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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

MODIFIED HOMOTOPY PERTURBATION METHOD BASED SOLUTION OF LINEARLY DAMPED DUFFING OSCILLATOR AND COMPARISON WITH SIMULATED SOLUTION*

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Abstract

The Duffing oscillator provides a basis for studying nonlinear dynamics as its phase space trajectory is fairly complex and depends on the parameter of the system viz., initial amplitude, phase, frequency, linear damping coefficient and non-linearity parameter. In order to understand the complexity of the system, three variable effective expansions have been introduced in the usual homotopy perturbation framework to obtain the solution of damped Duffing system which finds application in several areas in engineering sciences such as vibration of bars, plates and electronic circuits, etc. The necessity of the extended homotopy frame work has been further discussed for non-conservative system. Simulation results for different parameters of the systems, such as, linear damping coefficient (μ), amplitude (α) and nonlinearity parameter (ϵ) are compared with the corresponding results based on perturbative homotopy analysis up to third order by changing (i) the magnitude of linear damping coefficient (μ), (ii) the magnitude of the nonlinearity of the system (ϵ). Even though the simulated result matches satisfactorily with the perturbative solution over the entire evolutionary time scale, noticeable divergence and phase shift are observed only lately for increased value μ and ϵ, respectively.
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Keywords and Phrases: Dynamical system, Duffing Oscillator, regular solution, Homotopy method, three control parameter expansion.

1 Introduction

Many engineering applications, such as large amplitude magneto-elastic system, centrifugal governor, vibration of bars and plates involve the Duffing oscillator as basic nonlinear oscillator [10,14]. The Duffing oscillator has been used to explain many observed phenomena in science, engineering, biological systems in particular nano-tubes, microtubules and hence, dynamical analysis of this oscillator attracted many workers [1,3,10,13,15]. Numerous researchers contributed to both analytical and numerical solutions of the Duffing oscillator with and without damping [4,16,17,18]. In a similar way the problem related to synchronization of chaotic Duffing system has also been taken up in recent years in [1,15]. Duffing equation without a damping term represents a conservative system. In view of the nonlinear characteristic of the basic Duffing oscillator, several authors have developed different analytical methods to obtain approximate analytical solution so as to understand the complexity of the involved dynamics [10,15,18]. Interestingly, the solution of Duffing oscillator, in case of non-conservative system, involves intricacies that led to several methods for the situation when damping coefficient is large [10,12]. Among the various perturbative methods, the homotopy perturbation method (*HPM)* has been extensively used, in general, for finding analytical solution of nonlinear oscillators [2,5,6,7,8,9,12,19]. In this work, we revisited the *HPM* to investigate in detail the complexity of

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dynamics of duffing system in the presence of large damping coefficient. For this, a revised framework of the *HPM* involving three parameter expansion method has been used to elucidate the convergence of obtained solution with the numerically simulated solution for different values of parameters defining the system.

This paper is organized as follows: in section 2, overview of the scheme involved in Homotopy perturbation method (*HPM*) given by He [6] is revisited and three parameter expansion formalism due to He and El-Dib [4,7] is explained. Analytical and numerical solutions for non-conservative Duffing oscillator are obtained. In sections 3, we provide the results of numerical simulation for various control parameters of the system and compare them with those obtained using *HPM*.

2 Homotopy Perturbation Method to Solve Non-conservative Duffing system

In the following, we describe briefly *HPM* for solving nonlinear differential equation and in particular the one that governs the dynamics of a damped Duffing oscillator. Further, the methodology used provides a basis for using three parameter expansion.

2.1 Homotopy perturbation scheme

For a general nonlinear ordinary differential equation, we may write it as [5,12],

$$A(Q) - f(r) = 0, \;\; r \in \Omega, \tag{2.1}$$

with boundary conditions

$$B\left(Q, \frac{\partial Q}{\partial n}\right) = 0, \;\; r \in \Gamma, \tag{2.2}$$

where $A,\ B$ refer to general differential operator and boundary operator respectively and further $f(r)$, a known analytic function with Γ referring to boundary of the domain Ω. We may divide the operator A into linear (L) and nonlinear part (N) resulting in the following form.

