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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

FIXED POINT RESULTS IN ORDERED EXTENDED RECTANGULAR $b-$ METRIC SPACE WITH GERAGHTY-WEAK CONTRACTION

Mohammad Asim and Rachna Rathee
Department of Mathematics, Faculty of Science, SGT University, Gurugram (Haryana), India-122505.
Email: mailtoasim27@gmail.com, rachnarathee81@gmail.com

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Abstract

In this paper, we prove some ordered-theoretic fixed point results for a Geraghty-weak contraction on an ordered extended rectangular $b-$metric spaces. Our results generalize several core results of the existing literature especially involving Geraghty-weak contractions and the results proved in extended rectangular $b-$metric space. Some examples are also furnished to exhibits the utility of our main results.
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1 Introduction

In 1992, Banach [10] introduced the classical fixed point theorem which is known as Banach contraction principle. The concept of generalized metric space has been increased by adding new generalized metrics one after another. The class of $b-$metric spaces [12] is generalized by the classes of extended $b-$metric spaces [21] as well as rectangular $b-$metric spaces [15] and so on. Now days, it is not only the metric spaces that are generalized by time to time but mappings are also. For example contraction mapping is generalized by weak contractions Geraghty contractions [14] and many others. The importance of fixed theory is also increasing day by day. In 2008, George et al. [15] introduced rectangular $b-$metric with the combination of rectangular and $b-$metric. In 2019, Asim et al. [4] introduced extended rectangular $b-$metric space and prove some fixed points. Recently in 2021, sharma and Tiwari [31] established some fixed-point theorems for three functions on contraction and expansive mappings in rectangular $b-$metric spaces. Also, very recently in 2022, Joshi [22] established some common fixed-point theorems for generalized multi-valued contraction in $b-$metric and dislocated $b-$metric spaces. Now, we apply the concept of ordered on extended rectangular $b-$metric space by using the mapping Geraghty-weak contraction. We recall the Definition of extended rectangular $b-$metric space as follow:

2 Preliminaries

Definition 2.1. ([4]). *Let U be non-empty set. Also $\theta : U\times U \to [1,\infty)$. Let a mapping $r_\theta : U\times U \to \mathbb{R}^+$ will be extended rectangular $b-$metric on U if it satisfy following properties ($\forall\ u, v \in U$ and $a, b \in U \setminus \{a, b\},\ a \neq b$):*

(a) $r_\theta(u, v) = 0 \iff u = v$,
(b) $r_\theta(u, v) = r_\theta(v, u)$,
(c) $r_\theta(u, v) \leq \theta(u, v)[r_\theta(u, a) + r_\theta(a, b) + r_\theta(b, v)]$.

The pair (U, r_θ) is said to be extended rectangular $b-$metric space.

Example 2.1. ([4]). Let $U = \{1, 2, 3, 4, 5\}$. A mapping $\theta : U \times U \to [1,\infty)$ such that $\theta(u, v) = u + v + 1\ \forall\ u, v \in U$. Also $r_\theta : U \times U \to \mathbb{R}^+$. Now, we can see that '$r_\theta$' is an extended $b-$metric space.

Definition 2.2. ([4]). *Let (U, r_θ) be an extended rectangular $b-$metric space and consider a sequence $\{u_n\}$ of U. We say that*

(a) *$\{u_n\}$ is said to be Cauchy if for each $\epsilon > 0$ there exists a natural number N such that $r_\theta(u_n, u_m) < \epsilon\ \forall\ n > m > N$.*
(b) *$\{u_n\}$ is said to be convergent if for each $\epsilon > 0$ there exists a natural number N such that $r_\theta(u_n, u) < \epsilon\ \forall\ n > N$.*

(c) (U, r_θ) *is said to be complete if every Cauchy sequence is convergent in* (U, r_θ).

Remark 2.1 ([4]). *If we replace the function 'θ' with a variable $s \geq 1$ then it will result in rectangular $b-$metric space. We can conclude that extended rectangular $b-$metric space $\Longrightarrow$ rectangular $b-$metric space. Before chalking out our main result, we give the following definitions, notations and results for the setting of ordered relation in the framework of extended rectangular $b-$metric spaces. So, lets move to the some definition which is needed in our forthcoming discussion.*

Definition 2.3. *Let $(U, \preceq)$ be an ordered set and (U, r_θ) an extended rectangular $b-$metric space. Then a triplet $(U, r_\theta, \preceq)$ is called an ordered extended rectangular $b-$metric space.*

Definition 2.4. *Let $(U, r_\theta, \preceq)$ be an ordered extended rectangular $b-$metric space. Let T be a self-mapping. Then*

(a) *$(U, r_\theta, \preceq)$ is said to follow the property increasing-convergence-comparable (in short ICC-property) if each terms of $\{u_{n_k}\}$, any subsequence of increasing convergent sequence $\{u_n\}$ in U is comparable with the limit of $\{u_n\}$. In other words,*
$u_n \uparrow u$, there exists $\{u_{n_k}\}$ a subsequence of $\{u_n\}$ also $u_n \prec\succ u \;\forall\; k \in \mathbb{N}$.

