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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

ON THE CHARACTERISTIC POLYNOMIAL OF CHEBYSHEV MATRICES

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Abstract

We exhibit that the coefficients of the characteristic polynomial of any matrix $\mathbf{A}_{nxn}$ can be written in terms of the complete Bell polynomials, and this result is applied to Chebyshev matrices which generates the concept of Associated Polynomials of Chebyshev.

Keywords and Phrases: Bell and Chebyshev polynomials, Characteristic polynomial, Chebyshev matrices, Gauss hypergeometric function.

1 Introduction

For an arbitrary matrix $\mathbf{A}_{\text{nxn}} = (A^i{}_j)$ its characteristic polynomial [9, 10, 19]

(1.1) $$P_n(\lambda) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n,$$

can be obtained, through several procedures [11, 19, 25, 33, 34], directly from the condition

$$P_n(\lambda) = det(\lambda\delta^i_j - A^i{}_j).$$

The approach of Leverrier-Takeno [2, 8, 15, 20, 21, 32, 33, 35] is a simple and interesting technique to construct (1.1) based in the traces of the powers $A^r, r = 1, \ldots, n$. In fact, if we define the quantities

(1.2) $$a_0 = 1, s_k = tr\mathbf{A}^k, k = 1, 2, \ldots, n,$$

then by (1.2) the process of Leverrier-Takeno implies (1.1) wherein the a_i are determined with the recurrence relation

(1.3) $$\text{r } a_r + s_1 a_{r-1} + s_2 a_{r-2} + \ldots + s_{r-1} \text{ a}_1 + s_r = 0, r = 1, 2, \ldots, n,$$

therefore

(1.4) $$a_1 = -s_1, 2!a_2 = (s_1)^2 - s_2, 3!a_3 = -(s_1)^3 + 3s_1 \text{ s}_2 - 2s_3,$$
$$4!a_4 = (s_1)^4 - 6(s_1)^2 s_2 + 8s_1 \text{ s}_3 + 3(s_2)^2 - 6s_4,$$
$$5!a_5 = -(s_1)^5 + 10(s_1)^3 \text{ s}_2 - 20(s_1)^2 \text{ s}_3 - 15s_1(s_2)^2 + 30 \text{ s }_1 \text{ s }_4 + 20s_2s_3 - 24s_5, \ldots,$$

in particular, det $A = (-1)^n a_n$, that is, the determinant of any matrix only depends on the traces s_r, which means that A and its transpose have the same determinant.

2 Complete Bell polynomials in terms of the determinant

In this section we make an appeal to recurrence relations (1.3) and (1.4) and thus due to [1, 5, 22] find the general expression

(2.1) $$a_m = \frac{(-1)^m}{m!} \begin{vmatrix} s_1 & s_2 & s_3 & \cdots & s_{m-1} & s_m \\ m-1 & s_1 & s_2 & \cdots & s_{m-2} & s_{m-1} \\ 0 & m-2 & s_1 & \cdots & s_{m-3} & s_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & s_1 \end{vmatrix}, m = 1, \ldots, n.$$

which allows reproduce the expressions (1.4). The formula (2.1) permits relate the coefficients of the characteristic polynomial (1.1) with the complete Bell polynomials [3, 4, 29, 30, 36]. In [12, 23] we find the following expression for the Bell polynomials

$$B_m\left(x_1,x_2,\ldots,x_m\right)=\begin{vmatrix} \binom{m-1}{0}x_1 & \binom{m-1}{1}x_2 & \cdots & \binom{m-1}{m-2}x_{m-1} & \binom{m-1}{m-1}x_m \\ -1 & \binom{m-2}{0}x_1 & \cdots & \binom{m-2}{m-3}x_{m-2} & \binom{m-2}{m-2}x_{m-1} \\ 0 & -1 & \cdots & \binom{m-3}{m-4}x_{m-3} & \binom{m-3}{m-3}x_{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{1}{0}x_1 & \binom{1}{1}x_2 \\ 0 & 0 & \cdots & -1 & \binom{0}{0}x_1 \end{vmatrix}. \tag{2.2}$$

