ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(2) (2023), 177-190**
(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

CERTAIN INTEGRALS OF PRODUCT OF MITTAG-LEFFLER FUNCTION, M-SERIES AND I-FUNCTION OF TWO VARIABLES

Dheerandra Shanker Sachan and Giriraj Singh
St.Mary's Postgraduate College, Vidisha, Madhya Pradesh, India-464001
Email: sachan.dheerandra17@gmail.com, singh.giriraj392@gmail.com
(Received: August 30, 2023; In format: September 12, 2023; Revised: October 05, 2023; Accepted: November 16, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53221

Abstract

The object of this paper is to establish certain unified integrals associated with I-function of two variables. First, we have evaluated integrals whose integrand is the product of generalized Mittag-Leffler function, generalized M-series and I-function of two variables. Moreover, the integrand of the last integral is the product of generalized Mittag-Leffler function, generalized M-series, H-function of one variables and I-function of two variables. We have evaluated this integral by means of Mellin transform of H-function of one variables. In consequence of general nature of I-function of two variables, some special cases also have been considered.
2020 Mathematical Sciences Classification : 33B15, 33E12, 33C60, 44A20.
Keywords and Phrases : Generalized Mittag-Leffler function, Generalized M-series, Fox's H-function, Mellin transform of H-function, I-function of two variables, Mellin-Barnes type integrals.

1 Introduction

Using various special functions, numerous integrals have been established. For example, in 2003, Garg and Mittal [13] obtained new unified integrals whose integrands contain product of general class of polynomial and H-function having general arguments. Saha *et al.*[19] presented certain new type of integrals having the product of I-function with exponential function, hypergeometric function and H-function in 2011. In 2011, Agarwal *et al.*[1] established some new finite integrals containing Jacobi polynomials and I-function of one variable. In 2019, Agarwal *et al.*[2] established some new integral formulas with the involvement of $\aleph$-function associated with Laguerre-type polynomials. Abeye and Suthar[3] evaluated three definite integrals involving the $\overline{H}$-function together with the Srivastava's general class of polynomial in 2019. Ayant *et al.*[4] established two finite integrals containing the product of Legendre function, generalized hypergeometric function and the modified generalized multivariable I-function in 2020. For similar work, we may also refer to Kumar *et al.*[15], Suthar *et al.* [20], Bohara and Jain [6], Singh and Chandel [32], Goyal and Agrawal [12].

Motivated by these results, in this paper, we have established certain unified integrals associated with two variable's I-function defined by Goyal and Agrawal[11]. In first to sixth integrals, we have evaluated integrals whose integrand is the product of generalized Mittag-Leffler function, generalized M-series and I-function of two variables. The integrand of the seventh integral is the product of generalized Mittag-Leffler function, generalized M-series, H-function of one variables and I-function of two variables. We have evaluated this integral by means of Mellin transform of H-function of one variables. The results of all the integrals are expressed in terms of I-function of two variables.

The results evaluated here are quite general and a large number of known and new integrals can be evaluated as special cases by specializing the parameters in I-function of two variables. For the sake of illustrations, we have recorded some special cases of our main findings at the end of the paper.

The I-function of two variables defined by Goyal and Agrawal[11] in 1995 due to double Mellin-Barnes type contour integral and they discussed the asymptotic behavior and convergence conditions also. The I-function of two variables is very general in nature and specializing the parameters we obtain I-function of one variable, H-function of one variable, H-function of two variables and many more as its special case. For current research of I-function of two variables, see [26, 28].

The I-function of two variables is expressed in the following manner:

(1.1)

$$I^{m_1,n_1:m_2,n_2;m_3,n_3}_{p,q:p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r}\left[\begin{array}{c|l} z_1 & [(e_p:E_p,E_p^{'})]:[(a_\tau,\alpha_\tau)_{1,n_2}],[(a_{\tau i},\alpha_{\tau i})_{n_2+1,p_i^{(1)}}];[(c_\tau,\gamma_\tau)_{1,n_3}],[(c_{\tau i},\gamma_{\tau i})_{n_3+1,p_i^{(2)}}] \\ z_2 & [(f_q:F_q,F_q^{'})]:[(b_\tau,\beta_\tau)_{1,m_2}],[(b_{\tau i},\beta_{\tau i})_{m_2+1,q_i^{(1)}}];[(d_\tau,\delta_\tau)_{1,m_3}],[(d_{\tau i},\delta_{\tau i})_{m_3+1,q_i^{(2)}}] \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}\,d\xi\,d\eta,$$

where $\omega=\sqrt{-1}$ and $\phi_1(\xi)$, $\phi_2(\eta)$, $\psi(\xi,\eta)$ are given by

(1.2) $$\phi_1(\xi)=\frac{\prod_{\tau=1}^{m_2}\Gamma(\,b_\tau-\beta_\tau\xi\,)\prod_{\tau=1}^{n_2}\Gamma(\,1-a_\tau+\alpha_\tau\xi\,)}{\sum_{i=1}^{r}\left[\prod_{\tau=m_2+1}^{q_i^{(1)}}\Gamma(1-b_{\tau i}+\beta_{\tau i}\xi)\prod_{\tau=n_2+1}^{p_i^{(1)}}\Gamma(a_{\tau i}-\alpha_{\tau i}\xi)\right]},$$

(1.3) $$\phi_2(\eta)=\frac{\prod_{\tau=1}^{m_3}\Gamma(d_\tau-\delta_\tau\eta)\prod_{\tau=1}^{n_3}\Gamma(1-c_\tau+\gamma_\tau\eta)}{\sum_{i=1}^{r}\left[\prod_{\tau=m_3+1}^{q_i^{(2)}}\Gamma(1-d_{\tau i}+\delta_{\tau i}\eta)\prod_{\tau=n_3+1}^{p_i^{(2)}}\Gamma(c_{\tau i}-\gamma_{\tau i}\eta)\right]},$$

(1.4) $$\psi(\xi,\eta)=\frac{\prod_{\tau=1}^{m_1}\Gamma(f_\tau-F_\tau\xi-F_\tau^{'}\eta)\prod_{\tau=1}^{n_1}\Gamma(1-e_\tau+E_\tau\xi+E_\tau^{'}\eta)}{\prod_{\tau=m_1+1}^{q}\Gamma(1-f_\tau+F_\tau\xi+F_\tau^{'}\eta)\prod_{\tau=n_1+1}^{p}\Gamma(e_\tau-E_\tau\xi-E_\tau^{'}\eta)},$$

where an empty product is termed as unity, z_1, z_2 are two non zero complex variables, and L_1, L_2 are two Mellin-Barnes type contour integrals.

(i) $m_1,n_1;m_2,n_2;m_3,n_3$ and $p,q;\ p_i^{(1)},q_i^{(1)};\ p_i^{(2)},q_i^{(2)}$ are non-negative integers satisfying the conditions $0\le n_1\le p$, $0\le n_2\le p_i^{(1)}$, $0\le n_3\le p_i^{(2)}$, $0\le m_1\le q$, $0\le m_2\le q_i^{(1)}$, $0\le m_3\le q_i^{(2)}$ for all $i=1,2,3,\ldots,r$ where r is also a positive integer.

(ii) $\alpha_\tau(\tau=1,\ldots,n_2),\beta_\tau(\tau=1,\ldots,m_2),\gamma_\tau(\tau=1,\ldots,n_3),\delta_\tau(\tau=1,\ldots,m_3),\alpha_{\tau i}(\tau=n_2+1,\ldots,p_i^{(1)}),$ $\beta_{\tau i}(\tau=m_2+1,\ldots,q_i^{(1)}),\gamma_{\tau i}(\tau=n_3+1,\ldots,p_i^{(2)}),\delta_{\tau i}(\tau=m_3+1,\ldots,q_i^{(2)})$ are termed to be positive quantities for standardization purposes.$E_\tau,E_\tau^{'},F_\tau,F_\tau^{'}$ are also positive quantities.

(iii) $a_\tau(\tau=1,\ldots,n_2),b_\tau(\tau=1,\ldots,m_2),c_\tau(\tau=1,\ldots,n_3),d_\tau(\tau=1,\ldots,m_3),a_{\tau i}(\tau=n_2+1,\ldots,p_i^{(1)}),b_{\tau i}(\tau=m_2+1,\ldots,q_i^{(1)}),c_{\tau i}(\tau=n_3+1,\ldots,p_i^{(2)}),d_{\tau i}(\tau=m_3+1,\ldots,q_i^{(2)})$ are complex for all $i=1,2,3,\ldots,r$.

