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ON A CLASS OF SLANT WEIGHTED TOEPLITZ OPERATORS

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Abstract

If $\beta = \langle \beta_n \rangle_{n\in\mathbb{Z}}$ is a sequence of positive numbers with $\beta_0 = 1$, then a slant weighted Toeplitz operator A_ϕ is an operator on $L^2(\beta)$ defined as $A_\phi = WM_\phi$ where M_ϕ is the multiplication operator on $L^2(\beta)$ given by $M_\phi e_k(z) = \frac{1}{\beta_k}\sum_{n=-\infty}^{\infty} a_n \beta_{n+k} e_{n+k}(z)$. In this paper we investigate the closure of the set of these operators. We also discuss the C^*-algebra generated by a particular class of slant weighted Toeplitz operators and obtain the spectral radius for this class.

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1 Introduction and preliminaries

Toeplitz operators were introduced by Toeplitz [22] in the year 1911. Subsequently many mathematicians came up with different generalizations of the Toeplitz operators. In 1995, Ho [9] introduced the class of slant Toeplitz operators having the property that the matrices with respect to the standard orthonormal basis could be obtained by eliminating every alternate row of the matrices of the corresponding Toeplitz operators. These operators arise in plenty of applications like prediction theory [3], wavelet analysis [4], signal processing [17, 18, 19], and solution of differential equations [5]. However, these studies were made in the context of the usual Hardy spaces H^2 and H^p and the Lorentz spaces L^2 and L^p. Meanwhile the notion of the weighted sequence spaces $H^2(\beta)$ and $L^2(\beta)$ came up. A systematic study of the shift operator and the multiplication operator on $L^2(\beta)$ was made by Shields [20]. Lauric [13] studied particular cases of Toeplitz operators on $H^2(\beta)$.

Motivated by the increasing popularity of the spaces $L^2(\beta)$ and $H^2(\beta)$ and the diverse applications of the slant Toeplitz operators, we introduced and studied the notion of a weighted Toeplitz operator [1] and a slant weighted Toeplitz operator [2]. We also explored the properties of the k-th order slant weighted Toeplitz operator [3] and those of its compression on $H^2(\beta)$ [4]. Subsequently, others have studied the commutativity [5] and hyponormality [10] of these operators. Several approximations of related signals functions have also been explored [15] and [16] in Banach spaces and fuzzy normed spaces [14]. The essentially slant weighted Toeplitz operators and their generalisations have been studied by Gupta and Singh [7]. Amongst the recent advances in this direction is the study of a slant weighted Toeplitz operator in Calkin Algebra by Datt and Ohri [6]. The minimal reducing subspaces of the compression of a slant weighted Toeplitz operator have been explored by Hazarika [11]. The study of weighted Toeplitz operators and that of slant weighted Toeplitz operators is of interest to physicists, probalists and computer scientists. In this paper we study a particular class of the slant weighted Toeplitz operator and determine the spectral radius for it. We begin with the following preliminaries. Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$ and $0 < \frac{\beta_n}{\beta_{n+1}} \leq 1$ for every $n \geq 0$, $0 < \frac{\beta_n}{\beta_{n-1}} \leq 1$ for every $n \leq 0$. We also assume that $\frac{\beta_{2n}}{\beta_n} \leq M < \infty$. Consider the spaces [20]

$$L^2(\beta) = \left\{ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \,\middle|\, a_n \in \mathbb{C}, \|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\},$$

and [13]

$$H^2(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \,\middle|\, a_n \in \mathbb{C}, \|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}.$$

Then $(L^2(\beta), \|\cdot\|_\beta)$ is a Hilbert space [13] with an orthonormal basis given by $\left\{ e_k(z) = \dfrac{z^k}{\beta_k} \right\}_{k\in\mathbb{Z}}$ and with an inner product defined by

$$\left\langle \sum_{n=-\infty}^{\infty} a_n z^n, \sum_{n=-\infty}^{\infty} b_n z^n \right\rangle = \sum_{n=-\infty}^{\infty} a_n \bar{b}_n \beta_n^2 .$$

Further, $H^2(\beta)$ is a subspace of $L^2(\beta)$. Now, let

$L^\infty(\beta) = \{\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \,\big|\, \phi L^2(\beta) \subseteq L^2(\beta)$ and $\exists\, c \in \mathbb{R}$ such that $\|\phi f\|_\beta \le c\|f\|_\beta$ for all $f \in L^2(\beta)\}$.