$$L(Q) + N(Q) - f(r) = 0. \tag{2.3}$$

Homotopy method formulated earlier in [5] involves constructing a homotopy $q(r,p) : \Omega \times [0,1] \to R$ satisfying

$$\mathcal{H}(q,p) = (1-p)\left[L(q) - L(Q_0)\right] + p\left[A(q) - f(r)\right] = 0, \;\; p \in [0,1], \;\; r \in \Omega$$

or

$$\mathcal{H}(q,p) = L(q) - L(Q_0) + pL(Q_0) + p\left[N(q) - f(r)\right] = 0, \tag{2.4}$$

where $p \in [0,1]$ defines an embedding parameter and Q_0 refers to an initial approximate solution of equation (2.1) satisfying the boundary conditions. From equation (2.4), we observe that

$$\mathcal{H}(q,0) = L(q) - L(Q_0) = 0,$$

$$\mathcal{H}(q,1) = A(q) - f(r) = 0. \tag{2.5}$$

This implies that as p changes from $0 \to 1$, the homotopy q goes from $Q_0 \to Q$. If we write the solution of equation (2.4) as a power series in p as

$$Q = Q_0 + pq_1 + p^2 q_2 + p^3 q_3 + \cdots . \tag{2.6}$$

then the solution of equation (2.1) would be

$$Q = \lim_{p \to 1} Q = Q_0 + q_1 + q_2 + q_3 + \cdots . \tag{2.7}$$

It may be noted that use of standard *HPM* results in inconsistency, as described briefly in Box: 2.1.

2.2 Three parameter expansion formalism

In view of the description in Box: 2.1, a need for modification of the *HPM* method arises. In the context of Damped Duffing equation (*DDE*), we note that the solution comprising of three variables i.e., homotopy function, oscillation amplitude (A), and frequency ω. Following He and El. Dib [6] and He [7] Homotopy Perturbation Method (*HPM*), we write the homotopy equation corresponding to *DDE* as,

$$\ddot{Q}(t) + \omega_0^2 Q(t) + p\left\{\mu \dot{Q}(t) + \epsilon Q^3(t)\right\} = 0, \quad p \in [0,1]. \tag{2.8}$$

where $\omega_0,\ \mu,\ \epsilon$ refers to the natural frequency, linear damping coefficient and magnitude of nonlinearity of the system.

The Homotopy Q is now expressed as a power series in p, given as

$$Q = q_0(t) + p^1 q_1(t) + p^2 q_2(t) + p^3 q_3(t) + \cdots . \tag{2.9}$$

Substitution of equation (2.9) in (2.8) and equating coefficients of p^0 to zero gives us

$$p^0 \; : \; \ddot{q}_0 + \omega_0^2 q_0 = 0, \tag{2.10}$$

whose exact solution would be

$$q_0(t) = A\cos(\omega_0 t + \phi), \tag{2.11}$$

where A and ϕ are real constants. To solve the non-conservative nonlinear equation a further expansion of linear frequency ω_0 and the time dependent amplitude A in powers of p is suggested [4,6]. Expanding ω_0 and $A(t)$ as follows

$$\omega_0^2 = \omega^2 - p\omega_1 - p^2\omega_2 - p^3 om_3 - \cdots . \tag{2.12}$$

$$A(t) = \alpha(1 + pc_1 + p^2c_2 + p^3c_3 + \cdots), \tag{2.13}$$

which implies that

$$q_0(t) = \alpha\{1 + pc_1 + p^2c_2 + p^3c_3 + \cdots\}\cos(\omega t + \phi). \tag{2.14}$$

Box: 2.1

Considering the following *DDE* where fundamental frequency is taken as 1, *i.e.*, $\omega_0 = 1$

$$\ddot{Q} + \mu\dot{Q} + Q + \epsilon Q^3 = 0. \qquad (B.1)$$

Observe that

$$A(Q) = L(Q) + N(Q), \qquad (B.2)$$

where

$$L(Q) = \ddot{Q}(t) + Q(t); \quad L(q_0) = \ddot{q}_0(t) + q_0; \quad N(Q) = \mu\dot{Q}(t) + \epsilon Q^3.$$

Following the standard *HPM* framework, we may write equation $(B.1)$ as,

$$p^0 \; : \; \ddot{q}_0(t) + q_0(t) - \ddot{Q}_0(t) - Q_0(t) = 0, \qquad (B.3)$$

$$p^1 \; : \; \ddot{q}_1(t) + q_1 + \ddot{Q}_0(t) + Q_0(t) + \mu\dot{Q}_0(t) + \epsilon Q_0^3 = 0, \qquad (B.4)$$

For $p \to 0$, $Q_0 \to q_0$, we may write equation $(B.4)$ as

$$\ddot{q}_1(t) + q_1 + \ddot{q}_0(t) + q_0(t) + \mu\dot{q}_0(t) + \epsilon q_0^3 = 0. \qquad (B.5)$$

