ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(2) (2023), 151-160**
(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

A NEW α-LAPLACE TRANSFORM ON TIME SCALES

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(Received: July 22, 2023, In format: August 31, 2023; Revised: September 24, 2023; Accepted: October 02, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53218

Abstract

In this paper we introduce a new $\alpha-$ Laplace transform which is a generalization of nabla version of Laplace transform on time scales. In particular for $0 < \alpha < 1$ this transform will serve as fractional Laplace transform on time scales. Existence theorem and some important properties such as linearity, initial and final value theorem, transform of integral, shifting theorem, transform of derivative are proved. Additionally convolution theorem and formulae for fractional integral, Riemann-Liouville fractional derivative, Liouville-Caputo fractional derivative, Mittag Leffler function are given. At last for a suitable value of α a fractional dynamic equation with given initial condition is solved.

2020 Mathematical Sciences Classification: 26E70, 44A35, 26A33
Keywords and Phrases: Time scales, Integral transform, Dynamic equations.

1 Introduction and Motivation

Integral transforms are mathematical techniques that play a crucial role in various fields of Science and Engineering. The important significance of integral transform is their ability to simplify mathematical equations often involving derivatives and integrals, into algebraic equations or simpler differential equations. The theory of the Laplace transform has a great theoretical interest as it is one of the former integral transforms invented by Pierre-Simon Laplace. Furthermore many researchers have introduced several integral transforms such as Laplace-Carson, Sumudu, Elzaki, Natural and Shehu transforms [7, 10, 11, 12, 27], all of which represent a family of Laplace transforms. Subsequently an ample development regarding classical integral transforms in form of generalization in distribution spaces, formulation of multidimensional and fractional transform has been done due to [5, 16, 18, 23, 26]. Recently H.M.Srivastava [24] has explored recent developments in the Laplace and Hankel transforms and their extensions and variations. Using Srivastava's generalized Whittaker transform [17], Hardy's generalized Hankel transform [9] and Srivastava's $\epsilon-$ generalized Hankel transform [16], the properties, characteristics and relationships among integral transforms representing the family of Laplace transform are studied. Further eminent results regarding integral transforms and fractional calculus have been studied in [19, 20, 21, 22].

Time scale calculus is an unification tool that encompasses both continuous (*e.g.* $\mathbb{R}$) and discrete (*e.g.* $\mathbb{Z}$) domains. Integral transforms on time scales a mathematical framework that extends the concepts of classical integral transforms into functions defined on time scale domains. Thus far Laplace, Fourier, Sumudu and Shehu transforms have been introduced on time scales and have served as a powerful tool for modeling and solving problems that bridge the gap between continuous and discrete dynamic systems [1, 2, 3, 4, 6, 8, 15, 25, 28]. In 2016 Medina Gustavo et al.[13] introduced a new $\alpha-$integral Laplace transform which is a generalization of Laplace transform when $\alpha \to 1$. Subsequently, in 2007, a the fractional Laplace transform was formulated and is applied to solve fractional differential equations [14]. In our work we develop a new $\alpha-$ Laplace transform on time scales and discuss its fundamental properties. Using this transform, we solve the fractional dynamic equation on time scales with given initial conditions.

The next section is concerned with precursory concepts needed for comprehension of our work.

2 Preliminaries

Note that the discussion in this section follows from [2, 3, 4, 6, 15, 25]. Here we will assume that a time scale $\mathbb{T}$ is unbounded above and $t_0 \in \mathbb{T}$ is fixed. For $t \in \mathbb{T}$, the *forward jump operator* $\sigma(t) : \mathbb{T} \to \mathbb{T}$ is given as $\sigma(t) := inf\{s \in \mathbb{T} : s > t\}$. And the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is given as $\rho(t) := sup\{s \in \mathbb{T} : s < t\}$.

If $\sigma(t) > t$, t is said to be *right-scattered*, while if $\rho(t) < t$ then t is *left-scattered*. Also, if $t < sup\mathbb{T}$ and $\sigma(t) = t$,then t is called *right-dense*, and if $t > inf\mathbb{T}$ and $\rho(t) = t$, then t is *left-dense*.

For $t \in \mathbb{T}$ the *forward graininess* function $\mu : \mathbb{T} \to [0, \infty)$ is $\mu(t) = \sigma(t) - t$. And the *backward graininess* function $\nu : \mathbb{T} \to [0, \infty)$ is $\nu(t) = t - \rho(t)$.

Definition 2.1. *A function $h : \mathbb{T} \to \mathbb{C}$ is said to be* ld-continuous *if it is continuous at every* left-dense point*, and the right sided limit exists at every* right-dense point. *It is expressed as $h(t) \in \mathcal{C}_{ld}(\mathbb{T}, \mathbb{C})$.*

Note that if time scale $\mathbb{T}$ has a right scattered miminum m, the $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$

Definition 2.2. *An* ld-continuous *function $h : \mathbb{T} \to \mathbb{C}$ is called complex $\nu-regressive$ if $1 - \nu h \neq 0$ for all $t \in \mathbb{T}_k$. It is denoted as $\mathcal{R}^{\nu}_c(\mathbb{T}, \mathbb{C})$.*

For $h > 0$, we have $\mathbb{C}_h = \{z \in \mathbb{C} : z \neq \frac{1}{h}\}$ and $\mathbb{Z}_h = \{z \in \mathbb{C} : \frac{-\pi}{h} < Im(z) \leq \frac{\pi}{h}\}$ with $\mathbb{C}_0 = \mathbb{Z}_0 = \mathbb{C}$. Further the Hilger real part and imaginary part of a complex numbers are given by $\mathscr{R}e_h(z) = \frac{1}{h}(1 - |1 - hz|)$ and $\mathscr{I}m_h(z) = \frac{1}{h}Arg(1 - hz)$ respectively, where Arg denotes principal argument of a complex number. In particular we have, $\mathscr{R}e_0(z) = \mathscr{R}e(z)$ and $\mathscr{I}m_0(z) = \mathscr{I}m(z)$.

