CERTAIN PROPERTIES OF A NEW SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract

The purpose of this paper is to study a new subclass of close-to-convex functions associated with generalized Janowski’s function. Various properties such as coefficient estimates, inclusion relationship, distortion property, argument property and radius of convexity, are established for this class. The results mentioned here, generalize some earlier known results.

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1 Introduction

By \(A\), we denote the class of functions \(f\) of the form \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n\), which are analytic in the open unit disc \(E = \{z : |z| < 1\}\). Further, the class of functions \(f \in A\) and which are univalent in \(E\), is denoted by \(S\). A function \(w\) is said to be a Schwarz function if it has expansion of the form \(w(z) = \sum_{n=1}^{\infty} c_n z^n\) and satisfy the conditions \(w(0) = 0\) and \(|w(z)| \leq 1\). The class of Schwarz functions is denoted by \(\mathcal{U}\).

For two analytic functions \(f\) and \(g\) in \(E\), \(f\) is said to be subordinate to \(g\), if there exists a Schwarz function \(w \in \mathcal{U}\) such that \(f(z) = g(w(z))\). If \(f\) is subordinate to \(g\), then it is denoted by \(f \prec g\). Further, if \(g\) is univalent in \(E\), then \(f \prec g\) is equivalent to \(f(0) = g(0)\) and \(f(E) \subset g(E)\).

By \(S^*\) and \(K\), we denote the classes of starlike functions and of convex functions respectively, which are defined as follows:

\[
S^* = \left\{ f : f \in A, \text{Re}\left( \frac{zf'(z)}{f(z)} \right) > 0, z \in E \right\}
\]

and

\[
K = \left\{ f : f \in A, \text{Re}\left( \frac{(zf'(z))'}{f'(z)} \right) > 0, z \in E \right\}.
\]

A function \(f \in A\) is said to be close-to-convex function if there exists a function \(g \in S^*\) such that

\[
\text{Re}\left( \frac{zf'(z)}{g(z)} \right) > 0(z \in E).
\]

The class of close-to-convex functions is denoted by \(C\) and was given by Kaplan\(^6\). Several subclasses of close-to-convex functions were studied by various authors and recently by Singh and Singh\(^{14}\), but here we mention those which are relevant to our study.

Gao and Zhou\(^3\) studied the class \(K_S\) defined as

\[
K_S = \left\{ f : f \in A, \text{Re}\left( \frac{-z^2 f'(z)}{g(z)g(-z)} \right) > 0, g \in S^* \left( \frac{1}{2} \right), z \in E \right\}.
\]

Further, Kowalczyk and Les-Bomba\(^7\) extended the class \(K_S\) by introducing the class \(K_S(\gamma), (0 \leq \gamma < 1)\), which is mentioned below:

\[
K_{S}(\gamma) = \left\{ f : f \in A, \text{Re}\left( \frac{-z^2 f'(z)}{g(z)g(-z)} \right) > \gamma, g \in S^* \left( \frac{1}{2} \right), z \in E \right\}.
\]

For \(\gamma = 0\), the class \(K_S(\gamma)\) reduces to the class \(K_S\).
Later on, Seker [12] established the class $K_s^{(k)}(\gamma) \ (0 \leq \gamma < 1)$ of close-to-convex analytic functions $f \in \mathcal{A}$ which satisfy the condition
\[
\text{Re} \left( \frac{z^k f'(z)}{g_k(z)} \right) > \gamma,
\]
where
\[
g_k(z) = \Pi_{\nu=0}^{k-1} e^{-\nu} g(e^\nu z) (e^k = 1; k \geq 1),
\]
and $g \in S^* \left( \frac{k-1}{k} \right)$.

As a generalization, Seker and Cho [13] introduced the class $K_s^{(k)}(\gamma; \delta; \eta)$ of the functions $f \in \mathcal{A}$ which satisfy the condition
\[
\frac{z^k f'(z)}{g_k(z)} < \frac{1 + \eta [1 - (1 + \delta) \gamma] z}{1 - \eta \delta z},
\]
where $g_k$ is defined in (1.1) and $0 \leq \gamma < 1, 0 \leq \delta \leq 1$ and $0 < \eta \leq 1$.