Then, $L^2(\beta)$ is a Banach space with respect to the norm defined by

$$\|\phi\|_\infty = \inf \left\{ c \,\middle|\, \|\phi f\|_\beta \le c\|f\|_\beta \text{ for all } f \in L^2(\beta) \right\}.$$

Let $P : L^2(\beta) \to H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$. Let $\phi \in L^\infty(\beta)$, then the weighted multiplication operator [20] with symbol ϕ, that is $M_\phi : L^2(\beta) \to L^2(\beta)$ is given by $M_\phi e_k(z) = \dfrac{1}{\beta_k} \sum_{n=-\infty}^{\infty} a_n \beta_{n+k} e_{n+k}(z)$.

If we put $\phi_1(z) = z$, then $M_{\phi_1} = M_z$ is the operator defined as $M_z e_k(z) = w_k e_{k+1}(z)$, where $w_k = \dfrac{\beta_{k+1}}{\beta_k}$ for all $k \in \mathbb{Z}$, and is known as a weighted shift [20].

Further, the weighted Toeplitz operator T_ϕ [13] on $H^2(\beta)$ is defined as $T_\phi(f) = P(\phi f)$.

This mapping is well defined, for, if $f \in H^2(\beta) \subset L^2(\beta)$, then by definition, $\phi f \in L^2(\beta)$ and hence $P(\phi f) \in H^2(\beta)$.

The matrix of T_ϕ is:

$$\begin{bmatrix} a_0\dfrac{\beta_0}{\beta_0} & a_{-1}\dfrac{\beta_0}{\beta_1} & a_{-2}\dfrac{\beta_0}{\beta_2} & \dots \\ a_1\dfrac{\beta_1}{\beta_0} & a_0\dfrac{\beta_1}{\beta_1} & a_{-1}\dfrac{\beta_1}{\beta_2} & \dots \\ a_2\dfrac{\beta_2}{\beta_0} & a_1\dfrac{\beta_2}{\beta_1} & a_0\dfrac{\beta_2}{\beta_2} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}.$$

Hence the effect of T_ϕ on the orthonormal basis can be described by

$$T_\phi e_k(z) = \frac{1}{\beta_k} \sum_{n=0}^{\infty} a_{n-k} \beta_n e_n(z).$$

2 Slant Weighted Toeplitz Operator

Let $\phi \in L^\infty(\beta)$.

Definition 2.1 ([2])**.** *The slant weighted operator A_ϕ is an operator on $L^2(\beta)$ defined as $A_\phi : L^2(\beta) \to L^2(\beta)$ such that*

$$A_\phi e_k(z) = \frac{1}{\beta_k} \sum_{n=-\infty}^{\infty} a_{2n-k} \beta_n e_n(z).$$

If $W : L^2(\beta) \to L^2(\beta)$ such that

$$W e_{2n}(z) = \frac{\beta_n}{\beta_{2n}} e_n(z),$$

and

$$W e_{2n-1}(z) = 0 \quad \text{for all } \; n \in \mathbb{Z},$$

then an alternate definition of A_ϕ is given by

$$A_\phi(f) = WM_\phi(f) = W(\phi f) \quad \textit{for all} \quad f \in L^2(\beta).$$

Clearly, $W = A_1$. In [2] we have shown that

$$M_z W = W M_{z^2} \tag{2.1}$$

$$M_{\phi(z)} W = A_{\phi(z^2)} = W M_{\phi(z^2)} \tag{2.2}$$

$$\langle A_\phi e_{j+2}, e_{i+1} \rangle = \frac{w_i}{w_j w_{j+1}} \langle A_\phi e_j, e_i \rangle. \tag{2.3}$$

Now, let S denote the shift operator on $L^2(\beta)$ given by $Se_j = \dfrac{1}{w_j}\, e_{j+1}$.

Then $S^* e_j = \dfrac{1}{w_{j-1}}\, e_{j-1}$. Also, S is bounded as $\langle w_n \rangle$ is positive and bounded.

Lemma 2.1. $S^* = M_z^{-1}$.

Proof.

$$\begin{aligned} S^* M_z e_j &= S^* w_j e_{j+1} \\ &= \frac{w_j}{w_j}\, e_j = e_j, \quad j = 0, \pm 1, \pm 2 \dots . \end{aligned}$$

□

We now use Lemma 2.1 and equation (2.3) to prove the following:

Theorem 2.1. *A bounded operator A on $L^2(\beta)$ is a slant weighted Toeplitz operator on $L^2(\beta)$ if and only if $A = M_z^{-1} A M_{z^2}$ where M_z and M_{z^2} are the weighted multiplication operators an $L^2(\beta)$ induced by z and z^2 respectively.*

Proof. Let A be a slant weighted Toeplilz operator on $L^2(\beta)$. Then from equation (2.3) we get that

$$\begin{aligned} \langle Ae_j, e_i \rangle &= \frac{w_j w_{j+1}}{w_i} \langle Ae_{j+2}, e_{i+1} \rangle \\ &= \langle AM_{z^2} e_j, S_{e_i} \rangle \\ &= \langle S^* AM_{z^2} e_j, e_i \rangle \\ &= \langle M_z^{-1} AM_{z^2} e_j, e_i \rangle \qquad i, j = 0, \pm 1, \pm 2, \dots . \end{aligned}$$

Hence $A = M_z^{-1} A M_{z^2}$.