(iv) The contour L_1 lies in the complex ξ-plane which runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(\,b_\tau-\beta_\tau\xi\,)(\tau=1,\ldots,m_2),\Gamma(\,f_\tau-F_\tau\xi-F_\tau^{'}\eta\,)(\tau=1,\ldots,m_1)$ lies to the right and the poles of $\Gamma(\,1-a_\tau+\alpha_\tau\xi\,)(\tau=1,\ldots,n_2),\Gamma(\,1-e_\tau+E_\tau\xi+E_\tau^{'}\eta\,)(\tau=1,\ldots,n_1)$ to the left of the contour L_1.

(v) The contour L_2 lies in the complex η-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(\,d_\tau-\delta_\tau\eta)(\tau=1,\ldots,m_3),\Gamma(\,f_\tau-F_\tau\xi-F_\tau^{'}\eta)(\tau=1,\ldots,m_1)$ lies to the right and the poles of $\Gamma(\,1-c_\tau+\gamma_\tau\xi\,)(\tau=1,\ldots,n_3),\Gamma(\,1-e_\tau+E_\tau\xi+E_\tau^{'}\eta\,)(\tau=1,\ldots,n_1)$ to the left of the contour L_2. All the poles are simple poles.

Convergence conditions are as follows:

(1.5) $$|\arg\,z_1|<\frac{A_i\pi}{2},\ |\arg\,z_2|<\frac{B_i\pi}{2},$$

where

$$A_i=\sum_{\tau=1}^{n_1}E_\tau+\sum_{\tau=1}^{m_1}F_\tau-\sum_{\tau=n_1+1}^{p}E_\tau-\sum_{\tau=m_1+1}^{q}F_\tau+ \sum_{\tau=1}^{m_2}\beta_\tau+\sum_{\tau=1}^{n_2}\alpha_\tau-\sum_{\tau=m_2+1}^{q_i^{(1)}}\beta_{\tau i}-\sum_{\tau=n_2+1}^{p_i^{(1)}}\alpha_{\tau i}>0 \tag{1.6}$$

and

$$B_i=\sum_{\tau=1}^{n_1}E_\tau^{'}-\sum_{\tau=n_1+1}^{p}E_\tau^{'}+\sum_{\tau=1}^{m_1}F_\tau^{'}-\sum_{\tau=m_1+1}^{q}F_\tau^{'}+ \sum_{\tau=1}^{m_3}\delta_\tau-\sum_{\tau=m_3+1}^{q_i^{(2)}}\delta_{\tau i}+\sum_{\tau=1}^{n_3}\gamma_\tau-\sum_{\tau=n_3+1}^{p_i^{(2)}}\gamma_{\tau i}>0, \tag{1.7}$$

for $i=1,\ldots,r$.

For the sake of brevity throughout the paper, following notations will be used:
$P=m_2,n_2;m_3,n_3,$
$Q=p_i^{(1)},\,q_i^{(1)};\,p_i^{(2)},\,q_i^{(2)}:r,$
$U=[(a_\tau,\alpha_\tau)_{1,n_2}],[(a_{\tau i},\alpha_{\tau i})_{n_2+1,p_i^{(1)}}];[(c_\tau,\gamma_\tau)_{1,n_3}],[(c_{\tau i},\gamma_{\tau i})_{n_3+1,p_i^{(2)}}],$
$V=[(b_\tau,\,\beta_\tau)_{1,m_2}],[(b_{\tau i},\,\beta_{\tau i})_{m_2+1,q_i^{(1)}}];[(d_\tau,\delta_\tau)_{1,m_3}],[(d_{\tau i},\delta_{\tau i})_{m_3+1,q_i^{(2)}}],$

The generalized Mittag-Leffler function $E_{\theta,\varphi}(z)$ is a complex function involving two complex parameters θ and φ. It is defined by means of following series when $\Re(\theta)$ is strictly positive

$$E_{\theta,\varphi}\left(z\right)=\sum_{k\geq 0}\frac{z^k}{\Gamma\left(\theta k+\varphi\right)}. \tag{1.8}$$

If θ and φ are positive and real, the function converges for all z. By specializing the parameters, Mittag-Leffler function reduces to the exponential function, error function, hyperbolic sine function, hyperbolic cosine function.

This function was studied by Wiman [33] in 1905, Agrawal [5] in 1953, Humbert and Agrawal [14] in 1953 and Dzrbashjan [8, 9, 10]. Kilbas et al. [16] studied the several properties of the Mittag-Leffler function related to the generalized fractional calculus operators.

During the last some decades, the special importance to Mittag-Leffler function is given by the mathematicians due to its vast and vivid involvement to solve the problems of probability, engineering and statistical distribution theory. The solution of fractional order differential and integral equations occurs naturally in terms of Mittag-Leffler function.

A detailed description about the basic properties of Mittag-Leffler function has been described in the third volume of Batemann Manuscript Project which was written by Erdélyi et al in 1955. For current research of Mittag-Leffler function, see [29].

Sharma and Jain [24] introduced generalized M-series which is defined as

$$\begin{aligned}{}_pM_q^{\theta,\varphi}(z)&={}_pM_q^{\theta,\varphi}(c_1,\ldots,c_p;d_1,\ldots,d_q;z)\\&=\sum_{k\geq 0}\frac{(c_1)_k\ldots(c_p)_k}{(d_1)_k\ldots(d_q)_k}\frac{z^k}{\Gamma(\theta k+\varphi)},\end{aligned} \tag{1.9}$$

where $\theta,\varphi\in\mathbb{C}$, $z\in\mathbb{C}$, $\Re(\theta)>0$; $(c_\tau)_k$ $(\tau=1,\ldots,p)$ and $(d_\varsigma)_k$ $(\varsigma=1,\ldots,q)$ are Pochhammer symbols. The series (1.9) is defined when no parameters d_ς $(\varsigma=1,\ldots,q)$ is a negative integer or zero; if any numerator parameter c_τ is a negative integer or zero, then series terminates to a polynomial in z. The series (1.9) is convergent for all z if $p\leq q$; it is convergent for $|z|<\delta=\theta^\theta$ if $p=q+1$ and divergent if $p>q+1$. When $p=q+1$ and $|z|=\delta$, the series is convergent on conditions depending on the parameters. The detailed description of the M-Series can be seen in the paper [24]. The M-series has interesting relationship with various classical functions, for instance, see [25, 27, 30].

2 Required Results

We require following results for our study.

In view of Mellin inversion theorem and using the definition of H-function, The Mellin transform of H-function is given by

$$\int_0^\infty x^{s-1} H_{p,q}^{m,n}\left[ax\left| \begin{matrix} (a_\tau, \alpha_\tau)_{1,p} \\ (b_\tau, , \beta_\tau)_{1,q} \end{matrix}\right.\right] dx = a^{-s}\chi(-s) \tag{2.1}$$

$$= a^{-s} \frac{\prod_{\tau=1}^{m} \Gamma(b_\tau + \beta_\tau s) \prod_{\tau=1}^{n} \Gamma(1 - a_\tau - \alpha_\tau s)}{\prod_{\tau=m+1}^{q} \Gamma(1 - b_\tau - \beta_\tau s) \prod_{\tau=n+1}^{p} \Gamma(a_\tau + \alpha_\tau s)},$$

where

$$|\arg a| < \frac{\pi A}{2}, \; \delta = -\sum_{\tau=1}^{p} \alpha_\tau + \sum_{\tau=1}^{q} \beta_\tau > 0, \; A > 0,$$

$$A = \sum_{\tau=1}^{n} \alpha_\tau + \sum_{\tau=1}^{m} \beta_\tau - \sum_{\tau=n+1}^{p} \alpha_\tau - \sum_{\tau=m+1}^{q} \beta_\tau > 0,$$

and

$$-\min_{1\le\tau\le m} \Re\left(\frac{b_\tau}{\beta_\tau}\right) < \Re(s) < \min_{1\le\tau\le n} \Re\left(\frac{1-a_\tau}{\alpha_\tau}\right).$$

From Rainville [18], we have

$$\sum_{f\ge0}\sum_{u\ge0} A(u,f) = \sum_{f\ge0}\sum_{u=0}^{f} A(u, f-u), \tag{2.2}$$

$$\int_{-1}^{1} (1+x)^{\varsigma-1} (1-x)^{e-1} \, dx = 2^{\varsigma+e-1} B(\varsigma, e), \;\; \varsigma > 0, e > 0. \tag{2.3}$$

3 Main Results

In this section, we evaluate certain type of new unified integrals with the involvement of the product of I-function of two variables with generalized Mittag-Leffler function, generalized M-series and Fox's H-function.

