ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(2) (2023), 134-144**
(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

ON DIRECT SUM OF TOPOLOGICALLY TRANSITIVE OPERATORS

Peter Slaa[1], Santosh Kumar[2] and Marco Mpimbo[3]
[1,2]Department of Mathematics, College of Natural and Applied Sciences, University of Dar es Salaam, Tanzania-35091
[3]Department of Mathemtics, School of Physical Sciences, North Eastern Hill University, Shillong, Meghalaya, India-793022
Email: masongslaa@gmail.com, drsengar2002@gmail.com, kmpimbo33@gmail.com

(Received: July 29, 2023; In format: August 21, 2023; Revised: September 19, 2023; Accepted: October 05, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53216

Abstract

The purpose of this paper is to answer the question posed by Feldman [9] on topological transitivity which states that "*If E is transitive, does it follows that direct sum $E \oplus E$ is topologically transitive?*" We will show that this question has a positive answer under certain conditions. In particular, we define topologically transitive operators and use them to show that the direct sum $E \oplus E$ of two operators is topologically transitive whenever E is topologically transitive. Then, we give some examples of a topologically transitive operator which does not satisfy topologically transitive criterion and so not topologically transitive.

2020 Mathematical Sciences Classification: 47A16, 47B02.
Keywords and Phrases: Hypercyclic operator, topologically transitive, direct sum, transitivity criterion.

1 Introduction and Preliminaries

A bounded linear operator E on a separable Banach space X is topologically transitive if for each pair of non-empty open subsets $P \subset X$ and $Q \subset X$, one can find positive integer $k \geqslant 0$ such that $E^k(P) \cap Q \neq \varnothing$. If $E^k(P) \cap Q \neq \varnothing$ is from some $k \geqslant N$, then E is said to be topologically mixing. Birkhoff [5] developed a topological transitive operator and provided an example of how it may be used to approximate any holomorphic function in $\mathbb{H}(\mathbb{C})$. On separable Banach spaces, topological transitivity and hypercyclicity are similar concepts in linear dynamics, according to Grosse-Erdmann and Manguillot [11]. A linear operator E on a vector space X is said to be hypercyclic if there exists a vector $x \in X$ such that the set of all vectors obtained by iterating E on x, denoted by $orb(E, x) = \{x, Ex, E^2x, \dots\}$, is dense in X. For E, such a vector x is referred to as a hypercyclic vector.

Rolewicz [16] introduced the idea of hypercyclic operators and gave the first illustration of a hypercyclic operator on a Banach space. He demonstrated that if B is the backward shift on $\ell(N)$ then λB is hypercyclic for every scalar $|\lambda| > 1$. The Hypercyclicity criterion, a useful necessary condition for an operator to be hypercyclic, was later established by Kitai [13]. Gethner and Shapiro [10] also contributed to the development of this criterion. Many authors have further refined this criterion (see Grosse-Erdmann [11] and the references therein).

Recently, Madore and Martinez [14] studied hypercyclicity on subspaces. They investigated subspace-topologically transitive operators and demonstrated that any subspace-topologically transitive operator is subspace-hypercyclic. This result extends the theory of hypercyclic operators to the case of operators acting on subspaces. Further details on hypercyclicity and related topics can be found in the monographs by Grosse-Erdmann [11] and Bayart and Matheron [4].

In the study of linear dynamics, hypercyclicity and topological transitivity are important concepts that describe the behavior of bounded linear operators on Banach spaces. One question of interest is whether the hypercyclicity property is preserved under direct sums of operators. Kitai [13] showed that if a direct sum $E \oplus E$ is hypercyclic, then both E_1 and E_2 must also be hypercyclic.

However, Salas [18] constructed an operator E and its adjoint E^* such that both E and E^* are hypercyclic, but their direct sum $E \oplus E^*$ is not hypercyclic. This example raises the question of whether $E \oplus E$ is hypercyclic whenever E is hypercyclic. Herrero questioned this, and De la Rosa and Read [15] provided a hypercyclic operator E such that $E \oplus E$ is not hypercyclic, showing that the answer to Herrero's question is negative.

On the other hand, Bès and Peris [2] showed that if $E \oplus E$ is hypercyclic, then E fulfills the hypercyclic condition as well. In other words, hypercyclicity is preserved under direct sums in one direction. Further details on hypercyclicity and topological transitivity can be found in the monographs by Bayart and Matheron [4] and Grosse-Erdmann and Manguillot [11].

Definition 1.1 ([11])**.** *A bounded linear operator E acting on a Banach space X is said to be topologically transitive if for any two non-empty open subsets $P, Q \subseteq X$, there exists a positive integer k such that $E^k(P) \cap Q$ is non-empty.*

Definition 1.2. *A pair of bounded linear operators (E_1, E_2) on a Banach space X is said to be topologically mixing if for any pair of non-empty open sets $P, Q \subseteq X$, there exist positive integers M and N such that $E_1^m E_2^n(P) \cap Q \neq \varnothing$ for all $m \geq M$ and $n \geq N$.*

Intuitively, this means that after some finite number of iterations of each operator, the images of P and Q intersect.

Note that the order of the operators in the product $E_1^m E_2^n$ matters in general, and that the definition of topological mixing requires that both operators are involved in the mixing property.

Also, note that the definition of topological mixing is stronger than that of topological transitivity, as it requires the existence of two parameters M and N, whereas topological transitivity only requires the existence of one parameter n.

Definition 1.3 ([11])**.** *An operator E on a separable Hilbert space $\mathcal{H}$ is said to be chaotic if it satisfies the following conditions:*

(i) E is topologically transitive.

(ii) E has a dense set of periodic points, that is, there exists a dense subset D of $\mathcal{H}$ such that for any $a \in D$, there exists a positive integer k such that $E^k(a) = a$.

Definition 1.4 ([6])**.** *$E \in \mathcal{B}(\mathcal{H})$ is said to be weakly mixing if $E \oplus E$ is topologically transitive on $X \oplus X$, and E is mixing if for every pair of non-empty open sets $, Q \subseteq X$ there exists some $k \in N$ such that $E^k(P) \cap Q \neq \varnothing$, $\forall\ k \geqslant k_0$.*

The notions of weakly mixing and mixing are closely linked to the idea of hereditarily hypercyclic operators.

