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FUZZY CONTINUOUS AND FUZZY BOUNDED LINEAR OPERATORS OVER ANTI-FUZZY FIELDS

Parijat Sinha and Yogesh Chandra
Department of mathematics, V.S.S.D. College, Kanpur-208002, Uttar Pradesh, India
Email: parijatvssd@gmail.com,ycyogesh09@gmail.com

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Abstract

In this paper, we have studied the concept of anti-norm and anti-inner product function on anti-fuzzy linear space over anti-fuzzy field, we have also given fuzzy continuous linear operator from an anti-normed anti-fuzzy linear space to another anti-normed anti-fuzzy linear space and also introduced three types (strong, weak and sequential) of fuzzy bounded linear operators.

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1 Introduction

During the last few years there is a growing interest in the extension of fuzzy set theory which is a useful tool to describe the situation in which data are imprecise or vague or uncertain. Fuzzy set theory handles the situation, by attributing a degree of membership to which a certain object belongs to a set. The fundamental concept of fuzzy set theory was introduced by Zadeh [23] in 1965 and thereafter, the concept of fuzzy set theory applied on different branches of pure and applied mathematics in different ways. The fuzzy topology was introduced by Chang [4] in 1968, while the concept of fuzzy norm was introduced by Katsaras [9] in 1984. Thereafter Wu and Fang [20] introduced a fuzzy normed space. In 1991, Biswas [1] defined fuzzy norm and fuzzy inner product function on a linear space. In 1992, Felbin [8] introduced fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space. Another important approach of fuzzy norm on a linear space was introduced in 1994 by Cheng and Morderson [5], on a parallel line as the corresponding fuzzy metric due to Kramosil and Michelek [11] type. Krishna and Sarma [10], Xiao and Zhu [22] discussed fuzzy norms on linear spaces at different points of view. In 2005, Bag and Samanta [2], introduced an idea of fuzzy norm of a linear operator from a fuzzy normed linear space to another fuzzy normed linear space and defined various notions of continuities and boundedness of linear operators over fuzzy normed linear spaces such as fuzzy continuity, sequential fuzzy continuity, weakly fuzzy continuity, strongly fuzzy continuity, weakly and strongly fuzzy boundedness. All these Researchers have done their work in the area of crisp linear space. Wenxiang and Tu [21] were the first to introduce the concept of fuzzy fields and fuzzy linear spaces over fuzzy fields. In 2011, Santosh and Ramakrishnan [18] introduced norm and inner product on fuzzy linear spaces over fuzzy field. In 2012, Srinivas, Swamy and Nagaiah [19] introduced anti-fuzzy near-algebras over anti-fuzzy fields. In 2022, Barge and Yadav [3] defined (λ, μ)-anti-fuzzy linear spaces. In 2022, Chandra, Srivastava, and Sinha [6]; Srivastava, Sinha and Chandra [17] introduced 2-norm and 2-inner product on fuzzy linear spaces over fuzzy field. For more recent work of the area under study, we refer to [7,12,13,14,15,16]. In the present paper we introduce the idea of anti-norm and anti-inner product function on anti-fuzzy linear space over anti-fuzzy field and also given fuzzy continuous and fuzzy bounded linear operators on anti-fuzzy linear space over anti-fuzzy field.

2 Preliminaries

This section contains some definitions and preliminary results which are used in the paper.

Definition 2.1 ([21]). *Let X be a field and F a fuzzy set in X with the following conditions:*
(i) $F(x+y) \geq \min\{F(x), F(y)\}, x, y \in X,$
(ii) $F(-x) \geq F(x), x \in X,$

(iii) $F(xy) \geq \min\{F(x), F(y)\}, x, y \in X$,
(iv) $F\left(x^{-1}\right) \geq F(x), x(\neq 0) \in X$.
Then we call F *a fuzzy field in* X *and denoted by* (F, X) *and it is also called a fuzzy field of* X.

Theorem 2.1 ([21]). *If* (F, X) *is a fuzzy field of* X, *then*
(i) $F(0) \geq F(x), x \in X$.
(ii) $F(1) \geq F(x), x(\neq 0) \in X$.
(iii) $F(0) \geq F(1)$.

Theorem 2.2 ([21]). *Let* X *and* Y *be field and* f *a homomorphism of* X *into* Y *suppose that* (F, X) *is a fuzzy field of* X *and* (G, Y) *is a fuzzy field of* Y. *Then*
(i) $(f(F), Y)$ *is a fuzzy field of* Y.
(ii) $\left(f^{-1}(G), X\right)$ *is a fuzzy field of* X.