Result 3.1.

$$I_1 \equiv \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1} E_{\mu,\lambda}\{(t-x)z\}\, {}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\} \tag{3.1}$$

$$\times I_{p,q:Q}^{m_1,n_1:P}\left[\begin{matrix} z_1 x^{\mu_1}(t-x)^{\nu_1} \\ z_2 x^{\mu_2}(t-x)^{\nu_2}\end{matrix}\left| \begin{matrix} [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')] : V \end{matrix}\right.\right] dx$$

$$= t^{\rho_1+\sigma_1-1} \sum_{k\ge0}\sum_{m=0}^{k} f(m) z^{k-m} t^{(\rho_2+\sigma_2-1)m+k}$$

$$\times I_{p+2,q+1:Q}^{m_1,n_1+2:P}\left[\begin{matrix} z_1 t^{\mu_1+\nu_1} \\ z_2 t^{\mu_2+\nu_2}\end{matrix}\left| \begin{matrix} E_1, [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')], E_2 : V \end{matrix}\right.\right],$$

where

$$E_1 = [(1-\rho_1-\rho_2 m : \mu_1, \mu_2)], [(1-\sigma_1-(\sigma_2-1)m-k : \nu_1, \nu_2)],$$
$$E_2 = [(1-\sigma_1-\rho_1-(\rho_2+\sigma_2-1)m-k : \mu_1+\nu_1, \mu_2+\nu_1)],$$

and

$$f(m) = \frac{a^m (a_1')_m \ldots (a_u')_m}{(b_1')_m \ldots (b_u')_m \Gamma(\mu(k-m)+\lambda)\Gamma(Gm+T)},$$

provided

(i) $\Re(\mu) > 0$, $\Re(\lambda) > 0$, $\Re(G) > 0$, $\Re(T) > 0$,
(ii) $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\nu_1 \geq 0$, $\nu_2 \geq 0$. (Not all zero simultaneously),
(iii) ρ_2, σ_2 are positive integers such that $\rho_2 + \sigma_2 \geq 1$,
(iv) $A_i > 0$, $B_i > 0$, $|\arg z_1| < \frac{\pi A_i}{2}$, $|\arg z_2| < \frac{\pi B_i}{2}$,
(v) $\Re(\rho_1) + \mu_1 \min\limits_{1\leq\tau\leq m_2} \Re\left(\frac{b_\tau}{\beta_\tau}\right) + \mu_2 \min\limits_{1\leq\tau\leq m_3} \Re\left(\frac{d_\tau}{\delta_\tau}\right) > 0,$

$$\Re(\sigma_1) + \nu_1 \min_{1\leq\tau\leq m_2} \Re\left(\frac{b_\tau}{\beta_\tau}\right) + \nu_2 \min_{1\leq\tau\leq m_3} \Re\left(\frac{d_\tau}{\delta_\tau}\right) > 0.$$

Proof.

$$I_1 \equiv \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1} E_{\mu,\lambda}\{(t-x)z\}\, {}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\}$$
$$\times I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c} z_1 x^{\mu_1}(t-x)^{\nu_1} \\ z_2 x^{\mu_2}(t-x)^{\nu_2} \end{array}\middle| \begin{array}{c} [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')] : V \end{array}\right] dx.$$

Now expressing Mittag-Leffler function and M-series in summation form and I-function in its well known Mellin-Barnes contour integral, we get

$$I_1 = \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1} \sum_{k\geq 0} \frac{(t-x)^k z^k}{\Gamma(\mu k+\lambda)} \sum_{m\geq 0} \frac{(a_1')_m \cdots (a_u')_m}{(b_1')_m \cdots (b_v')_m} \frac{a^m x^{\rho_2 m}(t-x)^{\sigma_2 m}}{\Gamma(Gm+T)}$$

$$\times \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^\xi\, z_2^\eta x^{\mu_1\xi} x^{\mu_2\eta}(t-x)^{\nu_1\xi}(t-x)^{\nu_2\eta}\, d\xi\, d\eta\, dx$$

$$= \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1} \sum_{k\geq 0}\sum_{m\geq 0} \frac{(a_1')_m \cdots (a_u')_m}{(b_1')_m \cdots (b_v')_m} \frac{z^k a^m x^{\rho_2 m}}{\Gamma(\mu k+\lambda)} \frac{(t-x)^{\sigma_2 m+k}}{\Gamma(Gm+T)}$$

$$\times \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^\xi\, z_2^\eta x^{\mu_1\xi+\mu_2\eta}(t-x)^{\nu_1\xi+\nu_2\eta}\, d\xi\, d\eta\, dx.$$

Now by an application of (2.2), the above result turns to

$$I_1 = \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1} \sum_{k\geq 0}\sum_{m=0}^{k} \frac{(a_1')_m \cdots (a_u')_m}{(b_1')_m \cdots (b_v')_m} \frac{z^{k-m} a^m x^{\rho_2 m}}{\Gamma(\mu(k-m)+\lambda)} \frac{(t-x)^{\sigma_2 m+k-m}}{\Gamma(Gm+T)}$$

$$\times \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^\xi\, z_2^\eta x^{\mu_1\xi+\mu_2\eta}(t-x)^{\nu_1\xi+\nu_2\eta}\, d\xi\, d\eta\, dx.$$

Changing the order of integral and summation which is valid due to the conditions mentioned with the equation (3.1), we obtain

$$I_1 = \sum_{k\geq 0}\sum_{m=0}^{k} f(m) z^{k-m} \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^\xi\, z_2^\eta$$

$$\times \left\{ \int_0^t x^{\mu_1\xi+\mu_2\eta+\rho_2 m+\rho_1-1}(t-x)^{\nu_1\xi+\nu_2\eta+\sigma_2 m+k-m+\sigma_1-1}\, dx \right\} d\xi\, d\eta,$$

where $f(m)$ is given with the integral (3.1).

On putting $x = st$ in the x-integral, the above expression becomes

$$I_1 = t^{\rho_1+\sigma_1-1} \sum_{k\geq 0}\sum_{m=0}^{k} f(m) z^{k-m} t^{(\rho_2+\sigma_2-1)m+k} \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^\xi\, z_2^\eta$$

$$\times t^{(\mu_1+\nu_1)\xi+(\mu_2+\nu_2)\eta}\left\{\int_0^1 s^{\mu_1\xi+\mu_2\eta+\rho_2 m+\rho_1-1}(1-s)^{\nu_1\xi+\nu_2\eta+\sigma_2 m+k-m+\sigma_1-1}\,ds\right\}d\xi\,d\eta$$

$$= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}$$

$$\times\frac{\Gamma(\mu_1\xi+\mu_2\eta+\rho_2 m+\rho_1)\Gamma(\nu_1\xi+\nu_2\eta+\sigma_2 m+k-m+\sigma_1)}{\Gamma(\mu_1\xi+\mu_2\eta+\rho_2 m+\rho_1+\nu_1\xi+\nu_2\eta+\sigma_2 m+k-m+\sigma_1)}t^{(\mu_1+\nu_1)\xi+(\mu_2+\nu_2)\eta}\,d\xi\,d\eta,$$

Finally, by re-arranging the double Mellin-Barnes contour integrals by means of I-function of two variables represented by (1.1) , we get

$$I_1 = t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k}$$
$$\times I_{p+2,q+1:Q}^{m_1,n_1+2:P}\left[\begin{array}{c} z_1 t^{\mu_1+\nu_1}\\ z_2 t^{\mu_2+\nu_2}\end{array}\middle|\begin{array}{c} E_1,[(e_p:E_p,E_p')]:U\\ [(f_q:F_q,F_q')],E_2:V\end{array}\right],$$

where E_1 and E_2 are given with (3.1). Hence the desired result.