Raina et al. [10] established the class of strongly close-to-convex functions of order $\beta$, as below:
\[
C_\beta' = \left\{ f : f \in \mathcal{A}, \arg \left\{ \frac{z^k f'(z)}{g(z)} \right\} < \frac{\beta \pi}{2}, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E \right\},
\]
which can also be expressed as
\[
C_\beta' = \left\{ f : f \in \mathcal{A}, \frac{z^k f'(z)}{g(z)} < \left( \frac{1 + z}{1 - z} \right)^\beta, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E \right\}.
\]

For $-1 \leq B < A \leq 1$, Janowski [5] introduced the class of functions in $\mathcal{A}$ which are of the form
\[
p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n
\]
and satisfying the condition $p(z) < \frac{1 + A z}{1 + B z}$. This class plays an important role in the study of various subclasses of analytic-univalent functions. As a generalization of Janowski’s class, Polatoglu et al. [9] established the class $\mathcal{P}(A, B; \alpha) \ (0 \leq \alpha < 1)$, the subclass of $\mathcal{A}$ which consists of functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ such that $p(z) < \frac{1 + [B+(A-B)\alpha] z}{1 + B z}$. Also for $\alpha = 0$, the class $\mathcal{P}(A, B; \alpha)$ agrees with the class defined by Janowski [5].

Inspired by the above mentioned classes, now we define the following generalized class which is to study in this paper.

**Definition 1.1.** Let $K_s^{(k)}(A, B; \alpha; \beta)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the conditions,
\[
\frac{z^k f'(z)}{g(z)} < \left( \frac{1 + B + (A - B)(1 - \alpha) z}{1 + B z} \right)^\beta, -1 \leq B < A \leq 1, z \in E,
\]
where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^* \left( \frac{k-1}{k} \right)$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $-1 \leq B < A \leq 1$ and $g_k(z)$ is defined in (1.1).

The following observations are obvious:
(i) $K_s^{(k)}(\eta[1 - (1 + \delta) \gamma], -\eta \delta; 0; 1) \equiv K_s(\gamma, \delta, \eta)$, the class established by Seker and Cho [13].
(ii) $K_s^{(k)}(1 - 2\gamma, -1; 0; 1) \equiv K_s^{(k)}(\gamma)$, the class studied by Seker [12].
(iii) $K_s^{(2)}(1, -1; 0; 1) \equiv K_s$, the class introduced by Gao and Zhou [3].
(iv) $K_s^{(3)}(1 - 2\gamma, -1; 0; 1) \equiv K_s(\gamma)$, the class established by Kowalczyk and Les Bomba [7].

As $f \in K_s^{(k)}(A, B; \alpha; \beta)$, by definition of subordination, it follows that
\[
\frac{z^k f'(z)}{g_k(z)} = \left( \frac{1 + [B + (A - B)(1 - \alpha)] w(z)}{1 + B w(z)} \right)^\beta, w \in \mathcal{U}.
\]

We study various properties such as coefficient estimates, inclusion relationship, distortion theorem, argument theorem and radius of convexity for the functions in the class $K_s^{(k)}(A, B; \alpha; \beta)$. The results proved by various authors follow as special cases.

Throughout this paper, we assume that $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 \leq \gamma < 1$, $0 < \eta \leq 1$, $0 \leq \delta \leq 1$, $k \geq 1$, $z \in E$. 

162
2 Preliminary Results

For the derivation of our main results, we must require the following lemmas:

**Lemma 2.1** ([2, 11]). Let,

\begin{equation}
(1 + [B + (A - B)(1 - \alpha)]w(z))^{\beta} = (P(z))^\beta = 1 + \sum_{n=1}^{\infty} p_n z^n,
\end{equation}

then

\[|p_n| \leq \beta(1 - \alpha)(A - B), n \geq 1.\]

**Lemma 2.2** ([10]). Let \(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1\), then

\[\frac{1 + A_1 z}{1 + B_1 z} \leq \frac{1 + A_2 z}{1 + B_2 z}.\]

**Lemma 2.3** ([8]). If \(g \in S^*,\ then for \(|z| = r, 0 < r < 1\), we have

\[\frac{r}{(1+r)^2} \leq |g(z)| \leq \frac{r}{(1-r)^2}.\]

**Lemma 2.4** ([15]). For \(g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*\), then

\[G_k(z) = \frac{g_k(z)}{z^{k-1}} = z + \sum_{n=2}^{\infty} d_n z^n \in S^*.\]