Conversely, let A be a bounded operator on $L^2(\beta)$ such that $A = M_z^{-1} A M_{z^2}$. Then, for all $i, j = 0, \pm 1, \pm 2, \dots$ we have

$$\begin{aligned} \langle Ae_j, e_i \rangle &= \langle M_z^{-1} AM_{z^2} e_j, e_i \rangle \\ &= \langle S^* AM_{z^2} e_j e_i \rangle \\ &= \langle AM_{z^2} e_j, Se_i \rangle \\ &= \frac{w_j w_{j+1}}{w_i} \langle Ae_{j+2}, e_i \rangle. \end{aligned}$$

In [2] we have proved that the necessary and sufficient condition for a bounded operator A on $L^2(\beta)$ to be a slant weighted Toeplitz operator is that its matrix entries satisfy equation (2.3). Hence we may conclude that A is a slant weighted Toeplitz operator. □

Corollary 2.1. *A bounded operator A on $L^2(\beta)$is a slant weighted Toeplitz operator on $L^2(\beta)$ if and only if $A = S^* A M_{z^2}$*

3 C^*-Algebra of Slant Weighted Toeplitz Operators

Let $L^2(\beta)$ be a given space. Let $\mathcal{A}$ denote the set of all slant weighted Toeplitz operators on $L^2(\beta)$.

Theorem 3.1. *$\mathcal{A}$ is weakly closed and hence strongly closed.*

Proof. Let A_n be a sequence of slant weighted Toeplitz operators such that $\langle A_n f, g\rangle \to \langle Af, g\rangle$ for all $f, g \in L^2(\beta)$. Then $A_n = M_z^{-1} A_n M_{z^2}$ for all n.

Therefore, as $n \to \infty$, we have

$$\begin{aligned}\langle A_n f, g\rangle &= \langle M_z^{-1} A_n M_{z^2} f, g\rangle \\ &= \langle S^* A_n M_{z^2} f, g\rangle \\ &= \langle A_n M_{z^2} f, S_g\rangle \\ &\to \langle A M_{z^2} f, S_g\rangle \\ &= \langle S^* A M_{z^2} f, g\rangle \\ &= \langle M_z^{-1} A M_{z^2} f, g\rangle.\end{aligned}$$

Thus $M_z^{-1} A_n M_{z^2} \to M_z^{-1} A M_{z^2}$ weakly. Hence $A = M_z^{-1} A M_{z^2}$. Hence from Theorem 2.1, A is a slant weighted Toeplitz operator. □

Next, to study the C^*-algebra generated by slant weighted Toeplitz operators and to obtain the spectral radius of A_ϕ, we impose a restriction on the sequence $\langle \beta_n\rangle$. Hence forth we consider only those sequences $\langle \beta_n\rangle_{n\in\mathbb{Z}}$ such that

$$\left.\begin{aligned}\beta_n &= \alpha^n && \text{when } n \geq 0,\\ \beta_n &= \alpha^{-n} && \text{when } n < 0.\end{aligned}\right\} \quad \text{for } 1 < \alpha < \infty.$$

Then the weight sequence $\left\langle w_n = \dfrac{\beta_{n+1}}{\beta_n}\right\rangle$ is of the form

$$\begin{aligned} w_n &= \alpha \quad \text{for } n > 0,\\ w_n &= \frac{1}{\alpha} \quad \text{for } n \leq 0.\end{aligned}$$

In that case, the matrix of M_ϕ becomes

$$\left[\begin{array}{c|cccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \cdots & a_0 & a_{-1}\alpha & a_{-2}\alpha^2 & a_{-3}\alpha & a_{-4} & \cdots \\ \cdots & \dfrac{a_1}{\alpha} & a_0 & a_{-1}\alpha & a_{-2} & \dfrac{a_{-3}}{\alpha} & \cdots \\ \cdots & \dfrac{a_2}{\alpha^2} & \dfrac{a_1}{\alpha} & a_0 & \dfrac{a_{-1}}{\alpha} & \dfrac{a_{-2}}{\alpha^2} & \cdots \\ \cdots & \dfrac{a_3}{\alpha} & a_2 & a_1\alpha & a_0 & \dfrac{a_{-1}}{\alpha} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right].$$