(b) *$(U, r_\theta, \preceq)$ is said to follow the property of decreasing-convergence-comparable (in short DCC-property) if each terms of $\{u_{n_k}\}$, any subsequence of decreasing convergent sequence $\{u_n\}$ in U is comparable with the limit of $\{u_n\}$. In other words, $u_n \downarrow u$, there exists $\{u_{n_k}\}$ a subsequence of $\{u_n\}$ also $u_n \prec\succ u \;\forall\; k \in \mathbb{N}$.*

(c) *$(U, r_\theta, \preceq)$ is said to the property of follow monotone-convergence-comparable (in short MCC-property) if each terms of $\{u_{n_k}\}$, any subsequence of monotone convergent sequence $\{u_n\}$ in U is comparable with the limit of $\{u_n\}$. In other words, $u_n \uparrow\downarrow u$, there exists $\{u_{n_k}\}$ a subsequence of $\{u_n\}$ also $u_n \prec\succ u \;\forall\; k \in \mathbb{N}$.*

Definition 2.5. *Let $(U, r_\theta, \preceq)$ be an ordered extended rectangular $b-$metric space and T be a self-mapping on U. Then T is called $\overline{O} - r_\theta$-continuous (resp. $\underline{O} - r_\theta$-continuous, $O - r_\theta$-continuous) at point $u \in U$ if $T(u_n) \xrightarrow{r_\theta} T(u)$ $u_n \uparrow u$ (resp. $u_n \downarrow u, u_n \uparrow\downarrow u$) for any sequence $\{u_n\} \subset U$. Also, T is said to be $\overline{O} - r_\theta$-continuous (resp. $\underline{O} - r_\theta$-continuous, $O - r_\theta$-continuous) if T is $\overline{O} - r_\theta$-continuous (resp. $\underline{O} - r_\theta$-continuous, $O - r_\theta$-continuous) at each point of U.*

Remark 2.2. *In $(U, r_\theta, \preceq)$, continuity $\Longrightarrow O - r_\theta$-continuity $\Longrightarrow \overline{O} - r_\theta$-continuity also $\underline{O} - r_\theta$-continuity.*

Definition 2.6. *Let $\{u_n\}$ be a sequence in $(U, r_\theta, \preceq)$. Then $\{u_n\}$ will be $\overline{O} - r_\theta$-Cauchy (resp. $\underline{O} - r_\theta$-Cauchy, $O - r_\theta$-Cauchy) at point $u \in U$ if $\{u_n\}$ is an increasing sequence (resp. decreasing and monotone) and r_θ-Cauchy. Moreover, $\{u_n\}$ is called $\overline{O} - r_\theta$-convergent (resp. $\underline{O} - r_\theta$-convergent, $O - r_\theta$-convergent) at point $u \in U$ if $\{u_n\}$ is an increasing (resp. decreasing and monotone) r_θ-convergent sequence, abbreviated by $u_n \uparrow u$ (resp. $u_n \downarrow u, u_n \uparrow\downarrow u$).*

Definition 2.7. *Let $\{u_n\}$ be any sequence in $(U, r_\theta, \preceq)$. Then $\{u_n\}$ is said to be $\overline{O} - r_\theta$-complete (resp. $\underline{O} - r_\theta$-complete, $O - r_\theta$-complete) at point $u \in U$ if each $\overline{O} - r_\theta$-Cauchy (resp. $\underline{O} - r_\theta$-Cauchy, $O - r_\theta$-Cauchy) sequence in U if it converges to any point $u \in U$.*

Remark 2.3. *In ordered extended rectangular $b-$metric space, completeness $\Longrightarrow O - r_\theta$-completeness $\Longrightarrow \overline{O} - r_\theta$-completeness also $\underline{O} - r_\theta$-completeness.*
Now, we have all the definition regarding the topic in our minds. The first classic fixed point theory was given by S. Banach [10] known as Banach contraction principle. But as few decades passed away, it has been generalized number of ways one of them is Geraghty-weak contraction. Geraghty principle came into existence in 1973 when Geraghty generalized the Banach contraction principle. Later, in 2016 Roshan et al. by using Geraghty-weak contraction proved fixed point results in the b-metric space. Also in 2021, fixed point results in ordered partial rectangular $b-$metric space was proved by Asim et al. [5] with Geraghty-weak contraction theory. Now a day, many researchers are utilizing of this mapping in their research. In this chapter we are using Geraghty-weak contraction principle to prove fixed point results for ordered extended rectangular $b-$metric space by employing suitable conditions.

3 Main Results

Definition 3.1. *Let $\lambda : [0,\infty) \to [0, \frac{1}{\theta})$ $(\theta : U\times U \to [1,\infty))$ which satisfy the given condition for $u_n \in [0,\infty)$, any sequence:*

$$\lim_{n\to\infty} \sup \lambda(u_n) = \frac{1}{\theta} \Longrightarrow \lim_{n\to\infty}(u_n) = 0.$$

The collection of such functions of λ is denoted by Λ.