Considering the initial approximate solution as $Q_0 = q_0 = A\cos\omega t$, where A is the amplitude and ω is the frequency of the output, which when substituted back in last equation, results in

$$\ddot{q}_1 + q_1 + = A\left\{\omega^2 - 1 - \frac{3}{4}A^2\epsilon\right\}\cos\omega t - \mu\omega A\sin\omega t - \frac{1}{4}\epsilon A^3\cos 3\omega t, \qquad (B.6)$$

where we have two secular terms. One of them gives us frequency ω of the output, as

$$\omega = \sqrt{1 + \frac{3}{4}\epsilon A^2}. \qquad (B.7)$$

which is the frequency obtained up to first order for conservative Duffing system and the other indicates that $\mu = 0$, *i.e.*, no damping. It is to be noted that even a second expansion, *i.e.*, expansion of amplitude A does not work [4,7]. Keeping these facts, to deal with damped Duffing oscillator, the modified *HPM*, explained in *section 2.2*, is considered.

Making an application of equations (2.11)-(2.14) in equation (2.8) and equating coefficients of p^i, $i = 1, 2, 3, \cdots$, we obtain following equations for q_1, q_2, q_3, $\cdots$ as

$$\ddot{q}_1(t) + \omega^2 q_1(t) = \alpha\omega\{2\dot{c}_1 + \mu\}\sin(\omega t + \phi) + \alpha\left\{\omega_1 - \ddot{c}_1 - \frac{3}{4}\epsilon\alpha^2\right\}\cos(\omega t + \phi) + \\ -\frac{1}{4}\epsilon\alpha^3\cos 3(\omega t + \phi), \tag{2.15}$$

$$\ddot{q}_2(t) + \omega^2 q_2(t) = \omega_1 q_1 - \mu\dot{q}_1 + \alpha\omega(2\dot{c}_2 + \mu c_1)\sin(\omega t + \phi) - \frac{3}{4}\epsilon\alpha^3 c_1\cos 3(\omega t + \phi) + \\ -\frac{3}{2}\epsilon\alpha^2 q_1(1 + \cos 2(\omega t + \phi)) + \alpha\left\{\omega_2 - \ddot{c}_2 + \omega_1 c_1 - \mu\dot{c}_1 - \frac{9}{4}\epsilon\alpha^2 c_1\right\}\cos(\omega t + \phi), \tag{2.16}$$

$$\text{(2.17)}\quad \begin{aligned}\ddot{q}_3(t)+\omega^2 q_3(t) =&\omega_1 q_2+\omega_2 q_1-\mu\dot{q}_2-3\epsilon q_0 q_1(q_0+q_1)+\alpha\omega(2\dot{c}_3+\mu c_2)\sin(\omega t+\phi)\\ &+\alpha\left\{\omega_2 c_1+\omega_3+\omega_1 c_2-\mu\dot{c}_2-\ddot{c}_3-\frac{9}{4}\epsilon\alpha^3(c_1^2+c_2)-\frac{3}{64}\frac{\epsilon^2\alpha^5}{\omega^2}\ c_1\right\}\cos(\omega t+\phi)\\ &-\left\{\frac{3}{4}\epsilon\alpha^3(c_1^2+c_2)+\frac{3}{32}\frac{\epsilon^2\alpha^5}{\omega^2}c_1\right\}\cos 3(\omega t+\phi)-\frac{3}{64}\frac{\epsilon^2\alpha^5}{\omega^2}c_1\ \cos 5(\omega t+\phi).\end{aligned}$$

The solutions for $q_1(t),\ q_2(t)$ and $q_3(t)$ could easily be obtained by removing the secular terms in the respective equations. Removal of secular terms from equation (2.15) results in following conditions on c_1 and ω_1,

$$\text{(2.18)}\quad \dot{c}_1=-\frac{1}{2}\mu \quad\Longrightarrow c_1=-\frac{1}{2}\mu t,\quad \ddot{c}_1=0,\quad \text{and}\quad \omega_1=\frac{3}{4}\ \epsilon\alpha^2,$$

which further leads to

$$\text{(2.19)}\quad q_1(t)=\frac{1}{32}\ \frac{\epsilon\alpha^3}{\omega^2}\quad \cos 3(\omega t+\phi).$$

