ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(2) (2023), 126-133**
*(Dedicated to Professor V. P. Saxena on His $80^{th}$ Birth Anniversary Celebrations)*

# ON A DUAL CHARACTERIZATION OF THE ASYMPTOTIC CONE FOR THE SOLUTION SET OF A LINEAR OPTIMIZATION PROBLEM

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*(Received: July 13, 2023; In format : August 26, 2023; Revised : November 26, 2023; Accepted : November 28, 2023)*

DOI: https://doi.org/10.58250/jnanabha.2023.53215

**Abstract**

The concept of the asymptotic cone is very useful in various branches of pure and applied mathematics, especially in optimization and variational inequalities. In recent years, many authors and researchers have studied asymptotic directions and asymptotically convergent algorithms for unbounded solution sets. In this paper, we consider the asymptotic cone of the solution set $\Omega$ of a linear optimization problem and investigate various results on its asymptotic cone, asymptotic regularity, the dual and polar cones of the asymptotic cone, the support function of the solution set, etc. Finally, we present a dual characterization of the asymptotic cone $\Omega_\infty$ for the solution set of a linear optimization problem.
**2020 Mathematical Sciences Classification:** 90C05, 90C60, 46B06, 40A05.
**Keywords and Phrases:** Linear optimization, Asymptotic cones, Asymptotic regularity, Normalized set, Positive hull, Polar cone, Dual cone, Support function.

## 1 Introduction

The concept of an asymptotic cone appeared in the literature first time in 1913 in Steintiz [35] to deal with the unboundedness of sets, particularly unbounded convex sets. For further details on asymptotic cones for convex sets we refer to Auslender and Teboulle [8], Luc and Penot [26], and Petrovai [29] and various relevant references cited in each of them. For the notion of asymptotic cone for nonconvex sets we refer to Luc [23,24,25,26], Penot [28], and Stoker[36].

The purpose of this paper is to investigate various asymptotic properties of the solution set for a linear optimization problem and utilize them to provide a dual characterization of the asymptotic cone of the solution set.

Throughout the paper, an $n$-dimensional Euclidean space will be denoted by $R^n$. For a point or vector $x = (x_1, x_2, \dots, x_n) \in R^n$, the Euclidean norm of $x$ is given as $\|x\| = \left(\sum_{i=1}^n {x_i}^2\right)^{\frac{1}{2}}$. A sequence in $R^n$ is written as $\{x_k\}$ or sometimes $\{x_k\}_{k\in N}$, where $N$ is the set of natural numbers. A subsequence of this sequence is denoted by $\{x_k\}_{k\in K}$, and $K \subset N$. A sequence $\{x_k\}_{k\in N}$ is said to converge to $x \in R^n$, if $\|x_k - x\| \to 0$, as $k \to \infty$.

It is indicated by the notation $\lim_{k\to\infty} x_k = x$ or $x_k \to x$.

This is called a strong form of convergence. A sequence $\{x_k\}_{k\in N}$ in $R^n$ may converge to $x \in R^n$, linearly, quadratically, or super linearly. For further details on the order of convergence, we refer to Petrovai [29].

The Bolzano-Weierstrass theorem, which is a fundamental result of the convergence in a finitedimensional Euclidean space $R^n$, states that each bounded sequence in $R^n$ has a convergent subsequence. A point $x \in R^n$ is called a cluster point of the sequence $\{x_k\}_{k\in N}$, in $R^n$, if there exists a subsequence $\{x_k\}_{k\in K}$ that converges to $x$. Also, the sequence $\{x_k\}_{k\in N}$, in $R^n$ converges to a point $x \in R^n$ if and only if it is bounded and $x$ is its unique cluster point. We will make use of the Bolzano-Weierstrass theorem to prove some results associated with the asymptotic cone, asymptotic regularity, etc. of the solution set of the linear optimization problem.

Further details for dealing with the asymptotic behavior of sets and functions can be referred to [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 20, 21, 22, 26, 27, 31, 35, 36] and the relevant references cited in these papers.

### 1.1 Linear Optimization Problem

A linear optimization problem in standard form can be stated as

$$\text{Maximize } f(x) = c^T x = c_1x_1 + c_2x_2 + \cdots + c_nx_n, \tag{1.1}$$

such that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \\ x_i \geq 0, i = 1, 2, \ldots, n & \end{aligned}$$

where $A = [a_{ij}] \in R^{m\times n}, b = (b_1, b_2, \ldots, b_m)^T \in R^m, c = (c_1, c_2, \ldots, c_n)^T \in R^n$, and $x = (x_1, x_2, \ldots, x_n)^T \in R^n$.