Definition 2.2 ([21]). *Let* X *be a field and* (F, X) *be a fuzzy field of* X. *Let* Y *be a linear space over* X *and* V *a fuzzy set of* Y. *Suppose the following condition hold:*
(i) $V(x+y) \geq \min\{V(x), V(y)\}, x, y \in Y$,
(ii) $V(\lambda x) \geq \min\{F(\lambda), V(x)\}, \lambda \in X, x \in Y$,
(iii) $V(-x) \geq V(x), x \in Y$,
(iv) $F(1) \geq V(0)$.
Then (V, Y) *is called a fuzzy linear space over* (F, X).

Theorem 2.3 ([21]). *If* (V, Y) *is a fuzzy linear space over fuzzy field* (F, X), *then*
(i) $F(0) \geq V(0)$.
(ii) $V(0) \geq V(x), x \in Y$.
(iii) $F(0) \geq V(x), x \in Y$.

Theorem 2.4 ([21]). *Let* (F, X) *be a fuzzy field of* X *and* Y *a linear space over* X. *Let* V *be a fuzzy set of* Y. *Then* (V, Y) *is a fuzzy linear space over* (F, X) *if and only if*
(i) $V(\lambda x+\mu y) \geq \min\{F(\lambda), F(\mu), V(x), V(y)\}, \lambda, \mu \in X$ *and* $x, y \in Y$.
(ii) $F(1) \geq V(x), x \in Y$.

Definition 2.3 ([18]). *Let (* $F, K)$ *be a fuzzy field of* K *(* K *denotes either* R *or* C *),* X *be a linear space over* K *and* (V, X) *be a fuzzy linear space over* (F, K). *A norm on* (V, X) *is a function* $|| \, || : X \rightarrow [0, \infty)$ *such that*
(i) $F(\|x\|) \geq V(x)$ *for all* $x \in X$,
(ii) $\|x\| \geq 0 \ \forall \ x \in X$ *and* $\|x\|=0$ *if and only if* $x=0$,
(iii) $\|x+y\| \leq\|x\|+\|y\|$ *for all* $x, y \in X$,
(iv) $\|k x\|=|k|\|x\|$ *for all* $k \in K$ *and for all* $x \in X$.
Then $(V, X, \|\|)$ *is called a normed anti-fuzzy linear space (NFLS) over fuzzy field.*

Definition 2.4 ([18]). *An inner product on a fuzzy linear space* (V, X) *over a fuzzy field* (F, K) *is a function* $\langle,\rangle :, X \times X \rightarrow K$ *such that for all* $x, y, z \in X$ *and* $k \in K$,
(i) $F(\langle x, y\rangle) \geq V \times V(x, y)$,
(ii) $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle=0$ *if and only if* $x=0$,
(iii) $\langle x+y, z\rangle=\langle x, z\rangle+\langle y, z\rangle$ *and* $\langle k x, y\rangle=k\langle x, y\rangle$,
(iv) $\langle y, x\rangle=\overline{\langle x, y\rangle}$.
Thus, $(V, X,\langle,\rangle)$ *is called an inner product on fuzzy linear space over fuzzy field.*

Definition 2.5 ([19]). *Let* X *be a field and* F *a fuzzy set in* X *with the following conditions:*
(i) $F(x+y) \leq \max\{F(x), F(y)\}, x, y \in X$,
(ii) $F(-x) \leq F(x), x \in X$,
(iii) $F(xy) \leq \max\{F(x), F(y)\}, x, y \in X$,
(iv) $F\left(x^{-1}\right) \leq F(x), x(\neq 0) \in X$.
An anti-fuzzy field F *of* X *is denoted by* (F, X).

Theorem 2.5 ([19]). *If* (F, X) *is an anti-fuzzy field of* X, *then*

(i) $F(0) \leq F(x)$, *for any* $x \in X$.
(ii) $F(1) \leq F(x)$, *for any* $x(\neq 0) \in X$.
(iii) $F(0) \leq F(1)$.

Definition 2.6 ([19])**.** *Let* X *be a field and* (F, X) *be an anti-fuzzy field of* X. *Let* Y *be a linear space over* X *and* V *a fuzzy set of* Y. *Suppose the following condition hold:*
(i) $V(x+y) \leq \max\{V(x), V(y)\}, x, y \in Y$
(ii) $V(\lambda x) \leq \max\{F(\lambda), V(x)\}, \lambda \in X, x \in Y$,
(iii) $V(-x) \leq V(x), x \in Y$,
(iv) $F(1) \leq V(0)$.
Then (V, Y) *is called an anti-fuzzy linear space over* (F, X).

3 Anti-norm and anti-inner product function on anti-fuzzy linear space over anti-fuzzy field

In this section, we define anti-norm and anti-inner product function on anti-fuzzy linear space over anti-fuzzy field and also establish relationship between them.

Here, K denotes either R (set of real numbers) or C (set of complex numbers).

