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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

SOME RESULTS ON COUPLED FIXED POINT BY DARBO EXTENSION THEOREM

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Abstract

This paper is based on extension of some basic results on coupled fixed point using extended Kuratowski measure. In the first part of the paper we have extended coupled fixed point results on product Banach space by measure of non compactness defined in the paper of Banas (1980). The second part of the paper contains the results on existence of coupled fixed point based on generalized Theorem 3.2 of Samih et.al. (2016) to get coupled fixed point of set valued co set contraction map on complete metric type of space.
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1 Introduction

In nonlinear functional analysis many operator equation has the expression $T(x, y) = (f(x, y), f(y, x))$, where f is a map from $X \times X$ to X and T is map from $X \times X$ to $X \times X$ and X is a Banach space.In such cases the fixed the point of T is nothing but coupled fixed point of f. That is $f(x, y) = x, f(y, x) = y$. Bhaskar and Lakshmikantham [6] introduced the concept of coupled fixed point of function of several variable. He gave results on fixed point of each partial function of several variables in product space. He also generalized Banach contraction principle for such maps. It plays an important role in nonlinear functional analysis. Latter some authors [1,17] proved some results on fixed point and coupled fixed point on closed bounded convex set of Banach space of different nonlinear maps.

Every continuous map on a closed ball of R has a fixed point. Tarski [19] and Kanster [12] extended this result to complete lattice, that every monotone function on complete lattice has a fixed point. Brouwer [5] extended it to R^n to get fixed point and further Schauder [18] extended this result to topological vector space. He proved that every continuous map on compact set has a fixed point.The main drawback of Schuader fixed point result is that, it is not applicable when a set losses compactness and the loss of compactness always occurs on boundary. There are several problems related to this area like Integral equation with singular kernel, differential equation over unbounded domain, embedding theorem between Sobolev spaces. Therefore, three important measures of non compactness were developed
(i) Housdorff measure of non compactness,
(ii) Kuratowski measure of non compactness,
(iii) Istratescu measure of non compactness.
Darbo [8] gave a nice result on compact convex bounded subset of Banach space, known as Darbo fixed point theorem.

Banach gave fixed point results on metric space but some space like l^p, for $0 < p < 1$ is not a metric space. Some author gave fixed point results on those space using some weaker topological condition. In 1989 Baktin [4] has first introduced a new topological space called b-metric space almost similar to metric space in order to generalize Banach contraction principle. Some more fixed point results are added in distance space by Kirk [13].In this work we give some results on coupled fixed point of KKM type map on b-metric space.

Mathematical Preliminaries.

Definition 1.1 ([6]). *Let F be a map on X, then x is called fixed point of F, if $F(x) = x$.*

Definition 1.2 ([6]). *Let F be a map from $X \times X$ to X, then (x,y) is called coupled fixed point, if $F(x,y) = x$ and $F(y,x) = y$.*

Definition 1.3 ([17]). *Let F be a map from X to 2^X, then f is called set valued map, where 2^X is power set of X.*

Definition 1.4 ([17]). *Let F be a set valued map from X to 2^X, then x is called fixed point of F if $x \in F(x)$.*

Definition 1.5 ([1]). *Let F be a set map from $X \times X$ to 2^X, then (x,y) is called coupled fixed point of F if $x \in F(x,y)$ and $y \in F(y,x)$.*

Definition 1.6 ([20]). *Let X and Y be two topological space and T be a set valued map from X to 2^Y, then T is called :*
(i) closed if graph $G_T = \{(x,y) : y \in T(x)\}$ is closed,
(ii) compact if closure of $T(X) = \bigcup_{x\in X} T(x)$ is compact,
(iii) lower semi continuous if for every open subset B of Y, the set $T^{-1}(B) = \{x \in X : T(x) \bigcap B \neq \phi\}$ is open.

Definition 1.7 ([14]). *Let S be a bounded subset of metric space X, then*
$\delta(S) = inf\{\epsilon > 0 : A$ can be covered by finitely many sets of diameter less than or equal to $\epsilon\}$, where diameter of $S = sup\{d(x,y) : x, y \in S\}$, is called measure of noncompactness in X.