Definition 2.3. *If $f \in \mathcal{R}^{\nu}_c(\mathbb{T}, \mathbb{C})$, then the nabla exponential function is given by, $\hat{e}_f(t, t_0) := exp\big[\int_{t_0}^{t} \hat{\xi}_{\nu(s)}(f(s))\nabla s\big]$ for $t, t_0 \in \mathbb{T}$ where, the $\nu-$cylinder transformation $\hat{\xi}_h : \mathbb{C}_h \to \mathbb{Z}_h$ is $\hat{\xi}_h(z) = \frac{-1}{h}Log(1 - zh)$.*

Theorem 2.1. *Let the first-order linear dynamic equation $x^{\nabla} = f(t)x$ is $\nu-$regressive and $t_0 \in \mathbb{T}$ is fixed. Then $\hat{e}_f(\cdot, t_0)$ is the solution of the initial value problem $x^{\nabla} = f(t)x, \quad x(t_0) = 1$ on $\mathbb{T}$.*

Lemma 2.1. *If $f \in \mathcal{R}^{\nu}_c(\mathbb{T}, \mathbb{C})$ then, $\hat{e}^{\rho}_{\ominus f}(t, t_0) = \frac{\hat{e}_{\ominus f}(t,t_0)}{1-\nu(t)f}$.*

Definition 2.4. *A function h belongs to the space of functions $\mathcal{A}(\mathbb{T})$ if*

(1) h is *piecewise ld-continuous in every interval* $[t_0, \tau] \cap \mathbb{T}$.
(2) h is of *exponential order* k ($k \in \mathcal{R}^{+\nu}_c([t_0, \infty))$ on $[t_0, \infty)$, *that is there exists constant $M > 0$ such that $|f(t)| \leq Me_k(t, t_0)$ for all $t \in [t_0, \infty)$.*

The *minimal-graininess* function $\nu_* : \mathbb{T} \to \mathbb{R}^+_0$ is given as $\nu_*(t_0) = inf\ \nu(t)$ for $t \in [t_0, \infty)_{\mathbb{T}}$.

Theorem 2.2 (Decay of the nabla-exponential function). *For $sup\ \mathbb{T} = \infty$, let $t_0 \in \mathbb{T}$ and $\lambda \in \mathcal{R}^{+\nu}_c([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$. Then for any $z \in \mathbb{C}_{\nu_*(t_0)}(\lambda)$, we have the following properties,*

(1) $|\hat{e}_{\lambda\ominus z}(t, t_0)| \leq \hat{e}_{\lambda\ominus\mathscr{R}e_{\nu_*(t_0)}(z)}(t, t_0)$ *for all* $t \in [t_0, \infty)_{\mathbb{T}}$.
(2) $\lim_{t\to\infty} \hat{e}_{\lambda\ominus\mathscr{R}e_{\nu_*(t_0)}(z)}(t, t_0) = 0$.
(3) $\lim_{t\to\infty} \hat{e}_{\lambda\ominus z}(t, t_0) = 0$.

Definition 2.5. *Let $t_0, t \in \mathbb{T}$ and $\lambda_1, \lambda_2 > -1$. The time scale power functions $\hat{h}_{\lambda_1}(t, t_0)$ are defined as a family of non-negative functions satisfying,*

(1) $\int_{t_0}^{t} \hat{h}_{\lambda_1}(t, \rho(s))\hat{h}_{\lambda_2}(s, t_0)\nabla s = \hat{h}_{\lambda_1+\lambda_2+1}(t, t_0)$ *for* $t \geq t_0$.
(2) $\hat{h}_0(t, t_0) = 1$ *for* $t \geq t_0$.
(3) $\hat{h}_{\lambda_1}(t, t) = 0$ *for* $\lambda_1 \in (0, 1)$.

Definition 2.6. *Let $t_0, t' \in \mathbb{T}$ and $a \geq 0, \beta, \lambda > 0$, then for $h \in \mathcal{C}_{ld}([t_0, t']_{\mathbb{T}}, \mathbb{C})$ one defines,*

(1) *The Riemann-Liouville fractional integral of order $a > 0$ with the lower limit t_0 as*
$$_{t_0}(\nabla^{-a}h)(t) := \int_{t_0}^{t} \hat{h}_{a-1}(t, \rho(\tau))h(\tau)\nabla\ \tau$$
and for $a = 0$ one have $(_{t_0}\nabla^0 h)(t) = h(t)$.
(2) *The Riemann-Liouville fractional derivative of order $\beta > 0$ with lower limit t_0 as*
$$(_{t_0}\nabla^{\beta}h)(t) := [_{t_0}\nabla^{-(n-\beta)}h]^{\nabla^n} \quad t \in [\sigma(t_0), t']_{\mathbb{T}},$$
where $n = [\beta] + 1$.
(3) *The Caputo fractional derivative $^{\mathcal{C}}_{t_0}\nabla^{\lambda}h(t)$ on $[\sigma(t), t']_{\mathbb{T}}$ is defined via the Riemann-Liouville fractional derivative by,*
$$^{\mathcal{C}}_{t_0}\nabla^{\lambda}h(t) := (_{t_0}\nabla^{-(n-\lambda)}h^{\nabla^n})(t),$$
where $n = [\lambda] + 1$.