Definition 3.2. *Suppose $(U, r_\theta, \preceq)$ is an ordered extended rectangular $b-$metric space. Let T be a self-mapping, is called Geraghty-weak contraction if $\exists \ \lambda \in \Lambda$ we have $u \preceq v \ \ \forall\, u, v \in U)$ such that*

$$r_\theta(T(u, T(v))) \le \lambda(r_\theta(u,v))M(r_\theta(u,v)), \tag{3.1}$$

and

$$M(r_\theta(u,v)) = \max\left\{(r_\theta(u,v)), \frac{r_\theta(u,T(u))r_\theta(v,T(v))}{1+r_\theta(T(u),T(v))}, \frac{r_\theta(u,T(u))r_\theta(v,T(v))}{1+r_\theta(u,v)}, \right.$$
$$\left. \frac{r_\theta(u,T(u))r_\theta(u,T(v))}{1+r_\theta(u,T(v))+r_\theta(v,T(u))}\right\}.$$

Theorem 3.1. *Let $(U, r_\theta, \preceq)$ be an ordered extended rectangular b-metric space and $T : U \to U$ an increasing mapping. Suppose these conditions holds:*

(i) there exists an $u_0 \in U$ such that $u_0 \preceq T(u_0)$,
(ii) T is Geraghty-weak contraction,
(iii) $(U, r_\theta, \preceq)$ is $\overline{O} - r_\theta$-complete,
(iv) either
(a) T is $\overline{O} - r_\theta$-continuous or
(b) $(U, r_\theta, \preceq)$ have the ICC-property.

Then we assure that T has a fixed point.

Proof. Let $u_0 \in U$ such that $u_0 \preceq T(u_0)$. As we know the mapping T is an increasing hence, we can construct an increasing sequence $\{u_n\}$, then we have for all $n \in \mathbb{N}_0$

$$u_1 = T(u_0),\ u_2 = T(u_1),\ u_3 = T(u_2), \cdots, u_{n+1} = T(u_n).$$

If we have $r_\theta(u_n, u_{n+1}) = 0$ for some $n \in \mathbb{N}_0$, then we can say that $\{u_n\}$ is a fixed point of T and we get our required result. Now, we have to suppose that $r_\theta(u_n, u_{n+1}) > 0$ for all $n \in \mathbb{N}_0$. We assert that $\lim_{n\to\infty} r_\theta(u_n, u_{n+1}) = 0$. By placing $u = u_{n-1}$ with $v = u_n$ in (3.1), we have result

$$\begin{aligned} r_\theta(u_n, u_{n+1}) &= r_\theta(T(u_{n-1}), T(u_n)) \\ &\le \lambda(r_\theta(u_{n-1}, u_n))M(r_\theta(u_{n-1}, u_n)) \\ &< \frac{1}{\theta}M(r_\theta(u_{n-1}, u_n)) \le M(r_\theta(u_{n-1}, u_n)) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} M(r_\theta(u_{n-1}, u_n)) &= \max\left\{ r_\theta(u_{n-1}, u_n), \frac{r_\theta(u_{n-1}, T(u_{n-1}))r_\theta(u_n, T(u_n))}{1+r_\theta(T(u_{n-1}), T(u_n))}, \right. \\ &\quad \frac{r_\theta(u_{n-1}, T(u_{n-1}))r_\theta(u_n, T(u_n))}{1+r_\theta(u_{n-1}, u_n)}, \\ &\quad \left. \frac{r_\theta(u_{n-1}, T(u_{n-1}))r_\theta(u_{n-1}, T(u_n))}{1+r_\theta(u_{n-1}, T(u_n))+r_\theta(u_n, T(u_{n-1}))} \right\} \\ &= \max\left\{ r_\theta(u_{n-1}, u_n), \frac{r_\theta(u_{n-1}, u_n)r_\theta(u_n, u_{n+1})}{1+r_\theta(u_n, u_{n+1})}, \right. \\ &\quad \left. \frac{r_\theta(u_{n-1}, u_n)r_\theta(u_n, u_{n+1})}{1+r_\theta(u_{n-1}, u_n)}, \frac{r_\theta(u_{n-1}, u_n)r_\theta(u_{n-1}, u_{n+1})}{1+r_\theta(u_{n-1}, u_{n+1})+r_\theta(u_n, u_n)} \right\} \\ &\le \max\{r_\theta(u_{n-1}, u_n), r_\theta(u_{n-1}, u_n), r_\theta(u_n, u_{n+1}), r_\theta(u_{n-1}, u_n)\} \end{aligned}$$

$$= \max\{r_\theta(u_{n-1},u_n), r_\theta(u_n,u_{n+1})\}.$$

Now suppose that, $\max\{r_\theta(u_{n-1},u_n), r_\theta(u_n,u_{n+1})\} = r_\theta(u_n,u_{n+1})$, by using (3.2) we get,

$$r_\theta(u_n,u_{n+1}) < \frac{1}{\theta}M(r_\theta(u_{n-1},u_n)) \leq r_\theta(u_n,u_{n+1}).$$

which is a contradiction. Hence, $\max\{r_\theta(u_n,u_{n+1}, r_\theta(u_n,u_{n+1})\} = r_\theta(u_{n-1},u_n)$. Therefore, by using (3.2) we have,

(3.3) $$r_\theta(u_n,u_{n+1}) < r_\theta(u_{n-1},u_n).$$

Thus $\{r_\theta(u_n,u_{n+1})\}$ is the decreasing sequence of non-negative real numbers. Hence, there must exists $b \geq 0$ such that

$$\lim_{n\to\infty} r_\theta(u_n,u_{n+1}) = b.$$

Assume that $b > 0$. Then from (3.2), we get

$$\lim_{n\to\infty} r_\theta(u_n,u_{n+1}) \leq \lim_{n\to\infty}[\lambda(r_\theta(u_{n-1},u_n))M(r_\theta(u_{n-1},u_n)].$$