Result 3.2.

$$(3.2)\qquad I_2 \equiv \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\}$$
$$\times I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c} z_1 x^{-\mu_1}(t-x)^{-\nu_1}\\ z_2 x^{-\mu_2}(t-x)^{-\nu_2}\end{array}\middle|\begin{array}{c} [(e_p:E_p,E_p')]:U\\ [(f_q:F_q,F_q')]:V\end{array}\right]dx$$
$$= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k}$$
$$\times I_{p+1,q+2:Q}^{m_1+2,n_1:P}\left[\begin{array}{c} z_1 t^{-\mu_1-\nu_1}\\ z_2 t^{-\mu_2-\nu_2}\end{array}\middle|\begin{array}{c} [(e_p:E_p,E_p')],E_3:U\\ E_4,[(f_q:F_q,F_q')]:V\end{array}\right],$$

where

$$E_3 = [((\rho_2+\sigma_2-1)m+\rho_1+\sigma_1+k:\mu_1+\nu_1,\mu_2+\nu_2)],$$
$$E_4 = [(\rho_1+\rho_2 m:\mu_1,\mu_2)],[((\sigma_2-1)m+\sigma_1+k:\nu_1,\nu_2)],$$

provided

$$\Re(\rho_1)-\left[\mu_1\max_{1\leq\tau\leq n_2}\Re\left(\frac{a_\tau-1}{\alpha_\tau}\right)+\mu_2\max_{1\leq\tau\leq n_3}\Re\left(\frac{c_\tau-1}{\gamma_\tau}\right)\right]>0,$$
$$\Re(\sigma_1)-\left[\nu_1\max_{1\leq\tau\leq n_2}\Re\left(\frac{a_\tau-1}{\alpha_\tau}\right)+\nu_2\max_{1\leq\tau\leq n_3}\Re\left(\frac{c_\tau-1}{\gamma_\tau}\right)\right]>0,$$

and also satisfies the conditions (i) to (iv) (3.1) and $f(m)$ is given with (3.1).

Result 3.3.

$$(3.3)\qquad I_3 \equiv \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\}$$
$$\times I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c} z_1 x^{\mu_1}(t-x)^{-\nu_1}\\ z_2 x^{\mu_2}(t-x)^{-\nu_2}\end{array}\middle|\begin{array}{c} [(e_p:E_p,E_p')]:U\\ [(f_q:F_q,F_q')]:V\end{array}\right]dx$$
$$= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k}$$
$$\times I_{p+1,q+2:Q}^{m_1+1,n_1+1:P}\left[\begin{array}{c} z_1 t^{\mu_1-\nu_1}\\ z_2 t^{\mu_2-\nu_2}\end{array}\middle|\begin{array}{c} E_5,[(e_p:E_p,E_p')]:U\\ E_6,[(f_q:F_q,F_q')],E_7:V\end{array}\right],$$

where

$$E_5=[(1-\rho_1-\rho_2 m:\mu_1,\mu_2)],$$
$$E_6=[(\sigma_1+(\sigma_2-1)m+k:\nu_1,\nu_2)],$$
$$E_7=[(1-\rho_1-\sigma_1-(\rho_2+\sigma_2-1)m-k:\mu_1-\nu_1,\mu_2-\nu_2)],$$

provided $\mu_1>0,\mu_2>0,\nu_1\geq 0,\nu_2\geq 0$ such that $\mu_1-\nu_1\geq 0,\mu_2-\nu_2\geq 0$,

$$\Re(\rho_1)+\mu_1\min_{1\leq\tau\leq m_2}\Re\left(\frac{b_\tau}{\beta_\tau}\right)+\mu_2\min_{1\leq\tau\leq m_3}\Re\left(\frac{d_\tau}{\delta_\tau}\right)>0,$$

$$\Re(\sigma_1)-\left[\nu_1\max_{1\leq\tau\leq n_2}\Re\left(\frac{a_\tau-1}{\alpha_\tau}\right)+\nu_2\max_{1\leq\tau\leq n_3}\Re\left(\frac{c_\tau-1}{\gamma_\tau}\right)\right]>0,$$

and also satisfies the conditions (i) to (iv) of (3.1) and $f(m)$ is given with (3.1).
Result 3.4.

$$\begin{aligned}(3.4)\qquad I_4\equiv&\int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\}\\&\times I^{m_1,n_1:P}_{p,q:Q}\left[\begin{array}{c}z_1x^{\mu_1}(t-x)^{-\nu_1}\\z_2x^{\mu_2}(t-x)^{-\nu_2}\end{array}\middle|\begin{array}{c}[(e_p:E_p,E_p^{'})]:U\\ [(f_q:F_q,F_q^{'})]:V\end{array}\right]dx\\=&\,t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k}f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k}\\&\times I^{m_1+1,n_1+1:P}_{p+2,q+1:Q}\left[\begin{array}{c}z_1t^{\mu_1-\nu_1}\\z_2t^{\mu_2-\nu_2}\end{array}\middle|\begin{array}{c}E_8,[(e_p:E_p,E_p^{'})],E_9:U\\E_{10},[(f_q:F_q,F_q^{'})]:V\end{array}\right],\end{aligned}$$

where

$$E_8=[(1-\rho_1-\rho_2 m:\mu_1,\mu_2)],$$
$$E_9=[(\rho_1+\sigma_1+k+(\rho_2+\sigma_2-1)m:\nu_1-\mu_1,\nu_2-\mu_2)],$$
$$E_{10}=[(\sigma_1+(\sigma_2-1)m+k:\nu_1,\nu_2)],$$

provided $\mu_1\geq 0,\mu_2\geq 0,\nu_1>0,\nu_2>0$ such that $\nu_1-\mu_1\geq 0,\nu_2-\mu_2\geq 0$,

$$\Re(\rho_1)+\mu_1\min_{1\leq\tau\leq m_2}\Re\left(\frac{b_\tau}{\beta_\tau}\right)+\mu_2\min_{1\leq\tau\leq m_3}\Re\left(\frac{d_\tau}{\delta_\tau}\right)>0,$$

$$\Re(\sigma_1)-\nu_1\max_{1\leq\tau\leq n_2}\Re\left(\frac{a_\tau-1}{\alpha_\tau}\right)-\nu_2\max_{1\leq\tau\leq n_3}\Re\left(\frac{c_\tau-1}{\gamma_\tau}\right)>0,$$

and also satisfies the conditions (i) to (iv) of (3.1) and $f(m)$ is given with (3.1).
Result 3.5.

$$\begin{aligned}(3.5)\qquad I_5\equiv&\int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\}\\&\times I^{m_1,n_1:P}_{p,q:Q}\left[\begin{array}{c}z_1x^{-\mu_1}(t-x)^{\nu_1}\\z_2x^{-\mu_2}(t-x)^{\nu_2}\end{array}\middle|\begin{array}{c}[(e_p:E_p,E_p^{'})]:U\\ [(f_q:F_q,F_q^{'})]:V\end{array}\right]dx\\=&\,t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k}f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k}\\&\times I^{m_1+1,n_1+1:P}_{p+2,q+1:Q}\left[\begin{array}{c}z_1t^{-\mu_1+\nu_1}\\z_2t^{-\mu_2+\nu_2}\end{array}\middle|\begin{array}{c}E_{11},[(e_p:E_p,E_p^{'})],E_{12}:U\\E_{13},[(f_q:F_q,F_q^{'})]:V\end{array}\right],\end{aligned}$$

where

$$E_{11}=[(1-\sigma_1-(\sigma_2-1)m-k:\nu_1,\nu_2)],$$

$$E_{12} = [(\rho_1 + \sigma_1 + (\rho_2 + \sigma_2 - 1)m + k : \mu_1 - \nu_1, \mu_2 - \nu_2)],$$
$$E_{13} = [(\rho_1 + \rho_2 m : \mu_1, \mu_2)],$$

provided $\mu_1 > 0, \mu_2 > 0, \nu_1 \geq 0, \nu_2 \geq 0$ such that $\mu_1 - \nu_1 \geq 0, \mu_2 - \nu_2 \geq 0$,

$$\Re(\rho_1) - \mu_1 \max_{1\leq\tau\leq n_2} \Re\left(\frac{a_\tau - 1}{\alpha_\tau}\right) - \mu_2 \max_{1\leq\tau\leq n_3} \Re\left(\frac{c_\tau - 1}{\gamma_\tau}\right) > 0,$$

$$\Re(\sigma_1) + \nu_1 \min_{1\leq\tau\leq m_2} \Re\left(\frac{b_\tau}{\beta_\tau}\right) + \nu_2 \min_{1\leq\tau\leq m_3} \Re\left(\frac{d_\tau}{\delta_\tau}\right) > 0,$$

and also satisfies the conditions (i) to (iv) of (3.1) and $f(m)$ is given with (3.1).
Result 3.6.