3 Main Results

In this section we define $\alpha-$Laplace transform and give some of its salient properties.

Definition 3.1. *Let $h : \mathbb{T} \to \mathbb{C}$ is an ld-continuous function and α is a real number, then we define the $\alpha-$Laplace transform $\mathscr{L}^{\alpha}\{h(t)\} = \mathscr{H}_{\mathbb{T}}(z)$ of $h(t)$ of order α as*

$$\mathscr{H}_{\mathbb{T}}(z) = \mathscr{L}^{\alpha}\{h(t)\} = \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t,t_0)h(t)\ \nabla t \quad \textit{for all}\ \ t \in \mathcal{D}_{\nu},$$

where $\mathcal{D}_{\nu} \subset \mathbb{C}$ consists of all complex numbers $z \in \mathcal{R}_c(\mathbb{T},\mathbb{C})$ for which the improper integral converges.

Next we define fractional Laplace transform on time scales using above definition as.

Definition 3.2. *Let $h : \mathbb{T} \to \mathbb{C}$ is an ld-continuous function then for a real number $0 < \alpha < 1$, $\mathscr{L}^{\alpha}\{h(t)\} = \mathscr{H}_{\mathbb{T}}(z)$ will be the fractional Laplace transform of $h(t)$.*

Theorem 3.1 (**Existence Theorem**). *Let $h : \mathbb{T} \to \mathbb{C}$ is a function of class $\mathcal{A}(\mathbb{T})$ of exponential order k, then the α-Laplace transform $\mathscr{L}^{\alpha}\{h(t)\}$ of $h(t)$ exists for all $z \in \mathbb{C}_{(\nu_*(t_0))^{1/\alpha}}$ with $\mathscr{R}e_{(\nu_*(t_0))^{1/\alpha}}(z) > k$ and converges absolutely.*

Proof. We have

$$\begin{aligned}
|\mathscr{L}^{\alpha}\{h(t)\}| &= \left| \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t_0,t)h(t)\ \nabla t \right| \\
&\leq \int_{t_0}^{\infty} |\hat{e}^{\rho}_{\ominus z^{\alpha}}(t_0,t)h(t)\ \nabla t| \\
&\leq M \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t_0,t)\hat{e}_k(t,t_0)\ \nabla t \\
&= M \int_{t_0}^{\infty} \frac{\hat{e}_{\ominus z^{\alpha}}(t_0,t)\hat{e}_k(t,t_0)}{1-\nu(t)z^{\alpha}}\ \nabla t \\
&= M \int_{t_0}^{\infty} \frac{\hat{e}_{k\ominus z^{\alpha}}(t,t_0)}{1-\nu(t)z^{\alpha}}\ \nabla t \\
&= \frac{M}{k-z^{\alpha}} \int_{t_0}^{\infty} \frac{k-z^{\alpha}}{(1-\nu(t)z^{\alpha})}\hat{e}_{k\ominus z^{\alpha}}(t,t_0)\ \nabla t \\
&= \frac{M}{k-z^{\alpha}} \int_{t_0}^{\infty} (k\ominus z^{\alpha})\hat{e}_{k\ominus z^{\alpha}}(t,t_0)\ \nabla t \\
&= \frac{M}{k-z^{\alpha}} \int_{t_0}^{\infty} \hat{e}^{\nabla}_{k\ominus z^{\alpha}}(t,t_0)\nabla t \\
&= \frac{M}{z^{\alpha}-k}.
\end{aligned}$$

Last step follows from Theorem 2.2. □

Theorem 3.2 (**Linearity**). *Let $ah_1(t)$ and $bh_2(t)$ are functions of class $\mathcal{A}(\mathbb{T})$ for constants $a, b \in \mathbb{R}$ then,*

$$\mathscr{L}^{\alpha}\{ah_1(t) + bh_2(t)\} = a\mathscr{L}^{\alpha}\{h_1(t)\} + b\mathscr{L}^{\alpha}\{h_2(t)\}.$$

Proof. Let $ah_1(t)$ and $bh_2(t)$ are functions of class $\mathcal{A}(\mathbb{T})$ with exponential order k_1 and k_2 respectively, then $\mathscr{L}^{\alpha}\{ah_1(t)\}$ exists for all $z \in \mathbb{C}_{(\nu_*(t_0))^{1/\alpha}}(k_1)$ with $\mathscr{R}e_{(\nu_*(t_0))^{1/\alpha})}(z) > k_1$ and $\mathscr{L}^{\alpha}\{bh_2(t)\}$ exists for all $z \in \mathbb{C}_{(\nu_*(t_0)^{1/\alpha})}(k_2)$ with $\mathscr{R}e_{(\nu_*(t_0))^{1/\alpha}}(z) > k_2$. Then $\mathscr{L}^{\alpha}\{ah_1(t) + bh_2(t)\}$ exists for all $z \in \mathbb{C}_{(\nu_*(t_0))^{1/\alpha}}$ with $\mathscr{R}e_{(\nu_*(t_0))^{1/\alpha}}(z) > max\{k_1, k_2\}$. Thus,