Definition 3.1. *Let* (F, K) *be an anti-fuzzy field of* K, X *be a linear space over* K *and* (V, X) *be an anti-fuzzy linear space over* (F, K). *An anti-norm on* (V, X) *is function* $||.|| : X \to [0, \infty)$ *such that:*
(i) $F(\|x\|) \leq$ V(x) *for all* $x \in X$,
(ii) $\|x\| \geq 0 \ \forall \ x \in X$ *and* $\|x\| = 0$ *if and only if* $x = 0$,
(iii) $\|x+y\| \leq \|x\| + \|y\|$,
(iv) $\|kx\| \leq |k| \|x$, *for all* $k \in K$ *and for all* $x \in X$.
Then $(V, X, \|.\|)$ *is called anti-normed anti-fuzzy linear space (ANAFLS).*

Theorem 3.1. *Let* (V, X) *be an anti-fuzzy linear space over an anti-fuzzy field* $(F, K), Y$ *be a linear space over* K *and* T *be an isomorphism of* X *onto* Y. (V, X) *is an anti-normed anti-fuzzy linear space over* (F, K) *if and only if* $(T(V), Y)$ *is an anti-normed anti-fuzzy linear space over* (F, K).

Proof. Let $\| \cdot \|_{\mathrm{X}}$ be an anti-norm on (V, X). Let x $\in$ X so, T(x) $\in$ Y. Take T(x) = y. Now consider the anti-norm $\| \cdot \|_{Y}$ on Y defined $\|y\|_Y = \|x\|_X$. Then F $(\|y\|_Y) = F(\|x\|_X) \leq V(x) = T(V)T(x) = T(V)(y)$. Therefore $\| \cdot \|_{\mathrm{Y}}$ is an anti-norm on $(T(V), Y)$.

Conversely, assume that $\| \cdot \|_Y$ is an anti-norm on $(T(V), Y)$. Consider the anti-norm $\| \cdot \|_X$ on X as $\|x\|_X = \|Tx\|_Y$

Then $F(\|x\|_X) = F(\|Tx\|_Y) \leq T(V)(T(x)) = V(x)$.

Therefore, $\| \cdot \|_X$ is an anti-norm on (V, X). □

Theorem 3.2. *Let* X *be a linear space over* $K, (W, Y)$ *an anti-fuzzy linear space over an anti-fuzzy field* (F, K) *and* $T : X \to Y$ *be an injective linear transformation. If* (W, Y) *is an anti-normed anti-fuzzy linear space over* (F, K). *Then* $\left(T^{-1}(W), X\right)$ *is an anti-normed anti-fuzzy linear space over* (F, K).

Proof. Let $\|.\|_Y$ be an anti-norm on (W, Y). Consider the anti-norm $\|.\|_X$ on X as

$$\|x\|_X = \|Tx\|_Y \quad \text{Then}$$
$$F(\|x\|_X) = F(\|Tx\|_Y) \leq W\left(T(x) = T^{-1}W(x).\right.$$

Hence $\| \cdot \|_X$ is an anti-norm on $\left(T^{-1}(W), X\right)$. □

Theorem 3.3. *Let* (V, X) *be an anti-normed anti-fuzzy linear space over an anti-fuzzy field* (F, K) *and* $T : X \to X$ *be an injective linear transformation. Then* $\left(T^{-1}(V), X\right)$ *is an anti-normed anti-fuzzy linear space over* (F, K). *Proof. Obvious by Theorem 3.2.*

Definition 3.2. *An anti-inner product on an anti-fuzzy linear space* (V, X) *over an anti-fuzzy field* (F, K) *is a function* $\langle , \rangle : X \times X \to K$ *such that for all* $x, y, z \in X$ *and* $k \in K$,
(i) $F(\langle x, y \rangle) \leq V \times V(x, y)$
(ii) $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$
(iii) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ *and* $\langle kx, y \rangle = k \langle x, y \rangle$
(iv) $\langle y, x \rangle = \overline{\langle x, y \rangle}$.

Thus, $(V, X, \langle , \rangle)$ is called an anti-inner product on anti-fuzzy linear space over anti-fuzzy field.

Example 3.1. Let F be an anti-fuzzy field of R. The anti-inner product $\langle , \rangle$ on R^n defined by $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ is an anti-inner product on an anti-fuzzy linear space $(\underbrace{F \times F \times \ldots \times F}_{n\ times}, R^n)$.

