ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(2) (2023), 161-167**
(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

CERTAIN PROPERTIES OF A NEW SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

Gagandeep Singh[1], Gurcharanjit Singh[2], Navyodh Singh[3] and Navjeet Singh[4]
[1,3,4]Department of Mathematics, Khalsa College, Amritsar, Punjab, India-143001.
[2]Department of Mathematics, GNDU College, Chungh(TT), Punjab, India-143304. Email:
kamboj.gagandeep@yahoo.in; navyodh81@yahoo.co.in; navjeet8386@yahoo.com, dhillongs82@yahoo.com
(Received: August 02, 2023; In format: August 22, 2023; Revised: August 29, 2023; Accepted: October 30, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53219

Abstract

The purpose of this paper is to study a new subclass of close-to-convex functions associated with generalized Janowski's function. Various properties such as coefficient estimates, inclusion relationship, distortion property, argument property and radius of convexity, are established for this class. The results mentioned here, generalize some earlier known results.
2020 Mathematical Sciences Classification: 30C45, 30C50.
Keywords and Phrases: Analytic functions, Univalent functions, Subordination, Close-to-convex functions, Distortion theorem, Argument theorem.

1 Introduction

By $\mathcal{A}$, we denote the class of functions f of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which are analytic in the open unit disc $E = \{z :| z |< 1\}$. Further, the class of functions $f \in \mathcal{A}$ and which are univalent in E, is denoted by $\mathcal{S}$. A function w is said to be a Schwarz function if it has expansion of the form $w(z) = \sum_{n=1}^{\infty} c_n z^n$ and satisfy the conditions $w(0) = 0$ and $|w(z)| \leq 1$. The class of Schwarz functions is denoted by $\mathcal{U}$.

For two analytic functions f and g in E, f is said to be subordinate to g, if there exists a Schwarz function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$. If f is subordinate to g, then it is denoted by $f \prec g$. Further, if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

By $\mathcal{S}^*$ and $\mathcal{K}$, we denote the classes of starlike functions and of convex functions respectively, which are defined as follows:

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}$$

and

$$\mathcal{K} = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}.$$

A function $f \in \mathcal{A}$ is said to be close-to-convex function if there exists a function $g \in \mathcal{S}^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0 (z \in E).$$

The class of close-to-convex functions is denoted by $\mathcal{C}$ and was given by Kaplan [6]. Several subclasses of close-to-convex functions were studied by various authors and recently by Singh and Singh [14], but here we mention those which are relevant to our study.

Gao and Zhou [3] studied the class $\mathcal{K}_S$ defined as

$$\mathcal{K}_s = \left\{ f : f \in \mathcal{A}, Re\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > 0, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E \right\}.$$

Further, Kowalczyk and Les-Bomba [7] extended the class $\mathcal{K}_S$ by introducing the class $\mathcal{K}_S(\gamma), (0 \leq \gamma < 1)$, which is mentioned below:

$$\mathcal{K}_s(\gamma) = \left\{ f : f \in \mathcal{A}, Re\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > \gamma, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E \right\}.$$

For $\gamma = 0$, the class $\mathcal{K}_S(\gamma)$ reduces to the class $\mathcal{K}_S$.

Later on, Seker [12] established the class $\mathcal{K}_s^{(k)}(\gamma)$ $(0 \leq \gamma < 1)$ of close-to-convex analytic functions $f \in \mathcal{A}$ which satisfy the condition

$$Re\left(\frac{z^k f'(z)}{g_k(z)}\right) > \gamma,$$

where

$$g_k(z) = \Pi_{\nu=0}^{k-1} \epsilon^{-\nu} g(\epsilon^\nu z)(\epsilon^k = 1; k \geq 1), \tag{1.1}$$

and $g \in \mathcal{S}^*\left(\frac{k-1}{k}\right)$.

As a generalization, Seker and Cho [13] introduced the class $\mathcal{K}_s^{(k)}(\gamma;\delta;\eta)$ of the functions $f \in \mathcal{A}$ which satisfy the condition

$$\frac{z^k f'(z)}{g_k(z)} \prec \frac{1+\eta[1-(1+\delta)\gamma]z}{1-\eta\delta z}$$

where g_k is defined in (1.1) and $0 \leq \gamma < 1, 0 \leq \delta \leq 1$ and $0 < \eta \leq 1$.