$$\begin{aligned}
&\mathscr{L}^{\alpha}\{ah_1(t) + bh_2(t)\} \\
&= \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t,t_0)[ah_1(t) + bh_2(t)]\ \nabla t \\
&= a\int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t,t_0)h_1(t)\ \nabla t + b\int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^{\alpha}}(t,t_0)h_2(t)\ \nabla t \\
&= a\mathscr{L}^{\alpha}\{h_1(t)\} + b\mathscr{L}^{\alpha}\{h_2(t)\}.
\end{aligned}$$

□

We apply definition 3.1 to find transform of some elementary functions which are given in the table below

$h(t)$	1	$\hat{e}_a(t,t_0)$	$sin_a(t,t_0)$	$cos_a(t,t_0)$
$\mathscr{L}^\alpha\{h(t)\}$	$\frac{1}{z^\alpha}$	$\frac{a}{z^\alpha-a}$	$\frac{a}{z^{2\alpha}+a^2}$	$\frac{z^\alpha}{z^{2\alpha}+a^2}$

$h(t)$	$sinh_a(t,t_0)$	$cosh_a(t,t_0)$	$h_k(t,t_0)$	$\hat{h}_\lambda(t,t_0)$
$\mathscr{L}^\alpha\{h(t)\}$	$\frac{a}{z^{2\alpha}-a^2}$	$\frac{z^\alpha}{z^{2\alpha}-a^2}$	$\frac{1}{z^{(k+1)\alpha}}$	$\frac{1}{z^{(\lambda+1)\alpha}}$

.

Here $\hat{h}_\lambda(t,t_0)$ is the time scale power function defined in definition 2.5 and it's transform was found using Convolution theorem which we are going to prove further.

Theorem 3.3. *Assume that $h(t)$ is regulated function such that $H(t)=\int_{t_0}^{t}h(s)\nabla s$ for $t,t_0\in\mathbb{T}$ is of class $\mathcal{A}(\mathbb{T})$ then $\mathscr{L}^\alpha\{H(t)\}=\frac{1}{z^\alpha}\mathscr{L}^\alpha\{h(t)\}$.*

Proof.

$$\begin{aligned}
\mathscr{L}^\alpha\{H(t)\} &= \int_{t_0}^{\infty} H(t)\hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)\ \nabla t\\
&= -\int_{t_0}^{\infty} H(t)\frac{\ominus z^\alpha}{z^\alpha}\hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t\\
&= \frac{-1}{z^\alpha}\int_{t_0}^{\infty} H(t)\ominus z^\alpha\hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t\\
&= \frac{-1}{z^\alpha}\int_{t_0}^{\infty} H(t)\hat{e}^{\nabla}_{\ominus z^\alpha}(t,t_0)\ \nabla t
\end{aligned}$$

Using integration by parts

$$\begin{aligned}
\mathscr{L}^\alpha\{H(t)\} &= \frac{-1}{z^\alpha}\Big[\big[H(t)\hat{e}_{\ominus z^\alpha}(t,t_0)\big]_{t=t_0}^{t\to\infty} - \int_{t_0}^{\infty} H^{\nabla}(t)\hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t\Big]\\
&= \frac{-1}{z^\alpha}\Big[-H(t_0) - \int_{t_0}^{\infty} h(t)\hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t\Big]\\
&= \frac{1}{z^\alpha}\int_{t_0}^{\infty} h(t)\hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t\\
&= \frac{1}{z^\alpha}\mathscr{L}^\alpha\{h(t)\},
\end{aligned}$$

provided $\lim_{t\to\infty} H(t)\hat{e}_{\ominus z^\alpha}(t,t_0)=0$.

□

Theorem 3.4 (Second Shifting Theorem). *If $h(t)\in\mathcal{A}(\mathbb{T})$ and $u_a(t)=\begin{cases}1 & if\ \ t\in\mathbb{T}\cap(-\infty,a]\\ 0 & if\ \ t\in\mathbb{T}\cup(a,\infty)\end{cases}$ where $a\in\mathbb{T}$ with $a>0$, then $\mathscr{L}^\alpha\{u_a(t)h(t)\}=\hat{e}_{\ominus z^\alpha}(a,t_0)\mathscr{L}_\alpha\{h(t)\}$.*

Proof.

$$\begin{aligned}
&\mathscr{L}^\alpha\{u_a(t)h(t)\}\\
&= \int_{t_0}^{\infty}\hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)u_a(t)h(t)\ \nabla t\\
&= \int_{a}^{\infty}\hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)h(t)\ \nabla t\\
&= \int_{a}^{\infty}\frac{\hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)}{1-\nu(t)z^\alpha}h(t)\ \nabla t\\
&= \int_{a}^{\infty}\frac{\hat{e}_{\ominus z^\alpha}(t,a)\hat{e}_{\ominus z^\alpha}(a,t_0)}{1-\nu(t)z^\alpha}h(t)\ \nabla t
\end{aligned}$$

$$= \hat{e}_{\ominus z^\alpha}(a,t_0)\int_a^\infty \frac{\hat{e}_{\ominus z^\alpha}(t,a)}{1-\nu(t)z^\alpha}h(t)\ \nabla t$$
$$= \hat{e}_{\ominus z^\alpha}(a,t_0)\int_a^\infty \hat{e}^{\rho}_{\ominus z^\alpha}(t,a)h(t)\ \nabla t$$
$$= \hat{e}_{\ominus z^\alpha}(a,t_0)\mathscr{L}^\alpha\{h(t)\}.$$