$$(3.6) \qquad I_6 \equiv \int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1} E_{\mu,\lambda}\{(t-x)z\}\, {}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\}$$
$$\times I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c} z_1 x^{-\mu_1}(t-x)^{\nu_1} \\ z_2 x^{-\mu_2}(t-x)^{\nu_2} \end{array}\middle| \begin{array}{c} [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')] : V \end{array}\right] dx$$

$$= t^{\rho_1+\sigma_1-1} \sum_{k\geq 0}\sum_{m=0}^{k} f(m) z^{k-m} t^{(\rho_2+\sigma_2-1)m+k}$$
$$\times I_{p+1,q+2:Q}^{m_1+1,n_1+1:P}\left[\begin{array}{c} z_1 t^{-\mu_1+\nu_1} \\ z_2 t^{-\mu_2+\nu_2} \end{array}\middle| \begin{array}{c} E_{14}, [(e_p : E_p, E_p')] : U \\ E_{15}, [(f_q : F_q, F_q')], E_{16} : V \end{array}\right],$$

where

$$E_{14} = [(1 - \sigma_1 - (\sigma_2 - 1)m - k : \nu_1, \nu_2)],$$
$$E_{15} = [(\rho_1 + \rho_2 m : \mu_1, \mu_2)],$$
$$E_{16} = [(1 - \rho_1 - \sigma_1 - (\rho_2 + \sigma_2 - 1)m - k : \nu_1 - \mu_1, \nu_2 - \mu_2)],$$

provided $\mu_1 \geq 0, \mu_2 \geq 0, \nu_1 > 0, \nu_2 > 0$ such that $\nu_1 - \mu_1 \geq 0, \nu_2 - \mu_2 \geq 0$,

$$\Re(\rho_1) - \left[\mu_1 \max_{1\leq\tau\leq n_2} \Re\left(\frac{a_\tau - 1}{\alpha_\tau}\right) + \mu_2 \max_{1\leq\tau\leq n_3} \Re\left(\frac{c_\tau - 1}{\gamma_\tau}\right)\right] > 0,$$

$$\Re(\sigma_1) + \nu_1 \min_{1\leq\tau\leq m_2} \Re\left(\frac{b_\tau}{\beta_\tau}\right) + \nu_2 \min_{1\leq\tau\leq m_3} \Re\left(\frac{d_\tau}{\delta_\tau}\right)\Bigg] > 0,$$

and also satisfies the conditions (i) to (iv) of (3.1) and $f(m)$ is given with (3.1).

The integrals (3.2) to (3.6) can be established on similar lines as of integral (3.1).
Result 3.7.

$$(3.7) \qquad I_7 \equiv \int_0^\infty x^{l-1} E_{\mu,\lambda}(ax)\, {}_uM_v^{G,T}(ax^\rho) I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c} z_1 x^{\sigma_1} \\ z_2 x^{\sigma_2} \end{array}\middle| \begin{array}{c} [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')] : V \end{array}\right]$$
$$\times H_{p',q'}^{m,n}\left[wx\middle| \begin{array}{c} (c_\tau',\, \gamma_\tau')_{1,n}, (c_\tau',\, \gamma_\tau')_{n+1,p'} \\ (d_\tau',\, \delta_\tau')_{1,m}, (d_\tau',\, \delta_\tau')_{m+1,q'} \end{array}\right] dx$$

$$= w^{-l} \sum_{k\geq 0}\sum_{m=0}^{k} f(m) w^{-(\rho-1)m-k} I_{p+q',q+p':Q}^{m_1+n,n_1+m:P}\left[\begin{array}{c} z_1 w^{-\sigma_1} \\ z_2 w^{-\sigma_2} \end{array}\middle| \begin{array}{c} E_{17}, [(e_p : E_p, E_p')], E_{18} : U \\ E_{19}[(f_q : F_q, F_q')], E_{20} : V \end{array}\right],$$

where

$$E_{17} = [(1 - d_\tau' - \delta_\tau'(k + (\rho-1)m + l) : \sigma_1\delta_\tau', \sigma_2\delta_\tau')_{1,m}],$$
$$E_{18} = [(1 - d_\tau' - \delta_\tau'(k + (\rho-1)m + l) : \sigma_1\delta_\tau', \sigma_2\delta_\tau')_{m+1,q'}],$$
$$E_{19} = [(1 - c_\tau' - \gamma_\tau'(k + (\rho-1)m + l) : \sigma_1\gamma_\tau', \sigma_2\gamma_\tau')_{1,n}],$$
$$E_{20} = [(1 - c_\tau' - \gamma_\tau'(k + (\rho-1)m + l) : \sigma_1\gamma_\tau', \sigma_2\gamma_\tau')_{n+1,p'}],$$

and

$$f(m)=\frac{a^k(a_1')_m...(a_u')_m}{(b_1')_m...(b_u')_m\Gamma(\mu(k-m)+\lambda)\Gamma(Gm+T)},$$

provided

(i) $\Re(\mu)>0,\ \Re(\lambda)>0,\ \Re(G)>0,\ \Re(T)>0,$

(ii) $A_i>0, |\arg z_1|<\frac{\pi A_i}{2},$

(iii) $B_i>0, |\arg z_2|<\frac{\pi B_i}{2},$

(iv) $\Delta>0, |\arg w|<\frac{\pi\Delta}{2},$

(v) $\Delta\geq 0, |\arg w|\leq\frac{\pi\Delta}{2}, \Re(\Omega+1)<0,$

(vi) $\sigma_1>0,\sigma_2>0,-\sigma_1\min_{1\leq\tau\leq m_2}\Re\left(\frac{b_\tau}{\beta_\tau}\right)-\sigma_2\min_{1\leq\tau\leq m_3}\Re\left(\frac{d_\tau}{\delta_\tau}\right)-\min_{1\leq\tau\leq m}\Re\left(\frac{d_\tau'}{\delta_\tau'}\right),$

$$<\Re(l)<\sigma_1\min_{1\leq\tau\leq n_2}\Re\left(\frac{1-a_\tau}{\alpha_\tau}\right)+\sigma_2\min_{1\leq\tau\leq n_3}\Re\left(\frac{1-c_\tau}{\gamma_\tau}\right)+\min_{1\leq\tau\leq n}\Re\left(\frac{1-c_\tau'}{\gamma_\tau'}\right),$$

where

$$\Delta=\sum_{\tau=1}^{m}\delta_\tau'+\sum_{\tau=1}^{n}\gamma_\tau'-\sum_{\tau=m+1}^{q'}\delta_\tau'-\sum_{\tau=n+1}^{p'}\gamma_\tau',$$

$$\Omega=\frac{1}{2}(p'-q')+\sum_{\tau=1}^{q'}d_\tau'-\sum_{\tau=1}^{p'}c_\tau',$$

A_i and B_i are same as given in (1.6) and (1.7).