Raina et al. [10] established the class of strongly close-to-convex functions of order β, as below:

$$\mathcal{C}'_\beta = \left\{ f : f \in \mathcal{A}, \left| arg\left\{\frac{zf'(z)}{g(z)}\right\}\right| < \frac{\beta\pi}{2}, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E \right\},$$

which can also be expressed as

$$\mathcal{C}'_\beta = \left\{ f : f \in \mathcal{A}, \frac{zf'(z)}{g(z)} \prec \left(\frac{1+z}{1-z}\right)^\beta, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E \right\}.$$

For $-1 \leq B < A \leq 1$, Janowski [5] introduced the class of functions in $\mathcal{A}$ which are of the form $p(z) = 1 + \sum_{n=1}^\infty p_n z^n$ and satisfying the condition $p(z) \prec \frac{1+Az}{1+Bz}$. This class plays an important role in the study of various subclasses of analytic-univalent functions. As a generalization of Janowski's class, Polatoglu et al. [9] established the class $\mathcal{P}(A,B;\alpha)$ $(0 \leq \alpha < 1)$, the subclass of $\mathcal{A}$ which consists of functions of the form $p(z) = 1 + \sum_{n=1}^\infty p_n z^n$ such that $p(z) \prec \frac{1+[B+(A-B)(1-\alpha)]z}{1+Bz}$. Also for $\alpha = 0$, the class $\mathcal{P}(A,B;\alpha)$ agrees with the class defined by Janowski [5].

Inspired by the above mentioned classes, now we define the following generalized class which is to study in this paper.

Definition 1.1. *Let $\mathcal{K}_s^{(k)}(A,B;\alpha;\beta)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the conditions,*

$$\frac{z^k f'(z)}{g_k(z)} \prec \left(\frac{1+[B+(A-B)(1-\alpha)]z}{1+Bz}\right)^\beta, -1 \leq B < A \leq 1, z \in E,$$

where $g(z) = z + \sum_{n=2}^\infty b_n z^n \in \mathcal{S}^*\left(\frac{k-1}{k}\right)$, $0 \leq \alpha < 1, 0 < \beta \leq 1, -1 \leq B < A \leq 1$ and $g_k(z)$ is defined in (1.1).

The following observations are obvious:

(i) $\mathcal{K}_s^{(k)}(\eta[1-(1+\delta)\gamma], -\eta\delta; 0; 1) \equiv \mathcal{K}_s(\gamma,\delta,\eta)$, *the class established by Seker and Cho [13].*

(ii) $\mathcal{K}_s^{(k)}(1-2\gamma,-1;0;1) \equiv \mathcal{K}_s^{(k)}(\gamma)$, *the class studied by Seker [12].*

(iii) $\mathcal{K}_s^{(2)}(1,-1;0;1) \equiv \mathcal{K}_s$, *the class introduced by Gao and Zhou [3].*

(iv) $\mathcal{K}_s^{(2)}(1-2\gamma,-1;0;1) \equiv \mathcal{K}_s(\gamma)$, *the class established by Kowalczyk and Les Bomba [7].*

As $f \in \mathcal{K}s^{(k)}(A,B;\alpha;\beta)$, by definition of subordination, it follows that

$$\frac{z^k f'(z)}{g_k(z)} = \left(\frac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)}\right)^\beta, w \in \mathcal{U}. \tag{1.2}$$

We study various properties such as coefficient estimates, inclusion relationship, distortion theorem, argument theorem and radius of convexity for the functions in the class $\mathcal{K}_s^{(k)}(A,B;\alpha;\beta)$. The results proved by various authors follow as special cases.

Throughout this paper, we assume that $-1 \leq B < A \leq 1, 0 \leq \alpha < 1, 0 < \beta \leq 1, 0 \leq \gamma < 1, 0 < \eta \leq 1, 0 \leq \delta \leq 1, k \geq 1, z \in E$.