Here $f : R^n \to R$ is a linear map defined by $f(x) = c^T x$ and, $A : R^n \to R^m$, is also a linear map defined by $AX = b$.

If we define $m$ hyperplanes

$$H_i = \{x \in R^n : a_{11}x_1 + a_{11}x_1 + \cdots + a_{11}x_1 = b_i\}, i = 1, 2, \cdots, m. \tag{1.2}$$

Then

$$x \in \{x \in R^n : Ax = b\} \text{ if and only if } x \in \bigcap_{i=1}^{m} H_i. \tag{1.3}$$

Let $P_+ = \{x \in R^n : x \geq 0\}$ denotes the positive orthant of $R^n$ and

$$\Omega = [\cap_{i=1}^m H_i] \cap P_+. \tag{1.4}$$

Then the above linear optimization problem (1.1) can be stated as

$$\text{Maximize } f(x) = c^T x, \text{ such that } x \in \Omega. \tag{1.5}$$

Further, we assume that

a) $A \in {R_m}^{m\times n}$, that is $A$ is an $m \times n$ matrix of rank $m$.

b) $[A, b] \in {R_m}^{m\times(n+1)}$, that is, the augmented matrix $[A, b]$ is of order $m \times (n + 1)$, and rank m.

Thus, we have, $\text{rank}[A, b] = \text{rank}(A) = m$.

The solution set $\Omega$ is a nonempty closed subset of $R^n$. It is easy to see that it is also a convex subset of $R^n$.

## 2 Definitions and Notations

In this section, we explicate some definitions and related notations that will be used throughout this paper.

Let $x_k \in \Omega \subset R^n$, and $\|x_k\| \to \infty$, as $k \to \infty$. Then there exists a real sequence $\{\alpha_k\}_{k\in K}$, defined as $\alpha_k := \|x_k\|$, k $\in K, K \subset N$ such that $\lim_{k\in K} \alpha_k = +\infty$, and $\lim_{k\in K} \frac{x_k}{\alpha_k} = \beta$.

**Definition 2.1.** *(Nonnegative orthant). The nonnegative orthant of an $n$-dimensional Euclidean space is denoted by $R^n_+$ and is given by*

$R^n_+ = \{x \in R^n \mid x_i \geq 0, i = 1, 2, 3, \ldots, n\}$.

**Definition 2.2.** *(Cone or nonnegative homogeneous). A set $K$ is called a cone if $\forall x \in K$, and $\mu \geq 0, \mu x \in K$.*

**Definition 2.3.** *(Convex hull of a set). The convex hull of a set $K$ is denoted by conv $K$, is the set of all convex combinations of the points in $K$ :*

$$\text{conv}\, K = \left\{ \sum_{i=1}^{k} \mu_i x_i : x_i \in K, \mu_i \geq 0, \forall i, \sum_{i=1}^{k} \mu_i = 1 \right\}$$

**Definition 2.4.** *The sequence $\{x_k\}_{k\in N} \subset \Omega \subset R^n$ is said to converge to a direction $\beta_k \in R^n$, If there exists a real sequence $\{\alpha_k\}$, with $\alpha_k \to +\infty$ such that $\lim_{k\in K} \frac{x_k}{\alpha_k} = \beta$. The vector $\beta \in R^n$ is called the direction of convergence.*

**Definition 2.5.** *(Asymptotic Cone of the Solution set $\Omega$ ). The asymptotic cone of the solution set $\Omega$, denoted by $\Omega_\infty$ is the collection of the vector $\beta \in R^n$ that are limits in the direction of the sequence $\{x_k\}_{k\in N}$ contained in the solution set $S$. i.e.,*

$$\Omega_\infty = \left\{\beta \in R^n : \exists \alpha_k \to +\infty, \exists x_k \in \Omega, \text{ with } \lim_{k\to\infty} \frac{x_k}{\alpha_k} = \beta\right\} \tag{2.1}$$

**Definition 2.6.** *Let the solution set $\Omega$ of the linear optimization problem (1.1) be nonempty and define a set denoted by $\Omega^1_\infty$ as follows:*

$$\Omega^1_\infty = \left\{\beta \in R^n : \forall \alpha_k \to +\infty, \exists x_k \in \Omega, \text{ with } \lim_{k\to\infty} \frac{x_k}{\alpha_k} = \beta\right\} \tag{2.2}$$