By the definition of λ we get $b < \frac{1}{\theta}b$, a contraction. Thus,

(3.4) $$\lim_{n\to\infty} r_\theta(u_n,u_{n+1}) = 0.$$

Now, by taking $u = u_{n-1}$ with that we take $v = u_{n+1}$ in (3.1), we get

(3.5) $$\begin{aligned} r_\theta(u_n,u_{n+2}) &= r_\theta(T(u_{n-1}),T(u_{n+1})) \leq \lambda(r_\theta(u_{n-1},u_{n+1})M(r_\theta(u_{n-1},u_{n+1})) \\ &< \frac{1}{\theta}M(r_\theta(u_{n-1},u_{n+1})) \leq M(r_\theta(u_{n-1},u_{n+1})), \end{aligned}$$

where,

$$\begin{aligned} M(r_\theta(u_{n-1},u_{n+1})) &= \max\left\{r_\theta(u_{n-1},u_{n+1}), \frac{r_\theta(u_{n-1},T(u_{n-1}))r_\theta(u_{n+1},T(u_{n+1}))}{1+r_\theta(T(u_{n-1}),T(u_{n+1}))},\right. \\ &\quad \frac{r_\theta(u_{n-1},T(u_{n-1}))r_\theta(u_{n+1},T(u_{n+1}))}{1+r_\theta(u_{n-1},u_{n+1})}, \\ &\quad \left.\frac{r_\theta(u_{n-1},T(u_{n-1}))r_\theta(u_{n-1},T(u_{n+1}))}{1+r_\theta(u_{n-1},T(u_{n+1}))+r_\theta(u_{n+1},T(u_{n-1}))}\right\} \\ &= \max\left\{r_\theta(u_{n-1},u_{n+1}), \frac{r_\theta(u_{n-1},u_n)r_\theta(u_{n+1},u_{n+2})}{1+r_\theta(u_n,u_{n+2})},\right. \\ &\quad \left.\frac{r_\theta(u_{n-1},u_n)r_\theta(u_{n+1},u_{n+2})}{1+r_\theta(u_{n-1},u_{n+1})}, \frac{r_\theta(u_{n-1},u_n)r_\theta(u_{n-1},u_{n+2})}{1+r_\theta(u_{n-1},u_{n+2})+r_\theta(u_{n+1},u_n)}\right\} \\ &\leq \max\Big\{r_\theta(u_{n-1},u_{n+1}), [r_\theta(u_{n-1},u_n)r_\theta(u_{n+1},u_{n+2})] \\ &\quad [r_\theta(u_{n-1},u_n)r_\theta(u_{n+1},u_{n+2})], r_\theta(u_{n-1},u_n)\Big\}. \end{aligned}$$

Using (3.3) we get,

$$M(r_\theta(u_{n-1},u_{n+1})) \leq \max\{r_\theta(u_{n-1},u_{n+1}), r_\theta(u_{n-1},u_n), [r_\theta(u_{n-1},u_n)]^2\}.$$

First of all, let us suppose that

$$\max\{r_\theta(u_{n-1},u_{n+1}), r_\theta(u_{n-1},u_n), [r_\theta(u_{n-1},u_n)]^2\} = r_\theta(u_{n-1},u_n) or [r_\theta(u_{n-1},u_n)]^2.$$

As $\lim_{n\to\infty} r_\theta(u_{n-1},u_n) = 0$, by using (3.5), we get

$$\lim_{n\to\infty} r_\theta(u_n,u_{n+2}) = 0.$$

If the equation $\max\{r_\theta(u_{n-1},u_{n+1}), r_\theta(u_{n-1},u_n), [r_\theta(u_{n-1},u_n)]^2\} = r_\theta(u_{n-1},u_{n+1})$ is true, by using (3.5), we get

$$r_\theta(u_n,u_{n+2}) < r_\theta(u_{n-1},u_{n+1}).$$

Thus $\{r_\theta(u_n,u_{n+2})\}$ is a decreasing sequence of non-negative real numbers. Hence, there must exists $b \geq 0$ such that

$$\lim_{n\to\infty} r_\theta(u_n,u_{n+2}) = b.$$

Let us assume that $b > 0$. Then from (3.5), we get

$$\lim_{n\to\infty} r_\theta(u_n, u_{n+2}) \leq \lim_{n\to\infty} \lambda(r_\theta(u_{n-1}, u_{n+1}))M(r_\theta(u_{n-1}, u_{n+1})).$$

By using the definition of λ we get $b < \frac{1}{\theta} b$, a contradiction. Thus, we obtain

$$\lim_{n\to\infty} r_\theta(u_n, u_{n+2}) = 0. \tag{3.6}$$

Now, we have to show that $u_n \neq u_m$ for each $n = m$. On contrary we suppose that, $u_n = u_m$ for some $n > m$, then we get $u_{n+1} = T(u_n) = T(u_m) = x_{m+1}$. Then, from (3.2) we have

$$\begin{aligned} r_\theta(u_m, u_{m+1}) &= r_\theta(u_n, u_{n+1}) = r_\theta(T(u_{n-1}, T(u_n))) \\ &\leq \lambda(r_\theta(u_{n-1}, u_n))M(r_\theta(u_{n-1}, u_n)) \\ &< \frac{1}{\theta} M(r_\theta(u_{n-1}, u_n)) \leq M(r_\theta(u_{n-1}, u_n)) \\ &\leq \max\{r_\theta(u_{n-1}, u_n) r_\theta(u_n, u_{n+1})\}. \end{aligned}$$