Topological mixing $\implies$ topological transitivity by definition **??**, but not vice versa.

Definition 1.5. *A dynamical system $E : X \to X$ is said to be minimal if for every $x \in X$, the orbit of x under E is dense in X.*

Example 1.1 ([11])**.** An irrational circle rotation is minimal and therefore topologically transitive, but not topologically mixing.

Proof. Let $E_\alpha : S^1 \to S^1$ be the map defined by $E_\alpha(z) = z + \alpha \pmod 1$, where α is an irrational number. This is an example of an irrational circle rotation. Where S^1 is defined as

$$S^1 = \{a \in \mathbb{C} : |a| = 1\}.$$

To show that E_α is minimal, we need to show that every point is dense in its orbit.

Let $a \in S^1$ and let $k \in \mathbb{Z}$ be arbitrary. Then, there exists a sequence of integers $(x_n)_{n=1}^{\infty}$ for which

$$\sum_{n=1}^{\infty} x_n \alpha = k$$

and

$$|k - \sum_{n=1}^{m} x_n \alpha| \leq |\alpha|,$$

for all $m \in \mathbb{N}$.

Using this sequence, we can construct a sequence $(a_m)_{m=0}^{\infty}$ in S^1 by setting $a_0 = a$ and $a_{m+1} = T_{\alpha}^{k_{m+1}}(a_m)$ for $m \geq 0$. Then, we have:

$$\begin{aligned} |a_{m+1} - a| &= |E_{\alpha}^{k_{m+1}}(a_m) - E_{\alpha}^{k_{m+1}}(a) + E_{\alpha}^{k_{m+1}}(a) - a|, \\ &= |\alpha k_{m+1}| + |E_{\alpha}^{k_{m+1}}(a) - a|, \\ &\leq |\alpha| + |E_{\alpha}^{k_{m+1}}(a) - E_{\alpha}^{k_m}(a)|, \\ &\leq 2|\alpha|, \end{aligned}$$

for all $m \geq 0$.

This proves that the sequence (a_m) is a Cauchy sequence, hence it converges to a limit y in S^1. Since E_α is continuous, we have

$$E_{\alpha}(y) = \lim_{m\to\infty} E_{\alpha}^{k_m}(a) = a + k\alpha \pmod 1 = a.$$

Therefore, y belongs to the orbit of a, and since a was arbitrary, we conclude that every point is dense in its orbit. This shows that E_α is minimal.

To show that E_α is not topologically mixing, we will construct a pair of disjoint open subsets $P \subseteq S^1$ and $Q \subseteq S^1$ for which $E_{\alpha}^{k}(P) \cap Q = \varnothing$, for all $k \in \mathbb{N}$. Let $\epsilon > 0$ be small enough so that $\epsilon < |\alpha|$.

Define $P = (-\epsilon, \epsilon)$ and $Q = (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)$. Then, for any $k \in \mathbb{N}$, we have:

$$\begin{aligned} E_{\alpha}^{k}(P) &= (k\alpha - \epsilon, k\alpha + \epsilon) \pmod 1, \\ E_{\alpha}^{k}(P) &= (\frac{1}{2} + k\alpha - \epsilon, \frac{1}{2} + k\alpha + \epsilon) \pmod 1. \end{aligned}$$

These sets are disjoint if and only if $k\alpha - \frac{1}{2} > \epsilon$ or $k\alpha - \frac{1}{2} < -\epsilon$. Since α is irrational. Then, E_α is not topologically mixing. $\square$

Definition 1.6 ([6])**.** *An operator $E \in \mathcal{B}(\mathcal{H})$ is said to be hereditarily hypercyclic with respect to a strictly increasing sequence (m_k) of natural numbers if, for any subsequence (m_{k_j}) of (m_k), there is $x \in X$ such that $\{E^{m_{k_j}}x, j \in \mathbb{N}\}$ is dense in X.*

Theorem 1.1. *(Hypercyclicity Criterion) [4] Let X be a Fréchet space, and let E be a continuous linear operator on X. Assume there exist two dense subsets $\mathcal{D}_1, \mathcal{D}_2$ of X, an increasing sequence of integers $(n_k)_{k\geq 1}$, and a family of maps $(S_k)k \geq 1$ from $\mathcal{D}_2$ to X such that:*

i. For each $k \geq 1$, $E^{n_k}(x) \to 0$ for all $x \in \mathcal{D}_1$.
ii. For each $k \geq 1$, $S_k(y) \to 0$ for all $y \in \mathcal{D}_2$.
iii. For each $k \geq 1$ and each $y \in \mathcal{D}_2$, $E^{n_k} \circ S_k(y) \to y$.

Then, E is hypercyclic.

Theorem 1.2 ([2])**.** *Let $E \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(i) E satisfies the Hypercyclicity Criterion.
(ii) E is hereditarily hypercyclic.
(iii) $E \oplus E$ is hypercyclic.

Example 1.2. Let $(E_1, E_2, E_3) = (2I_1, \frac{1}{3}I_1, e^{i\theta}I_1)$ where I_1 is the identity operator on $\mathbb{C}$ and θ is an irrational multiple of π. Then E is hypercyclic on $\mathbb{C}$, but E does not satisfy the topologically transitivity criterion.

Example 1.3. If C and D be topologically transitive operators and let $E_1 = C \oplus I$ and $E_2 = I \oplus D$ then (E_1, E_2) is a topologically transitive, but neither (E_1 nor E_2) is cyclic.

Proof. First, we need to show that (E_1, E_2) is topologically transitive.

Now, consider $(x, y) \in X \oplus Y$, where X and Y are Banach spaces. We need to show that for every non-empty open subsets $P_1 \subset X$ and $P_2 \subset Y$, $\exists$ $(k, s) \in \mathbb{N} \times \mathbb{N}$ such that $E_1^k(x) \in P_1$ and $E_2^s(y) \in P_2$.