**Definition 2.7.** *The solution set $\Omega$ of the linear optimization problem (1.1) is called asymptotically regular, if*

$$\Omega_\infty = \Omega^1_\infty \tag{2.3}$$

**Definition 2.8.** *The normalized set of $\Omega$ ). Let the Solution set $\Omega$ of the linear optimization problem (1.1) be nonempty, then the normalized set of $\Omega$ is denoted as $\Omega_N$, and is defined as*

$$\Omega_N = \left\{\beta \in R^n : \exists \{x_k\} \in \Omega, \|x_k\| \to +\infty, \text{ with } \beta = \lim_{k\to\infty} \frac{x_k}{\|x_k\|}\right\}. \tag{2.4}$$

**Definition 2.9.** *(Support Function of $\Omega$ ). Let the solution set $\Omega$ of the linear optimization problem (1.1) be a nonempty, closed convex set in $R^n$ then the support function of $\Omega$ is a map $\sigma_\Omega(x) : R^n \to \mathrm{R}$ defined by*

$$\sigma_\Omega(x) = \sup\left\{x^T y : y \in \Omega\right\}. \tag{2.5}$$

*If* A *and $B$ are two convex sets in $R^n$. Then $\sigma_A(x) = \sigma_B(x) \Leftrightarrow A = B$.*

**Definition 2.10.** *(The Housdorff distance). The Houdorff distance between two nonempty compact convex sets $A$ and $B$ can be expressed in terms of support functions as follows:*

$$d_H(A, B) = \|\sigma_A - \sigma_B\|_\infty, \text{ where } \|.\| \text{ denotes the uniform norm.} \tag{2.6}$$

**Definition 2.11.** *(The Domain of the support function of $\Omega$ ). The domain of the support function of the solution set $\Omega$ is given as*

$$\mathrm{Dom}\,\sigma_\Omega = \left\{x : \sup_{y\in\Omega} x^T y < \infty\right\} \tag{2.7}$$

**Definition 2.12.** *(The Dual cone of $\Omega_\infty$ ). The Dual cone of $\Omega_\infty$ is the set*

$$\Omega^*_\infty = \left\{y : y^T x \geq 0, \forall x \in \Omega_\infty\right\} \tag{2.8}$$

**Definition 2.13.** *(The Polar cone of $\Omega_\infty$ ). The polar cone of $\Omega_\infty$ is the set*

$$\Omega^p_\infty == \left\{y : y^T x \leq 0, \forall x \in \Omega_\infty\right\} \tag{2.9}$$

**Remark 2.1.** *The polar cone $\Omega^p_\infty$ is just the negative of the polar cone $\Omega^*_\infty$.*

## 3 Main Results

In this section, we will prove some theorems related to the asymptotic cone, asymptotic regularity, and the normalized set of the solution set $\Omega$. Finally, we present a dual characterization of the asymptotic cone $\Omega_\infty$ of the solution set for the linear optimization problem (1.1) in terms of the polar cone and the support function.

**Theorem 3.1.** *The necessary and sufficient condition for the solution set $\Omega$ of the linear optimization problem (1.1) is bounded is that the asymptotic cone of $\Omega$ does not contain any nonzero vector. i. e., if $\Omega_\infty = \{0\}$.*

*Proof.* It is obvious that if the solution set $\Omega$ of the linear optimization problem (1.1) is bounded then there does not exist a direction $\beta \in \Omega_\infty$ with $\beta \neq 0$.

Conversely, suppose, if possible, $\Omega$ is unbounded, and $\Omega_\infty = \{0\}$. As $\Omega$ is unbounded so, $\exists$ a sequence $\{x_k\}$ contained in the solution set $\Omega$, such that $x_k \neq 0$, and $\forall k \in N$ $\alpha_k := \|x_k\| \to \infty$.

Now we have, $\beta_k := {\alpha_k}^{-1} x_k$.