Proof. Let $V = (\underbrace{F \times F \times \ldots \times F}_{n\ times}, R^n)$

$$\begin{aligned} F(\langle x, y \rangle) &= F(x_1y_1 + x_2y_2 + \cdots + x_ny_n) \\ &\leq \max\{F(x_1y_1), F(x_2y_2), \ldots, F(x_ny_n)\} \\ &\leq \max\{\max\{F(x_1), F(y_1)\}, \ldots, \max\{F(x_n), F(y_n)\}\} \\ &= \max\{\max\{F(x_1), \ldots, F(x_n)\}, \max\{F(y_1), \ldots, F(y_n)\}\} \\ &= \max\{V(x), V(y)\} \\ &= V \times V(x, y). \end{aligned}$$

So, $\langle , \rangle$ is an anti-inner product on $(\underbrace{F \times F \times \ldots \times F}_{n\ times}, R^n)$. □

Theorem 3.4. *If $\langle , \rangle$ is an anti-inner product on the anti-fuzzy linear space (V, X) over the anti-fuzzy field (F, K), then for all $x, y, z \in X$ and $k \in K$.*
(i) $F\langle \underline{x + y}, z \rangle \leq V \times V(x + y, z)$
(ii) $F(\overline{\langle x, y \rangle}) \leq V \times V(y, x)$
(iii) $F(\lambda \langle x, y \rangle) \leq V \times V(\lambda x, y)$.

Proof.

$$\begin{aligned} \text{(i) } F\langle x + y, z \rangle &= F\{\langle x, z \rangle + \langle y, z \rangle\} \\ &= F\langle x, z \rangle + F\langle y, z \rangle \\ &\leq V \times V(x, z) + V \times V(y, z) \\ &\leq V \times V(x + y, z). \\ \text{(ii) } \quad F(\overline{\langle x, y \rangle}) &= F(\langle y, x \rangle) \\ &\leq V \times V(y, x). \\ \text{(iii) } F(\lambda \langle x, y \rangle) &= F(\langle \lambda x, y \rangle). \\ &\leq V \times V(\lambda x, y). \end{aligned}$$

□

Theorem 3.5. *If $\langle , \rangle$ is an anti-inner product on the anti-fuzzy linear space (V, X) over the anti-fuzzy field (F, K), then*
(i) $F(\langle x + y, z \rangle) \leq \max\{V(x), V(y), V(z)\}$,
(ii) $F(\langle kx, y \rangle) \leq \max\{F(k), V(x), V(y)\}$.

Proof.

$$\begin{aligned} \text{(i) } F(\langle x + y, z \rangle) &\leq V \times V(x + y, z) \\ &= \max\{V(x + y), V(z)\} \\ &\leq \max\{\max\{V(x), V(y), V(z)\} \\ &= \max\{V(x), V(y), V(z)\}. \\ \text{(ii) } \quad F(\langle kx, y \rangle) &\leq V \times V(kx, y) \\ &= \max\{V(kx), V(y)\} \\ &\geq \max\{\max\{F(k), V(x), V(y)\} \\ &= \max\{F(k), V(x), V(y)\}. \end{aligned}$$

□

Theorem 3.6. *Let (V, X) be an anti-fuzzy linear space over an anti-fuzzy field $(F, K), Y$ a linear space over K and T is an isomorphism of X onto Y. Then there exists an anti-inner product on $(V,\ X)$ if and only if there exists an anti-inner product on $(T(V), Y)$.*

Proof. (⇒) Let $\langle , \rangle_X$ be an anti-inner product on (V, X). Consider the anti-inner product $\langle , \rangle_Y$ on Y defined by $\langle y_1, y_2\rangle_Y = \langle x_1, x_2\rangle_X$ where $y_1 = Tx_1$ and $y_2 = Tx_2$.
$F(\langle y_1, y_2\rangle_Y) = F(\langle x_1, x_2\rangle_X) \leq V \times V(x_1, x_2) = T(V) \times T(V)(Tx_1, Tx_2) = T(V) \times T(V)(y_1, y_2)$.

So $\langle , \rangle_Y$ is an anti-inner product on $(T(V), Y)$.

(⇐) Assume that $\langle , \rangle_Y$ is an anti-inner product on $(T(V), Y)$. Consider, the anti-inner product $\langle , \rangle_X$ on X defined by $\langle x_1, x_2\rangle_X = \langle Tx_1, Tx_2\rangle_Y$.

$$F\left(\langle x_1, x_2\rangle_X\right) = F\left(\langle Tx_1, Tx_2\rangle_Y \leq T(V) \times T(V)\left(Tx_1, Tx_2\right) = V \times V\left(x_1, x_2\right).\right.$$

So, $\langle , \rangle_X$ is an anti-inner product on (V, X). □

Theorem 3.7. *Let X be a linear space over $K, (W, Y)$ be an anti-fuzzy linear space over an anti-fuzzy field (F, X) and $T : X \to Y$ be an injective linear transformation. If there exists an anti-inner product on (W, Y), then there exists an anti-inner product on $\left(T^{-1}(W), X\right)$.*

Proof. Let $\langle , \rangle_Y$ be an anti-inner product on (W, Y). Consider the anti-inner product $\langle , \rangle_X$ on X defined by $\langle x_1, x_2\rangle_X = \langle Tx_1, Tx_2\rangle_Y$.