□

Theorem 3.5 (**Transform of derivative**). *Let $h, h^\nabla \in \mathcal{A}(\mathbb{T})$, then $\mathscr{L}^\alpha\{h^\nabla(t)\} = z^\alpha \mathscr{L}^\alpha\{h(t)\} - h(t_0)$ for those regressive $z \in \mathbb{C}$ satisfying $\lim_{t\to\infty}\{h(t)\hat{e}_{\ominus z^\alpha}(t,t_0)\} = 0$*

Proof.

$$\begin{aligned}
\mathscr{L}^\alpha\{h^\nabla(t)\} &= \int_{t_0}^\infty \hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)h^\nabla(t)\ \nabla t \\
&= \int_{t_0}^\infty \Big[[h(t)\hat{e}_{\ominus z^\alpha}(t,t_0)]^\nabla - h(t)\hat{e}^\nabla_{\ominus z^\alpha}(t,t_0)\Big]\ \nabla t \\
&= [h(t)\hat{e}_{\ominus z^\alpha}(t,t_0)]_{t=t_0}^{t\to\infty} - \int_{t_0}^\infty h(t)\hat{e}^\nabla_{\ominus z^\alpha}(t,t_0)\ \nabla t \\
&= -h(t_0) - \int_{t_0}^\infty h(t)\ominus z^{1/\alpha}\hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t \\
&= -h(t_0) + z^\alpha\int_{t_0}^\infty h(t)\frac{-\ominus z^\alpha \hat{e}_{\ominus z^\alpha}(t,t_0)}{z^\alpha}\ \nabla t \\
&= -h(t_0) + z^\alpha\int_{t_0}^\infty h(t)\hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)\ \nabla t \\
&= z^\alpha\mathscr{L}^\alpha\{h(t)\} - h(t_0),
\end{aligned}$$

provided, $\lim_{t\to\infty} h(t)\hat{e}_{\ominus z^\alpha}(t,t_0) = 0$. □

In a similar way for $h, h^\nabla, h^{\nabla\nabla} \in \mathcal{A}(\mathbb{T})$ then

$$\begin{aligned}
\mathscr{L}^\alpha\{h^{\nabla\nabla}(t)\} &= \int_{t_0}^\infty \hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)h^{\nabla\nabla}(t)\ \nabla t \\
&= \int_{t_0}^\infty \Big[[h^\nabla(t)\hat{e}_{\ominus z^\alpha}(t,t_0)]^\nabla - h^\nabla(t)\hat{e}^\nabla_{\ominus z^\alpha}(t,t_0)\Big]\ \nabla t \\
&= [h^\nabla(t)\hat{e}_{\ominus z^\alpha}(t,t_0)]_{t=t_0}^{t\to\infty} - \int_{t_0}^\infty h^\nabla(t)\hat{e}^\nabla_{\ominus z^\alpha}(t,t_0)\ \nabla t \\
&= -h^\nabla(t_0) + z^\alpha\int_{t_0}^\infty h^\nabla(t)\frac{\ominus z^\alpha \hat{e}_{\ominus z^\alpha}(t,t_0)}{-z^\alpha}\ \nabla t \\
&= -h^\nabla(t_0) + z^\alpha\int_{t_0}^\infty h^\nabla\hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)\ \nabla t \\
&= -h^\nabla(t_0) + z^\alpha\mathscr{L}^\alpha\{h^\nabla(t)\} \\
&= -h^\nabla(t_0) + z^\alpha[z^\alpha\mathscr{L}^\alpha\{h(t)\} - h(t_0)] \\
&= z^{2\alpha}\mathscr{L}^\alpha\{h(t)\} - z^\alpha h(t_0) - h^\nabla(t_0).
\end{aligned}$$

More generally we get, $\mathscr{L}^\alpha\{h^{\nabla^n}(t)\} = z^{n\alpha}\mathscr{L}^\alpha\{h(t)\} - \sum_{k=0}^{n-1} z^{(n-(k+1))\alpha}h^{\nabla^k}(t_0)$.

Theorem 3.6 (**Initial and Final Value Theorem**). *$h, h' \in \mathcal{A}(\mathbb{T})$ with $H_{\mathbb{T}}(z) = \mathscr{L}^\alpha\{h(t)\}$ then $h(t_0) = \lim_{z\to\infty} z^\alpha H_{\mathbb{T}}(z)$ and $\lim_{t\to\infty} h(t) = \lim_{z\to 0} z^\alpha H_{\mathbb{T}}(z)$.*