2 Preliminary Results

For the derivation of our main results, we must require the following lemmas:

Lemma 2.1 ([2, 11]). *Let,*

$$\left(\frac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)}\right)^{\beta} = (P(z))^{\beta} = 1+\sum_{n=1}^{\infty} p_n z^n, \tag{2.1}$$

then

$$|p_n| \le \beta(1-\alpha)(A-B), n \ge 1.$$

Lemma 2.2 ([10]). *Let* $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$, *then*

$$\left(\frac{1+A_1 z}{1+B_1 z}\right)^{\beta} \prec \left(\frac{1+A_2 z}{1+B_2 z}\right)^{\beta}.$$

Lemma 2.3 ([8]). *If* $g \in \mathcal{S}^*$, *then for* $|z| = r, 0 < r < 1$, *we have*

$$\frac{r}{(1+r)^2} \le |g(z)| \le \frac{r}{(1-r)^2}.$$

Lemma 2.4 ([15]). *For* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*\left(\frac{k-1}{k}\right)$, *then*

$$G_k(z) = \frac{g_k(z)}{z^{k-1}} = z + \sum_{n=2}^{\infty} d_n z^n \in \mathcal{S}^*.$$

Lemma 2.5 ([1, 2]). *If* $P(z) = \frac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)}, -1 \le B < A \le 1, w \in \mathcal{U}$, *then for* $|z| = r < 1$, *we have*

$$Re\frac{zP'(z)}{P(z)} \ge \begin{cases} -\frac{(A-B)(1-\alpha)r}{(1-[B+(A-B)(1-\alpha)]r)(1-Br)}, & \text{if } R_1 \le R_2, \\ 2\frac{\sqrt{(1-B)(1-[B+(A-B)(1-\alpha)])(1+[B+(A-B)(1-\alpha)]r^2)(1+Br^2)}}{(A-B)(1-\alpha)(1-r^2)} \\ -\frac{(1-[B+(A-B)(1-\alpha)]Br^2)}{(A-B)(1-\alpha)(1-r^2)} + \frac{(A+B)-\alpha(A-B)}{(A-B)(1-\alpha)}, & \text{if } R_1 \ge R_2, \end{cases}$$

where $R_1 = \sqrt{\frac{(1-[B+(A-B)(1-\alpha)])(1+[B+(A-B)(1-\alpha)]r^2)}{(1-B)(1+Br^2)}}$ *and* $R_2 = \frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}$.

3 Main Results

Theorem 3.1. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_s^{(k)}(A, B; \alpha; \beta)$, *then*

$$|a_n| \le 1 + \frac{\beta(1-\alpha)(n-1)(A-B)}{2}. \tag{3.1}$$

Proof. As $f \in \mathcal{K}_s^{(k)}(A, B; \alpha; \beta)$, therefore (1.2) can be written as

$$\frac{z^k f'(z)}{g_k(z)} = (P(z))^{\beta},$$

which can be further expressed as

$$\frac{zf'(z)}{G_k(z)} = (P(z))^{\beta}, \tag{3.2}$$

where

$$G_k(z) = \frac{g_k(z)}{z^{k-1}} = z + \sum_{n=2}^{\infty} d_n z^n. \tag{3.3}$$

By Lemma 2.4, we have $G_k \in \mathcal{S}^*$.

Using (2.1) and (3.3) in (3.2), it yields

$$1 + \sum_{n=2}^{\infty} n a_n z^{n-1} = \left(1 + \sum_{n=2}^{\infty} n d_n z^{n-1}\right)\left(1 + \sum_{n=1}^{\infty} p_n z^n\right). \tag{3.4}$$

As $G_k(z) = z + \sum_{n=2}^{\infty} d_n z^n \in \mathcal{S}^*$, it is well known that $|d_n| \le n$.

Comparing the coefficients of z^{n-1} in (3.4), we have

$$n a_n = d_n + d_{n-1}p_1 + d_{n-2}p_2 + ... + d_2 p_{n-2} + p_{n-1}. \tag{3.5}$$

Applying triangle inequality, using Lemma 2.1 and the inequality $|d_n| \le n$ in (3.5), it gives

$$n|a_n| \le n + \beta(1-\alpha)(A-B)[(n-1) + (n-2) + ... + 2 + 1], \tag{3.6}$$

which proves Theorem 3.1. □

For $A=\eta[1-(1+\delta)\gamma], B=-\eta\delta, \alpha=0, \beta=1$, Theorem 3.1 gives the following result:

Corollary 3.1. *If $f \in \mathcal{K}_s^{(k)}(\gamma;\delta;\eta)$, then*

$$|a_n| \le 1 + \frac{\eta(n-1)(1+\delta)(1-\gamma)}{2}.$$

Putting $A=1-2\gamma$, $B=-1$, $\alpha=0$ and $\beta=1$ in Theorem 3.1, the following result is obvious:

Corollary 3.2. *If $f \in \mathcal{K}_s^{(k)}(\gamma)$, then*

$$|a_n| \le n-(n-1)\gamma.$$

Substituting for $k=2, A=1-2\gamma$, $B=-1$, $\alpha=0$ and $\beta=1$ in Theorem 3.1, we can easily obtain the following result:

Corollary 3.3. *If $f \in \mathcal{K}_s(\gamma)$, then*

$$|a_n| \le n-(n-1)\gamma.$$

Taking $k=2, A=1$, $B=-1$, $\alpha=0$ and $\beta=1$, Theorem 3.1 yields the following result:

Corollary 3.4. *If $f \in \mathcal{K}_s$, then*

$$|a_n| \le n.$$

Theorem 3.2. *If $-1 \le B_2 = B_1 < A_1 \le A_2 \le 1$ and $0 \le \alpha_2 \le \alpha_1 < 1$, then*

$$\mathcal{K}_s^{(k)}(A_1,B_1;\alpha_1;\beta) \subset \mathcal{K}_s^{(k)}(A_2,B_2;\alpha_2;\beta).$$

Proof. As $f \in \mathcal{K}_s^{(k)}(A_1,B_1;\alpha_1;\beta)$, so

$$\frac{z^k f'(z)}{g_k(z)} \prec \left(\frac{1+[B_1+(A_1-B_1)(1-\alpha_1)]z}{1+B_1 z}\right)^{\beta}.$$

As $-1 \le B_2 = B_1 < A_1 \le A_2 \le 1$ and $0 \le \alpha_2 \le \alpha_1 < 1$, we have

$$-1 \le B_1+(1-\alpha_1)(A_1-B_1) \le B_2+(1-\alpha_2)(A_2-B_2) \le 1.$$

Thus by Lemma 2.2, it yields

$$\frac{z^k f'(z)}{g_k(z)} \prec \left(\frac{1+[B_2+(A_2-B_2)(1-\alpha_2)]z}{1+B_2 z}\right)^{\beta},$$

which implies $f \in \mathcal{K}_s^{(k)}(A_2,B_2;\alpha_2;\beta)$. □

Theorem 3.3. *If $f \in \mathcal{K}_s^{(k)}(A,B;\alpha;\beta)$, then for $|z|=r, 0<r<1$, we have*

$$\left(\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}\right)^{\beta} \cdot \frac{1}{(1+r)^2} \le |f'(z)| \le \left(\frac{1+[B+(A-B)(1-\alpha)]r}{1+Br}\right)^{\beta} \cdot \frac{1}{(1-r)^2} \tag{3.7}$$

and

$$\int_0^r \left(\frac{1-[B+(A-B)(1-\alpha)]t}{1-Bt}\right)^{\beta} \cdot \frac{1}{(1+t)^2} dt \le |f(z)| \le \int_0^r \left(\frac{1+[B+(A-B)(1-\alpha)]t}{1+Bt}\right)^{\beta} \cdot \frac{1}{(1-t)^2} dt. \tag{3.8}$$

Proof. From (3.2), we have

$$|f'(z)| = \frac{|G_k(z)|}{|z|}(P(z))^{\beta}. \tag{3.9}$$

Aouf [2] proved that

$$\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br} \le |P(z)| \le \frac{1+[B+(A-B)(1-\alpha)]r}{1+Br},$$

which implies

$$\left(\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}\right)^{\beta} \le |P(z)|^{\beta} \le \left(\frac{1+[B+(A-B)(1-\alpha)]r}{1+Br}\right)^{\beta}. \tag{3.10}$$

Since $G_k \in \mathcal{S}^*$, so by Lemma 2.3, we have

$$\frac{r}{(1+r)^2} \le |G_k(z)| \le \frac{r}{(1-r)^2}. \tag{3.11}$$

Relation (3.9) together with (3.10) and (3.11) yields (3.7). On integrating (3.7) from 0 to r, (3.8) follows.