So, $\|\beta_k\| = \left\|\alpha_k^{-1} x_k\right\| = \left\|\alpha_k^{-1}\right\| \|x_k\| = \left\|\frac{\|x_k\|}{\|x_k\|}\right\| = 1$, so the sequence $\{\beta_k\}_{k\in N}$ is bounded. Now using the Bolzano-Weierstrass theorem we can pull a subsequence $\{\beta_k\}_{k\in K}, K \subset N$ , out of this sequence such that $\lim_{k\in K} \beta_k = \beta, K \subset N$, and $\|\beta\| = 1$. Thus $\exists$ a nonzero direction $\beta \in \Omega_\infty$, which contradicts the fact that $\Omega_\infty = \{0\}$. ☐

**Theorem 3.2.** *If the solution set $\Omega \neq \varnothing$, and convex then $\Omega$ is asymptotically regular.*

*Proof.* It is easy to verify that $\Omega$ is a convex set. If $x_1, x_2 \in \Omega$ then both vectors will satisfy the equation $AX = b$ and therefore $Ax_1 = b$, and $Ax_2 = b$, $x_1, x_2 \geq 0$. Now we consider a convex combination of $x_1$ and $x_2$, as $\mu x_1, +(1-\mu)x_2$, with $0 \leq \mu \leq 1$. Clearly $\mu x_1, +(1-\mu)x_2 \geq 0$. and $A\left[\mu x_1, +(1-\mu)x_2\right] = \mu Ax_1 + (1-\mu)Ax_2 = \mu b + (1-\mu)b = b$. So $\Omega$ is a convex set.

Now it follows from the definitions of $\Omega_\infty$ and $\Omega^1_\infty$ that

$$\Omega^1_\infty \subseteq \Omega_\infty. \tag{3.1}$$

Our next goal is to show that $\Omega_\infty \subseteq \Omega^1_\infty$.

Let $\beta \in \Omega$. Then it follows from the definition of $\Omega_\infty$ that there exists a sequence $\{x_k\}_{k\in N} \in \Omega$, and $\exists$ a sequence of real numbers $\{p_k\}_{k\in N}$ such that $p_k \to \infty$, and

$$\beta = \lim_{k\to\infty} p_k^{-1} x_k. \tag{3.2}$$

For, $x \in \Omega$, we define a sequence of directions $\{\beta_k\}_{k\in N} \in R^n$ as

$$\beta_k = p_k^{-1}\left(x_k - x\right). \tag{3.3}$$

Now $\beta_k = p_k^{-1}\left(x_k - x\right) \Rightarrow p_k\beta_k = x_k - x \Rightarrow x_k = x + p_k\beta_k$, As $x_k \in \Omega$, so, $x + p_k\beta_k \in \Omega$, and $\beta = \lim_{k\to\infty} \beta_k$.

Let $\{\delta_k\}_{k\in N}$ be a sequence of real numbers such that $\lim_{k\to\infty} \delta_k = +\infty$.

Now for a fixed natural number $m$, there exists $k(m)$ with

$$\lim_{m\to\infty} k(m) = +\infty, \text{ such that } \delta_m \leq p_{k(m)}. \tag{3.4}$$

As $\Omega$ is convex, we have $x^*_m = x + \delta_m p_{k(m)} \in \Omega$, therefore

$$\beta = \lim_{m\to\infty} \delta_m \beta_{k(m)}. \tag{3.5}$$

This implies that $\beta \in \Omega$, so we have

$$\Omega_\infty \subseteq \Omega^1_\infty. \tag{3.6}$$

Hence, it follows from (3.1) and (3.6) that

$\Omega_\infty = \Omega^1_\infty$.

Thus, the solution set $\Omega$, of the linear optimization problem (1.1) is asymptotically regular. □

**Theorem 3.3.** *Let the solution set $\Omega \neq \emptyset$ and define a normalized set of $\Omega$, as*

$\Omega_N := \left\{\beta \in R^n : \exists \{x_k\} \in \Omega, \|x_k\| \to +\infty, \text{ with } \beta = \lim_{k\to\infty} \frac{x_k}{\|x_k\|}\right\}$.

*Then, $\Omega_\infty = pos\ \Omega_N$, where $pos\ \Omega_N = \{\lambda x : x \in \Omega, \lambda \geq 0\}$ is the positive hull of $\Omega$.*

*Proof.* From the definitions of $\Omega_\infty$ and $\Omega_N$ it follows that

$$\Omega_N \subseteq \Omega_\infty. \tag{3.7}$$

To prove that $\Omega_\infty \subseteq \Omega_N$, let $\beta \in \Omega_\infty$ and $\beta \neq 0$. Then from the definition of $\Omega_\infty$ there exists a real sequence

$\{\alpha_k\}_{k\in N}$ with $\lim_{k\to\infty} \alpha_k = +\infty$. Now for $x_k \in \Omega$, we have

$$\beta = \lim_{m\to\infty}\left[\alpha_k^{-1} x_k\right] = \lim_{m\to\infty}\left[\alpha_k^{-1}\|x_k\|\frac{x_k}{\|x_k\|}\right]. \tag{3.8}$$