Therefore

$$B_0=1, B_1=x_1, B_2=x_1^2+x_2, B_3=x_1^3+3x_1x_2+x_3, B_4=x_1^4+6\ x_1^2\ x_2+4\ x_1x_3+3\ x_2^2+x_4,$$
$$B_5=x_1^5+10\ x_1^3\ x_2+10\ x_1^2\ x_3+15x_1\ x_2^2+5\ x_1x_4+10x_2x_3+x_5,\ldots \tag{2.3}$$

We see that with (2.3) we can deduce (1.4) if we employ $x_1=-s_1, x_2=-s_2, x_3=-2s_3, x_4=-6s_4, x_5=-24s_5, \ldots$, that is

$$a_m=\frac{1}{m!}B_m(-0!s_1,-1!s_2,-2!s_3,-3!s_4,\ldots,-(m-2)!s_{m-1},-(m-1)!s_m). \tag{2.4}$$

In fact, it is simple to prove that (2.2) with $x_k=-(k-1)!s_k$ implies (2.1), thus the coefficients of the characteristic polynomial (1.1) are generated by the complete Bell polynomials [3, 4, 12, 23, 29, 30, 36].

3 Chebyshev matrices

The first-kind Chebyshev polynomials $T_n(x), |x|\le 1$, verify the differential equation [6, 17, 18, 19, 26, 28]

$$\left(1-x^2\right)\frac{d^2}{dx^2}T_n-x\frac{d}{dx}T_n+n^2T_n=0, n=0,1,2,\ldots \tag{3.1}$$

which is equivalent to the following expression in terms of the Gauss hypergeometric function [7, 24, 31]

$$T_n(x)={}_2F_1\left(-n,n;\frac{1}{2};\frac{1-x}{2}\right), \tag{3.2}$$

thus

$$T_0=1, T_1=x, T_2=2x^2-1, T_3=4x^3-3x, T_4=8x^4-8x^2+1,\ldots \tag{3.3}$$

Alternatively, we can employ the Chebyshev matrices [27]

$$A_{nxn}(x)=\begin{pmatrix} x & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2x & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2x & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2x & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & 1 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2x \end{pmatrix}, \tag{3.4}$$

whose determinant generates the Chebyshev polynomials

$$T_n(x)=\det A_{nxn}(x). \tag{3.5}$$

That is

$$T_1=det(x), T_2=det\begin{pmatrix} x & 1 \\ 1 & 2x \end{pmatrix}, T_3=det\begin{pmatrix} x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{pmatrix}, T_4=det\begin{pmatrix} x & 1 & 0 & 0 \\ 1 & 2x & 1 & 0 \\ 0 & 1 & 2x & 1 \\ 0 & 0 & 1 & 2x \end{pmatrix},\ldots. \tag{3.6}$$

Therefore from (1.4), (2.4) and (3.5)

$$T_n(x) = \frac{(-1)^n}{n!} B_n\left(-0!\, s_1, -1! s_2, -2! s_3, -3! s_4, \ldots, -(n-2)! s_{n-1}, -(n-1)!\, s_n\right), \tag{3.7}$$

where s_j are the traces of the powers of the matrix (3.4); hence the complete Bell polynomials allow construct the Chebyshev polynomials of the first kind.

It is natural to investigate the characteristic polynomial of (3.4) for several values of n, thus