Definition 1.8 ([9]). *Let $(X, \| \ . \ \|)$ be a Banach space and β be the family of bounded subset of X. The function δ from β to $R+$ defined for every $B \in \beta$ by*
$\delta(B) = inf\{\epsilon > 0 : B$ can be covered by finitely many sets of diameter less than or equal to $\epsilon\}$, where diameter of $S = sup\{\|x - y : x, y \in S\|\}$, is called measure of noncompactness in X.

2 Some Basic definitions and results

In this section we give some definitions and results on coupled fixed point by contraction with Kuratowski measure on Banach space.

Theorem 2.1 ([18]). *(Schauder fixed point theorem) Everey continuous and compact map on closed bounded convex set has fixed point.*

Theorem 2.2 ([8]). *(Darbo fixed point theorem) Let E be closed bounded convex subset of a Banach space X and $F : E \to E$ be continuous such that $\mu(F(E)) \leq k\mu(E)$, where $k \in (0,1)$ and μ is the Kuratowski measure of non-compactness in X, then F has fixed point in E.*

Definition 2.1 ([21]). *Let X be a real Banach space, Y be the set of all bounded subsets of X and let $\mu : Y \to R$ be a map satisfying:*
(i) $Ker\mu$ (zero set of μ) is a non empty subset of Y,
(ii) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$,
(iii) $\mu(C) = \mu(\overline{C})$,
(iv) $\mu(C) = \mu(coC)$,
(v) $\mu(rA + (1-r)(B)) \leq r\mu(A) + (1-r)\mu(B))$, for all $A, B \in Y$, for all $r \in (0,1)$,
(vi) If (A_n) be a sequence of sets in Y with $A_{n+1} \subset A_n$ and $lim_{n\to\infty}\mu(A_n) = 0$, then $\bigcap A_n$ is non empty, μ is called Kuratowski measure of non-compactness.

Now we give our first result on coupled fixed point based on Kuratowski measure on product space.

Theorem 2.3. *Suppose X be a real Banach space. Let μ_1, μ_2 be two Kuratowski measures of noncompactness on X. Let the measures μ and $\mu^{'}$ be defined on the product space $X \times X$ by*
(i) $\mu(Y) = \mu_1(Y_1) + \mu_2(Y_2)$,
(ii) $\mu^{'}(Y) = max(\mu_1(Y_1), \mu_2(Y_2))$, where
Y_1 and Y_2 are natural projections of Y on X and Y be a arbitrary subset of $X \times X$.
Let f be a continuous map from $X \times X$ to X such that for every bounded closed convex subset C of $X \times X$ satisfies the following set valued contraction:
(1) $\mu_1(f(C)) \leq a\mu(C)$ and $\mu_2(f^{'}(C)) \leq b\mu(C)$, for $a + b \in (0,1)$,
(2) $\mu_1(f(C)) \leq \mu^{'}(C)$ and $\mu_2(f^{'}(C)) \leq \mu^{'}(C)$,
where $f^{'} : X \times X \to X$ and $f^{'}(x,y) = f(y,x)$,
then f has a coupled fixed point on $X \times X$.

Proof. From [21] we can conclude μ and μ' are Kuratowski measure. Since X is a real Banach space, then $X \times X$ is also a real Banach space. In product space $X \times X$ we define
$T(x, y) = (f(x, y), f^{'}(x, y))$, where $f^{'}(x, y) = f(y, x)$. Hence T is a map from $X \times X$ to $X \times X$.
Let C be a arbitrary subset of $X \times X$. Now we claim that

$$T(C) \subset f(C) \times f^{'}(C).$$

Let $(z, w) \in T(C) \Rightarrow$ there exist $(x, y) \in C$ such that $T(x, y) = (z, w)$. Then it follows that $(z, w) = T(x, y) = (f(x, y), f^{'}(x, y))$, then $z = f(x, y)$ and $w = f(y, x)$
$\Rightarrow (z, w) \in f(C) \times f^{'}(C)$
$\Rightarrow T(C) \subset f(C) \times f^{'}(C)$.
Now
$\mu(T(C)) \leq \mu(f(C) \times f^{'}(C))$
$= (\mu_1(f(C) + \mu_2(f^{'}(C)))$
$\leq a\mu(C) + b\mu(C)$
$= (a + b)\mu(C)$. Therefore, T has a fixed point in $X \times X$.