Proof. We have,

$$\mathscr{L}^\alpha\{h^\nabla(t)\}$$

$$= \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0) h^{\nabla}(t)\ \nabla t$$
$$= \int_{t_0}^{\infty} \left[(h(t)\hat{e}_{\ominus z^\alpha}(t,t_0))^{\nabla} - h(t)\hat{e}^{\nabla}_{\ominus z^\alpha}(t,t_0) \right]\ \nabla t$$
$$= \left[h(t) e_{\ominus z^\alpha}(t,t_0) \right]_{t=t_0}^{\infty} - \int_{t_0}^{\infty} h(t)\hat{e}^{\nabla}_{\ominus z^\alpha}(t,t_0)\ \nabla t$$
$$= -h(t_0) - \int_{t_0}^{\infty} h(t) \ominus z^\alpha \hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t$$
$$= -h(t_0) + z^\alpha \int_{t_0}^{\infty} h(t) \frac{\ominus z^\alpha}{-z^\alpha} \hat{e}_{\ominus z^\alpha}(t,t_0)\ \nabla t$$
$$= -h(t_0) + z^\alpha \int_{t_0}^{\infty} h(t)\hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)\ \nabla t$$
$$= z^\alpha \mathscr{L}^\alpha\{h(t)\} - h(t_0).$$

Provided $\lim_{t\to\infty} \hat{e}_{\ominus z^\alpha}(t,t_0) = 0$.
Taking $\lim_{z\to\infty}$ on both sides,

$$\lim_{z\to\infty} \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0) h^{\nabla}(t) = 0 = \lim_{z\to\infty} z^\alpha \mathscr{L}^\alpha h(t) - \lim_{z\to\infty} h(t_0)$$
$$\lim_{z\to\infty} z^\alpha H_{\mathbb{T}}(z) = h(t_0).$$

Now taking $\lim_{z\to 0}$ on both sides we get,

$$\lim_{z\to 0} \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0) h^{\nabla}(t) = \lim_{z\to 0} z^\alpha \mathscr{L}^\alpha h(t) - \lim_{z\to 0} h(t_0)$$
$$\int_{t_0}^{\infty} h^{\nabla}(t) \nabla t = \lim_{z\to 0} z^\alpha \mathscr{L}^\alpha\{h(t)\} - h(t_0)$$
$$\lim_{t\to\infty} h(t) - h(t_0) = \lim_{z\to 0} z^\alpha \mathscr{L}^\alpha\{h(t)\} - h(t_0)$$
$$\lim_{t\to\infty} h(t) = \lim_{z\to 0} z^\alpha H_{\mathbb{T}}(z).$$

□

Definition 3.3 ([28]). *For given functions $h_1, h_2 : \mathbb{T} \to \mathbb{C}$ their convolution $h_1 * h_2$ is defined by,*

$$(h_1 * h_2)(t) = \int_{t_0}^{t} \tilde{h_1}(t, \rho(\tau)) h_2(\tau) \nabla\tau \qquad t \in \mathbb{T}$$

where $\tilde{h}$ is the shift of $h : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$ is the solution of the initial value problem

$$g^{\nabla_t}(t, \rho(s)) = -g^{\nabla_s}(t,s), \qquad t, s \in \mathbb{T},\ t \le s \le t_0$$
$$g(t, t_0) = h(t) \qquad t \in \mathbb{T},\ t \ge t_0.$$

Theorem 3.7 (**Convolution theorem**). *If $h_1(t), h_2(t) \in \mathcal{A}(\mathbb{T})$ having $\alpha-$Laplace transforms $\mathscr{L}^\alpha\{h_1(t)\}$ and $\mathscr{L}^\alpha\{h_2(t)\}$ respectively, then*

$$\mathscr{L}^\alpha\{h_1(t) * h_2(t)\} = \mathscr{L}^\alpha\{h_1(t)\} \cdot \mathscr{L}^\alpha\{h_2(t)\}.$$

Proof.

$$\mathscr{L}^\alpha\{h_1(t) * h_2(t)\}$$
$$= \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)[h_1(t) * h_2(t)]\ \nabla t$$
$$= \int_{t_0}^{\infty} \hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0) \Big[\int_{t_0}^{t} h_1(t, \rho(\tau)) h_2(\tau) \nabla\tau \Big] \nabla t$$
$$= \int_{t_0}^{\infty} h_2(\tau) \Big[\int_{\rho(\tau)}^{\infty} h_1(t, \rho(\tau)) \hat{e}^{\rho}_{\ominus z^\alpha}(t,t_0)\ \nabla t \Big] \nabla\tau$$

$$
\begin{aligned}
&= \int_{t_0}^{\infty} \Big[\int_{t_0}^{\infty} h_1(t,\rho(\tau)) u_{\rho(\tau)}(t) \hat{e}^{\rho}_{\ominus z^{\alpha}}(t,t_0)\nabla t \Big] \nabla \tau \\
&= \int_{t_0}^{\infty} h_2(\tau) \mathscr{L}^{\alpha}\{u_{\rho(\tau)}(t) h_1(t)\} \nabla \tau \\
&= \int_{t_0}^{\infty} h_2(\tau) \hat{e}_{\ominus z^{\alpha}}(\rho(t),t_0)\ \mathscr{L}^{\alpha}\{h_1(t)\} \nabla \tau \\
&= \mathscr{L}^{\alpha}\{h_1(t)\} \int_{t_0}^{\infty} h_2(\tau) \hat{e}^{\rho}_{\ominus z^{\alpha}}(\tau,t_0) \nabla \tau \\
&= \mathscr{L}^{\alpha}\{h_1(t)\} \cdot \mathscr{L}^{\alpha}\{h_2(t)\}.
\end{aligned}
$$