$$\begin{aligned} F\left(\langle x_1, x_2\rangle_X\right) &= F\left(\langle Tx_1, Tx_2\rangle_Y\right) \leq W \times W\left(Tx_1, Tx_2\right) = \max\left\{W\left(Tx_1\right), W\left(Tx_2\right)\right\} \\ &= \max\left\{T^{-1}(w)\left(x_1\right), T^{-1}(w)\left(x_2\right)\right\} = T^{-1}(w) \times T^{-1}(w)\left(x_1, x_2\right). \end{aligned}$$

Therefore $\langle , \rangle_X$ is an anti-inner product on $\left(T^{-1}(W), X\right)$. □

Theorem 3.8. *Let (V, X) be an anti-fuzzy linear space over (F, K) and $T : X \to X$ be an injective linear transformation. If there exists an anti-inner product on (V, X) then there exists an anti-inner product on $\left(T^{-1}(V), X\right)$.*

Proof. Let $\langle , \rangle_X$ be an anti-inner product on (V, X). Consider the anti-inner product $\langle , \rangle_X$ on X defined by $\langle x_1, x_2\rangle_X = \langle Tx_1, Tx_2\rangle_X$.

$$\begin{aligned} F\left(\langle x_1, x_2\rangle_X\right) &= F\left(\langle Tx_1, Tx_2\rangle_X\right) \leq V \times V\left(Tx_1, Tx_2\right) = \max\left\{V\left(Tx_1\right), V\left(Tx_2\right)\right\} \\ &= \max\left\{T^{-1}(v)\left(x_1\right), T^{-1}(v)\left(x_2\right)\right\} = T^{-1}(v) \times T^{-1}(v)\left(x_1, x_2\right). \end{aligned}$$

Therefore $\langle , \rangle_X$ is an anti-inner product on $\left(T^{-1}(V), X\right)$. □

Theorem 3.9. *Let (V, X) be an anti-fuzzy linear space over (F, K). An anti-norm on (V, X) satisfying the parallelogram law induces an anti-inner product on (V, X) if $F(4), F(i) \leq V(x)$ for all $x \in X$.*

Proof. If $\|.\|$ is an anti-norm on (V, X) satisfying the parallelogram law, then $\mathrm{F}(\|x\|) \leq V(x)$ for all $x \in X$ and $\|.\|$ induces an anti-inner product $\langle , \rangle$ on X given by

$$\begin{aligned} \langle x, y\rangle &= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right) \\ F(\langle x, y\rangle) &= F\left(\frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)\right) \\ &\leq \max\left\{F\left(\frac{1}{4}\right), F\left(\|x+y\|^2\right), \mathrm{F}\left(-\|x-y\|^2\right), \mathrm{F(i)}, \mathrm{F}\left(\|x+iy\|^2\right), \mathrm{F}\left(-\|x-iy\|^2\right)\right\} \\ &= \max\left\{F(4), \mathrm{F(i)}, F\left(\|x+y\|^2\right), \mathrm{F}\left(\|x-y\|^2\right), \mathrm{F}\left(\|x+iy\|^2\right), \mathrm{F}\left(\|x-iy\|^2\right)\right\} \\ &\leq \max\{F(4), \mathrm{F(i)}, F(\|x+y\|), \mathrm{F}(\|x-y\|), \mathrm{F}(\|x+iy\|), \mathrm{F}(\|x-iy\|)\} \\ &\leq \max\{F(4), F(i), V(x+y), V(x-y), V(x+iy), V(x-iy)\} \\ &\leq \max\{F(4), F(i), V(x), V(y)\} \\ &= \max\{V(x), V(y)\} \text{ if } \mathrm{F(4), F(i)} \leq \mathrm{V(x)} \text{ for all } x \in X \\ &= V \times V(x, y). \end{aligned}$$

Hence anti-norm induces an anti-inner product on (V, X) if $F(4), F(i) \leq V(x)$ for all $x \in X$. □

4 Fuzzy continuous mapping and fuzzy bounded linear operators

In this section we define different types of continuity such as weak fuzzy continuity, strong fuzzy continuity and sequential fuzzy continuity of an operator over anti-normed anti-fuzzy linear spaces. The notion of weakly fuzzy boundedness and strongly fuzzy boundedness are defined for linear operators over anti-normed anti-fuzzy linear spaces.

Definition 4.1. *A mapping T from $(V_1, X, \|.\|_1)$ to $(V_2, Y, \|.\|_2)$ is said to be weakly fuzzy continuous at $x_0 \in X$ if for each $\epsilon > 0, \exists\ \delta > 0$ such that $\forall\ x \in X$*

$\|T(x) - T(x_0)\|_2 < \epsilon$ *whenever* $\|x - x_0\|_1 < \delta$,

and $F\|x_0\|_1 \leq V_1(x_0)$ *and* $F\|Tx_0\|_2 \leq V_2 T(x_0)$.