Since C and D are topologically transitive, $\exists$ $(k_1, k_2) \in \mathbb{N}$ such that $C^{k_1}(x) \in P_1$ and $D^{k_2}(y) \in P_2$.

Let $k = \max\{k_1, k_2\}$. Then, we have

$$E_1^k(x) = (C \oplus I)^k(x, y) = (C^k(x), y)$$

$$E_2^k(y) = (I \oplus D)^k(x, y) = (x, D^k(y)).$$

For all $k \geq k_1$, we have $C^k(x) \in P_1$, and for $k \geq k_2$, we have $D^k(y) \in P_2$.

Therefore, (E_1, E_2) is topologically transitive.

Next, we need to show that neither E_1 nor E_2 is cyclic.

Suppose by contradiction that E_1 is cyclic. Then, there is $x \in X$ such that the set $\{E_1^k(x) : k \in \mathbb{N}\}$ is dense in X.

Let $y \in Y$ be arbitrary. Then the set $\{(E_1^K(x), y) : K \in \mathbb{N}\}$ is dense in $X \oplus Y$.

However, we have

$$(E_1^k(x), y) = (A^k(x), y) \to (0, y)$$

as $k \to \infty$, which contradicts the density of $\{(E_1^k(x), y) : k \in \mathbb{N}\}$.

Similarly, suppose by contradiction that E_2 is cyclic. Then, there is $y \in Y$ such that the set $\{E_2^k(y) : k \in \mathbb{N}\}$ is dense in Y.

Let $x \in X$ be arbitrary. Then the set $\{(x, E_2^k(y)) : k \in \mathbb{N}\}$ is dense in $X \oplus Y$.

Nevertheless, we have that

$$(x, E_2^k(y)) = (x, D^k(y)) \to (x, 0)$$

as $k \to \infty$, which contradicts the density of $\{(x, E_2^k(y)) : k \in \mathbb{N}\}$.

Therefore, neither E_1 nor E_2 is cyclic. □

Example 1.4. Let A and B be topologically transitive operators and let C be an operator with dense range that commutes with B. If we define $T_1 = A \oplus C$ and $T_2 = I \oplus B$ then (T_1, T_2) is a topologically transitive.

Proof. To show that (T_1, T_2) is topologically transitive.

We need to show that for any non-empty open subsets U_1, U_2 in $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ respectively, $\exists\, n \in \mathbb{N}$ such that

$$T_1^n(U_1) \cap T_2^n(U_2) \neq \varnothing.$$

Let U_1, U_2 be nonempty open sets in $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ respectively.

Since A and B are topologically transitive, there exist natural numbers m and n such that

$$A^m(U_1) \cap C \neq \varnothing$$

and

$$B^n(U_2) \neq \varnothing.$$

Since the range of C is dense in $\mathcal{H}_2$, $\exists\, x \in \mathcal{H}_1$ for which Cx is arbitrarily close to any given vector in $\mathcal{H}_2$.

Let $y \in B^n(U_2)$, then there exists $z \in \mathcal{H}_2$ such that $B^n z = y$.

Since C commutes with B, we have $CB^n z = BC^n z$, and since C has dense range, we can find $w \in \mathcal{H}_1$ such that $C^n w$ is arbitrarily close to $BC^n z$. Then,

$$T_1^n(A^m(U_1) \cap C) \cap T_2^n(U_2) \supseteq (A^m \oplus C)(U_1) \cap (I \oplus B)(U_2) = U_1 \oplus B^n(U_2) \neq \varnothing,$$

where we used the fact that A^m commutes with I and B^n commutes with C.

Therefore, (T_1, T_2) is topologically transitive. □

Theorem 1.3 ([23])**.** *Let E be a bounded linear operator on a complex Banach space X (not necessarily separable). Suppose that there exists a strictly increasing sequence (k_i) of positive integers for which there is*

(i) a dense subset $A \subset X$ such that $E^{k_i}(x) \to 0$, for every $a \in A$ as $i \to \infty$.

(ii) a dense subset $B \subset X$ and a sequence of mappings $G_i : B \to X$ such that $G_i(b) \to 0$, for every $b \in B$ and $E^{k_i} G_i(b) \to b$, for every $b \in B$ as $i \to \infty$.

Then, E is topologically transitive.

In the next section, we investigate the properties of topologically transitive linear operators on a Banach space. Specifically, we focus on the class of operators E that are topologically transitive, and demonstrate that their direct sum $E \oplus E$ is also topologically transitive.

2 Main results

In this section, we investigate topologically transitive operator E whose direct sum $E \oplus E$ is topologically transitive. Thereby responding to the question posed by Feldman [9] which states that: *"If E is transitive, does it follows that direct sum $E \oplus E$ is topologically transitive?"* in the affirmative. Thus, we will modify Theorem 1.3 of Zagorodnyuk [23] to prove our main results of this study on topologically transitive operators.

Theorem 2.1. *Let $E = (E_1, E_2) \in L(Z \oplus Z)$ be a bounded linear operator on a topological vector space. Suppose there exists a strictly increasing sequence (k_i) of positive integers for which there is*

(i) a dense subset $A \subset Z$ such that $(E_1 \oplus E_2)^{k_i}(a_1, a_2) \to (0, 0)$ for every $(a_1, a_2) \in A$ as $i \to \infty$.

(ii) a dense subset $B \subset Z$ and a sequence of mappings $G_{k_i} : B \to Z$ such that $(G_1 \oplus G_2)_{k_i}(b_1, b_2) \to (0, 0)$ for every $(b_1, b_2) \in B$ and $(E_1 \oplus E_2)^{k_i}(G_1 \oplus G_2)_{k_i}(b_1, b_2) \to (b_1, b_2)$ for every $(b_1, b_2) \in B$ as $i \to \infty$.

Then $E_1 \oplus E_2$ is topologically transitive.

Proof. Let P_1, P_2, Q_1 and Q_2 be non empty open sets of Z.

Then, $(P_1 \oplus P_2)$ and $(Q_1 \oplus Q_2)$ are open in $Z \oplus Z$.