**Lemma 2.5** ([1, 2]). If \(P(z) = \frac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)}, -1 \leq B < A \leq 1, w \in \mathcal{U},\ then for \(|z| = r < 1\), we have

\[R = \frac{zP'(z)}{P(z)} \geq \begin{cases} \frac{(A-B)(1-\alpha)r}{2(1-B)(1-[B+(A-B)(1-\alpha)]r^2/(1+Br^2)} & \text{if } R_1 \leq R_2, \\ \frac{-(1-[B+(A-B)(1-\alpha)]Br^2)^2}{(A-B)(1-\alpha)(1-r^2)} & \text{if } R_1 \geq R_2, \end{cases}\]

where \(R_1 = \sqrt{\frac{(1-[B+(A-B)(1-\alpha)])(1+B+(A-B)(1-\alpha)r^2)/(1+Br^2)}{(1-B)(1+Br^2)}}\) and \(R_2 = \frac{1-[B+(A-B)(1-\alpha)]r^2}{1-Br}.

3 Main Results

**Theorem 3.1.** If \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K_s^{(k)}(A, B; \alpha; \beta)\), then

\begin{equation}
|a_n| \leq 1 + \frac{\beta(1 - \alpha)(n - 1)(A - B)}{2}.
\end{equation}

**Proof.** As \(f \in K_s^{(k)}(A, B; \alpha; \beta),\ therefore (1.2) can be written as

\[z^k f'(z) = g_k(z),\]

which can be further expressed as

\begin{equation}
z^k f'(z) = \frac{g_k(z)}{G_k(z)} = (P(z))^\beta,
\end{equation}

where

\begin{equation}
G_k(z) = \frac{g_k(z)}{z^{k-1}} = z + \sum_{n=2}^{\infty} d_n z^n.
\end{equation}

By Lemma 2.4, we have \(G_k \in S^*\).

Using (2.1) and (3.3) in (3.2), it yields

\begin{equation}
1 + \sum_{n=2}^{\infty} na_n z^{n-1} = \left(1 + \sum_{n=2}^{\infty} nd_n z^{n-1}\right) \left(1 + \sum_{n=1}^{\infty} p_n z^n\right).
\end{equation}

As \(G_k(z) = z + \sum_{n=2}^{\infty} d_n z^n \in S^*\), it is well known that \(|d_n| \leq n\).

Comparing the coefficients of \(z^{n-1}\) in (3.4), we have

\begin{equation}
a_n = d_n + d_{n-1}p_1 + d_{n-2}p_2 + \ldots + d_{n-2}p_2 + \ldots + d_{n-2}p_2 + p_{n-1}.
\end{equation}

Applying triangle inequality, using Lemma 2.1 and the inequality \(|d_n| \leq n\) in (3.5), it gives

\begin{equation}
n|a_n| \leq n + \beta(1 - \alpha)(A - B)[(n - 1) + (n - 2) + \ldots + 2 + 1],
\end{equation}

which proves Theorem 3.1.
For $A = \eta[1 - (1 + \delta)\gamma], B = -\eta\delta, \alpha = 0, \beta = 1$, Theorem 3.1 gives the following result:

**Corollary 3.1.** If $f \in K_s^{(k)}(\gamma; \delta; \eta)$, then

$$ |a_n| \leq 1 + \frac{\eta(n - 1)(1 + \delta)(1 - \gamma)}{2}. $$

**Corollary 3.2.** If $f \in K_s^{(k)}(\gamma)$, then

$$ |a_n| \leq n - (n - 1)\gamma. $$

Putting $A = 1 - 2\gamma, B = -1, \alpha = 0$ and $\beta = 1$ in Theorem 3.1, the following result is obvious:

**Corollary 3.3.** If $f \in K_s(\gamma)$, then

$$ |a_n| \leq n - (n - 1)\gamma. $$

Taking $k = 2, A = 1, B = -1, \alpha = 0$ and $\beta = 1$, Theorem 3.1 yields the following result:

**Corollary 3.4.** If $f \in K_s$, then

$$ |a_n| \leq n. $$

**Theorem 3.2.** If $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$ and $0 \leq \alpha_2 \leq \alpha_1 < 1$, then $K_s^{(k)}(A_1, B_1; \alpha_1; \beta) \subset K_s^{(k)}(A_2, B_2; \alpha_2; \beta)$.