By Darbos theorem there exist a point $(x, y) \in X \times X$ such that $T(x, y) = (x, y)$
$\Rightarrow f(x, y) = x$ and $f(y, x) = y$. So f has a coupled fixed point.
Further using condition (2), we get

$$\mu^{'}(T(C)) \leq \mu^{'}(f(C) \times f^{'}(C)) = \max(\mu_1(f(C), \mu_2(f^{'}(C))) \leq \mu^{'}(C).$$

Proceding similarly as done under condition (1), we get f has a coupled fixed point in X.
Remark 2.1. *Theorem2.3 generalizes Darbo fixed point theorem on product space by Kuratowski measures μ and μ'.*

3 b-metric space

Definition 3.1 ([20]). *Let X be a non empty set equipped with a map d from $X \times X$ to $\mathbb{R}$ is called b-metric space or metric type space, if it satisfies the following conditions:*
(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq k\,(d(x, z) + d(z, y))$, for all $k > 1$, for all $x, y, z \in X$.

Note:We have used the word d-topology as synonym of b-metric.

Definition 3.2 ([20]). *Let (X, d) be a b-metric space, then we define*
(i) a sequence(x_n) in X converges to x, if $lim_{n\to\infty} d(x_n, x) = 0$,
(ii) a sequence(x_n) in X is called Cauchy, if $lim_{n\to\infty} lim_{m\to\infty} d(x_n, x_m) = 0$,
(iii) The space (X, d) is complete, if every Cauchy sequence is convergent.

Definition 3.3 ([20]). *Let (X, d) be a b-metric space and $A \subset X$, then we define*
(i) $\bar{A}$ = Intersection of all closed set containing A.
(ii) co(A)= Intersection of all closed ball containing A.

Theorem 3.1 ([20]). *Let (X, d) be a b-metric space, then following results hold:*
(i) $A \subset X$ is closed $\Leftrightarrow$ every sequence $(x_n) \in A$ converges to x, then $x \in A$,
(ii) For $x \in \overline{A}$, we have $B(x, r) \bigcap A \neq \phi$,
for every $r > 0$, where $\overline{A}$ is the intersection of all closed sets containing A,
(iii) A is called totally bounded, if for every $r > 0$ there exist $x_1, x_2, ..., x_n \in A$ such that,

$$A \subset B(x_1, r) \bigcup B(x_2, r) \bigcup ... \bigcup B(x_n, r),$$

(iv) Every compact set is sequentially compact but the converse is not true.

Definition 3.4 ([20]). *Let (X, d) be a b-metric space. Let $A \subset X$ is called admissible if $co(A) = A$ and it is called sub admissible, if for every finite subset B of A, $co(B) \subset A$.*

Definition 3.5 ([20]). *Let (X, d) be a b-metric space $A \subset X$ is called nearly sub-admissible, if for each compact subset B of A and each $r \geq 0$, there exists a function $f : B \to A$ such that $x \in B(f(x), r)$, for each $x \in B$ and $co(f(B)) \subset A$.*

Definition 3.6 ([20]). *Let(X,d) be a b-metric space. A set valued map $T : A \to 2^X$ is called KKM map, if for every finite subset B of A the $co(B) \subset T(B) = \bigcup_{x\in B} T(x)$.*

Theorem 3.2 ([20]). *Let (X,d) be a b-metric space and A be a nonempty subset of X. Suppose Y be a topological space, then the following properties of KKM map hold:*
(i) if $T \in KKM(A,Y)$ and $F \in C(Y,X)$, then $FoT \in KKM(A,X)$,
(ii) if B is a nonempty subset of A, then $T|B \in KKM(B,Y)$.

Definition 3.7 ([20]). *Let (X,d) be a $b-metricspace$.*
$A \subset X$, then $diam(A) = sup\{d(x,y) : x, y \in A\}$.

In 2016, Samih et al. [17] extended Kuratowski measure on b-metric space. He proved results on existence of fixed point on b-metric space using generalized Kuratowski measure. We are looking extension of generalized Kuratowski measure on product of b-metric space to get coupled fixed point of KKM type function of double variable map.

Definition 3.8 ([20]). *Let (X,d) be a b-metric space with coefficient k and A be a subset of X, then Kuratowski measure of A denoted as $\alpha(A)$, is given by*
$\alpha(A) = inf\{\epsilon > 0 : A$ can be covered by finitely many sets of diameter less than or equal to $\epsilon\}$.