Therefore, we get

$$\max\{r_\theta(u_{n-1}, u_n), r_\theta(u_n, u_{n+1})\} = r_\theta(u_n, u_{n+1}),$$

so that

$$r_\theta(u_m, u_{m+1}) < r_\theta(u_n, u_{n+1}),$$

which is a contradiction. Suppose

$$\max\{r_\theta(u_{n-1}, u_n), r_\theta(u_n, u_{n+1})\} = r_\theta(u_{n-1}, u_n),$$

we have

$$r_\theta(u_m, u_{m+1}) = r_\theta(u_n, u_{n+1}) < r_\theta(u_{n-1}, u_n) < r_\theta(u_{n-2}, u_{n-1}) < \cdots < r_\theta(u_m, u_{m+1})),$$

which is a contradiction. So, we can take $u_n \neq u_m \ \ \forall\ n \neq m$. Now its turn to prove that $\{u_n\}$ is $\overline{O} - r_\theta$-Cauchy sequence in $(U, r_\theta, \preceq)$. On contrary suppose that, $\{u_n\}$ is not $\overline{O} - r_\theta$-Cauchy sequence. So there must exist $\epsilon > 0$ and also two subsequences $\{n_k\}$ and $\{m_k\}$ such that $\{n_k\}$ is the index which is smallest for that

$$\{n_k\} > \{m_k\} > k \text{ and } r_\theta(u_{m_k}, u_{n_k}) \geq \frac{\epsilon}{2}, \tag{3.7}$$

which implies that

$$r_\theta(u_{m_k}, u_{n_{k-1}}) < \frac{\epsilon}{2}. \tag{3.8}$$

Now, on using rectangular inequality, we have

$$\frac{\epsilon}{2} \leq r_\theta(u_{m_k}, u_{n_k}) \leq \theta(r_\theta(u_{m_k}, u_{n_{k-1}}) + \theta(r_\theta(u_{n_{k-1}}, u_{n_{k+1}})) + \theta(r_\theta(u_{n_{k+1}}, u_{n_k}). \tag{3.9}$$

Now, using (3.4) (3.6) (3.8) and also taking limit as $k \to \infty$, we have

$$\frac{\epsilon}{2} \leq \lim_{k\to\infty} \sup r_\theta(u_{m_k}, u_{n_k}) \leq \theta\left(\frac{\epsilon}{2}\right). \tag{3.10}$$

On using (3.1) and definition of r_θ, we have

$$\begin{aligned} \lim_{k\to\infty} r_\theta(u_{m_k}, u_{n_k}) &\leq \lim_{k\to\infty} \theta(r_\theta(x_{m_{k+1}}, u_{m_k})) + \lim_{k\to\infty} \theta(r_\theta(x_{m_{k+1}}, u_{n_{k+1}}) \\ &+ \lim_{k\to\infty} \theta(r_\theta(u_{n_{k+1}}, u_{n_k}), \\ &\leq \theta \lim_{k\to\infty} \lambda(r_\theta(u_{m_k}, u_{n_k}))M(r_\theta(u_{m_k}, u_{n_k})), \end{aligned} \tag{3.11}$$

where,

$$\begin{aligned} M(r_\theta(u_{m_k}, u_{n_k})) &= \max\Bigg\{ r_\theta(u_{m_k}, u_{n_k}), \frac{r_\theta(u_{m_k}, T(u_{m_k}))r_\theta(u_{n_k}, T(u_{n_k}))}{1 + r_\theta(T(u_{m_k}), T(u_{n_k}))}, \\ &\frac{r_\theta(u_{m_k}, T(u_{m_k}))r_\theta(u_{n_k}, T(u_{n_k}))}{1 + r_\theta(u_{m_k}, u_{n_k})}, \\ &\frac{r_\theta(u_{m_k}, T(u_{m_k}))r_\theta(u_{m_k}, T(u_{n_k}))}{1 + r_\theta(u_{m_k}, T(u_{n_k})) + r_\theta(u_{n_k}, T(u_{m_k}))} \Bigg\} \end{aligned} \tag{3.12}$$

$$= \max\left\{r_\theta(u_{m_k},u_{n_k}), \frac{r_\theta(u_{m_k},u_{m_{k+1}})r_\theta(u_{n_k},u_{n_{k+1}})}{1+r_\theta(x_{m_{k+1}},u_{n_{k+1}})}, \frac{r_\theta(u_{m_k},u_{m_{k+1}})r_\theta(u_{n_k},u_{n_{k+1}})}{1+r_\theta(u_{m_k},u_{n_k})}, \frac{r_\theta(u_{m_k},u_{m_{k+1}})r_\theta(u_{m_k},u_{n_{k+1}})}{1+r_\theta(u_{m_k},u_{n_{k+1}})+r_\theta(u_{n_k},u_{m_{k+1}})}\right\}.$$