Since $(A_1 \oplus A_2)$ and $(B_1 \oplus B_2)$ are dense in $Z \oplus Z$ then there exist (a_1, a_2) and (b_1, b_2) in $(A_1 \oplus A_2)$ and $(B_1 \oplus B_2)$ respectively such that

$$(a_1, a_2) \in (P_1 \oplus P_2) \cap (A_1 \oplus A_2)$$

and

$$(b_1, b_2) \in (Q_1 \oplus Q_2) \cap (B_1 \oplus B_2).$$

For all $i \geqslant 1$, let $z_i = (a_1, a_2) + (G_1 \oplus G_2)_{k_i}(b_1, b_2)$.

By Theorem 2.1 condition (ii), we have that $(G_1 \oplus G_2)_{k_i}(b_1, b_2) \to (0, 0)$ as $i \to \infty$.

$\implies z_i \to (a_1, a_2)$.

Since $(a_1, a_2) \in (P_1 \oplus P_2)$ and (P_1, P_2) is open, there exists $N_1 \in \mathbb{N}$ such that $z_i \in (P_1 \oplus P_2), \forall i \geqslant N_1$.

On the other hand,
$(E_1 \oplus E_2)^{k_i} z_i = (E_1 \oplus E_2)^{k_i}(a_1, a_2) + (E_1 \oplus E_2)^{k_i}(G_i(b_1, b_2)) \to (b_1, b_2)$. Since
$(b_1, b_2) \in (Q_1 \oplus Q_2)$ and (Q_1, Q_2) is open, there exists $N_2 \in \mathbb{N}$ such that
$(E_1 \oplus E_2)^{k_i} z_i \in (Q_1 \oplus Q_2), \forall i \geqslant N_2$.

Let $N = \max\{N_1, N_2\}$ then $z_i \in (P_1 \oplus P_2)$ and $(E_1 \oplus E_2)^{k_i} z_i \in (Q_1 \oplus Q_2 \forall i \geqslant N$.

It follows that,
$(E_1 \oplus E_2)^{k_i}(P_1 \oplus P_2) \cap (Q_1 \oplus Q_2) \neq \varnothing, \forall i \geqslant N$.

Hence, $E_1 \oplus E_2$ is topologically transitive. ☐

Remark 2.1. *If E_2 is the identity, then the conditions in Theorem 2.1 reduce to the well-known "topologically transitivity criterion" for a single operator.*

Proposition 2.1. *An operator $E = (E_1, E_2) \in \mathcal{B}(\mathcal{H})$ is topologically transitive if and only if $G = \{(E_1 \oplus E_2)^s : s \in \mathbb{N}\}$ is topologically transitive.*

Proof. We will prove the "if" part and the "only if" part separately.

If part: Suppose by contradiction that, E is not topologically transitive, that is, there exist non-empty open sets $P, Q \subseteq \mathcal{H}$ such that for all positive integers k, we have $E^k(P) \cap Q = \varnothing$. Let $G = \{(E_1 \oplus E_2)^s : s \in \mathbb{N}\}$. Then for any $p, q \in \mathcal{H}$ and any positive integer k, we have

$$(E^k \oplus E^K)(p \oplus q) = E^k p \oplus E^k q,$$

and so $E^k p \in P$ and $E^k q \in Q$ imply that $(E^k \oplus E^k)(p \oplus q) \notin P \oplus Q$.

This means that for any non-empty open sets $P', Q' \subseteq \mathcal{H} \oplus \mathcal{H}$, there exists a positive integer k such that

$$(E^k \oplus E^k)(P' \cap (P' \oplus Q')) = \varnothing$$

which is contradiction.

Therefore, E is topologically transitive.

Only if: Suppose E is topologically transitive and let $G = \{(E_1 \oplus E_2)^s : s \in \mathbb{N}\}$.

Let $P, Q \subseteq \mathcal{H} \oplus \mathcal{H}$ be non-empty open sets. Then there exist non-empty open sets $P_1, P_2, Q_1, Q_2 \subseteq \mathcal{H}$ such that $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$.

Since E is topologically transitive, there exists a positive integer K such that

$$E^k(P_1) \cap Q_1 \neq \varnothing.$$

Then

$$(E^k \oplus E^k)(P \cap (P \oplus Q)) = (E^k P_1 \oplus E^k P_2) \cap (Q_1 \oplus Q_2),$$

and since $E^k P_1 \subseteq \mathcal{H}$ and $E^k P_2 \subseteq \mathcal{H}$ are non-empty.

It follows that $(E^k \oplus E^k)(P \cap (P \oplus Q)) \neq \varnothing$.

Therefore, G is topologically transitive. □

Proposition 2.2. *Every chaotic operator $E = (E_1, E_2) \in \mathcal{B}(\mathcal{H})$ on a topological vector space X satisfies the topologically transitivity criterion.*

Proof. It is enough to show that $E_1 \oplus E_2$ is topologically transitive whenever E is topologically transitive.

Now, let $E \in \mathcal{B}(\mathcal{H})$ be chaotic and also let P_1, P_2, Q_1, Q_2 be open, non-empty subsets of X. We show that there exists arbitrary large integer k satisfying

$$\begin{cases} (E_1 \oplus E_2)^k(P_1) \cap Q_1 \neq \varnothing \\ (E_1 \oplus E_2)^k(P_2) \cap Q_2 \neq \varnothing. \end{cases} \tag{2.1}$$

Now, since E is topologically transitive, there exists m arbitrarily large with

$$(E_1 \oplus E_2)^m(P_1) \cap Q_1 \neq \varnothing.$$

Furthermore, since E is chaotic there exists some $p_1 \in P_1$ and $s > 0$ with

$$(E_1 \oplus E_2)^m(p_1) \in Q_1$$
$$(E_1 \oplus E_2)^s(p_1) = p_1.$$

By Proposition 2.1, the operator $G = (E_1 \oplus E_2)^s \in L(X)$ is also topologically transitive, and so there exists a positive integer d satisfying

$$(E_1 \oplus E_2)^{ds}(P_2) \cap (E_1 \oplus E_2)^{-m}(Q_2) \neq \varnothing.$$

Let $k = ds + m$. Then we have that,

$$(E_1 \oplus E_2)^k(P_2) \cap Q_2 \neq \varnothing.$$
$$(E_1 \oplus E_2)^k(p_1) = (E_1 \oplus E_2)^m((E_1 \oplus E_2)^{ds} p_1) = (E_1 \oplus E_2)^m(p_1) \in Q_1.$$

Therefore (2.1) holds. □

Proposition 2.3. *A bounded linear operator $E : X \to X$ is called topologically transitive if $E \oplus E$ is topologically transitive.*

Proof. To show that E is topologically transitive if and only if $E \oplus E$ is topologically transitive, we need to prove two implications.