Then the following properties hold:
(i) $\alpha(A) = 0 \Leftrightarrow A$ is totally bounded,
(ii) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$,
(iii) If B is finite subset of X, then $\alpha(A \bigcup B) = \alpha(A)$,
(iv) $\alpha(A) \leq \alpha(\overline{A}) \leq k^2\alpha(A)$,

Definition 3.9 ([20]). *Let (X,d) be a b-metric space. Let $A \subset X$ and f be a map on A, is said to be co contraction on A, if for every bounded subset $B \subset A$ with $f(B)$ bounded, then $\alpha(f(coB)) \leq k\alpha(B))$, for $0 < k < 1$.*

Theorem 3.3 ([20]). *Let (X,d) be a complete b-metric space. Let A be a nonempty bounded nearly sub-admissible subset of X and F be a map from A to 2^A with closed, co set contraction and KKM on A and $\overline{F(A)} \subset A$, then F has a fixed point on A.*

4 Main results

Theorem 4.1. *Let (X,d) be a b-metric space. Let D be a map from $X' \times X'$ to $[0,\infty)$ defined by $D((x_1,y_1),(x_2,y_2)) = max\{d((x_1,x_2),(y_1,y_2)\}$, then (X',D) is a b-metric space, where $X' = X \times X$.*

Proof. Let $p_1 = (x_1,y_1)$, $p_2 = (x_2,y_2)$, $p_3 = (x_3,y_3)$, where $p_i \in X'$, $i = 1,2,3$, then we have
(i) $D(p_1,p_2) = D(p_2,p_1)$ is obvious,
(ii) Symmetric property is obvious,
(iii) we wish to show k-triangular property.
From definition

$$d(x_1,x_2) \leq k[d(x_1,x_3) + d(x_3,x_2)]$$

and

$$d(y_1,y_2) \leq k[d(y_1,y_3) + d(y_3,y_2)].$$

Now

$$\begin{aligned}
&D(p_1,p_2) = max\,[d((x_1,x_2), d(y_1,y_2)]\\
&\leq k([d(x_1,x_3) + d(x_3,x_2)] + [d(y_1,y_3) + d(y_3,y_2)])\\
&\leq k([d(x_1,x_3) + d(y_1,y_3)] + [d(x_2,x_3) + d(y_2,y_3)])\\
&\leq 2k\ max[d(x_1,x_3), d(y_1,y_3)] + 2k\ max[d(x_2,x_3), d(y_2,y_3)]\\
&= 2k[D(p_1,p_3) + D(p_3,p_2)].
\end{aligned}$$

Taking $K = 2k$, we get the result.
Note: Since both X and $X^{'}$ are b-metric space, therefore, we have taken the notation d-topology for X and D-topology for product space $X^{'}$ in this section.

Theorem 4.2. *Let (X, d) be a b-metric space, then the following results hold:*
(i) If A and B be two closed subsets of X with respect to d-topology, then $A \times B$ is closed in $X \times X$ with respect to D-topology,
(ii) If A and B are two subsets of X, then $\overline{A \times B} \subset \overline{A} \times \overline{B}$,
(iii) $B_d(a, r) \times B_d(b, r) = B_D((a, b), r)$, for every $a, b \in X$ and every $r > 0$,
(iv) if X is complete w.r.t d-topology, then $X \times X$ is complete w.r.t D-topology.

Proof. (i) Let A and B be two closed subset of X with respect to d-topology we wish to show $A \times B$ is closed with respect to D-topology. Let $(z_n) = (x_n, y_n)$ converge to some (a, b) with respect to D-topology. Now

$$lim_{n\to\infty} D(((x_n, y_n), (a, b)) = 0$$
$$\Rightarrow lim_{n\to\infty}\ max[d(x_n, a), d(y_n, b)] = 0,$$
$$\Rightarrow lim_{n\to\infty} d(x_n, a) = 0 \ and \ lim_{n\to\infty} d(y_n, b) = 0$$

$\Rightarrow (x_n)$ converges to a and (y_n) converges to b.
Since A and B are closed in X, then $a \in A$ and $b \in B \Rightarrow (a, b) \in A \times B \Rightarrow A \times B$ is closed in $X \times X$.
(ii) Let A and B be two subsets of X, then

$$A \subset \overline{A} \text{ and } B \subset \overline{B} \Rightarrow A \times B \subset \overline{A} \times \overline{B}.$$