Taking the limit $k\to\infty$ and using (3.12), we have

$$\lim_{n\to\infty}\sup M(r_\theta(u_{m_k},u_{n_k})) = \lim_{n,m\to\infty}\sup(r_\theta(u_{m_k},u_{n_k})).$$

By using (3.11), we get

$$\lim_{n\to\infty}\sup r_\theta(u_{m_k},u_{n_k}) \le \theta \lim_{n,m\to\infty}\sup\lambda(r_\theta(u_{m_k},u_{n_k})) \lim_{n,m\to\infty}\sup(r_\theta(u_{m_k},u_{n_k})).$$

As we have supposed that $\lim_{k\to\infty}\sup r_\theta(u_{m_k},u_{n_k}) \neq 0$, then from above inequality, we have

$$\frac{1}{\theta} \le \lim_{k\to\infty}\sup \lambda r_\theta(u_{m_k},u_{n_k}).$$

Since $\lambda\in\Lambda$, so that $\lim_{n,m\to\infty} r_\theta(u_{m_k},u_{n_k}) = 0$, which is in general a contradiction. Hence, we assure that $\{u_n\}$ is $\overline{O}-r_\theta$-Cauchy sequence in $(U,r_\theta,\preceq)$. As $(U,r_\theta,\preceq)$ is $\overline{O}-r_\theta$-complete so there must exist $u\in U$ such that $u_n\uparrow u$ and also,

$$\lim_{n,m\to\infty} r_\theta(u_n,u_m) = 0.$$

Now coming to last condition, first of all suppose that T is $\overline{O}-r_\theta$- continuous then we will show that x is a fixed point of T.

$$x = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} T(u_n) = T(\lim_{n\to\infty} u_n) = T(u).$$

Now, we take second condition, i.e., $(U,r_\theta,\preceq)$ follows ICC-property. So there must exist a subsequence of $\{u_n\}$ which is $\{u_{n_k}\}$ such that $\{u_{n_k}\} \prec\succ\ u\ \forall\ k\in\mathbb{N}$. First we take $\{u_{n_k}\ \preceq\ x\ \forall\ k\in\mathbb{N}$ (proof for both case are alike). So by using (3.1), we get

$$\begin{aligned}\lim_{k\to\infty} r_\theta(u_{n_{k+1}},T(u)) &= \lim_{k\to\infty} r_\theta(T(u_{n_k}),T(u))\\ &\le \lim_{k\to\infty}\lambda(r_\theta((u_{n_k}),u)) \lim_{k\to\infty} M(r_\theta((u_{n_k}),u)\end{aligned}$$

where

$$\begin{aligned}\lim_{k\to\infty} M(r_\theta((u_{n_k}),u) &= \lim_{k\to\infty}\left(\max\left\{r_\theta((u_{n_k}),u), \frac{r_\theta(u_{n_k},T(u_{n_k})),r_\theta(u,T(u))}{1+r_\theta(T(u_{n_k}),T(u))}, \frac{r_\theta(u_{n_k},T(u_{n_k})),r_\theta(u,T(u))}{1+r_\theta(u_{n_k},u)}, \frac{r_\theta(u_{n_k},T(u_{n_k})),r_\theta(u_{n_k},T(u))}{1+r_\theta(u_{n_k},T(u))+r_\theta(u,T(u_{n_k}))}\right\}\right)\\ &= \lim_{k\to\infty}\left(\max\left\{r_\theta((u_{n_k}),u), \frac{r_\theta(u_{n_k},u_{n_{k+1}}),r_\theta(u,T(u))}{1+r_\theta(u_{n_{k+1}},T(u))}, \frac{r_\theta(u_{n_k},u_{n_{k+1}}),r_\theta(u,T(u))}{1+r_\theta(u_{n_k},u)}, \frac{r_\theta(u_{n_k},u_{n_{k+1}}),r_\theta(u_{n_k},T(u))}{1+r_\theta(u_{n_k},T(u))+r_\theta(u,u_{n_{k+1}})}\right\}\right)\\ &= \lim_{k\to\infty} r_\theta((u_{n_k}),u).\end{aligned}$$

Therefore

$$\lim_{k\to\infty} r_\theta((u_{n_k}),u.) = \lim_{k\to\infty} r_\theta(u_{n_{k+1}},T(u)) = 0.$$

We assert that the terms u_{n_k} and $u_{n_{k+1}}$ $\forall\ k \in \mathbb{N}$ are distinct from u and $T(u)$ both. By definition ,we have

$$r_\theta(u, T(u)) \leq \theta[r_\theta(u, u_{n_k}) + r_\theta(u_{n_k}, u_{n_{k+1}}) + r_\theta(u_{n_{k+1}}, T(u))]. \tag{3.13}$$

By taking $k \to \infty$ and using (3.4) and (3.13), we have $r_\theta(u, T(u)) = 0$. Hence we can say that $T(u) = u$. So, u is a fixed point of T. □

Example 3.1. Consider $U = (-1, 0]$. Define $r_\theta : U \times U \to \mathbb{R}_+$ by (for all $u, v \in U$):

$$r_\theta(u, v) = |u - v|^2.$$

Notice that, every increasing Cauchy sequence is convergent in U. Therefore, $(U, r_\theta, \preceq)$ is an $\overline{\text{O}}$-complete $r_\theta-$metric space with coefficient $\theta(u, v) = 2$ for all $u, v \in U$.