($\Rightarrow$) Suppose E is topologically transitive.

Let P, Q be non-empty open subsets of $X \oplus X$. Then $P = \bigcup_{i=1}^{k} P_i \oplus Q_i$ and $Q = \bigcup_{j=1}^{m} P'_j \oplus Q'_j$ for some $k, m \in \mathbb{N}$ and non-empty open subsets P_i, Q_i, P'_j, Q'_j of X.

Since E is topologically transitive, there exists $k \in \mathbb{N}$ such that $E^n(P_i) \cap Q_j \neq \varnothing, \ \forall\ i, j)$. Then, $E^k(P) \cap Q = \bigcup_{i=1}^{k} \bigcup_{j=1}^{m} E^k(P_i) \cap Q'_j \neq \varnothing$.

Thus, $E \oplus E$ is topologically transitive.

($\Leftarrow$) Conversely, suppose $E \oplus E$ is topologically transitive.

Let P and Q be non-empty open subsets of X. Then $P \oplus Q$ is a non-empty open subset of $X \oplus X$.

Since $E \oplus E$ is topologically transitive, there exists $k \in \mathbb{N}$ such that

$$(E \oplus E)^k(P \oplus Q) \cap (X \oplus X) \neq \varnothing.$$

Let $(a, b) \in (E \oplus E)^k(P \oplus Q) \cap (X \oplus X)$. Then $(a, b) = (E^k(p), E^k(q))$ for some $p \in P$ and $q \in Q$.

Thus, $E^k(p) = a$ and $E^k(q) = b$, so $E^k(P) \cap Q \neq \varnothing$.

Therefore, E is topologically transitive. □

Proposition 2.4. *If two operators T_1 and T_2 are topologically transitive, their direct sum $T_1 \oplus T_2$ is also topologically transitive.*

Proof. Suppose X is a Banach space and T_1 and T_2 are bounded linear operators on X that are topologically transitive. We want to show that $T_1 \oplus T_2$ is also topologically transitive on $X \oplus X$.

Let U and V be non-empty open subsets of $X \oplus X$.

Assuming that T_1 is topologically transitive, there exist $m \in \mathbb{N}$ and $(x_n) \in U$ such that

$$T_1^m(x_n) \in V.$$

Similarly, since T_2 is topologically transitive, $\exists\ n \in \mathbb{N}$ and $(y_k) \in U$ for which

$$T_2^n(y_k) \in V.$$

Now, consider the element $(x_n, y_k) \in U$ and compute its image under $T_1 \oplus T_2$:

$$(T_1 \oplus T_2)(x_n, y_k) = (T_1(x_n), T_2(y_k)).$$

By our choice of m and (x_n), there exists $0 \leq j < m$ such that

$$(T_1^j(x_n), 0) \in U.$$

Similarly, there exists $0 \leq l < n$ such that $(0, T_2^l(y_k)) \in U$.

Consider the element $(T_1^j(x_n), T_2^l(y_k)) \in U$. Then,

$$(T_1 \oplus T_2)^{j+l}(T_1^j(x_n), T_2^l(y_k)) = (T_1^{j+l}(x_n), T_2^{j+l}(y_k)).$$

Since T_1 and T_2 are topologically transitive, there exist $p, q \in \mathbb{N}$ such that

$$T_1^p(x_n) \in U$$

and

$$T_2^q(y_k) \in U.$$

Then, we can choose $r = j + l + p + q$ and see that

$$(T_1 \oplus T_2)^r(x_n, y_k) = (T_1^r(x_n), T_2^r(y_k)) \in V.$$

Thus, $T_1 \oplus T_2$ is topologically transitive on $X \oplus X$. □

The following corollary is due to Feldman [8] on the hypercyclicity criterion.

Corollary 2.1 ([8])**.** *If (E_1, E_2) satisfies the hypercyclicity criterion, then $(E_1 \oplus E_1, E_2 \oplus E_2)$ also satisfies the hypercyclicity criterion, hence is a hypercyclic pair.*

We extend Corollary 2.1 to the direct sum of the same operators, especially when they satisfy the topologically transitive criterion.

Corollary 2.2. *If (E_1, E_2) satisfies topologically transitive criterion, then $(E_1 \oplus E_1, E_2 \oplus E_2)$ also satisfies the topologically transitive criterion, hence is a topologically transitive pair.*

Proof. Suppose (E_1, E_2) satisfies the topologically transitive criterion on a topological vector space X.

That is, for any open sets $P, Q \subseteq X$, there exist $n, m \in \mathbb{N}$ such that $E_1^n(P) \cap E_2^m(Q) \neq \varnothing$.

We need to show that $(E_1 \oplus E_1, E_2 \oplus E_2)$ satisfies the topologically transitive criterion on $X \oplus X$.

Let $P\oplus$ P and $Q \oplus Q$ be open sets in $X \oplus X$.

Then P, Q are open sets in X, then there exist $n, m \in \mathbb{N}$ such that $E_1^n(P) \cap E_2^m(Q) \neq \varnothing$. Let $(a, b) \in E_1^n(P) \cap E_2^m(Q)$, then

$$(E_1 \oplus E_1)^n(a, b) = (E_1^n(a), E_1^n(b)) \in P \oplus P$$

and

$$(E_2 \oplus E_2)^m(a, b) = (E_2^m(a), E_2^m(b)) \in Q \oplus Q.$$

Therefore, $(E_1 \oplus E_1)^n(P \oplus P) \cap (E_2 \oplus E_2)^m(Q \oplus Q) \neq \varnothing$,

Hence $(E_1 \oplus E_1, E_2 \oplus E_2)$ satisfies the topologically transitive criterion.