Removing secular terms from equation (2.16), we get

$$\text{(2.20)}\quad \dot{c}_2=\frac{1}{4}\ \mu^2 t \quad\Longrightarrow \ddot{c}_2=\frac{1}{4}\mu^2,\ c_2=\frac{1}{8}\mu^2t^2\ \text{ and }\ \omega_2=-\frac{1}{4}\mu^2-\frac{3}{4}\epsilon\alpha^2\mu t+\frac{3}{128}\frac{\epsilon^2\alpha^4}{\omega^2},$$

and the solution for q_2 could be written as,

$$\text{(2.21)}\quad \begin{aligned} q_2(t) \quad = \quad & \frac{3}{64}\frac{\epsilon\alpha^3}{\omega^2}\left[\left\{\frac{1}{16}\frac{\epsilon\alpha^2}{\omega^2}-\mu t\right\}\cos 3(\omega t+\phi)+\frac{1}{2}\mu\sin 3(\omega t+\phi)\right.\\ + \quad & \left.\frac{1}{16}\frac{\epsilon\alpha^2}{\omega^4}\cos 5(\omega t+\phi)\right].\end{aligned}$$

Similarly removal of secular terms in equation (2.17), results in following conditions on c_3 and ω_3 as,

$$\text{(2.22)}\quad \begin{aligned} &\dot{c}_3=-\frac{1}{2}\mu c_2+\frac{9}{1024}\frac{\epsilon^2\alpha^4}{\omega^5},\ \mu=-\frac{1}{16}\mu^3t^2+\frac{9}{1024}\frac{\epsilon^2\alpha^4}{\omega^4}\mu\\ \Longrightarrow\quad &\ddot{c}_3=-\frac{1}{8}\mu^3 t \qquad \& \qquad \mathrm{c}_3=-\frac{1}{48}\mu^3\mathrm{t}^3+\frac{9}{1024}\frac{\epsilon^2\alpha^4}{\omega^4}\mu,\\ \text{and}\quad &\omega_3=\frac{3}{8}\epsilon\alpha^2\mu t\left\{\frac{1}{16}\epsilon\alpha^2+\mu t\right\}-\frac{3}{4096}\frac{\epsilon^3\alpha^6}{\omega^4},\end{aligned}$$

and solution for $q_3(t)$ may be written as,

$$\text{(2.23)}\quad \begin{aligned} q_3(t) \quad = \quad & -\frac{1}{8\omega^2}\left\{(X_1+X_2t)-\frac{7}{16\omega^2}(X_2+2X_3)+(X_4+X_5t)\right\}\ \cos 3(\omega t+\phi)\\ + \quad & \frac{1}{8\omega^2}\left\{\frac{3}{4\omega}(X_2+X_5+2X_3t)\right\}\ \sin 3(\omega t+\phi)-\frac{1}{24\omega^2}\left\{(X_6+X_7t+X_8)\right\}\ \cos 5(\omega t+\phi)\\ + \quad & \frac{5}{288\omega^3}X_7\ \sin 5(\omega t+\phi)-\frac{1}{48\omega^2}X_9\ \cos 7(\omega t+\phi).\end{aligned}$$

Therefore, from equation (2.9), the solution of non-conservative duffing oscillator up to third order would be

$$\text{(2.24)}\quad \begin{aligned} q(t)=&\lim_{p\to 1}\ Q=q_0+q_1+q_2+q_3\\ =&\alpha\left\{1-\frac{1}{2}\mu t+\frac{1}{8}\mu^2t^2-\frac{1}{48}\mu^3t^3+X_{13}\ t\right\}\cos(\omega t+\phi)+\left[X_{14}\left\{1-\frac{3}{2}\mu t+\frac{9}{8}\mu^2t^2\right\}\right.\\ &\left.-\frac{1}{8\omega^2}\left\{(X_1+X_2t)-\frac{7}{16\omega^2}(X_2+2X_3)+X_4+X_{10}+X_5\ t\right\}\right]\ \cos 3(\omega t+\phi)\\ &+\frac{3}{32\omega^2}\left\{X_2+X_5+2X_3\ t+X_{11}\right\}\ \sin 3(\omega t+\phi)-\frac{1}{24\omega^2}\left\{X_6+X_7\ t+X_8+X_{13}\right\}\ \cos 5(\omega t+\phi)\\ &+\frac{5}{288\omega^3}X_7\ \sin 5(\omega t+\phi)-\frac{1}{48\omega^2}X_9\ \cos 7(\omega t+\phi),\end{aligned}$$

where

$$X_1=-\left\{\frac{\epsilon\alpha^3}{32\omega^2}\ \mu^2+\frac{9\epsilon^3\alpha^7}{4096\omega^4}\right\},\qquad X_2=\frac{15\epsilon^2\alpha^5}{256\omega^2}\ \mu,\quad X_3=-\frac{9}{32}\epsilon\alpha^3\ \mu^2,\quad X_4=-\frac{9\epsilon^2\alpha^5}{1024\omega^3},$$