$$\begin{aligned}
&P_1 = \lambda - T_1, P_2 = \lambda^2 - 3x\lambda + T_2, P_3 = \lambda^3 - 5x\lambda^2 + \left(8x^2 - 2\right)\lambda - T_3,\\
&P_4 = \lambda^4 - 7x\lambda^3 + \left(18x^2 - 3\right)\lambda^2 - \left(20x^3 - 10x\right)\lambda + T_4,\\
&P_5 = \lambda^5 - 9x\lambda^4 + \left(32x^2 - 4\right)\lambda^3 - \left(56x^3 - 21x\right)\lambda^2 + \left(48x^4 - 36x^2 + 3\right)\lambda - T_5,\\
&P_6 = \lambda^6 - 11x\lambda^5 + \left(50x^2 - 5\right)\lambda^4 - \left(120x^3 - 36x\right)\lambda^3 + \left(160x^4 - 96x^2 + 6\right)\lambda^2 -\\
&\qquad - \left(112x^5 - 112x^3 + 21x\right)\lambda + T_6,\\
&P_7 = \lambda^7 - 13x\lambda^6 + \left(72x^2 - 6\right)\lambda^5 - \left(220x^3 - 55x\right)\lambda^4 + \left(400x^4 - 200x^2 + 10\right)\lambda^3 -\\
&\qquad - \left(432x^5 - 360x^3 + 54x\right)\lambda^2 + \left(256x^6 - 320x^4 + 96x^2 - 4\right)\lambda - T_7,\\
&P_8 = \lambda^8 - 15x\lambda^7 + \left(98x^2 - 7\right)\lambda^6 - \left(364x^3 - 78x\right)\lambda^5 + \left(840x^4 - 360x^2 + 15\right)\lambda^4 -\\
&\qquad - \left(1232x^5 - 880x^3 + 110x\right)\lambda^3 + \left(1120x^6 - 1200x^4 + 300x^2 - 10\right)\lambda^2 -\\
&\qquad - \left(576x^7 - 864x^5 + 360x^3 - 36\right)\lambda + T_8, \ldots .
\end{aligned} \tag{3.8}$$

That is

$$P_n(\lambda) = \sum_{m=0}^{n} T_n^m(x)\lambda^{n-m},\ T_n^0 = 1, T_n^n = (-1)^n T_n. \tag{3.9}$$

Then it is clear that $T_n^m(x), m = 0, 1, \ldots, n$ is a polynomial in x of degree m, and they may be named as Associated Polynomials of Chebyshev.

We know that if the operator $\frac{d^N}{\mathrm{dx}^N}$ is applied to the Legendre polynomials we obtain their associated polynomials, then now we shall show that this process can be employed for the first-kind Chebyshev polynomials $T_n(x)$ to construct the new polynomials $T_n^m(x)$ in terms of the Gauss hypergeometric function. In fact, we know the property

$$\frac{d^N}{dx^N}\ {}_2F_1(a, b; c; z) \propto {}_2F_1(a + N, b + N; c + N; z), \tag{3.10}$$

then we apply the operator $\frac{d^{n-m}}{\mathrm{dx}^{n-m}}$ to (3.2) and we use (3.10) with an adequate factor of proportionality to obtain the expression

$$\begin{aligned}
T_n^m(x) &= (-1)^m \binom{2n-m}{m}\, {}_2F_1\left(-m, 2n - m; n - m + \frac{1}{2}; \frac{1-x}{2}\right),\\
&= 2^{m-1}\frac{(n-1)!(2n-m)}{m!(n-m)!} \sum_{k=0}^{m} (-1)^{k-m} \binom{m}{k}\, {}_2F_1(k - m, -1 - 2m; -2m; 1)x^k
\end{aligned} \tag{3.11}$$

$m = 0, 1, \ldots, n$, verifying the differential equation

$$\left(1 - x^2\right)\frac{d^2}{dx^2}T_n^m - (2n - 2m + 1)x\frac{d}{dx}T_n^m + m(2n - m)T_n^m = 0; \tag{3.12}$$

with (3.11) it is simple to calculate these associated polynomials of Chebyshev, for example

$$T_3^1 = -5x, T_3^2 = 8x^2 - 2, T_5^2 = 32x^2 - 4, T_5^3 = -56x^3 + 21x, T_5^4 = 48x^4 - 36x^2 + 3,\ \text{etc.}$$

in accordance with (3.8). The relations (3.11) and (3.12) reproduce (3.1) and (3.2) for the case $m = n$.

Finally, it is easy to show that the associated polynomials (3.11) can generate the other types of Chebyshev polynomials [13, 14, 16, 26, 27]

$$U_n(x) = \frac{2(-1)^n}{2+n}T_{n+1}^n(x), V_n(x) = \frac{(-1)^n}{n+1}T_{2n+1}^{2n}\left(\sqrt{\frac{1-x}{2}}\right), W_n(x) = \frac{1}{n+1}T_{2n+1}^{2n}\left(\sqrt{\frac{1+x}{2}}\right). \tag{3.13}$$

4 An Abel type integral equation representation involving Chebyshev determinants (3.5)

In this section, on using orthogonal property of the Chebyshev polynomials [26, 27], we obtain orthogonal property of product of two Chebyshev determinants (3.5). Then we derive an Abel type integral equation representation involving these Chebyshev determinants.