If T is weakly fuzzy continuous at each point of X then we say that T is weakly fuzzy continuous on X.

Example 4.1. Let $T : (V_1, X, \|.\|_1) \to (V_2, Y, \|.\|_2)$ be a mapping where $(V_1, X, \|.\|_1)$ and $(V_2, Y, \|.\|_2)$ are anti-normed anti-fuzzy linear spaces where $\|x\|_1 = |x|$ and $\|x\|_2 = \frac{|x|}{2}$, and consider $T(x) = x$, here T is weakly fuzzy continuous.

Definition 4.2. *A mapping T from $(V_1, X, \|.\|_1)$ to $(V_2, Y, \|.\|_2)$ is said to be strongly fuzzy continuous at $x_0 \in X$ if for each $\epsilon > 0, \exists\ \ \delta > 0$ such that $\forall\ x \in X$*

$\|T(x) - T(x_0)\|_2 < \epsilon$ *whenever* $\|x - x_0\|_1 < \delta$,

and $\max\{F\|x_0\|_1, F\|Tx_0\|_2\} \leq \max\{V_1(x_0), V_2(Tx_0)\}$.

If T is strongly fuzzy continuous at each point of X then T is said to be strongly fuzzy continuous on X.

Example 4.2. Let $(V, X, \|.\|)$ be an anti-normed anti-fuzzy linear space where X = R and $\|x\| = |x|\ \forall\ x \in R$. Define two functions $\|.\|_1 \& \|.\|_2 : X \times R \to [0, 1]$ by $\|x\|_1 = |x|, \|x\|_2 = 2|x|$.

Then it can be easily verified that $\|x\|_1$ and $\|x\|_2$ are anti-norms on X and thus $(V, X\|x\|_1)$ and $(V, X\|x\|_2)$ are anti-normed anti-fuzzy linear spaces.

Now we consider a function $T(x) = 4x$. Therefore

$$\begin{aligned}
\|Tx - Tx_0\|_2 &= \|4x - 4x_0\|_2 \\
&= 2|4x - 4x_0| \\
&= 8|x - x_0| < \epsilon \\
&= |x - x_0| < \frac{\epsilon}{8} \\
&\|x - x_0\|_1 = |x - x_0| < \delta, \quad \text{Take } \delta = \frac{\epsilon}{8}. \\
&F(\|x_0\|_1) = F(|x_0|), \\
&F(\|Tx_0\|_2 = F(\|4x_0\|_2) = F(8|x_0|), \\
&V_1(x_0) \geq F(\|x_0\|_1) \quad \text{(by Def. 3.1(i))}. \\
&V_2(Tx_0) = V_2(4x_0) \geq F\|Tx_0\|_2 = F(\|4x_0\|_2) = F(8x_0), \\
&V_2(4x_0) \geq F(8x_0). \\
\max\{F(\|x_0\|_1), F(\|Tx_0\|_2)\} &\leq \max\{V_1(x_0), V_2 T(x_0)\} \\
&\leq \max\{V_1(x_0), V_2(Tx_0)\}.
\end{aligned}$$

Hence it is strongly fuzzy Continuous.

Definition 4.3. *A mapping T from anti-normed anti-fuzzy linear space $(V_1, \|.\|_1, X)$ to anti-normed anti-fuzzy linear space $(V_2, \|.\|_2, Y)$ over (F, K) is said to be sequentially fuzzy continuous at x_0 if for any sequence $\{x_n\}$ with $x_n \to x_0 \Rightarrow T(x_n) \to T(x_0)$*

$$\begin{aligned}
&\|T(x_n) - T(x_0)\|_2 \to 0 \ \textit{Whenever}\ \|x_n - x_0\|_1 \to 0 \ \textit{and} \\
&F\|x_n - x_0\|_1 \leq V_1(x_n - x_0), \\
&F\|T(x_n) - T(x_0)\|_2 \leq V_2(T(x_n) - T(x_0)).
\end{aligned}$$

If T is sequentially fuzzy continuous at each point of X then T is said to be sequentially fuzzy Continuous on X.