Proof.

$$I_7\equiv\int_0^\infty x^{l-1}E_{\mu,\lambda}(ax)\,{}_uM_v^{G,T}(ax^\rho)I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c}z_1x^{\sigma_1}\\ z_2x^{\sigma_2}\end{array}\middle|\begin{array}{c}[(e_p:E_p,E_p')]:U\\ [(f_q:F_q,F_q')]:V\end{array}\right]$$
$$\times H_{p',q'}^{m,n}\left[wx\middle|\begin{array}{c}(c_\tau',\gamma_\tau')_{1,n},(c_\tau',\gamma_\tau')_{n+1,p'}\\ (d_\tau',\delta_\tau')_{1,m},(d_\tau',\delta_\tau')_{m+1,q'}\end{array}\right]dx,$$

now expressing Mittag-Leffler function and M-series in summation form and I-function in its well known Mellin-Barnes contour integral, we get

$$I_7=\int_0^\infty x^{l-1}\sum_{k\geq 0}\frac{a^kx^k}{\Gamma(\mu k+\lambda)}\sum_{m\geq 0}\frac{(a_1')_m\cdots(a_u')_m}{(b_1')_m\cdots(b_v')_m}\frac{a^mx^{\rho m}}{\Gamma(Gm+T)}$$
$$\times\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^\xi\,z_2^\eta x^{\sigma_1\xi}x^{\sigma_2\eta}$$
$$\times H_{p',q'}^{m,n}\left[wx\middle|\begin{array}{c}(c_\tau',\gamma_\tau')_{1,n},(c_\tau',\gamma_\tau')_{n+1,p'}\\ (d_\tau',\delta_\tau')_{1,m},(d_\tau',\delta_\tau')_{m+1,q'}\end{array}\right]d\xi\,d\eta\,dx$$
$$=\int_0^\infty x^{l-1}\sum_{k\geq 0}\sum_{m\geq 0}\frac{(a_1')_m\cdots(a_u')_m}{(b_1')_m\cdots(b_v')_m}\frac{a^{k+m}}{\Gamma(\mu k+\lambda)}\frac{x^{k+\rho m}}{\Gamma(Gm+T)}$$
$$\times\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^\xi\,z_2^\eta x^{\sigma_1\xi}x^{\sigma_2\eta}$$
$$\times H_{p',q'}^{m,n}\left[wx\middle|\begin{array}{c}(c_\tau',\gamma_\tau')_{1,n},(c_\tau',\gamma_\tau')_{n+1,p'}\\ (d_\tau',\delta_\tau')_{1,m},(d_\tau',\delta_\tau')_{m+1,q'}\end{array}\right]d\xi\,d\eta\,dx.$$

Now with an appeal to (2.2), the above mentioned result reduces to

$$I_7 = \int_0^\infty \sum_{k\geq 0}\sum_{m=0}^{k} \frac{(a'_1)_m\cdots(a'_u)_m}{(b'_1)_m\cdots(b'_v)_m}\frac{a^k}{\Gamma(\mu(k-m)+\lambda)}\frac{x^{k-m+\rho m+l-1}}{\Gamma(Gm+T)}$$
$$\times\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}x^{\sigma_1\xi}x^{\sigma_2\eta}$$
$$\times H^{m,n}_{p',q'}\left[wx\left|\begin{matrix}(c'_\tau,\,\gamma'_\tau)_{1,n},(c'_\tau,\,\gamma'_\tau)_{n+1,p'}\\(d'_\tau,\,\delta'_\tau)_{1,m},(d'_\tau,\,\delta'_\tau)_{m+1,q'}\end{matrix}\right.\right]d\xi\,d\eta\,dx.$$

Changing the order of integral and summation which is valid due to the convergence conditions given with (3.7),

$$I_7=\sum_{k\geq 0}\sum_{m=0}^{k} f(m)\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}\left\{\int_0^\infty x^{k+(\rho-1)m+l+\sigma_1\xi+\sigma_2\eta-1}\right.$$
$$\left.\times H^{m,n}_{p',q'}\left[wx\left|\begin{matrix}(c'_\tau,\,\gamma'_\tau)_{1,n},(c'_\tau,\,\gamma'_\tau)_{n+1,p'}\\(d'_\tau,\,\delta'_\tau)_{1,m},(d'_\tau,\,\delta'_\tau)_{m+1,q'}\end{matrix}\right.\right]dx\right\}d\xi\,d\eta,$$

where $f(m)$ is given with (3.7).
Now using Mellin transform of H-function by means of (2.1), we obtain

$$I_7=w^{-l}\sum_{k\geq 0}\sum_{m=0}^{k} f(m)w^{-(\rho-1)m-k}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,(z_1w^{-\sigma_1})^{\xi}\,(z_2w^{-\sigma_2})^{\eta}$$
$$\times\frac{\prod_{\tau=1}^{m}\Gamma(d'_\tau+\delta'_\tau(l+(\rho-1)m+k+\sigma_1\xi+\sigma_2\eta))}{\prod_{\tau=m+1}^{q'}\Gamma(1-d'_\tau-\delta'_\tau(l+(\rho-1)m+k+\sigma_1\xi+\sigma_2\eta))}$$
$$\times\frac{\prod_{\tau=1}^{n}\Gamma(1-c'_\tau-\gamma'_\tau(l+(\rho-1)m+k+\sigma_1\xi+\sigma_2\eta))}{\prod_{\tau=n+1}^{p'}\Gamma(c'_\tau+\gamma'_\tau(l+(\rho-1)m+k+\sigma_1\xi+\sigma_2\eta))}$$

$$=w^{-l}\sum_{k\geq 0}\sum_{m=0}^{k} f(m)w^{-(\rho-1)m-k}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,(z_1w^{-\sigma_1})^{\xi}\,(z_2w^{-\sigma_2})^{\eta}$$
$$\times\frac{\prod_{\tau=1}^{m}\Gamma(d'_\tau+\delta'_\tau(l+(\rho-1)m+k)+\sigma_1\delta'_\tau\xi+\sigma_2\delta'_\tau\eta)}{\prod_{\tau=m+1}^{q'}\Gamma(1-d'_\tau-\delta'_\tau(l+(\rho-1)m+k)-\sigma_1\delta'_\tau\xi-\sigma_2\delta'_\tau\eta)}$$
$$\times\frac{\prod_{\tau=1}^{n}\Gamma(1-c'_\tau-\gamma'_\tau(l+(\rho-1)m+k)-\sigma_1\gamma'_\tau\xi-\sigma_2\gamma'_\tau\eta))}{\prod_{\tau=n+1}^{p'}\Gamma(c'_\tau+\gamma'_\tau(l+(\rho-1)m+k)+\sigma_1\gamma'_\tau\xi+\sigma_2\gamma'_\tau\eta)}.$$

Finally, by re-arranginging the double MB contour integrals by means of two variables I-function defined by (1.1), we establish

$$I_7=w^{-l}\sum_{k\geq 0}\sum_{m=0}^{k} f(m)w^{-(\rho-1)m-k}I^{m_1+n,n_1+m:P}_{p+q',q+p':Q}\left[\begin{matrix}z_1w^{-\sigma_1}\\z_2w^{-\sigma_2}\end{matrix}\left|\begin{matrix}E_{17},[(e_p:E_p,E'_p)],E_{18}:U\\E_{19}[(f_q:F_q,F'_q)],E_{20}:V\end{matrix}\right.\right],$$

where E_{17}, E_{18},E_{19} and E_{20} are given with (3.7). Hence the desired result.

4 Special Cases

I-function of two variables is of very general nature, it can be reduced in a large number of special functions by suitably specializing the parameters involved in the function. Here we record some special cases of main results.

(i) If we set $m_1 = 0$ and $r = 1$ in integral (3.1), the I-function of two variables occurring in integral (3.1) reduces into two variable's H-function [23] then we have following result

$$\int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\} \tag{4.1}$$
$$\times H^{0,n_1:m_2,n_2;m_3,n_3}_{p,q:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1x^{\mu_1}(t-x)^{\nu_1} \\ z_2x^{\mu_2}(t-x)^{\nu_2}\end{array}\middle| \begin{array}{c}[(e_p:E_p,E_p^{'})]:T_1 \\ [(f_q:F_q,F_q^{'})]:T_2\end{array}\right]dx$$
$$= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k}$$
$$\times H^{0,n_1+2:m_2,n_2;m_3,n_3}_{p+2,q+1:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1t^{\mu_1+\nu_1} \\ z_2t^{\mu_2+\nu_2}\end{array}\middle| \begin{array}{c}E_1,[(e_p:E_p,E_p^{'})]:T_1 \\ [(f_q:F_q,F_q^{'})],E_2:T_2\end{array}\right],$$

where
$$T_1 = [(a_\tau,\alpha_\tau)_{1,p_1^{(1)}}];[(c_\tau,\gamma_\tau)_{1,p_1^{(2)}}], \quad T_2 = [(b_\tau,\beta_\tau)_{1,q_1^{(1)}}];[(d_\tau,\delta_\tau)_{1,q_1^{(2)}}].$$

Also E_1, E_2 and $f(m)$ are similar as given with integral (3.1).