Making an use of the Eqn. (3.5) and the orthogonal property of the Chebyshev polynomials [26, 27] given by

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}T_n(x)T_m(x)\mathrm{dx}=\begin{cases}0, m\neq n;\\ \frac{\pi}{2}, m=n\neq 0;\\ \pi, m=n=0,\end{cases}\tag{4.1}$$

due to (4.1), we get an interesting orthogonal property in terms of product of two Chebyshev determinants as

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\left\{\det A_{nxn}(x)\det A_{mxm}(x)\right\}\mathrm{dx}=\begin{cases}0, m\neq n;\\ \frac{\pi}{2}, m=n\neq 0;\\ \pi, m=n=0.\end{cases}\tag{4.2}$$

Theorem 4.1. *For $x>0$, if $|xt|\leq 1$, and any function $f:(x,\ t)\rightarrow\mathbb{R}$, is defined by*

$$f(x,t)=\sum_{n=1}^{\infty}C_n\left\{\det A_{nxn}\left(xt\right)\right\},\ C_n \text{ an arbitrary constant,}\tag{4.3}$$

may be represented as an integral equation

$$f(x,t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\det A_{nxn}\left(xt\right)\right\}\int_{-1}^{1}\frac{f\left(x,ux^{-1}\right)}{\sqrt{1-u^2}}\det A_{nxn}(u)du.\tag{4.4}$$

Proof. Consider a function in terms of the series of Chebyshev determinants (3.5) as

$$f(x,t)=\sum_{n=1}^{\infty}C_n\left\{\det A_{nxn}\left(\mathrm{xt}\right)\right\}.\tag{4.5}$$

Then in both sides of Eqn. (4.5) multiply by $\frac{\det A_{mxm}(\mathrm{xt})}{\sqrt{1-(\mathrm{xt})^2}}$ and thus integrate that sides with respect to t from $t=-\frac{1}{x}$ to $t=\frac{1}{x}$, $\forall x>0$, we obtain

$$\int_{-\frac{1}{x}}^{\frac{1}{x}}\frac{f(x,\ t)}{\sqrt{1-(xt)^2}}\det A_{mxm}\left(xt\right)dt=\sum_{n=1}^{\infty}C_n\int_{-\frac{1}{x}}^{\frac{1}{x}}\frac{1}{\sqrt{1-(xt)^2}}\det A_{nxn}\left(xt\right)\det A_{mxm}\left(xt\right)dt.\tag{4.6}$$

After some manipulations in (4.6), we find that

$$\int_{-1}^{1}\frac{f\left(x,\ ux^{-1}\right)}{\sqrt{1-u^2}}\det A_{mxm}(u)du=\sum_{n=1}^{\infty}C_n\int_{-1}^{1}\frac{1}{\sqrt{1-u^2}}\det A_{nxn}(u)\det A_{mxm}(u)du.\tag{4.7}$$

Now in the Eqn. (4.8) use the orthogonality formula (4.2) we derive the coefficients

$$C_n=\frac{2}{\pi}\int_{-1}^{1}\frac{f\left(x,\ ux^{-1}\right)}{\sqrt{1-u^2}}\det A_{nxn}(u)du\ \forall n=1,2,3,\ldots.\tag{4.8}$$

Finally, with the aid of the formulae (4.5) and (4.9), we get an integral equation (4.4).

Specially, by Eqn. (4.4) for $n=1$ we find an Abel type integral equation

$$f(x,t)=\frac{2xt}{\pi}\int_{-1}^{1}\frac{f\left(x,\ ux^{-1}\right)}{\sqrt{1-u^2}}udu,\ \forall x>0.\tag{4.9}$$

□

5 Conclusions

In the Section 2, complete Bell polynomials are expressed in terms of determinant. The Section 3 consists of Chebyshev matrices. In the Section 4, on using orthogonal property of the Chebyshev polynomials [26, 27], an orthogonal property of product of two Chebyshev determinants (3.5) is derived. Again an integral equation representation involving these Chebyshev determinants is also obtained. The results obtained in the Eqns. (3.13) and (4.9) are very applicable in computational work of various scientific problems consisting of Abel's type integrals and Chebyshev polynomials.

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