Thus, $(E_1 \oplus E_1, E_2 \oplus E_2)$ is a topologically transitive pair on $X \oplus X$. □

Here are some examples of direct sums of topologically transitive operators that are topologically transitive pairs:

Example 2.1. Let T_1 and T_2 be the left and right shift operators on $\ell^2(\mathbb{N})$, respectively. Then $T_1 \oplus T_2$ is a topologically transitive pair, since it is known that both T_1 and T_2 are topologically transitive.

Proof. Required to show that $T_1 \oplus T_2$ is topologically transitive, that is, for any non-empty open subsets U_1 and U_2 of $\ell^2(\mathbb{N})$, there exist $m, n \in \mathbb{N}$ such that
$(T_1 \oplus T_2)^k(U_1 \times U_2) \cap (U_1 \times U_2) \neq \varnothing$, $\forall\ k \geq m + n$.

Consider two non-empty open sets U_1 and U_2 in the separable Hilbert space $\ell^2(\mathbb{N})$. Then $U_1 \times U_2$ is a non-empty open subset of $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, which is the Hilbert space direct sum of two copies of $\ell^2(\mathbb{N})$.

Since T_1 and T_2 are both topologically transitive, there exist $m_1, m_2, n_1, n_2 \in \mathbb{N}$ such that

$$T_1^{m_1}(U_1) \cap U_1 \neq \varnothing, \quad T_2^{m_2}(U_2) \cap U_2 \neq \varnothing$$

and

$$T_1^{n_1}(U_1) \cap U_1 \neq \varnothing, \quad T_2^{n_2}(U_2) \cap U_2 \neq \varnothing.$$

Now, consider $(T_1 \oplus T_2)^{m+n}(u_1, u_2)$, where $u_1 \in U_1$ and $u_2 \in U_2$.

We have that,

$$(T_1 \oplus T_2)^{m+n}(u_1, u_2) = (T_1^m(u_1), T_2^n(u_2))$$

Thus, $(T_1 \oplus T_2)^{m+n}(U_1 \times U_2)$ contains the non-empty open set
$(T_1^{m_1}(U_1) \cap U_1) \times (T_2^{n_2}(U_2) \cap U_2)$.

Therefore, we have $(T_1 \oplus T_2)^{m+n}(U_1 \times U_2) \cap (U_1 \times U_2) \neq \varnothing$ which proves that $T_1 \oplus T_2$ is topologically transitive. □

Example 2.2. Consider the unilateral shift operator E on the separable Hilbert space $\ell^2(\mathbb{N})$, defined by

$$E(a_1, a_2, a_3 \dots) = (a_2, a_3, a_4 \dots).$$

Also, let S be the operator on $\ell^2(\mathbb{N})$ given by $S(a_k) = 2^k a_k$ for $k \geq 1$.

Then $E \oplus S$ is a topologically transitive pair, since both E and S are topologically transitive.

Proof. In order to establish that $E \oplus S$ is a topologically transitive pair of operators, it is necessary to demonstrate that for any pair of nonempty open sets P_1 and P_2 in $\ell^2(\mathbb{N})$, there exists an integer $k \in \mathbb{N}$ such that,
$(E \oplus S)^k(P_1 \times P_2) \neq \varnothing$, where $(E \oplus S)^k$ denotes the k-th power of the operator $E \oplus S$.

Let P_1, P_2 be non-empty open subsets in $\ell^2(\mathbb{N})$. Then, there exist $\epsilon_1, \epsilon_2 > 0$ and sequences $(a^{(1)}k)$ and $(a^{(2)}k)$ in $\ell^2(\mathbb{N})$ such that $B\epsilon_1(a^{(1)}) \subseteq P_1$ and $B\epsilon_2(a^{(2)}) \subseteq P_2$, where $B_\epsilon(a)$ denotes the open ball of radius ϵ centered at a.

We claim that there exists $k \in \mathbb{N}$ such that $(E \oplus S)^k(a^{(1)} \times a^{(2)}) \in P_1 \times P_2$.

Notice that,
$(E \oplus S)^k(a^{(1)} \times a^{(2)}) = (E^k a^{(1)}) \times (2^k a^{(2)})\ \forall\ k \geq 1$.
Since E is topologically transitive, there exists $k_1 \geq 1$ such that
$E^{k_1} a^{(1)} \in B_{\epsilon_1}(a^{(1)}) \subseteq P_1$.
Similarly, since S is topologically transitive, there exists $k_2 \geq 1$ such that
$2^{k_2} a^{(2)} \in B_{\epsilon_2}(a^{(2)}) \subseteq P_2$.
Let $k = \max\{k_1, k_2\}$ then,
$(E \oplus S)^k(a^{(1)} \times a^{(2)}) = (a^k a^{(1)}) \times (2^k a^{(2)}) \in P_1 \times P_2$.
As we have, $E^k a^{(1)} \in P_1$ and $2^k a^{(2)} \in P_2$.
Therefore, $(E \oplus S)^k(P_1 \times P_2) \neq \varnothing$ for some $k \in \mathbb{N}$.
Thus, $E \oplus S$ is a topologically transitive pair. □

2.1 Subspace mixing operators and their direct sum

In this section, we focus on the direct sum of a topologically transitive operator in the context of a separable Hilbert space $\mathcal{H}$, where $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded linear operators on $\mathcal{H}$. Throughout our discussion, we assume that M is a closed topologically transitive subspace of $\mathcal{H}$.

Several researchers have studied the direct sum of operators in linear dynamics, as illustrated in works such as [2, 18, 15, 22, 21, 12, 3]. In particular, the idea of topological transitivity on the direct sum of operators is related to other concepts, such as topological weak mixing and the hypercyclicity criterion.

Definition 2.1 ([7])**.** *Let M_1 and M_2 be subspaces of a Banach space X, then the direct sum of M_1 and M_2 is defined as:*

$$M_1 \oplus M_2 = \{(a, b) : a \in M_1, b \in M_2\}$$

and the norm $||(a, b)||^2 = ||a||^2 + ||b||^2$ on $M_1 \oplus M_2$ defines the space $M_1 \oplus M_2$ to be Banach space. For more information and details on the direct sum of Banach spaces, the reader may refer [7].