The validity conditions of above mentioned result easily followed from integral (3.1).

(ii) If we set $m_1 = n_1 = p = q = 0$ in the integral (3.1) then we have following result in terms of product of I-function of one variable introduced by Saxena [31].

$$\int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\} \tag{4.2}$$
$$\times I^{m_2,n_2}_{p_i^{(1)},q_i^{(1)}:r}\left[z_1x^{\mu_1}(t-x)^{\nu_1}\middle| \begin{array}{c}(a_\tau,\alpha_\tau)_{1,n_2},(a_{\tau i},\alpha_{\tau i})_{n_2+1,p_i^{(1)}} \\ (b_\tau,\beta_\tau)_{1,m_2},(b_{\tau i},\beta_{\tau i})_{m_2+1,q_i^{(1)}}\end{array}\right]$$
$$\times I^{m_3,n_3}_{p_i^{(2)},q_i^{(2)}:r}\left[z_2x^{\mu_2}(t-x)^{\nu_2}\middle| \begin{array}{c}(c_\tau,\gamma_\tau)_{1,n_3},(c_{\tau i},\gamma_{\tau i})_{n_3+1,p_i^{(2)}} \\ (d_\tau,\delta_\tau)_{1,m_3},(d_{\tau i},\delta_{\tau i})_{m_3+1,q_i^{(2)}}\end{array}\right]dx$$
$$= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k} I^{0,2:m_2,n_2;m_3,n_3}_{2,1:p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r}\left[\begin{array}{c} z_1t^{\mu_1+\nu_1} \\ z_2t^{\mu_2+\nu_2}\end{array}\middle| \begin{array}{c}E_1,...,:U \\ ...,E_2:V\end{array}\right],$$

where E_1, E_2 and $f(m)$ are similar as given in integral (3.1).

The validity conditions of above mentioned result easily followed from integral (3.1).

(iii) If we set $m_1 = n_1 = p = q = 0$ and $r = 1$ in the integral (3.1) then we have following result in terms of product of H-function of one variable [23].

$$\int_0^t x^{\rho_1-1}(t-x)^{\sigma_1-1}E_{\mu,\lambda}\{(t-x)z\}\,{}_uM_v^{G,T}\{ax^{\rho_2}(t-x)^{\sigma_2}\} \tag{4.3}$$
$$\times H^{m_2,n_2}_{p_1^{(1)},q_1^{(1)}}\left[z_1x^{\mu_1}(t-x)^{\nu_1}\middle| \begin{array}{c}(a_\tau,\alpha_\tau)_{1,p_1^{(1)}} \\ (b_\tau,\beta_\tau)_{1,q_1^{(1)}}\end{array}\right]$$
$$\times H^{m_3,n_3}_{p_1^{(2)},q_1^{(2)}}\left[z_2x^{\mu_2}(t-x)^{\nu_2}\middle| \begin{array}{c}(c_\tau,\gamma_\tau)_{1,p_1^{(2)}} \\ (d_\tau,\delta_\tau)_{1,q_1^{(2)}}\end{array}\right]dx$$
$$= t^{\rho_1+\sigma_1-1}\sum_{k\geq 0}\sum_{m=0}^{k} f(m)z^{k-m}t^{(\rho_2+\sigma_2-1)m+k} H^{0,2:m_2,n_2;m_3,n_3}_{2,1:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1t^{\mu_1+\nu_1} \\ z_2t^{\mu_2+\nu_2}\end{array}\middle| \begin{array}{c}E_1,...,:T_1 \\ ...,E_2:T_2\end{array}\right],$$

where E_1, E_2 and $f(m)$ are same as given in integral (3.1). T_1 and T_2 are also same as given in (4.1).
The validity conditions of above mentioned result easily followed from integral (3.1).
Special cases of the integral (3.2) to integrals (3.6) can be obtained on following similar procedure but we do not mention them here.
(iv) If we set $r = 1$ in integral (3.7), we obtain following result in terms of two variable H-function introduced by Prasad and Gupta[17],

$$(4.4)\qquad \int_0^\infty x^{l-1} E_{\mu,\lambda}(ax)\, {}_uM_v^{G,T}(ax^\rho) H^{m_1,n_1:m_2,n_2;m_3,n_3}_{p,q:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1x^{\sigma_1} \\ z_2x^{\sigma_2}\end{array}\middle| \begin{array}{c} [(e_p:E_p,E_p')]:T_1 \\ [(f_q:F_q,F_q')]:T_2\end{array}\right]$$
$$\times H^{m,n}_{p',q'}\left[wx\middle| \begin{array}{c}(c_\tau',\gamma_\tau')_{1,n},(c_\tau',\gamma_\tau')_{n+1,p'} \\ (d_\tau',\delta_\tau')_{1,m},(d_\tau',\delta_\tau')_{m+1,q'}\end{array}\right] dx$$

$$= w^{-l}\sum_{k\geq 0}\sum_{m=0}^{k} f(m) w^{-(\rho-1)m-k}$$
$$\times H^{m_1+n,n_1+m:m_2,n_2;m_3,n_3}_{p+q',q+p':p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1w^{-\sigma_1} \\ z_2w^{-\sigma_2}\end{array}\middle| \begin{array}{c} E_{17},[(e_p:E_p,E_p')],E_{18}:T_1 \\ E_{19}[(f_q:F_q,F_q')],E_{20}:T_2\end{array}\right],$$

where $E_{17}, E_{18}, E_{19}, E_{20}$ and $f(m)$ are same as given in integral (3.7). T_1 and T_2 are also similar as given in (4.1).
The validity conditions of above mentioned result easily followed from integral (3.7).
(v) If we set $m_1 = n_1 = p = q = 0$ in the integral (3.7) then we have following result in terms of product of I-function and H-function of one variable.

$$(4.5)\qquad \int_0^\infty x^{l-1} E_{\mu,\lambda}\{ax\}\, {}_uM_v^{G,T}\{ax^\rho\} \times I^{m_2,n_2}_{p_i^{(1)},q_i^{(1)}:r}\left[z_1x^{\sigma_1}\middle| \begin{array}{c}(a_\tau,\alpha_\tau)_{1,n_2},(a_{\tau i},\alpha_{\tau i})_{n_2+1,p_i^{(1)}} \\ (b_\tau,\beta_\tau)_{1,m_2},(b_{\tau i},\beta_{\tau i})_{m_2+1,q_i^{(1)}}\end{array}\right]$$

$$\times I^{m_3,n_3}_{p_i^{(2)},q_i^{(2)}:r}\left[z_2x^{\sigma_2}\middle| \begin{array}{c}(c_\tau,\gamma_\tau)_{1,n_3},(c_{\tau i},\gamma_{\tau i})_{n_3+1,p_i^{(2)}} \\ (d_\tau,\delta_\tau)_{1,m_3},(d_{\tau i},\delta_{\tau i})_{m_3+1,q_i^{(2)}}\end{array}\right] \times H^{m,n}_{p',q'}\left[wx\middle| \begin{array}{c}(c_\tau',\gamma_\tau')_{1,n},(c_\tau',\gamma_\tau')_{n+1,p'} \\ (d_\tau',\delta_\tau')_{1,m},(d_\tau',\delta_\tau')_{m+1,q'}\end{array}\right] dx$$

$$= w^{-l}\sum_{k\geq 0}\sum_{m=0}^{k} f(m) w^{-(\rho-1)m-k} I^{n,m:m_2,n_2;m_3,n_3}_{q',p':p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r}\left[\begin{array}{c} z_1w^{-\sigma_1} \\ z_2w^{-\sigma_2}\end{array}\middle| \begin{array}{c} E_{17},\cdots,E_{18}:U \\ E_{19},\cdots,E_{20}:V\end{array}\right],$$

where $E_{17}, E_{18}, E_{19}, E_{20}$ and $f(m)$ are similar as given in integral 3.7.
The validity conditions of above mentioned result easily followed from integral (3.7).
(vi) If we set $m_1 = n_1 = p = q = 0$ and $r = 1$ in the integral (3.7) then we have following result in terms of product of three H-function of one variable.