Since $\overline{A} \times \overline{B}$ is closed in X' with respect to D-topology and $\overline{A \times B}$ is smallest closed set in $X \times X$, then $\overline{A \times B} \subset \overline{A} \times \overline{B}$.
(iii) Let

$$(x, y) \in B_D((a, b), r)$$
$$\Rightarrow D((a, b), (x, y)) \leq r$$
$$\Rightarrow max[d(a, x), d(b, y)] \leq r$$
$$\Rightarrow d(a, x) \leq r \text{ and } d(b, y) \leq r$$
$$\Rightarrow x \in B_d(a, r) \text{ and } y \in B_d(b, r)$$
$$\Rightarrow (x, y) \in B_d(a, r) \times B_d(b, r)$$
$$\Rightarrow B_D((a, b), r) \subset B_d(a, r) \times B_d(b, r).$$

(iv) Proof is easy so we omit it.
For reverse inclusion the proof is similar.
Hence, $B_D((a, b), r) = B_d(a, r) \times B_d(b, r)$.

Theorem 4.3. *Let (X, d) be metric a type of space, which induces D-topology on $X \times X$. Suppose α and β are Kurtoswki measures on X and $X \times X$ respectively, then following conclusion hold:*
(i) for a subset A of X and $x \in X, \beta(A \times \{x\}) = \alpha(A)$,
(ii) $\beta(A_1 \times A_2) \leq max[\alpha(A_1), \alpha(A_2)]$, for all A_1, A_2 subsets of X,
(iii) $\beta(A) = \beta(A')$, where $A' = \{(x, y) : (y, x) \in A\}$.

Proof. (i) Let $\epsilon > 0$, then there exist $\bigcup A_i$ $(i = 1, 2, ..., n)$, such that

$$A \subset \bigcup A_i \text{ and } \alpha(A) \leq \ diam(A_i) \leq \alpha(A) + \epsilon.$$
$$Now$$
$$A_i \times \{x\} \text{ covers } A \times \{x\}, \text{ for } i = 1, 2, .., n.$$
$$diam[A_i \times \{x\}] = sup\{D[(a_1, x), (a_2, x)] : a_1, a_2 \in A_i\}$$
$$= sup\{d(a_1, a_2) : a_1, a_2 \in A_i\}$$
$$= diam A_i \leq \alpha(A) + \epsilon$$
$$\Rightarrow \beta(A \times \{x\}) \leq \alpha(A) + \epsilon.$$

Since ϵ is arbitary, so we get $\beta(A \times \{x\}) \leq \alpha(A)$.

Now we the show reverse inequality.
Take ϵ (> 0). Then there exist $\bigcup C_i \subset X \times X$, $i = 1, 2, ..., n$ such that $A \times \{x\} \subset \bigcup C_i$ and

$$\beta(A \times \{x\}) \leq diam(C_i), \quad i = 1, 2, ..., n$$

$\leq \beta(A \times \{x\}) + \epsilon.$

Let

$C_i{}^x = \{c : (c, x) \in C_i\}.$

$\bigcup C_i^x$ covers A.

Therefore,

$$\begin{aligned}
&diam C_i^x \\
&= sup\{d(r,s) : r, s \in C_i{}^x\} \\
&\leq sup\{D((r_1, s_1), (r_2, s_2)) : (r_1, s_1), (r_2, s_2) \in C_i\} \\
&\leq \beta(A \times \{x\}) + \epsilon \\
&\Rightarrow \alpha(A) \leq \beta(A \times \{x\}) + \epsilon,
\end{aligned}$$

since ϵ is arbitrary we get $\alpha(A) \leq \beta(A \times \{x\})$. Hence $\alpha(A) = \beta(A \times \{x\})$.
(ii) Let $\epsilon > 0$, then there exist $\bigcup P_i$ and $\bigcup Q_j$, $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$ such that