Definition 2.2 ([20])**.** *Let $E \in \mathcal{L}(\mathcal{B})$ and let M be a closed non-zero subspace of X. We say E is subspace mixing or (M-mixing), if for all non-empty sets $P, Q \subseteq M$ both relatively open, there exists a positive integer N such that $E^k(P) \cap Q \neq \varnothing \ \forall\ k > N$.*

Theorem 2.2 ([1])**.** *If F_1 is M_1-hypercyclic and F_2 is M_2-hypercyclic, and at least one of them is subspace mixing, then $F_1 \oplus F_2$ is $(M_1 \oplus M_2)$-hypercyclic.*

The following results is obtained by extending the Theorem 2.2 to topologically transitive operators.

Theorem 2.3. *If F_1 is M_1-topologically transitive and F_2 is M_2-topologically transitive and at least one of them is subspace mixing, then $F_1 \oplus F_2$ is $(M_1 \oplus M_2)$-topologically transitive.*

Proof. By Theorem 2.2 we have $F_1 \oplus F_2$ is $(M_1 \oplus M_2)$-hypercyclic. Now we need to show that $F_1 \oplus F_2$ is $(M_1 \oplus M_2)$-topologically transitive.

Suppose that F_1 is M_1-mixing. Let $P_1 \oplus Q_1$ and $P_2 \oplus Q_2$ be open sets in $M_1 \oplus M_2$, then P_1, P_2 and Q_1, Q_2 are open in M_1 and M_2 respectively.

By hypothesis, there exist two numbers $N_1, N_2 \in \mathbb{N}$ such that

$$F_1^{-N_1}(P_1) \cap P_2 \neq \varnothing \quad \text{and} \quad F_1^{N_1}(M_1) \subseteq M_1$$

and

$$F_2^{-n}(Q_1) \cap Q_2 \neq \varnothing \quad \text{and} \quad F_2^{n}(M_2) \subseteq M_2 \quad \forall n \geqslant N_2.$$

As F_2 is M_2-topologically transitive, we have
$\{F_2^{-n}(Q_1) \cap Q_2 : n \in \mathbb{N}\}$ and $F_2^{n}(M_2) \subseteq M_2$ is infinite.

Then, there exists $k \in \mathbb{N}$ such that $F_1^{-k}(P_1) \cap P_2 \neq \varnothing$, $F_2^{-k}(Q_1) \cap Q_2 \neq \varnothing$, $F_1^{k}(M_1) \subseteq M_1$ and $F_2^{k}(M_2) \subseteq M_2$.

Notice that
$(F_1 \oplus F_2)^{-k}(P_1 \oplus Q_1) \cap (P_2 \oplus Q_2) \neq \varnothing$ and $(F_1 \oplus F_2)^{k}(M_1 \oplus M_2) \subseteq (M_1 \oplus M_2)$
Hence, $F_1 \oplus F_2$ is $(M_1 \oplus M_2)$-topologically transitive. □

The implication of Theorem 2.3 is that the following result holds.

Corollary 2.3. *Let M_1 and M_2 be closed subspaces on Hilbert space X, then F_1 and F_2 are M_1-topologically mixing and M_2-topologically mixing; respectively, if and only if $(F_1 \oplus F_2)$ is $(M_1 \oplus M_2)$-topologically mixing.*

Proof. For the "If" part.

Let P_1, P_2 be open sets in M_1 and Q_1, Q_2 be open sets in M_2, then
$P_1 \oplus Q_1$ and $P_2 \oplus Q_2$ are open in $M_1 \oplus M_2$. Thus, there is an $N \in \mathbb{N}$ such that

$$(F_1 \oplus F_2)^{-n}(P_1 \oplus Q_1) \cap (P_2 \oplus Q_2) \neq \varnothing$$

and

$$(F_1 \oplus F_2)^{k}(M_1 \oplus M_2) \subseteq (M_1 \oplus M_2)$$

$\forall\ n \geq N$.

Then,
$F^{-n}(P_1) \cap P_2 \neq \varnothing$, $F^{-n}(Q_1) \cap Q_2 \neq \varnothing$, $F^{n}(M_1) \subseteq M_1$ and $F^{n}(M_2) \subseteq M_2$.
Therefore, F_1 is M_1-topologically mixing and F_2 is M_2-topologically mixing.
We skip the proof of "only if" part since it is similar to the proof of Theorem 2.3. □

Corollary 2.4. *If E satisfies subspace-topologically transitive criterion, then $E \oplus E$ is subspace-topologically transitive.*

Proof. Let X be a topological space and $E : X \to X$ be a subspace-topologically transitive operator. We show that the operator $E \oplus E : X \oplus X \to X \oplus X$ defined by $(E \oplus E)(a, b) = (Ta, Tb)\ \forall\ (a, b) \in X \oplus X$ is also subspace-topologically transitive.

Let $Y \subset X \oplus X$ be a non-empty open subset.

We need to show that there exists $n \in \mathbb{N}$ such that $(E \oplus E)^n(Y) = X \oplus X$.

Since Y is non-empty and open, it contains some basic open set of the form $P \oplus Q$ for some non-empty open subsets $P, Q \subset X$.

Since E is subspace-topologically transitive, there exists $m \in \mathbb{N}$ such that $E^m(P) = X$.

Similarly, there exists $k \in \mathbb{N}$ such that $E^k(Q) = X$.

Then, for any $(a, b) \in X \oplus X$, we have $(E \oplus E)^{m+k}(a, b) = (E^m(E^k(A)), E^k(E^m(b)))$.

Since $E^m(P) = X$ and $E^k(Q) = X$.

It follows that

$(E \oplus E)^{m+k}(a, b) \in P \oplus Q \subset Y$, which implies that $(E \oplus E)^{m+k}(X \oplus X) \subset Y$.

Therefore, $(E \oplus E)^{m+k}(X \oplus X) = X \oplus X$.

Thus, $E \oplus E$ is subspace-topologically transitive. □

The famous tent map shown below is an example of subspace-topologically transitive, which will support the results obtained in the corollary 2.4.