Thus, the sequence $\left\{\alpha_k^{-1}\|x_k\|\right\}_{k\in N}$ is a nonnegative bounded sequence, so by Bolzano-Weierstrass theorem $\exists$ a convergent subsequence $\left\{\alpha_k^{-1}\|x_k\|\right\}_{k\in K}, K \subset N$ such that

$$\lim\left[\underset{k\to\infty}{\left[\alpha_k^{-1}\|x_k\|\right]} = \lambda \geq 0\right. \tag{3.9}$$

So, from (3.8) we have

$$\beta = \lim_{m\to\infty}\left[\alpha_k^{-1}\|x_k\|\frac{x_k}{\|x_k\|}\right] = \lim\left[\underset{k\to\infty}{\alpha_k^{-1}\|x_k\|}\right]\lim_{k\to\infty}\frac{x_k}{\|x_k\|} = \lambda\beta_N\ . \tag{3.10}$$

With normalized direction $\beta_N$ and $x \in \Omega$, so $\beta \in \operatorname{pos}\Omega$

$$\Omega_\infty \subseteq \Omega_N \tag{3.11}$$

Therefore, it follows from (3.7) and (3.11) that

$$\Omega_\infty = \operatorname{pos}\Omega_N.$$

□

**Theorem 3.4.** *(Dual Characterization Theorem). If the solution set $\Omega$ of the linear optimization problem is nonempty and $\Omega_\infty$ and $\Omega_\infty^P$ denote the asymptotic cone of $\Omega$ and the polar cone of $\Omega_\infty$ respectively. Then the following relations hold:*

*a) If $\sigma_\Omega$ denotes the support function for the solution set $\Omega$ of the linear optimization problem, dom $\sigma_\Omega \subset \Omega_\infty^P$.*

*b) If the interior of the polar cone $\Omega_\infty^P$ is nonempty, $\Omega_\infty^P \subset$ dom $\sigma_\Omega$.*

*c) For the solution set $\Omega$ of the linear optimization problem, $(\text{ dom } \sigma_\Omega)^P = \Omega_\infty$.*

*Proof.*

a) From the definitions 2.11 and 2.13 of $\operatorname{dom} \sigma_\Omega$ and $\Omega_\infty^P$, it follows that $\operatorname{dom} \sigma_\Omega \cap \Omega_\infty^P \neq \phi$.
Let $y \notin \Omega_\infty^P$. Then from the definition 2.13, $\exists \in \Omega_\infty$ such that $y^T\beta > 0$. As $\beta \in \Omega_\infty$
It follows from the definition of $\Omega_\infty$ that $\exists$ a sequence sequence $\{x_k\}_{k\in N} \subset \Omega$ in $R^n$ and sequence $\{\alpha_k\}_{k\in N}$ in $R$ such that $\alpha_k \to +\infty$, with $\alpha_k^{-1}x_k \to \beta$, and satisfying, the inequality $y^T\beta > 0$. Hence it follows that $y^Tx_k \to +\infty$.
This implies that $y \notin \operatorname{dom} \sigma_\Omega$, so $\operatorname{dom} \sigma_\Omega \subset \Omega_\infty^P$.

b) Let $y \notin$ dom $\sigma_\Omega$. Then $\exists \beta \in \Omega_\infty$ such that $y^T\beta > 0$ and $\beta \neq 0$. As $\beta \in \Omega_\infty$
$\exists$ a sequence sequence $\{x_k\}_{k\in N} \subset$ in $\Omega$ with

$$\left\{x_k^T y\right\}_{k\in N} \to +\infty$$

Considering subsequences, if necessary, without any loss of generality, we can assume that $\frac{x_k}{\|x_k\|} \longrightarrow \beta$, and $\beta \neq 0$, and $\beta \in \Omega_\infty$. Hence it follows that

$$\left(\frac{x_k}{\|x_k\|}\right)^T y \geq 0. \tag{3.12}$$

Hence for $\epsilon > 0$, we have $\beta^T(y + \epsilon\beta) \geq \epsilon\|\beta\|^2$
This implies that $y + \epsilon\beta \notin \Omega_\infty^P$.
That is, $y \notin$ int $\Omega_\infty^P$. Hence it follows that

$$\Omega_\infty^P \subset \operatorname{dom} \sigma_\Omega.$$

c) The set $\Omega$ is a closed convex set in $R^n$, so $\Omega_\infty$ is a closed convex cone then it follows from the definition of the polar cone that

