's formula, we establish

$$k_n^2(x) + k_{n+r}(x)k_{n-r}(x) = [A(\alpha^n + \beta^n)]^2 + A(\alpha^{n+r} + \beta^{n+r})A(\alpha^{n-r} + \beta^{n-r})$$
$$= [A(\alpha^{2n} + \beta^{2n} + 2\alpha^n\beta^n)]^2 + A^2(\alpha^{2n} + \alpha^{n+r}\beta^{n-r} + \alpha^{n-r}\beta^{n+r} + \beta^{2n})$$
$$= 2A^2(\alpha^{2n} + \beta^{2n}) + 2A^2(\alpha\beta)^n + A^2(\alpha\beta)^{n-r}(\alpha^{2r} + \beta^{2r})$$
$$= 2Ak_{2n}(x) + 2A^2(-1)^n + A(-1)^{n-r}k_{2r}(x)$$
$$= ak_{2n}(x) + \frac{a^2}{2}(-1)^n + \frac{a}{2}(-1)^{n-r}k_{2r}(x).$$

Proof (ii). With the help of Binet's formula, we derive

$$k_m(x)k_{n+1}(x) + k_{m+1}(x)k_n(x)$$
$$= A(\alpha^m + \beta^m)A(\alpha^{n+1} + \beta^{n+1}) + A(\alpha^{m+1} + \beta^{m+1})A(\alpha^n + \beta^n)$$
$$= A\{2A(\alpha^{m+n+1} + \beta^{m+n+1}) + (\alpha\beta)^{n+1}A(\alpha^{m-n-1} + \beta^{m-n-1}) + (\alpha\beta)^{m+1}A(\alpha^{n-m-1} + \beta^{n-m-1})\}$$
$$= A\{2k_{m+n+1}(x) + (-1)^{n+1}k_{m-n-1}(x) + (-1)^{m+1}k_{n-m-1}(x)\}$$
$$= \frac{a}{2}\{2k_{m+n+1}(x) + (-1)^{n+1}k_{m-n-1}(x) + (-1)^{m+1}k_{n-m-1}(x)\}.$$

□

## 7 Conclusion

In this paper, we have defined generalized Fibonacci and generalized Lucas polynomials. We have stated and derived many properties of our generalized Fibonacci polynomial and generalized Lucas polynomial through generating function and Binet's formula. Many other identities like Catalan's identity and d'Ocagne's identity can be derived easily from our generalized Fibonacci polynomial. Similarly, identities proved in section 5 for our generalized Fibonacci polynomial can also be proved for the generalized Lucas polynomial.

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