$$(4.6)\qquad \int_0^\infty x^{l-1} E_{\mu,\lambda}\{ax\}\, {}_uM_v^{G,T}\{ax^\rho\} \times H^{m_2,n_2}_{p_1^{(1)},q_1^{(1)}}\left[z_1x^{\sigma_1}\middle| \begin{array}{c}(a_\tau,\alpha_\tau)_{1,p_1^{(1)}} \\ (b_\tau,\beta_\tau)_{1,q_1^{(1)}}\end{array}\right]$$

$$\times H^{m_3,n_3}_{p_1^{(2)},q_1^{(2)}}\left[z_2x^{\sigma_2}\middle| \begin{array}{c}(c_\tau,\gamma_\tau)_{1,p_1^{(2)}} \\ (d_\tau,\delta_\tau)_{1,q_1^{(2)}}\end{array}\right] \times H^{m,n}_{p',q'}\left[wx\middle| \begin{array}{c}(c_\tau',\gamma_\tau')_{1,n},(c_\tau',\gamma_\tau')_{n+1,p'} \\ (d_\tau',\delta_\tau')_{1,m},(d_\tau',\delta_\tau')_{m+1,q'}\end{array}\right] dx$$

$$= w^{-l}\sum_{k\geq 0}\sum_{m=0}^{k} f(m) w^{-(\rho-1)m-k} H^{n,m:m_2,n_2;m_3,n_3}_{q',p':p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1w^{-\sigma_1} \\ z_2w^{-\sigma_2}\end{array}\middle| \begin{array}{c} E_{17},\cdots,E_{18}:T_1 \\ E_{19},\cdots,E_{20}:T_2\end{array}\right],$$

where $E_{17}, E_{18}, E_{19}, E_{20}$ and $f(m)$ are similar as given in integral (3.7). T_1 and T_2 are also similar as given in (4.1).
The validity conditions of above mentioned result easily followed from integral (3.7).

5 Conclusion

This paper has successfully achieved its objective of establishing unified integrals associated with the I-function of two variables. Through a systematic exploration, the authors have derived integrals encompassing a wide range of mathematical functions, including the generalized Mittag-Leffler function, generalized M-series, and H-function of one variable in addition to the I-function of two variables. The utilization of the Mellin transform technique for the evaluation of the integral (3.7) demonstrates the versatility and effectiveness of the methods employed in this study.

Moreover, the authors have highlighted the generality of the I-function of two variables, allowing for the consideration of various special cases. This not only adds depth to the understanding of these integrals but also opens the door to potential applications in diverse areas of mathematics and science.

Acknowledgement. We are very much thankful to the Editor and Referee for valuable suggestions to prepare the paper in its present form.

References

[1] P. Agarwal, S. Jain and M. Chand, Finite integrals involving Jacobi polynomials and I-function, *Theoretical Mathematics & Applications*, **1** (1) (2011), 115-123.

[2] P. Agarwal , M. Chand and J. Choi, Some integrals involving ℵ-function and Laguerre polynomials, *Ukrainian Mathematical Journal*, **71** (9) (2019), 1321-1340.

[3] N. Abeye and D.L. Suthar, The $\overline{H}$-function and Srivastava's polynomials involving the generalized Mellin-Barnes contour integrals, *Jour. of Frac. Calc. and Appl.* **10** (2) (2019), 290-297.

[4] F. Ayant, Y.P. Kumar, N. Srimannarayana and B. Satyanarayana, Certain integrals and series expansions involving modified generalized I-function of Prasad, *Adv. Math. Sci. Journal*, **9** (8) (2020), 5835-5847.

[5] R.P. Agrawal, A propos d'une note M. Pierre Humbert, *C.R. Math. Acad. Sci., Paris* **236** (1953), 2031-2032.

[6] R.C. Bohara and U.C. Jain, Integrals involving the H-function of several variables II, *Jñānābha*, **12** (1982), 97-104.

[7] F. Brafman, Generating functions of Jacobi and related polynomials, *Proc. Amer. Math. Soc.,* **2** (1951), 942-949.

[8] M.M. Dzrbashjan, On the integral representation and uniqueness of some classes of entire functions, *Mat. Sb. (N.S.),* **33** (75) (1953), 485-530.

[9] M.M. Dzrbashjan, On the integral transformations generated by the generalized Mittag-Leffler functions, *Izv. AN Arm. SSR*, **13** (3) (1960), 21-63.

[10] M.M. Dzrbashjan, *Integral Transforms and Representations of Functions in the Complex Domain,* Nauka, Moscow, 1966.

[11] A. Goyal and R.D. Agrawal, Integrals involving the product of I-function of two variables, *Journal of Maulana Azad College of Technology*, **28** (1995), 147–155.

[12] A. Goyal and R.D. Agrawal, Integration of I-function of two variables with respect to parameters, *Jñānābha* **25** (1995), 87-91.

[13] M. Garg and S. Mittal, On a new unified integral, *Proc. Indian Acad. Sci. (Math. Sci.)*, **114** (2) (2003), 99-101.

[14] P. Humbert and R.P. Agrawal, Sur la fonction de Mittag-Leffler et quelques-unes de ses généralisations, *Bull. Sci. Math.,* **2** (77) (1953), 180-185.

[15] D. Kumar , F.Ayant and D.Kumar, A new class of integrals involving generalized hypergeometric function and multivariable Aleph-function, *Kragujevac Journal of Mathematics*, **44** (4) (2020), 539-550.

[16] A.A. Kilbas, M. Saigo and R.K. Saxena, Generalized Mittag-Leffler function and generalized fractional calculus operators, *Integral Transform and special Functions,* **15** (2004), 31-39.

[17] Y.N. Prasad and R.K. Gupta, An expansion formula for H-function of two variables and its applications, *Vijnana Parishad Anusandhan Patrika,* **19** (1976), 39-45.

[18] E.D. Rainville, *Special Functions*, Macmillan, New York, 1960; Reprinted by Chelsea Publ. Co., Bronx, New York, 1971.

[19] U.K. Saha, L.K. Arora and B.K. Dutta, Integrals involving I-function, *Gen. Math. Notes*, **6** (1) (2011), 1-14.

[20] D.L. Suthar , S. Agarwal and D. Kumar, Certain integrals involving the product of Gaussian hypergeometric function and Aleph function, *Honam Mathematical J.,* **41** (1) (2019), 1-17.

[21] H.M. Srivastava, A multilinear generating function for the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials, *Pacific J. Math.,* **117** (1985), 183-191.

[22] H.M. Srivastava and N.P. Singh, The integration of certain products of the multivariable H-function with a general class of polynomials, *Rend. Circ. Mat. Palermo Ser.* 2, **32** (1983), 157-187.

[23] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-function of one and two Variables with Applications,* South Asian Publications. New Delhi, Madras., 1982.

[24] M. Sharma and R. Jain, A Note on a Generalized M-Series as a special function of fractional calculus, *Fract. Calc. Appl. Anal.,* **12** (2009), 449-452.

[25] D.S. Sachan, D. Kumar and K.S. Nisar, Certain properties associated with generalized M-Series using Hadamard Product, *Sahand Commun. Math. Anal.,* (accepted).

[26] D.S. Sachan and S. Jaloree, Generalized fractional calculus of I-function of two variables, *Jñānābha,* **50** (1) (2020), 164-178.

[27] D.S. Sachan and S. Jaloree, Integral transforms of generalized M-Series, *Jour. of Frac. Calc. and Appl.*, **12** (1) (2021), 213–222.

[28] D.S. Sachan, H. Jalori and S. Jaloree, Fractional calculus of product of M-series and I-function of two variables, *Jñānābha,* **52** (1) (2022), 189-202.

[29] D.S. Sachan, S. Jaloree and J. Choi, Certain recurrence relations of two parametric Mittag-Leffler function and their application in fractional calculus, *Fractal Fract.*, **5** (2021), Article ID 215.

[30] D.S. Sachan, S. Jaloree, K.S. Nisar and A. Goyal, Some integrals involving generalized M-Series using Hadamard product, *Palest. J. Math.,* (accepted).

[31] V.P. Saxena, *The I-function.* Anamaya Publishers, New Delhi, 2008.

[32] R. Singh and R.S. Chandel, Integrals involving modified multivariable H-function, *Jñānābha,* **39** (2009), 57-66.

[33] A. Wiman, Über die nullsteliun der fuctionen $E_\alpha(x)$, *Acta Math.,* **29** (1905), 217-234.