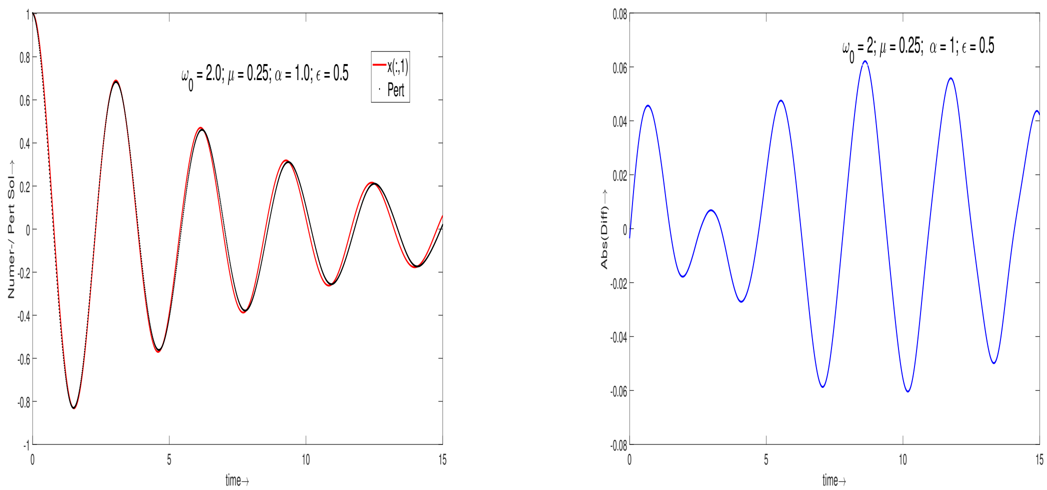
$$X_5 = -\frac{9\epsilon\alpha^3}{64\omega}\ \mu^2, \quad X_6 = -\frac{15\epsilon^3\alpha^7}{4096\omega^4}, \quad X_7 = \frac{15\epsilon^2\alpha^5}{1024\omega^2}\ \mu, \quad X_8 = -\frac{13\epsilon^2\alpha^5}{1024\omega^3}\ \mu, \quad X_9 = -\frac{3\epsilon^3\alpha^7}{4096\omega^4},$$

$$X_{10} = -\frac{3\epsilon^2\alpha^5}{128\omega^4}, \quad X_{11} = \frac{\epsilon\alpha^3}{4}\ \mu, \quad X_{12} = \frac{3\epsilon^2\alpha^5}{128\omega^2}, \quad X_{13} = \frac{9\epsilon^2\alpha^4}{1024\omega^4}\ \mu, \quad X_{14} = \frac{\epsilon\alpha^3}{32\omega^2}.$$

Following [4,6], we may rewrite equation (2.24) in a compact form as

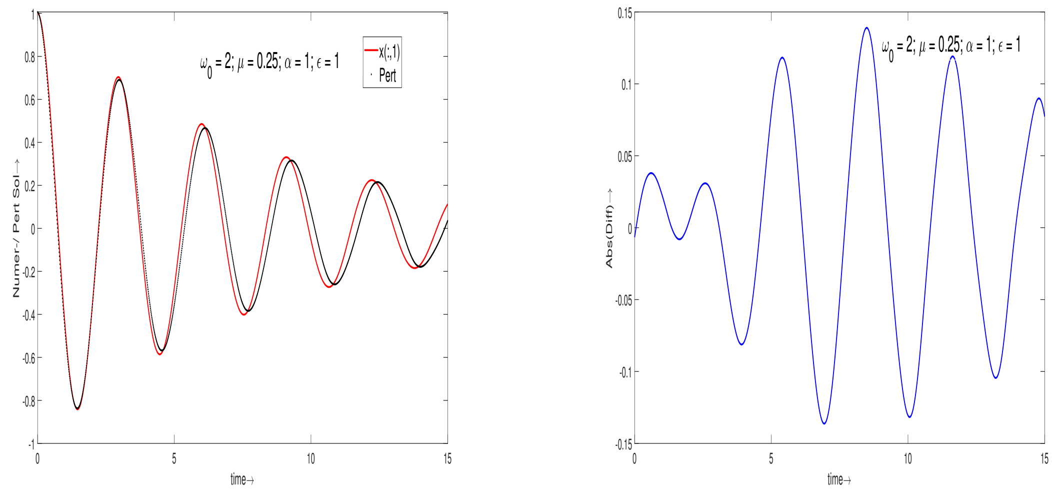
$$\begin{aligned} q(t) =& \alpha \left[e^{-\mu t/2} + X_{13}\ t\right] \cos(\omega t + \phi) \\ &+ \left[X_{14}\ e^{-3\mu t/2} - \frac{1}{8\omega^2}\left\{(X_1 + X_2\ t) - \frac{7}{16\omega^2}\,(X_2 + 2X_3) + X_4 + X_{10} + X_5\ t\right\}\right]\ \cos 3(\omega t + \phi) \\ &+ \frac{3}{32\omega^2}\left\{X_2 + X_5 + 2X_3\ t + X_{11}\right\}\ \sin 3(\omega t + \phi) \\ &- \frac{1}{24\omega^2}\left\{X_6 + X_7\ t + X_8 + X_{13}\right\}\ \cos 5(\omega t + \phi) \\ &+ \frac{5}{288\omega^3} X_7\ \sin 5(\omega t + \phi) - \frac{1}{48\omega^2} X_9\ \cos 7(\omega t + \phi). \end{aligned} \tag{2.25}$$