Hence the matrix of M_ϕ^* is given by

$$\left[\begin{array}{c|ccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \cdots & \bar{a}_0 & \dfrac{\bar{a}_1}{\alpha} & \dfrac{\bar{a}_2}{\alpha^2} & \dfrac{\bar{a}_3}{\alpha} & \cdots \\ \cdots & \bar{a}_{-1}\alpha & \bar{a}_0 & \dfrac{\bar{a}_1}{\alpha} & \bar{a}_2 & \cdots \\ \cdots & \bar{a}_{-2}\alpha^2 & \bar{a}_{-1}\alpha & \bar{a}_0 & \bar{a}_1\alpha & \cdots \\ \cdots & \bar{a}_{-3}\alpha & \bar{a}_{-2} & \dfrac{\bar{a}_{-1}}{\alpha} & \bar{a}_0 & \cdots \\ \cdots & \bar{a}_{-4} & \dfrac{\bar{a}_{-3}}{\alpha} & \dfrac{\bar{a}_{-2}}{\alpha^2} & \dfrac{\bar{a}_{-1}}{\alpha} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right].$$

It is observed that the matrix entries $\langle \lambda_{ij}\rangle$ of M_ϕ^* satisfy the relation

$$\lambda_{i+1,j+1} = \frac{w_i}{w_j}\lambda_{i,j}\,. \tag{3.1}$$

We have proved in [1] that equation (3.1) is the necessary and sufficient condition for the corresponding operator to be a weighted multiplication operator.

Hence M_ϕ^* is also a weighted multiplication operator. Further, the product of two weighted multiplication operators is also a weighted multiplication operator [1]. Hence we get that

$$M_\phi M_\phi^* = M_\psi \quad \text{for some} \quad \psi \in L^\infty(\beta). \tag{3.2}$$

We suppose that $\psi = \sum_{n=-\infty}^{\infty} b_n z^n$.

Theorem 3.2. *$A_\psi W^*$ is a weighted multiplication operator.*

Proof. For each $k \in \mathbb{Z}$, consider

$$\begin{aligned} A_\psi W^* e_k(z) &= \frac{\beta_k}{\beta_{2k}} A_\psi e_{2k}(z) \\ &= \frac{\beta_k}{\beta_{2k}} \frac{1}{\beta_{2k}} \sum_{n=-\infty}^{\infty} b_{2n-2k} \beta_n e_n(z) \\ &= \frac{1}{\beta_k} \sum_{n=-\infty}^{\infty} b_{2(n-k)} \frac{\beta_k^2}{\beta_{2k}^2} \beta_n e_n(z) \\ &= M_{\theta_k} e_k(z), \end{aligned}$$

where

$$\theta_k(z) = \sum_{n=-\infty}^{\infty} \left(b_{2n} \frac{\beta_k^2}{\beta_{2k}^2} \right) z^n \quad \text{is in} \quad L^\infty(\beta).$$

We therefore conclude that

$$A_\psi W^* = M_{\theta_k}. \tag{3.3}$$

Hence the theorem. □

Corollary 3.1. $A_\phi A_\phi^* = M_{\theta_k}$.

Proof.

$$\begin{aligned} A_\phi A_\phi^* &= W M_\phi M_\phi^* W^* \\ &= W M_\psi W^* && \text{using (3.2)} \\ &= A_\psi W^* \\ &= M_{\theta_k} && \text{using (3.3)} \end{aligned}$$

Finally $A_\phi A_\phi^* = M_{\theta_k}$. □

We now prove the main result of this paper:

Let $\mathcal{A}$ denote the C^*-algebra generated by all slant weighted Toeplitz operators A_ϕ on $L^2(\beta)$ with the sequence $\langle \beta_n \rangle$ discussed in this section.

Also, let $\mathcal{M}$ denote the C^*-algebra generated by all weighted multiplication operators on $L^2(\beta)$. We have proved in [2] that W does not commute with M_z. We now prove the following:

Lemma 3.1. *W commutes with the multiplication operator M_ψ if and only if $\psi =$ constant.*

Proof. Let $\psi \in L^\infty(\beta)$ be a constant. Then $M_\psi W = \alpha$ W for some constant α. Therefore

$$\begin{aligned} M_\psi W e_{2n}(z) &= \alpha W e_{2n}(z) \\ &= \alpha \frac{\beta_n}{\beta_{2n}} e_n(z) \\ &= W \alpha e_{2n}(z) \\ &= W M_\psi e_{2n}(z). \end{aligned} \tag{3.4}$$