For $A=\eta[1-(1+\delta)\gamma], B=-\eta\delta, \alpha=0, \beta=1$, Theorem 3.3 gives the following result: □

Corollary 3.5. *If $f \in \mathcal{K}_s^{(k)}(\gamma;\delta;\eta)$, then*

$$\left(\frac{1-\eta[1-(1+\delta)\gamma]r}{1+\eta\delta r}\right)\cdot\frac{1}{(1+r)^2} \le |f'(z)| \le \left(\frac{1+\eta[1-(1+\delta)\gamma]r}{1-\eta\delta r}\right)^{\beta}\cdot\frac{1}{(1-r)^2}$$

and

$$\int_0^r \left(\frac{1-\eta[1-(1+\delta)\gamma]t}{1+\eta\delta t}\right)\cdot\frac{1}{(1+t)^2}dt \le |f(z)| \le \int_0^r \left(\frac{1+\eta[1-(1+\delta)\gamma]t}{1-\eta\delta t}\right)\cdot\frac{1}{(1-t)^2}dt.$$

Putting $A = 1-2\gamma$, $B=-1$, $\alpha=0$ and $\beta=1$ in Theorem 3.3, the following result is obvious:

Corollary 3.6. *If $f \in \mathcal{K}_s^{(k)}(\gamma)$, then*

$$\frac{2\gamma r}{(1+r)^3} \le |f'(z)| \le \frac{2(1-\gamma)r}{(1-r)^3}.$$

and

$$\int_0^r \left(\frac{2\gamma t}{(1+t)^3}\right)dt \le |f(z)| \le \int_0^r \left(\frac{2(1-\gamma)t}{(1-t)^3}\right)dt.$$

Substituting for $k=2, A=1-2\gamma$, $B=-1$, $\alpha=0$ and $\beta=1$ in Theorem 3.3, we can easily obtain the following result:

Corollary 3.7. *If $f \in \mathcal{K}_s(\gamma)$, then*

$$\frac{1-(1-2\gamma)r}{(1+r)^3} \le |f'(z)| \le \frac{1+(1-2\gamma)r}{(1-r)^3}.$$

and

$$\int_0^r \left(\frac{1-(1-2\gamma)t}{(1+t)^3}\right)dt \le |f(z)| \le \int_0^r \left(\frac{1+(1-2\gamma)t}{(1-t)^3}\right)dt.$$

Taking $k=2, A=1$, $B=-1$, $\alpha=0$ and $\beta=1$, Theorem 3.3 yields the following result:

Corollary 3.8. *If $f \in \mathcal{K}_s$, then*

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}.$$

and

$$\int_0^r \left(\frac{1-t}{(1+t)^3}\right)dt \le |f(z)| \le \int_0^r \left(\frac{1+t}{(1-t)^3}\right)dt.$$

Theorem 3.4. *Let $f \in \mathcal{K}_s^{(k)}(A,B;\alpha;\beta)$, then*

$$Re\frac{(zf'(z))'}{f'(z)} \ge \begin{cases} \frac{1-r}{1+r} - \beta\frac{(A-B)(1-\alpha)r}{(1-[B+(A-B)(1-\alpha)]r)(1-Br)}, & \text{if } R_1 \le R_2, \\ \frac{1-r}{1+r} + \frac{(A+B)-\alpha(A-B)}{(A-B)(1-\alpha)} & \\ +2\frac{\sqrt{(1-B)(1-[B+(A-B)(1-\alpha)])(1+[B+(A-B)(1-\alpha)]r^2)(1+Br^2)}}{(A-B)(1-\alpha)(1-r^2)} & \\ -2\frac{(1-[B+(A-B)(1-\alpha)]Br^2)}{(A-B)(1-\alpha)(1-r^2)}, & \text{if } R_1 \ge R_2, \end{cases}$$

where R_1 and R_2 are defined in Lemma 2.5.