□

Theorem 3.8 (**$\alpha-$Laplace transform of Riemann-Liouville Fractional Integral**). *For $h \in \mathcal{C}_{ld}([t_0,t']_{\mathbb{T}},\mathbb{C})$ and $a > 0$, the $\alpha-$ Laplace transform of Riemann-Liouville fractional integral $({}_{t_0}\nabla^{\alpha} f)(t)$ is $\mathscr{L}^{\alpha}\{{}_{t_0}\nabla^{-a} f\}(t)$ and is given by $\mathscr{L}^{\alpha}\{{}_{t_0}\nabla^{-a} f\}(t) = z^{-a\alpha}\mathscr{L}^{\alpha}\{f(t)\}$.*

Proof. From Definition 2.6, Riemann-Liouville fractional integral can be written in form of convolution as

$$
\begin{aligned}
({}_{t_0}\nabla^{a} f)(t) &= \hat{h}_{a-1} * h(t) \\
\text{Thus,}\quad \mathscr{L}^{\alpha}\{({}_{t_0}\nabla^{-a} h)(t)\} &= \mathscr{L}^{\alpha}\{\hat{h}_{a-1}(t,t_0) * h(t)\} \\
&= \mathscr{L}^{\alpha}\{\hat{h}_{a-1}(t,t_0)\}\mathscr{L}^{\alpha}\{h(t)\} \\
&= \frac{1}{z^{a\alpha}}\mathscr{L}^{\alpha}\{h(t)\} \\
&= z^{-a\alpha}\mathscr{L}^{\alpha}\{h(t)\}.
\end{aligned}
$$

□

Theorem 3.9 (**$\alpha-$Laplace transform of Riemann-Liouville Fractional derivative**). *For $h \in \mathcal{C}_{ld}([t_0,t']_{\mathbb{T}},\mathbb{C})$ and $\beta > 0$, the $\alpha-$ Laplace transform of Riemann-Liouville fractional derivative $({}_{t_0}\nabla^{\beta} h)(t)$ is $\mathscr{L}^{\alpha}\{({}_{t_0}\nabla^{b} h)(t)\}$ and is given by*

$$
\mathscr{L}^{\alpha}\{({}_{t_0}\nabla^{\beta} f)(t)\} = z^{\beta\alpha}\mathscr{L}^{\alpha}\{h(t)\} - \sum_{k=0}^{m-1} z^{(m-k-1)\alpha}\big[{}_{t_0}\nabla^{-(m-\beta)} h\big]^{\nabla^k}(t_0).
$$

Proof. From Definition 2.6 the Riemann-Liouville fractional derivative can be written as,

$$
\begin{aligned}
({}_{t_0}\nabla^{\beta} h)(t) = (\chi^{\nabla^m})(t) \quad &\text{where}\quad \chi(t) = ({}_{t_0}\nabla^{-(m-\beta)} h)(t), \\
\mathscr{L}^{\alpha}\{({}_{t_0}\nabla^{\beta} h)(t)\} &= \mathscr{L}^{\alpha}\{\chi^{\nabla^m}(t)\} \\
&= z^{m\alpha}\mathscr{L}^{\alpha}\{\chi(t)\} - \sum_{k=0}^{m-1} z^{(m-(k+1))\alpha}\chi^{\nabla^k}(t_0) \\
&= z^{m\alpha}z^{(\beta-m)\alpha}\mathscr{L}^{\alpha}\{h(t)\} - \sum_{k=0}^{m-1} z^{(m-k-1)\alpha}\chi^{\nabla^k}(t_0) \\
&= z^{\beta\alpha}\mathscr{L}^{\alpha}\{h(t)\} - \sum_{k=0}^{m-1} z^{(m-k-1)\alpha}\big[{}_{t_0}\nabla^{-(m-\beta)h}\big]^{\nabla^k}(t_0).
\end{aligned}
$$

This is equivalent to

$$
\mathscr{L}^{\alpha}\{({}_{t_0}\nabla^{\beta} h)(t)\} = z^{\beta\alpha}\mathscr{L}^{\alpha}\{h(t)\} - \sum_{j=1}^{l} z^{(j-1)\alpha}({}_{t_0}\nabla^{\beta-j} h)(t_0) \qquad l-1 < \beta < l.
$$

□

Theorem 3.10 (**$\alpha-$Laplace transform of Liouville-Caputo fractional derivative**). *For $h \in \mathcal{C}_{ld}([t_0,t']_{\mathbb{T}},\mathbb{C})$ and $\lambda > 0$, the $\alpha-$ Laplace transform of Liouville-Caputo fractional derivative ${}^{\mathcal{C}}_{t_0}\nabla^{\lambda} h(t)$ is $\mathscr{L}^{\alpha}\{{}^{\mathcal{C}}_{t_0}\nabla^{\lambda} h(t)\}$ and is given by*

$$
\mathscr{L}^{\alpha}\{{}^{\mathcal{C}}_{t_0}\nabla^{\lambda} h(t)\} = z^{\lambda\alpha}\mathscr{L}^{\alpha}\{h(t)\} - \sum_{k=0}^{m-1} z^{(\lambda-k-1)\alpha}\ h^{\nabla^k}(t_0).
$$