Example 4.3. Let $T:(V_1,X,\|.\|_1)\rightarrow(V_2,Y,\|.\|_2)$ be a mapping where $(V_1,X,\|.\|_1)$ and $(V_2,Y\|.\|_2)$ are anti-normed anti-fuzzy linear spaces and $\|x\|_1=|x|$ and $\|x\|_2=\frac{|x|}{2}$ and consider the function $T(x)=x$

So whenever $\|x_n-x\|_1\rightarrow 0,\Rightarrow|x_n-x|\rightarrow 0$

Then $\|T(x_n)-T(x)\|_2=\|x_n-x\|_2=\frac{1}{2}|x_n-x_0|\rightarrow 0.$

Also, $F(\|x_n-x_0\|_1)\leq V_1(x_n-x_0)$ (by Def. 3.1. (i)).

$$F(\|Tx_n-Tx_0\|_2)=F(\|x_n-x_0\|_2)=F\left(\frac{|x_n-x_0|}{2}\right)\leq\max\left\{F\left(2^{-1}\right),F(x_n-x_0)\right\}$$
$$=\max\left\{F(2),F(x_n-x_0)\right\}.$$

As $n\rightarrow\infty$ then $F(x_n-x_0)\rightarrow F(0)$

$=\max\{F(2),F(0)\}=F(2).$

$V_2(Tx_n-Tx_0)=V_2(x_n-x_0).$

But as $n\rightarrow\infty$ then $V_2(x_n-x_0)\rightarrow V_2(0),$

While as $V_2(0)\geq F(2)$

$V_2(Tx_n-Tx_0)\geq F(\|Tx_n-Tx_0\|_2).$

Hence sequentially fuzzy continuous.

Definition 4.4. *Let us denote the set of all fuzzy bounded linear operators from anti- normed anti-fuzzy linear space* $(V_1,X,\|x\|_1)$ *to* $(V_2,Y,\|x\|_2)$ *by* B(X, Y).

$\|Tx\|_2\leq k\|x\|_1,\quad V_1(x)\geq F\|x\|_1$

and $V_2(T(x))\geq F\|x\|_2.$

Example 4.4. Let us take, $\|x\|_1=|x|,\quad\|x\|_2=4|x|$

Define, a linear map $T(x)=\frac{x}{2}$,

Now,

$$\|Tx\|_2=\left\|\frac{x}{2}\right\|_2=2|x|,$$
$$\|Tx\|_2\leq k|x|,\quad\text{For } k\geq 2.$$
$$V_1(x)\geq F(\|x\|_1),\quad\text{(by Definition 4.4)}$$
$$V_2(x)\geq F(\|x\|_2),\quad\text{(by Definition 4.4).}$$

From above it is clear that, the set $B(X,Y)$ is bounded linear operator on anti-normed anti-fuzzy linear space over anti-fuzzy field.

Theorem 4.1. *Let* $(V_1,X,\|.\|_1)$ *and* $(V_2,Y,\|.\|_2)$ *be two anti-normed anti-fuzzy linear spaces and* T *is a linear operator from* X *to* Y *then*

T *is weakly fuzzy continuous iff it is fuzzy bounded.*

Proof. Let T be fuzzy bounded.

$$\|T(x-x_0)\|_2\leq k\|x-x_0\|_1$$
$$\|Tx-Tx_0\|_2\leq\frac{\epsilon}{\delta}\|x-x_0\|_1\cdot\quad\text{Take } k=\frac{\epsilon}{\delta}.$$
$$\|Tx-Tx_0\|_2\leq\epsilon,\ \text{whenever } \|x-x_0\|_1\leq\delta.$$

So, T is weakly fuzzy continuous.

Now, T take weakly fuzzy continuous

$\|Tx-Tx_0\|_2<\epsilon$, whenever $\|x-x_0\|_1<\delta.$

Let $y\in X$, $x_1=x_0+\frac{\delta}{2}\frac{y}{\|y\|_1}$,

$$x_1-x_0=\frac{\delta}{2}\frac{y}{\|y\|_1}$$
$$\Rightarrow\|x_1-x_0\|_1=\left\|\frac{\delta}{2}\frac{y}{\|y\|_1}\right\|_1=\frac{\delta}{2}<\delta$$
$$\Rightarrow\|Tx_1-Tx_0\|_2<\varepsilon\quad\text{(given)}$$

$$\Rightarrow \|T(x_1 - x_0)\|_2 = \left\|T\frac{\delta}{2}\frac{y}{\|y\|_1}\right\|_2 = \frac{\delta}{2\|y\|_1}\|Ty\|_2 < \varepsilon$$

$$\Rightarrow \|Ty\|_2 \leq \frac{2\varepsilon}{\delta}\|y_1\|$$

$$\Rightarrow \|Ty\|_2 \leq k\|y_1\| \quad (\text{Taking } k = \frac{2\varepsilon}{\delta}).$$

Therefore T is fuzzy bounded. □

5 Conclusion

In this paper, we developed a theory of anti-norm, anti-inner product on anti-fuzzy linear space over anti-fuzzy field and relation between them. We proved fuzzy continuity theory and their related examples. In the future we will work on open mapping theorem and uniform boundedness principle over anti-fuzzy field.

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References

[1] R. Biswas, Fuzzy inner product space and fuzzy norm functions, *Information Sciences,* **53** (2) (1991), 185-190.