where $X's$ are defined as mentioned above.



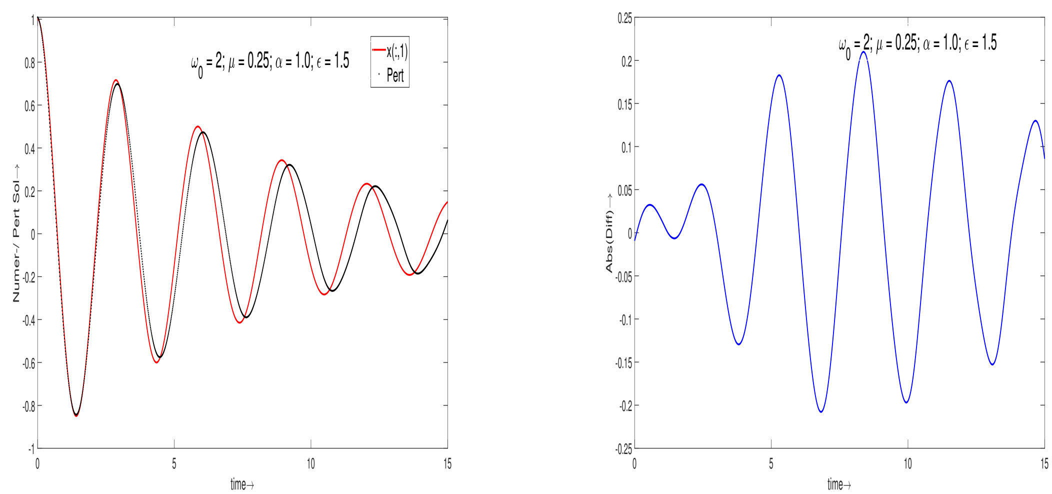
(a) **(b)**

***Figure* 2.1:** (a) Comparison of simulated result (shown in red color) and the homotopy based solution (shown in black color) of Duffing oscillator up to third order. (b) the evolution of absolute error.



(a) **(b)**

***Figure* 2.2:** (a) Comparison of simulated result (shown in red color) and the homotopy based solution (shown in black color) of Duffing oscillator up to third order. (b) the evolution of absolute error.



(a) **(b)**

***Figure* 2.3:** (a) Comparison of simulated result (shown in red color) and the homotopy based solution (shown in black color) of Duffing oscillator up to third order. (b) the evolution of absolute error.

2.3 Amplitude, Frequency and Stability condition with $\mu > 0$

Applying conditions (2.18), (2.20) and (2.22), in equations (2.12) and (2.13), the frequency ω and the amplitude A are obtained in terms of damping coefficient μ and nonlinearity parameter ϵ, respectively, as

$$\omega^2 = \omega_0^2 + \frac{3}{4}\,\epsilon\alpha^2 - \frac{1}{4}\mu^2 - \frac{3}{4}\epsilon\alpha^2\mu t + \frac{3}{128}\frac{\epsilon^2\alpha^4}{\omega^2} + \frac{3}{8}\epsilon\alpha^2\mu t\left\{\frac{1}{16}\epsilon\alpha^2 + \mu t\right\} - \frac{3}{4096}\frac{\epsilon^3\alpha^6}{\omega^4}. \tag{2.26}$$