Proof. From Definition 2.6 the Caputo fractional derivative can be written as,

$$ {}^{\mathcal{C}}_{t_0}\nabla^{\lambda}h(t) = ({}_{t_0}\nabla^{-(m-\lambda)}\chi)(t) \quad \text{where} \;\; \chi(t) = h^{\nabla^m}(t), $$

$$ \begin{aligned} \mathscr{L}^{\alpha}\{{}^{\mathcal{C}}_{t_0}\nabla^{\lambda}h(t)\} &= \mathscr{L}^{\alpha}\{{}_{t_0}\nabla^{-(m-\lambda)}\chi(t)\} \\ &= z^{-(m-\lambda)\alpha}\mathscr{L}^{\alpha}\{h^{\nabla^m}(t)\} \\ &= z^{-(m-\lambda)\alpha}\big[z^{m\alpha}\mathscr{L}^{\alpha}\{h(t)\sum_{k=0}^{m-1} z^{(m-k-1)\alpha}h^{\nabla^k}(t_0)\}\big] \\ &= z^{\lambda\alpha}\mathscr{L}^{\alpha}\{h(t)\} - \sum_{k=0}^{m-1} z^{(m-k-1)\alpha}h^{\nabla^k}(t_0). \end{aligned} $$

□

Definition 3.4 ([15]). *For $n > 0, m, \lambda \in \mathbb{R}$ and $t, t_0 \in \mathbb{T}$. The time scale Mittag-Leffler function is defined as*

$$ E^{\lambda}_{n,m}(t,t_0) = \sum_{k=0}^{\infty} \lambda^k \hat{h}_{nk+m-1}(t,t_0) $$

provided the right hand side series is convergent.

Theorem 3.11 (**$\alpha-$ Laplace transform of Mittag-Leffler function**). *For $n, m, \lambda \in \mathbb{T}$ and $t_0, t \in \mathbb{T}$*

$$ \mathscr{L}^{\alpha}\{E^{\lambda}_{n,m}(t,t_0)\} = \frac{z^{(m-n)\alpha}}{z^{m\alpha}-\lambda}. $$

Proof.

$$ \begin{aligned} &\mathscr{L}^{\alpha}\big\{E^{\lambda}_{n,m}(t,t_0)\big\} \\ &= \mathscr{L}^{\alpha}\big\{\sum_{k=0}^{\infty}\lambda^k\hat{h}_{nk+m-1}(t,t_0)\big\} \\ &= \sum_{k=0}^{\infty}\lambda^k\mathscr{L}^{\alpha}\{\hat{h}_{nk+m-1}(t,t_0)\} \\ &= \sum_{k=0}^{\infty}\lambda^k\frac{1}{z^{(mk+n)\alpha}} \\ &= \frac{1}{z^{n\alpha}}\sum_{k=0}^{\infty}\frac{\lambda^k}{z^{mk\alpha}} \\ &= \frac{1}{z^{n\alpha}}\Big[1+\frac{\lambda}{z^{n\alpha}}+\frac{\lambda^2}{z^{2m\alpha}}+...\Big] \\ &= \frac{1}{z^{n\alpha}}\Big[\frac{1}{1-\frac{\lambda}{m\alpha}}\Big] \qquad\qquad \text{provided} \quad \Big|\frac{\lambda}{z^{m\alpha}}\Big| < 1 \\ &= \frac{1}{z^{n\alpha}}\Big[\frac{z^{m\alpha}}{z^{m\alpha}-\lambda}\Big] \\ &= \frac{z^{(m-n)\alpha}}{z^{m\alpha}-\lambda}. \end{aligned} $$

□

In this last section we will solve a fractional dynamic equation using our defined transform.

4 Application

Consider the following fractional dynamic equation with initial condition,

$$ {}_0\nabla^{1/2}g(t) + ag(t) = 0, \qquad ({}_0\nabla^{-1/2}g)(0) = k. $$

Applying the $\alpha-$ Laplace transform with $\alpha = \frac{1}{2}$, we obtain

$$ \mathscr{L}^{\frac{1}{2}}\big\{{}_0\nabla^{\frac{1}{2}}g(t) + ag(t)\big\} = 0, $$

$$z\mathscr{L}^{\frac{1}{2}}\{g(t)\} - ({}_0\nabla^{-\frac{1}{2}}g)(0) + a\mathscr{L}^{\frac{1}{2}}\{g(t)\} = 0,$$
$$z\mathscr{L}^{\frac{1}{2}}\{g(t)\} - k + a\mathscr{L}^{\frac{1}{2}}\{g(t)\} = 0,$$
$$(z+a)\mathscr{L}^{\frac{1}{2}}\{g(t)\} = k,$$
$$\mathscr{L}^{\frac{1}{2}}\{g(t)\} = \frac{k}{(z+a)}.$$

Taking inverse required solution is,

$$g(t) = kE^{-a}_{\frac{1}{2},\frac{1}{2}}(t,0).$$

5 Conclusion

In this paper, we introduce a new α-Laplace transform on time scales. This transform for $\alpha = 1$ coinsides with a nabla Laplace transform on time scales and for $0 < \alpha < 1$ will serve as a fractional Laplace transform on time scales. Accompained by the existence theorem we have proved some of its important properties, including the convolution theorem and found, transform of the Riemann-Liouville fractional integral, Riemamm-Liouville fractional derivative, Liouville-Caputo derivative and Mittag Leffter function on time scales. A fractional dynamic equation with a given initial condition is solved for a suitable value of α showing efficiency of this integral transform.

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