Proof. Proof. As $f \in \mathcal{K}_s^{(k)}(A,B;\alpha;\beta)$, we have

$$zf'(z) = G_k(z)(P(z))^{\beta}.$$

Differentiating logarithmically, we get

$$\frac{(zf'(z))'}{f'(z)} = \frac{zG_k'(z)}{G_k(z)} + \beta\frac{zP'(z)}{P(z)}. \tag{3.12}$$

As $G_k \in \mathcal{S}^*$, so by the result due to Mehrok [8], we have

$$Re\left(\frac{zG_k'(z)}{G_k(z)}\right) \ge \frac{1-r}{1+r}. \tag{3.13}$$

Hence, using (3.13) and Lemma 2.5 in (3.12), the proof of Theorem 3.4 is obvious. □

Theorem 3.5. *If $f \in \mathcal{K}_s^{(k)}(A, B; \alpha; \beta)$ and let $F(z) = zf'(z)$, then for $|z| = r, 0 < r < 1$, we have*

$$\left|arg\frac{F(z)}{z}\right| \leq \beta sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right) + 2sin^{-1}r.$$

Proof. Proof. From (3.2), we have

$$\frac{zf'(z)}{G_k(z)} = (P(z))^{\beta},$$

which can be expressed as

$$F(z) = G_k(z)(P(z))^{\beta}.$$

Therefore, we have

$$\left|arg\frac{F(z)}{z}\right| \leq \beta|argP(z)| + \left|arg\frac{G_k(z)}{z}\right|. \tag{3.14}$$

It is well known that

$$|argP(z)| \leq sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right). \tag{3.15}$$

It was proved by Goel and Mehrok [4] that, for $G_k(z) \in S^*$,

$$\left|arg\frac{G_k(z)}{z}\right| \leq 2sin^{-1}r. \tag{3.16}$$

Using (3.15) and (3.16) in (3.14), Theorem 3.5 is obvious. □

4 Conclusion and Open Problems

Close-to-convex functions are of great importance in the study of univalent functions. In the present paper, we introduce a new and generalized subclass of close-to-convex functions using subordination and established various properties for this class. Many earlier known results follow as particular cases of our results. This study will motivate the other researchers to investigate other such classes and to discuss their properties.

Acknowledgement. The authors are very greatful to the editor and referees for their valuable suggestions to revise the paper.

References

[1] V. V. Anh and P. D. Tuan, On β-convexity of certain starlike functions, *Rev. Roum. Math. Pures et Appl.*, **25**(1979), 1413-1424.

[2] M. K. Aouf, On a class of p-valent starlike functions of order α, *Int. J. Math. Math. Sci.*, **10**(4)(1987), 733-744.

[3] C.Y. Gao and S.Q. Zhou, On a class of analytic functions related to the starlike functions, *Kyungpook Math. J.*, **45**(2005), 123-130.

[4] R.M. Goel and B.S. Mehrok, On a class of close-to-convex functions, *Indian J. Pure Appl. Math.*, **12**(5)(1981), 648-658.

[5] W. Janowski, Some extremal problems for certain families of analytic functions, *Ann. Pol. Math.*, **28**(1973), 297-326.

[6] W. Kaplan, Close-to-convex schlicht functions, *Michigan Math. J.*, **1**(1952), 169-185.

[7] J. Kowalczyk and E. Les-Bomba, On a subclass of close-to-convex functions, *Appl. Math. Letters*, **23**(2010), 1147-1151.

[8] B. S. Mehrok, A subclass of close-to-convex functions, *Bull. Inst. Math. Acad. Sin.*, **10**(4)(1982), 389-398.

[9] Y. Polatoglu, M. Bolkal, A. Sen and E. Yavuz, A study on the generalization of Janowski function in the unit disc, *Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis*, **22**(2006), 27-31.

[10] R. K. Raina, P. Sharma and J. Sokol, A class of strongly close-to-convex functions, *Bol. Soc. Paran. Mat.*, **38**(6)(2020), 9-24.

[11] W. Rogosinski, On the coefficients of subordinate functions, *Proc. Lond. Math. Soc.*, **48**(2)(1943), 48-82.

[12] B. Seker, On certain new subclass of close-to-convex functions, *App. Math. Comput.*, **218**(3)(2011), 1041-1045.

[13] B. Seker and N. E. Cho, A subclass of close-to-convex functions, *Hacettepe J. Math. Stat.*, **42**(4)(2013), 373-379.

[14] G. Singh and G. Singh, Coefficient bounds for certain subclasses of close-to-convex and quasi-convex functions with fixed point, *Jñānābha* , **52**(1)(2022), 141-148.
[15] Z. G. Wang, C. Gao and S. M. Yuan, On certain subclass of close-to-convex functions, *Mat. Vesnik*, **58**(2006), 119-124.