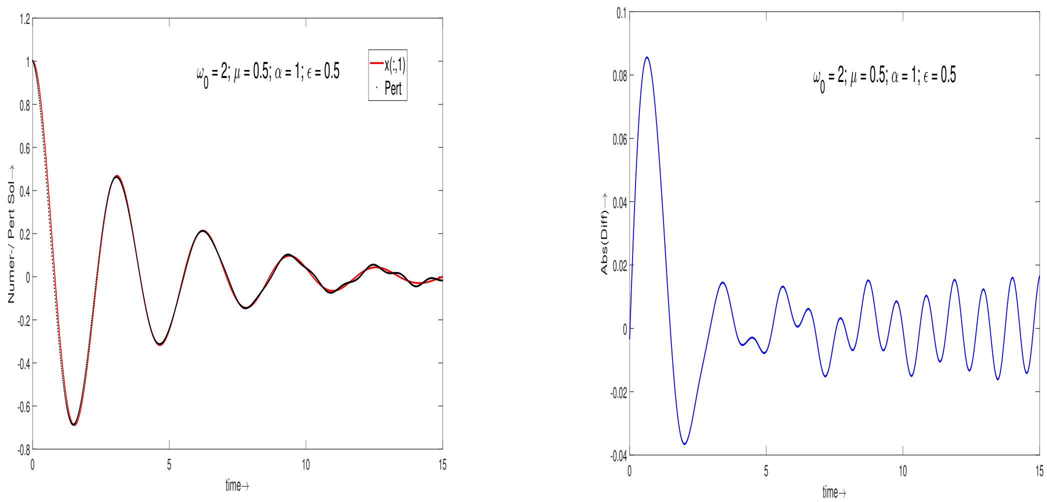
$$\implies \omega^2 = \omega_0^2 - \frac{1}{4}\mu^2 + \frac{3}{4}\,\epsilon\alpha^2\left\{1 - \mu\, t + \frac{1}{2}\mu^2 t^2\right\} + \frac{3}{128}\frac{\epsilon^2\alpha^4}{\omega^2} + \frac{3}{128}\epsilon^2\alpha^4\mu t - \frac{3}{4096}\frac{\epsilon^3\alpha^6}{\omega^4}.$$

$$A(t) = \alpha \left\{ 1 - \frac{1}{2}\mu\, t \frac{1}{8}\mu^2\, t^2 - \frac{1}{48}\mu^3\, t^3 + \frac{9}{1024}\, \frac{\epsilon^2\alpha^4}{\omega^4}\mu\, t \right\}. \tag{2.27}$$

Following [5], equation (2.26) and (2.27) may be written as

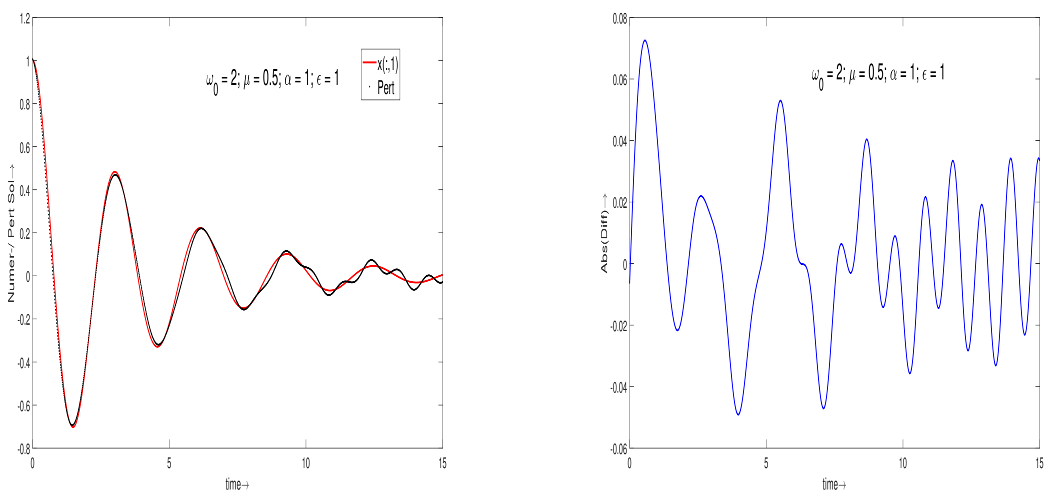
$$\omega^2 = \omega_0^2 - \frac{1}{4}\mu^2 + \frac{3}{4}\,\epsilon\alpha^2\, e^{-\mu\, t} + \frac{3}{128}\epsilon^2\alpha^4\mu t + \frac{3}{128}\,\frac{\epsilon^2\alpha^4}{\omega_0^2} - \frac{3}{128}\,\frac{\epsilon^2\alpha^4}{\omega_0^4}\omega_1 - \frac{3}{4096}\,\frac{\epsilon^3\alpha^6}{\omega_0^4}, \tag{2.28}$$

$$A(t) = \alpha \left\{ e^{-\frac{1}{2}\mu\, t} + \frac{9}{1024}\, \frac{\epsilon^2\alpha^4}{\omega^4}\mu\, t \right\}. \tag{2.29}$$