$$A_1 \subset \bigcup P_i,\ A_2 \subset \bigcup Q_j.$$

So we have

$$\alpha(A_1) \leq diam(P_i) \leq \alpha(A_1) + \epsilon$$

and

$$\alpha(A_2) \leq diam(Q_j) \leq \alpha(A_2) + \epsilon.$$

Take $R_{ij} = P_i \times Q_j$ clearly we can see

$$A_1 \times A_2 \subset \bigcup (P_i \times Q_j).$$

Now,

$$\begin{aligned}
&diam((P_i \times Q_j)) = sup\{D[(x_1, y_1), (x_2, y_2)] : x_1, x_2 \in P_i \ and\ y_1, y_2 \in Q_j\} \\
&= sup\{max(d(x_1, x_2), d(y_1, y_2)) : x_1, x_2 \in P_i \ and\ y_1, y_2 \in Q_j\} \\
&\leq sup[diam(P_i), diam(Q_j)] \\
&\leq sup[\alpha(A_1) + \epsilon, \alpha(A_2) + \epsilon].
\end{aligned}$$

Since ϵ is arbitrary, we obtain

$$\beta(A_1 \times A_2) \leq max[\alpha(A_1), \alpha(A_2)].$$

(iii) The proof is obvious, so we omit it.

Theorem 4.4. *Let (X, d) be a complete b-metric space and A be a nonempty bounded subset of X such that $A \times A$ is nearly sub admissible with respect to D-topology on $X \times X$ and f be a map from $A \times A$ to $2^{A \times A}$ satisfying:*
(i) For every finite subset E of $A \times A$, we have $co(E) \subset f(A) \times f^{'}(A)$,where $f^{'}(x, y) = f(y, x)$,
(ii)$f(A \times A) \subset A$,
(iii) $co(f(C) \times f^{'}(C)) = co(f(C)) \times co(f^{'}(C))$ and $\alpha(co(f(C))) \leq k\beta(C)$, for $0 < k < 1$, for every bounded subset C of $A \times A$ with $f(C)$ bounded,
(iv) the graph f and graph $f^{'}$ are closed, where$f^{'}(x, y) = f(y, x)$, then f has a coupled fixed point on $A \times A$.

Proof. From Theorem 4.2(iv) $X \times X$ is complete w.r.t D-topology.
Let T be a map from $A \times A$ to $2^{A \times A}$ defined by $T(x, y) = f(x, y) \times f(y, x)$.
Now we claim that for every subset C of $A \times A$,
$T(C) = f(C) \times f^{'}(C)$.
Now

$$\begin{aligned}
T(C) &= \bigcup_{(x,y)\in C} T(x, y) \\
&= \bigcup_{(x,y)\in C} f(x, y) \times f^{'}(x, y)
\end{aligned}$$

$$= \bigcup_{(x,y)\in C} f(x,y) \times \bigcup_{(x,y)\in C} f^{'}(x,y).$$

Hence, $T(C) = f(C) \times f^{'}(C)$.
Form condition (i), we conclude that T is KKM on $A \times A$.
From Theorem 3.3, we can conclude

$$\begin{aligned} &\overline{T(A \times A)} = \overline{f(A \times A) \times f'(A \times A)} \\ &\subset \overline{f(A \times A)} \times \overline{f'(A \times A)} \\ &\subset A \times A \end{aligned}$$

From condition-(iv), we can conclude T is closed.
Next we show that T has set valued co contraction on every bounded subset of $A \times A$.
Let C be arbitrary bounded subset of $A \times A$ with $f(C)$ bounded, then

$$\beta(coT(C) = \beta(co(f(C) \times f^{'}(C))) = \beta(co(f(C)) \times co(f^{'}(C))).$$

and $T(C)$ is bounded. From theorem 4.3 we can conclude that

$$\begin{aligned} &\beta(coT(C)) \\ &= \beta(co(f(C) \times f^{'}(C))) \\ &\leq max\{\alpha((co(f(C)), \alpha(co(f^{'}(C))\} \\ &\Rightarrow \beta(coT(C) \leq k\beta(C), \end{aligned}$$

Using Theorem 3.3 we obtain T has a fixed point in $A \times A$.
So there exist (x, y) such that

$$\begin{aligned} &(x, y) \in T(x, y) \Rightarrow (x, y) \in f(x, y) \times f(y, x) \\ &\Rightarrow x \in f(x, y) \ \text{ and } y \in f(y, x). \end{aligned}$$

Then f has a coupled fixed point on $A \times A$.

5 Conclusion

Theorem 4.4 generalizes Theorem 3.3 of the results of Samih et.al on the product space of fixed point of set valued map.

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