Further

$$\begin{aligned} M_\psi W e_{2n-1}(z) &= M_\psi 0 \\ &= 0 = W M_\psi e_{2n-1}(z). \end{aligned}$$

Thus

$$M_\psi W e_n(z) = W M_\psi e_n(z), \quad n = 0, \pm 1, \pm 2 \dots .$$

Conversely, suppose that $M_\psi W = W M_\psi$ for some

$$\psi = \sum_{i=-\infty}^{\infty} b_i z^i \in L^\infty(\beta).$$

From equations (2.1) and (2.2) we infer that $b_i = 0$ for all $i \neq 0$. So, $\psi = b_0 =$ constant. Hence the result. □

Theorem 3.3. $\mathcal{A}' = (I)$.

Proof. Consider the equation $A_\phi A_\phi^* = M_{\theta_k}$. This suggests that every weighted multiplication operator M_{θ_k} can be written as the product of some slant weighted Toeplitz operator A_ϕ and its adjoint A_ϕ^*. Hence $\mathcal{M} \subseteq \mathcal{A}$. We know that $\mathcal{M}$ is maximal abelian [20]. Hence $\mathcal{A}' \subseteq \mathcal{M}' = \mathcal{M}$, where $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$. Hence for a given $B \in \mathcal{A}'$ we get $B \in \mathcal{M}$. That is $B = M_\psi$ for some $\psi \in L^\infty(\beta)$. Also, $W = A_1$. Hence $(W) \subseteq \mathcal{A}$. Therefore $\mathcal{A}' \subseteq (W')$.

This implies that $B = M_\psi$ commutes with W. From the above lemma we get that $\psi =$ constant, and this is true for an arbitrary operator $B \in \mathcal{A}'$, Hence we get that $\mathcal{A}' = (I)$. □

As another consequence of Corollary 3.3, we now derive the spectral radius for a slant weighted Toeplitz operator belonging to this class. For this, we use the spectral radius formula $r(T) = \lim_{n\to\infty} (\|T^n\|)^{1/n}$ and proceed as follows.

Theorem 3.4. $r(A_\phi) = \lim_{n\to\infty} (\|\theta_n\|_\infty)^{1/2}$.

Proof. We know that $A_\phi {A_\phi}^* = M_{\theta_k}$. Taking norm on both sides we get

$$\|A_\phi A_\phi^*\| = \|M_{\theta_k}\| = \|\theta_k\|_\infty.$$

So,

$$\begin{aligned}\|A_\phi\|^2 &= \|\theta_k\|_\infty \\ \|A_\phi\| &= \sqrt{\|\theta_k\|_\infty} = (\|\theta_k\|_\infty)^{1/2}.\end{aligned}$$

Now

$$\begin{aligned}A_\phi^2 A_\phi^{*2} &= W M_\phi W M_\phi M_\phi^* W^* M_\phi^* W^* \\ &= W M_\phi W M_\psi W^* M_\phi^* W^* \\ &= W M_\phi A_\psi W^* M_\phi^* W^* \\ &= W M_\phi M_{\theta_k} M_\phi^* W^* \\ &= W M_{\phi_2} W^* \qquad \text{where} \quad M_{\phi_2} = M_\phi M_{\theta_k} M_\phi^* \\ &= A_{\phi_2} W^* \\ &= M_{\theta_2} \quad \text{(say)}.\end{aligned}$$

Proceeding in this manner, we can show that for each n, $A_\phi^n A_\phi^{*n}$ is a multiplication operator M_{θ_n}. Hence

$$\|A_\phi^n\|^2 = \|A_\phi^n A_\phi^{*n}\| = \|M_{\theta_n}\| = \|\theta_n\|_\infty.$$

Finally,

$$\begin{aligned}r(A_\phi) &= \lim_{n\to\infty} (\|A_\phi^n\|)^{1/n} \\ &= \lim_{n\to\infty} (\|\theta_n\|_\infty)^{1/2n}.\end{aligned}$$

□

4 Conclusion

In this paper we have proved that the set of all slant weighted Toeplitz operators on $L^2(\beta)$ is weakly closed and hence strongly closed. By considering a sequence of the type $\langle\beta_n\rangle_{n\in\mathbb{Z}}$ such that $\beta_n = \alpha^n$ when $n \geq 0$ and $\beta_n = \alpha^{-n}$ when $n < 0$ we have shown that $M_\phi M_\phi^*$ is also a weighted multiplication operator. Further, for such a sequence, every weighted multiplication operator can be written as the product of some slant weighted Toeplitz operator and its adjoint.

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