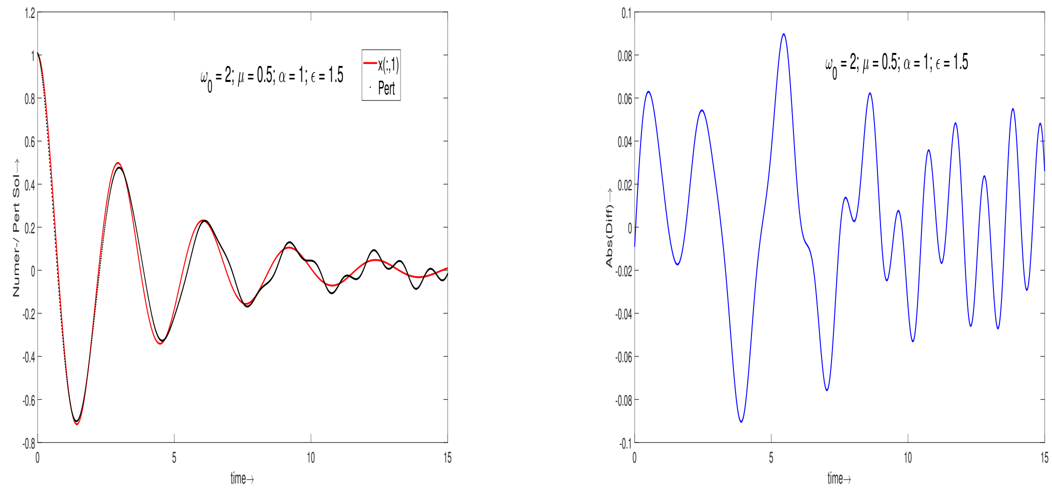
(a) **(b)**

***Figure* 2.4:** (a) Comparison of simulated result (shown in red color) and the homotopy based solution (shown in black color) of Duffing oscillator up to third order. (b) the evolution of absolute error.



(a) **(b)**

***Figure* 2.5:** (a) Comparison of simulated result (shown in red color) and the homotopy based solution (shown in black color) of Duffing oscillator up to third order. (b) the evolution of absolute error.



(a) **(b)**

***Figure* 2.6:** (a) Comparison of simulated result (shown in red color) and the homotopy based solution (shown in black color) of Duffing oscillator up to third order. (b) the evolution of absolute error.

For the values of parameters considered for numerical simulation, it is observed that the third term on the right hand side decreases at a faster rate than the corresponding rise in the fourth term and thereby resulting in a constant value of the frequency on larger time scale, say, $\omega_f > 0$, which may be considered also as a stability condition for the system.

3 Conclusion

In the present work, numerical simulation of the linearly damped Duffing system has been carried out by fixing the values of the initial amplitude, $\alpha = 1$ and the frequency, $\omega_0 = 2.0$, keeping the damping parameter (i) $\mu = 0.25$ and (ii) $\mu = 0.5$ while varying the nonlinearity parameter, ϵ. The modified version of the homotopy based perturbative solution, as obtained in equation (2.25), for various parameters are subsequently compared with the direct numerical simulation results. It is observed that the *HPM* based solutions compares well with those obtained numerically (Figs.2.1a-2.6a). The magnitude of errors between simulated and *HPM* based solution are observed to be nominal for $\epsilon < 1.0$. However, it is also observed that for lower values of damping parameter μ, noticeable changes in phase relationship between the numerical and *HPM* based solution occurs for moderately higher values of the nonlinearity parameter *i.e.*, $\epsilon \sim 1.5$. (Figs. 2.1b-2.6b) further illustrate the time variation of small deviation between the *HPM* and simulated solutions for various control parameters.

The foregoing *HPM* method allows one to obtain solution of the non-conservative Duffing system with larger damping coefficient (μ) and nonlinearity parameter (ϵ). We plan to use it subsequently to analyze the complex response of micro-nanosystems *i.e.*, , resonator used for mass detection, vibration of carbon nanotube, micro tubules, etc., which play important role in biological system [3].

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