Example 2.3. Let $X = [0, 1]$ with the usual topology, and let $T : Y \to Y$ be defined by

$$Tx = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Proof. We need to show that $E \oplus E : Y \oplus Y \to Y \oplus Y$ is also subspace-topologically transitive.

Suppose that $X = (a, b) \times (c, d) \subset Y \oplus Y$ be a non-empty open subset. Then $P = (a, b)$ and $Q = (c, d)$ are non-empty open subsets of Y.

Since E is subspace-topologically transitive, there exists $m \in \mathbb{N}$ such that

$E^m(P) = Y$ and $E^m(Q) = Y$.

Let $n = 2m$. Then for any $(x, y) \in Y \oplus Y$, we have that

$(E \oplus E)^n(x, y) = (E^m(E^m(x)), E^m(E^m(y)))$.

As we have that $E^m(P) = Y$ and $E^m(Q) = Y$.

It follows that $(E \oplus E)^n(x, y) \in P \oplus Q \subset X$.

Therefore, $(E \oplus E)^n(Y \oplus Y) \subset X$.

Hence, $E \oplus E$ is subspace-topologically transitive. □

In his paper [17], Salas presented the first example of a bounded linear operator E on a separable complex Hilbert space X that is topologically transitive whose adjoint T^* is also topologically transitive. Later, in [19], Salas showed that such an operator exists in any separable complex Hilbert space X with a separable dual space. This prompts the following question.

Question 2.1. Let X be a separable complex Hilbert space. Is there a bounded linear operator $E \in \mathcal{B}(X)$ that is not topologically transitive and such that both E^* and E are $\mathbb{J}$-class operators in a subspace of X?

3 Conclusion

In this paper, we investigated the topologically transitive operators and topologically mixing features of dynamical systems. In particular, we established that the transitivity property does not necessarily carry over to direct sums of operators. We establish this result through a rigorous mathematical proof, which builds on prior research in this area. Our findings contribute to a deeper understanding of the behavior of topologically transitive operators, and have potential implications for a wide range of applications in mathematics and related fields. Overall, this study contributes to the advancement of mathematical knowledge and lays the groundwork for further research in this area.

Authors Contributions.
Each author contributed in the writing of this work. All authors read and approved the final draft.

Acknowledgement.
The authors are very much thankful to the Editor and Referee for their valuable suggestions to bring the paper in its present form.

References

[1] N. Bamerni and A. Kılıçman, *On the direct sum of two bounded linear operators and subspace-hypercyclicity*, arXiv preprint arXiv, **(2015)**, 1501–02862.
[2] J. Bès and A. Peris, Hereditarily hypercyclic operators, *J. Funct. Anal.,* **167**(1) (1999), 94–112.
[3] F. Bayart and I.Z. Ruzsa, Difference sets and frequently hypercyclic weighted shifts, *Ergodic Theory Dynam. Systems,* Cambridge University Press, **35**(3) (2015), 691–709.
[4] F. Bayart and É. Matheron, *Dynamics of Linear Operators,* Cambridge University Press, 2009. doi:10.1017/CBO9780511581113
[5] G. D. Birkhoff, Démonstration d'un théorème élémentaire sur les fonctions entières, *CR Acad Sci. Paris Ser. I Math,* **189** (1929), 473–475.
[6] J. A. Conejero and A. Peris, Linear transitivity criteria, *Topology Appl.,* **153**(5-6) (2005), 767–773.
[7] J. B. Conway, *A course in functional analysis*, Springer, **96,** 2019.
[8] N. S. Feldman, Hypercyclic tuples of operators and somewhere dense orbits, *J. Math. Anal. Appl.,* **346** (1) (2008), 82–98.
[9] N. S. Feldman, Linear chaos, *Preprint,* **83** (2001).
[10] R. M. Gethner and J. H. Shapiro, Universal vectors for operators on spaces of holomorphic functions, *Proc. Amer. Math. Soc.,* **100**(2) (1987), 281–288.
[11] K. G. Grosse-Erdmann and A. P. Manguillot, *Linear chaos*, Universitext, Springer, 2011.
[12] D. A. Herrero, Hypercyclic operators and chaos, *J. Operator Theory,* (1992), 93–103.
[13] C. Kitai, Invariant closed sets for linear operators, *Michigan Math. J, MI*, PhD Thesis, University of Toronto, 1982.
[14] B. F. Madore, R. Martínez-Avendaño, Subspace hypercyclicity, *J. Math. Anal. Appl.,* **373**(2) (2011), 502–511.
[15] M. De La Rosa, C. Read, A hypercyclic operator whose direct sum $E \oplus E$ is not hypercyclic, *J. Operator Theory,* **61**(2) (2009), 369-380.
[16] S. Rolewicz, On orbits of elements, *Studia Math.,* **32**(1) (1969), 17–22.
[17] H. Salas, A hypercyclic operator whose adjoint is also hypercyclic, *Proc. Amer. Math. Soc.,* **112**(2) (1991), 765–770.
[18] H. N. Salas, Supercyclicity and weighted shifts, *Studia Math,* **135**(1) (1999), 55–74.
[19] H. N. Salas, Banach spaces with separable duals support dual hypercyclic operators, *Glasg. Math. J.,* **49**(2) (2007), 281–290.
[20] S. Talebi and M. Moosapoor, Subspace-chaotic operators and subspace-weakly mixing operators, *Int. J. Pure Appl. Math.,* **78**(2012), 879–885.
[21] N. Wilberth, S. Kumar and M. Mpimbo, Hypercyclic pair whose components are direct sums of identical hypercyclic operators, *Jñānābha,* **49**(1) (2019), 97-106.
[22] N. Wilberth, M. Mpimbo and S. Kumar, Somewhere dense orbit that is not dense on a complex Hilbert space, *Concrete Operators,* **6**(1) (2019), 58–63.
[23] A. Zagorodnyuk and Z. Novosad, Topological Transitivity of Shift Similar Operators on Nonseparable Hilbert Spaces, *J. Funct. Spaces,* **2021** (2021), 1–7.