Now, we define an ordered relation on U as under:

$$u, v \in U,\ u \preceq v \Leftrightarrow\ u = v \text{ or } \left(u, v \in \{0\} \cup \left\{\frac{-1}{n} : n = 2, 3, \cdots\right\} \text{ and } u \leq v\right),$$

where $\leq$ is the usual order. Define the mappings $T : U \to U$ as follows:

$$Tu = \begin{cases} 0, & \text{if } u = 0 \\ \frac{-1}{2n}, & \text{if } u = -1/n, n = 2, 3, \cdots. \\ -0.5, & \text{otherwise} \end{cases}$$

Observe that, T is increasing and U has the ICC-property. We distinguish two cases:

Case 1. Taking $u = -1/n$, (wherein $n = 3, 4, \cdots$) and $v = 0$. Then, from (3.1), we have

$$r_\theta(Tu, Tv) = \left|\frac{-1}{2n} - 0\right|^2 = \frac{1}{4}\left|\frac{-1}{n} - 0\right|^2 = \frac{1}{4} r_\theta(u, v). \tag{3.14}$$

Case 2. Taking $u = -1/n, v = -1/m\ m > n \geq 3$. Then, we have

$$r_\theta(Tu, Tv) = \left|\frac{-1}{2n} - \frac{-1}{2m}\right|^2 = \frac{1}{4}\left|\frac{-1}{n} - \frac{-1}{m}\right|^2 = \frac{1}{4} r_\theta(u, v). \tag{3.15}$$

If $u = v$, then condition (3.1) holds trivially. Thus, all the conditions of Theorems 3.1 are satisfied and the mapping T has a unique fixed point (namely $u = 0$).

Example 3.2. Let $U = \{1, 2, 3, 4, 5\}$ be equipped with the order relation $\preceq$ given by

$$\preceq = \{(1,1), (2,2), (3,3), (4,4), (5,5), (4,1), (4,2), (4,3), (4,5), (1,3), (2,3), (5,3)\}$$

and let $r_\theta : U \times U \to \mathbb{R}^+$ is defined by:

$$\begin{aligned} r_\theta(u, u) &= 0, \quad \text{for all } u \in U; \\ r_\theta(u, v) &= r_\theta(v, u), \quad \text{for all } u, v \in U; \\ r_\theta(1, 3) &= r_\theta(1, 5) = r_\theta(2, 3) = r_\theta(3, 5) = 3t; \\ r_\theta(1, 4) &= r_\theta(2, 4) = r_\theta(2, 5) = r_\theta(3, 4) = r_\theta(4, 5) = 4t; \\ r_\theta(1, 2) &= 5t; \end{aligned}$$

where $0 < t < -\ln(3/4)$, that is, $e^{-t} > 3/4$. Therefore, $(U, r_\theta, \preceq)$ is an $\overline{\text{O}}$-complete $r_\theta-$metric space with coefficient $\theta(u, v) = 3$ for all $u, v \in U$. Consider a mapping $T : U \to U$ defined by:

$$T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 1 & 3 \end{pmatrix}.$$

It is easy to check that all the conditions of Theorem 3.1 are fulfilled with $\lambda(u) = e^{-u}$ for each $u > 0$ and $\lambda(0) \in [0, 1/3)$. In particular, by choosing $u, v \in \{1, 2, 3, 5\}$ such that $u \preceq v$, then $Tu = Tv = 3$ implies the condition (3.1) is trivially holds. Now, if we take $u = 4$ and $v \in \{1, 2, 3, 5\}$, such that $u \preceq v$, we obtain $Tu = 1$ and $Tv = 3$. Then by (3.1), we have

$$\begin{aligned} r_\theta(Tu, Tv) &= r_\theta(1, 3) = 3t = \frac{3}{4} 4t < e^{-t}.4t \\ &= \lambda(t) d(x, y) \leq \lambda\big(r_\theta(x, y)\big) M(x, y). \end{aligned}$$

It follows that T has a unique fixed point (which is $x = 3$).

If we replace $\overline{O}$-completeness of U and $\overline{O}$-continuity of T in Theorem 3.1, then it remains a new version as given follows:

Corollary 3.1. *Let $(U, r_\theta, \preceq)$ be an ordered extended rectangular $b-$metric space and $T : U \to U$ be an increasing mapping. Suppose these conditions holds:*
1. *T follow Geraghty-weak contraction,*
2. *$(U, r_\theta, \preceq)$ is complete,*
3. *T is continuous.*

Then we assure that T has a fixed point.

If we replace Geraghty-weak-contraction by contraction condition in Theorem 3.1, then it remains a new version of the Theorem 3.1 due to Asim et al. [4].

Corollary 3.2. *Let $(U, r_\theta, \preceq)$ be an ordered extended rectangular $b-$metric space and $T : U \to U$ be an increasing mapping. Suppose these conditions holds:*
1. *there exists an $u_0 \in U$ such that $u_0 \preceq T(u_0)$,*
2. *If $u \preceq v \ \ \forall u, v \in U$, then we get*
$$r_\theta(T(u), T(v)) \leq LM(r_\theta(u, v)),$$
where, $L \in [0, \infty)$.
3. *$(U, r_\theta, \preceq)$ is $\overline{O} - r_\theta$-complete,*
4. *either*
   (a) T is $\overline{O} - r_\theta$-continuous or follow
   (b) $(U, r_\theta, \preceq)$ have the ICC-property.