**Proof.** As $f \in K_s^{(k)}(A_1, B_1; \alpha_1; \beta)$, so

$$ \frac{z^k f'(z)}{g_k(z)} < \left(1 + \frac{[B_1 + (A_1 - B_1)(1 - \alpha_1)]z}{1 + B_1 z}\right)^\beta. $$

As $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$ and $0 \leq \alpha_2 \leq \alpha_1 < 1$, we have

$$ -1 \leq B_1 + (1 - \alpha_1)(A_1 - B_1) \leq B_2 + (1 - \alpha_2)(A_2 - B_2) \leq 1. $$

Thus by Lemma 2.2, it yields

$$ \frac{z^k f'(z)}{g_k(z)} < \left(1 + \frac{[B_2 + (A_2 - B_2)(1 - \alpha_2)]z}{1 + B_2 z}\right)^\beta, $$

which implies $f \in K_s^{(k)}(A_2, B_2; \alpha_2; \beta)$. □

**Theorem 3.3.** If $f \in K_s^{(k)}(A, B; \alpha; \beta)$, then for $|z| = r, 0 < r < 1$, we have

$$ \left(1 - \frac{|B + (A - B)(1 - \alpha)|r}{1 - Br}\right)^\beta \cdot \frac{1}{(1 - r)^2} \leq |f'(z)| \leq \left(1 + \frac{|B + (A - B)(1 - \alpha)|r}{1 + Br}\right)^\beta \cdot \frac{1}{(1 - r)^2} $$

and

$$ \int_0^r \left(1 - \frac{|B + (A - B)(1 - \alpha)|t}{1 - Bt}\right)^\beta \cdot \frac{1}{(1 - t)^2} dt \leq |f(z)| \leq \int_0^r \left(1 + \frac{|B + (A - B)(1 - \alpha)|t}{1 + Bt}\right)^\beta \cdot \frac{1}{(1 - t)^2} dt. $$

**Proof.** From (3.2), we have

$$ |f'(z)| = \frac{|G_k(z)|}{|z|} |P(z)|^\beta. $$

Aouf [2] proved that

$$ \frac{1 - |B + (A - B)(1 - \alpha)|r}{1 - Br} \leq |P(z)| \leq \frac{1 + |B + (A - B)(1 - \alpha)|r}{1 + Br}, $$

which implies

$$ \left(1 - \frac{|B + (A - B)(1 - \alpha)|r}{1 - Br}\right)^\beta \leq |P(z)|^\beta \leq \left(1 + \frac{|B + (A - B)(1 - \alpha)|r}{1 + Br}\right)^\beta. $$

Since $G_k \in S^*$, so by Lemma 2.3, we have

$$ \int_0^r \frac{1}{(1 + r)^2} \leq |G_k(z)| \leq \frac{r}{(1 - r)^2}. $$

Relation (3.9) together with (3.10) and (3.11) yields (3.7). On integrating (3.7) from 0 to $r$, (3.8) follows.

For $A = \eta[1 - (1 + \delta)\gamma], B = -\eta\delta, \alpha = 0, \beta = 1$, Theorem 3.3 gives the following result: □
Corollary 3.5. If \( f \in \mathcal{K}_s^{(k)}(\gamma; \delta; \eta) \), then
\[
\frac{1}{(1+r)^2} \leq |f'(z)| \leq \left( 1 + \eta |1 - (1 + \delta)\gamma r| \right) \beta \cdot \frac{1}{(1-r)^2}
\]
and
\[
\int_0^r \left( 1 + \eta |1 - (1 + \delta)\gamma t| \right) \cdot \frac{1}{(1+t)^2} dt \leq |f(z)| \leq \int_0^r \left( 1 + \frac{1}{(1-t)^2} \right) dt.
\]

Putting \( A = 1 - 2\gamma, B = -1, \alpha = 0 \) and \( \beta = 1 \) in Theorem 3.3, the following result is obvious:

Corollary 3.6. If \( f \in \mathcal{K}_s^{(k)}(\gamma) \), then
\[
\frac{2\gamma}{(1+r)^3} \leq |f'(z)| \leq \frac{2(1 - \gamma)}{(1-r)^3}.
\]
and
\[
\int_0^r \left( \frac{2\gamma t}{(1+t)^3} \right) dt \leq |f(z)| \leq \int_0^r \left( \frac{2(1 - \gamma) t}{(1-t)^3} \right) dt.
\]

Substituting for \( k = 2, A = 1 - 2\gamma, B = -1, \alpha = 0 \) and \( \beta = 1 \) in Theorem 3.3, we can easily obtain the following result:

Corollary 3.7. If \( f \in \mathcal{K}_s(\gamma) \), then
\[
\frac{1 - (1 - 2\gamma)}{(1+r)^3} \leq |f'(z)| \leq \frac{1 + (1 - 2\gamma)}{(1-r)^3}.
\]
and
\[
\int_0^r \left( \frac{1 - (1 - 2\gamma) t}{(1+t)^3} \right) dt \leq |f(z)| \leq \int_0^r \left( \frac{1 + (1 - 2\gamma) t}{(1-t)^3} \right) dt.
\]

Taking \( k = 2, A = 1, B = -1, \alpha = 0 \) and \( \beta = 1 \), Theorem 3.3 yields the following result:

Corollary 3.8. If \( f \in \mathcal{K}_s \), then
\[
\frac{1 - r}{(1+r)^3} \leq |f'(z)| \leq \frac{1 + r}{(1-r)^3}.
\]
and
\[
\int_0^r \left( \frac{1 - t}{(1+t)^3} \right) dt \leq |f(z)| \leq \int_0^r \left( \frac{1 + t}{(1-t)^3} \right) dt.
\]

Theorem 3.4. Let \( f \in \mathcal{K}_s^{(k)}(A, B; \alpha; \beta) \), then
\[
\text{Re}\left( \frac{zf'(z)}{f(z)} \right)' \geq \begin{cases} 
\frac{1-r}{1+r} - \beta \frac{A-B(1-\alpha)}{1+B+(A-B)(1-\alpha)[1-r](1-Br)}, & \text{if } R_1 \leq R_2, \\
\frac{1-r}{1+r} + \beta \frac{A-B(1-\alpha)}{1+B+(A-B)(1-\alpha)} + 2 \sqrt{(1-B)(1-B+(A-B)(1-\alpha))(1+B+(A-B)(1-\alpha)r^2)(1+Br^2)} (A-B)^{(1-\alpha)(1-r^2)} & \text{if } R_1 \geq R_2
\end{cases}
\]
where \( R_1 \) and \( R_2 \) are defined in Lemma 2.5.

Proof. Proof. As \( f \in \mathcal{K}_s^{(k)}(A, B; \alpha; \beta) \), we have
\[
zf'(z) = G_k(z)(P(z))^\beta.
\]
Differentiating logarithmically, we get
\[
\frac{(zf'(z))'}{f'(z)} = \frac{zG_k'(z)}{G_k(z)} + \beta \frac{zP'(z)}{P(z)}.
\]
As \( G_k \in \mathcal{S}^* \), so by the result due to Mehrok [8], we have
\[
\text{Re}\left( \frac{zG_k'(z)}{G_k(z)} \right) \geq \frac{1-r}{1+r}.
\]
Hence, using (3.13) and Lemma 2.5 in (3.12), the proof of Theorem 3.4 is obvious.  
\[\square\]
Theorem 3.5. If \( f \in K^k_s(A, B; \alpha; \beta) \) and let \( F(z) = zf'(z) \), then for \( |z| = r, 0 < r < 1 \), we have
\[
\left| \arg \frac{F(z)}{z} \right| \leq \beta \sin^{-1} \left( \frac{(A - B)r}{1 - ABr^2} \right) + 2\sin^{-1}r.
\]

Proof. From (3.2), we have
\[
\frac{zf'(z)}{G_k(z)} = (P(z))^\beta,
\]
which can be expressed as
\[
F(z) = G_k(z)(P(z))^\beta.
\]
Therefore, we have
\[
(3.14) \quad \left| \arg \frac{F(z)}{z} \right| \leq \beta |\arg P(z)| + \left| \arg \frac{G_k(z)}{z} \right|.
\]
It is well known that
\[
(3.15) \quad |\arg P(z)| \leq \sin^{-1} \left( \frac{(A - B)r}{1 - ABr^2} \right).
\]
It was proved by Goel and Mehrok [4] that, for \( G_k(z) \in S^* \),
\[
(3.16) \quad \left| \arg \frac{G_k(z)}{z} \right| \leq 2\sin^{-1}r.
\]
Using (3.15) and (3.16) in (3.14), Theorem 3.5 is obvious. \( \square \)

4 Conclusion and Open Problems

Close-to-convex functions are of great importance in the study of univalent functions. In the present paper, we introduce a new and generalized subclass of close-to-convex functions using subordination and established various properties for this class. Many earlier known results follow as particular cases of our results. This study will motivate the other researchers to investigate other such classes and to discuss their properties.

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