[2] T. Bag, and S.K. Samanta, Fuzzy bounded linear operators, *Fuzzy Set and System,* **151**(3) (2005), 513-547.

[3] S. B. Barge and J. D. Yadav, (λ, μ)-Anti-fuzzy linear spaces, *Journal of Hyperstructures,* **11**(1) (2022), 1-19.

[4] C.L. Chang, Fuzzy topological spaces, *Journal of Mathematical Analysis and Application,* **24**(1)(1968), 182-190.

[5] S.C. Cheng and J.N. Mordeson, Fuzzy linear operators and fuzzy normed linear spaces, *Bulletin of Calcutta Mathematical Society,* **86**(5) (1994), 429-436.

[6] Y. Chandra, M. Srivastava and P. Sinha, 2-inner Product on fuzzy linear spaces over fuzzy fields, *J. Math. Comput. Sci.,* **12** (2022), Article ID 166.

[7] Deepmala, M. Jain, L.N. Mishra and V.N. Mishra, A note on the paper "Hu et al., Common coupled fixed point theorems for weakly compatible mappings in fuzzy metric spaces, Fixed Point Theory and Applications 2013, 2013:220", *Int. J. Adv. Appl. Math. and Mech.,* **5**(2) (2017), 51-52.

[8] C. Felbin, Finite dimensional fuzzy normed linear space, *Fuzzy Sets and Systems,* **48**(2) (1992), 239-248.

[9] A.K. Katsaras, Fuzzy topological vector spaces II, *Fuzzy Sets and Systems,* **12**(2)(1984), 143-154.

[10] S.V. Krishna and K.K. Sarma, Seperation of fuzzy normed linear spaces, *Fuzzy Sets and Systems,* **63**(2)(1994), 207-217.

[11] I. Kramosil and J. Michalek, Fuzzy metrics and statistical metric spaces, *Kybernetica,* **11**(5)(1975), 336-344.

[12] L.N. Mishra, M. Raiz, L. Rathour and V.N. Mishra, Tauberian theorems for weighted means of double sequences in intuitionistic fuzzy normed spaces, *Yugoslav Journal of Operations Research,* **32**(3) (2022), 377-388.
DOI: https://doi.org/10.2298/YJOR210915005M.

[13] V.N. Mishra, K. Khatri, L.N. Mishra and Deepmala, Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators, *Journal of Inequalities and Applications,* **2013** (2013),586.
doi:10.1186/1029-242X-2013-586.

[14] D. Rai, N. Subramanian and V.N. Mishra, The Generalized difference of $\int \chi^{2I}$ of fuzzy real numbers over $p-$metric spaces defined by Musielak Orlicz function, *New Trends in Math. Sci.,* **4**(3) (2016), 296-306.
DOI: 10.20852/ntmsci.2016320385

[15] M.K. Sharma, Sadhna, A.K. Bhargava, S. Kumar, L. Rathour, L.N. Mishra and S. Pandey, A fermatean fuzzy ranking function in optimization of intuitionistic fuzzy transportation problems, *Advanced Mathematical Models & Applications,* **7**(2) (2022), 191-204.

[16] M.K. Sharma, N. Dhiman, S. Kumar, L. Rathour and V.N. Mishra, Neutrosophic Monte Carlo Simulation Approach for Decision Making In Medical Diagnostic Process Under Uncertain Environment, *International Journal of Neutrosophic Science,* **22**(1) (2023), 08-16.
DOI: https://doi.org/10.54216/IJNS.220101

[17] M. Srivastava, P. Sinha and Y. Chandra, 2-norm on fuzzy linear spaces over fuzzy fields, *Ganita,* **72**(1)(2022), 311-317.
[18] C.P. Santhosh and T.V. Ramakrishnan, Norm and inner product on fuzzy linear spaces Over fuzzy fields, *Iranian Journal of Fuzzy Systems,* **8**(1) (2011), 135-144.
[19] T. Srinivas, P. Narasimha Swamy, and T. Nagaiah, Anti fuzzy near-algebras over anti fuzzy fields, *Annals of Fuzzy Mathematics and Informatics,* **4**(2) (2012), 243-252.
[20] C. Wu and J. Fang, Fuzzy generalization of Kolmogoroffs theorem, *J. Harbin Inst. Technol.,* **1**(1985), 1-7.
[21] G. Wenxiang and Lu Tu, Fuzzy linear spaces, *Fuzzy Sets and Systems,* **49**(3) (1992), 377- 380 .
[22] J. Xiao and X. Zhu, On linearly topological structure and property of fuzzy normed linear space, *Fuzzy Sets and Systems,* **125**(2) (2002), 153-161.
[23] L. A. Zadeh, Fuzzy Sets, *Information and Control,* **8**(3) (1965), 338-353.