$$\left(\Omega_\infty^P\right)^P = \Omega_\infty. \tag{3.13}$$

Now from (a) we have

$$\operatorname{dom} \sigma_\Omega \subset \Omega_\infty^P. \tag{3.14}$$

This implies that $\left(\Omega_\infty^P\right)^P \subset (\operatorname{dom} \sigma_\Omega)^P$. Using equation (3.13) we have

$$\Omega_\infty \subset (\operatorname{dom} \sigma_\Omega)^P. \tag{3.15}$$

Now in order to prove that $(\text{ dom } \sigma_\Omega)^P \subset \Omega_\infty$, suppose that $\beta \in (\operatorname{dom} \sigma_\Omega)^P$, for a real number $\alpha > 0$ and an arbitrary point $\bar{x}$ in $\Omega$, $\alpha\beta \in (\operatorname{dom} \sigma_\Omega)^P$, so for an arbitrary $y \in \operatorname{dom} \sigma_\Omega$, we have

$$\begin{aligned}(\bar{x} + \alpha\beta)^T y &= \bar{x}^T y + (\alpha\beta)^T y \\ &\leq \bar{x}^T y \\ &\leq \sup\left\{x^T y : x \in \Omega\right\} \\ &= \sigma_\Omega(y)\end{aligned} \tag{3.16}$$

Thus, for an arbitrary $y \notin dom\sigma_\Omega$, we have

$$\sigma_\Omega(y) = +\infty.$$

The inequality (3.15) remains valid $\forall y$ in $R^n$.
Therefore, $\forall \alpha > 0, \bar{x} + \alpha\beta \in \Omega$, where $\operatorname{cl} \Omega$ denotes the closure of the solution set $\Omega$.
We know that for any convex set $\Omega$ in $R^n, \Omega_\infty$ is a closed convex cone and

$$\Omega_\infty = D = \left\{\beta \in R^n : \mathrm{x} + \alpha\beta \in \operatorname{cl} \Omega, \forall \alpha > 0, \text{ and } \forall x \in \Omega\right\}. \tag{3.17}$$

Thus, $\beta \in \Omega_\infty$, and

$$(\operatorname{dom}\sigma_\Omega)^P \subset \Omega_\infty. \tag{3.18}$$

Now it follows from (3.15) and (3.17) that

$$(\operatorname{dom}\sigma_\Omega)^P = \Omega_\infty$$

Further details of the asymptotic properties of the sets and the functions can be referred to [1,2,4,5,9,10,11,12,14,16,20,21,22,23,25,26,27,31,35,36].

□

## 4 Concluding Remarks

The concept of the asymptotic cone is enormously useful in the study of the behavior of both convex and nonconvex sets. For example, in [26] Luc and Penot have investigated various properties of the asymptotic directions of unbounded sets to examine the perturbation of the data. Petrovai [29] has investigated the notion of asymptotic convergence which is extremely useful for the algorithms dealing with nonlinear mathematical programming Problems. The fundamental problem of linear optimization is to arrive at the best possible decision in any given set of circumstances when the functions to be optimized and the constraints are both linear. These days linear optimization is one of the most frequently used decisionmaking tools in the industry, administration, banking, finance, marketing, and various other spheres of life. A desirable property of an algorithm for solving a linear optimization problem is that it generates a well-defined solution at each iteration of the algorithm and its solution set remains bounded all the time. However, in several situations, the sequence of iterates may not remain bounded, and consequently, we get an unbounded solution set. The results obtained in this paper, together with the BolzanoWeierstrass theorem, and the notion of asymptotic convergence will be useful to deal with the unbounded solution sets of mathematical optimization problems. The results of this paper can be extended for the solution sets of the other conic optimization problems like semidefinite programming (SDP) and second-order cone programming (SOCP). These results can help to obtain some characterization results for the asymptotic cones of the solution sets for SDP and SOCP. The notion of asymptotic, polar cones and asymptotic regularity plays a considerable role in various disciplines of Mathematical Sciences.

The various applications of the asymptotic cones, polar cones, dual cones, and associated asymptotic functions in various areas of mathematical sciences can be referred to [4,5,9,10,11, 12,14,16,20,21,22,23,27,31,35,36], and the references cited in these papers.

**Acknowledgement.** We are very much grateful to the Editor and Reviewer for their valuable suggestions for improving the paper in its present form.

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