Then we assure that T has a fixed point.

Corollary 3.3. *Let (U, r_θ) be a complete extended rectangular $b-$metric space and T be a continuous and self-mapping. Also suppose T follows the property of Geraghty-weak contraction. Then we assure that T has a fixed point.*

Proposition 3.1. ([5]). *Let $(U, r_\theta, \preceq)$ be an ordered extended rectangular $b-$metric space and $T : U \to U$ is a Geraghty-weak contraction. If $u \prec\succ v$ then $u = v \ \ \forall \ u, v \in Fix(f)$.*

Definition 3.3. ([20]). *Suppose $(U, \preceq)$ is an ordered set and let T be an self-mapping. Then we define*
$$U_T = \{u \in U : u \prec\succ T(u)\}.$$
Then $(U, \preceq)$ is said to be $T-$ directed if there exists $a \in U_T$ such that $u \prec\succ a \prec\succ v \ \ \forall u, v \in U$.

Theorem 3.2. *If with all the conditions of Theorem 3.1 we add that $(U, \preceq)$ is $T-$directed. Then we assure that T has a unique fixed point.*

Proof. Let us suppose that u and v be two different points of T. Also as $(U, \preceq)$ is $T-$directed then there must exists $a \in U_T$ such that $u \prec\succ a \prec\succ v$. If we take $a = u$ or $a = v$ then by above preposition we have $u = v$, which is a contradiction. Hence, we have to suppose that $u \neq a, v \neq a$. As we know $a \in U_T$ then we get $a \prec\succ T(a)$. By putting $a = a_0$ where $a_0 \preceq T(a_0)$ we define a sequence $\{a_n\}$ as follow
$$a_{n+1} = T(a_n), n \in \mathbb{N}_0.$$
As we know that T is an increasing mapping and $u \prec\succ a \prec\succ v$, we get
$$u \prec\succ a_n \prec\succ v, \quad n \in \mathbb{N}_0.$$
If we put $a_n = a_{n+1}$ for any $n \in \mathbb{N}_0$, then we have, a_n is the fixed point of T and by relation and above preposition we get $u = a_n = v$, which is a contradiction. So, we can't say $a_n = a_{n+1}$ for all $n \in \mathbb{N}_0$. Now, by proceeding the proof of Theorem 3.1 we can prove that

$$\lim_{n,m \to \infty} r_\theta(a_n, a_m) = 0. \tag{3.16}$$

Using (3.1), we get

$$\begin{aligned} r_\theta(u, a_n) &= r_\theta(T(u), T(u_{n+1})) \leq \lambda(r_\theta(u, a_{n-1}))M(r_\theta(u, a_{n-1})) \\ &< \frac{1}{\theta}M(r_\theta(u, a_{n-1})) \leq (r_\theta(u, a_{n-1}), \end{aligned} \tag{3.17}$$

and

$$
\begin{aligned}
M(r_\theta(u,a_{n-1})) &= \max\left\{(r_\theta(u,a_{n-1})), \frac{r_\theta(u,T(u))r_\theta(a_{n-1},T(a_{n-1}))}{1+r_\theta(T(u),T(a_{n-1}))},\right.\\
&\quad \frac{r_\theta(u,T(u))r_\theta(a_{n-1},T(a_{n-1}))}{1+r_\theta(u,a_{n-1})},\\
&\quad \left.\frac{r_\theta(u,T(u))r_\theta(u,T(a_{n-1}))}{1+r_\theta(u,T(a_{n-1}))+r_\theta(a_{n-1},T(u))}\right\}\\
&= \max\left\{r_\theta(u,a_{n-1}), \frac{r_\theta(a_{n-1},(a_n))}{1+r_\theta(u,(a_n))},\right.\\
&\quad \frac{r_\theta(a_{n-1},a_n)}{1+r_\theta(u,a_{n-1})},\\
&\quad \left.\frac{r_\theta(u,a_n)}{1+r_\theta(u,a_n)+r_\theta(a_{n-1},u)}\right\}\\
&= r_\theta(u,a_{n-1}).
\end{aligned}
$$

As we know that $\{r_\theta(u,a_n)\}$ is the decreasing sequence of positive real numbers. Then we choose $b \geq 0$ such that

$$\lim_{n\to\infty} r_\theta(u,a_n) = b.$$

Then suppose that $b > 0$. Then from (3.17), we get

$$\lim_{n\to\infty} r_\theta(u,a_n) \leq \lim_{n\to\infty} \lambda(r_\theta(u,a_{n-1})r_\theta(u,u_{n-1}).$$

By the definition of λ we get $r < \frac{1}{\theta}r$, which is a contradiction. Hence

$$\lim_{n\to\infty} r_\theta(u,a_n) = 0. \tag{3.18}$$

Similarly, we can prove that

$$\lim_{n\to\infty} r_\theta(v,a_n) = 0. \tag{3.19}$$

Now, using rectangle inequality, we have

$$r_\theta(u,v) \leq \theta[r_\theta(u,a_n) + r_\theta(a_n,a_{n+1}) + r_\theta(a_{n+1},v)].$$

At $n \to \infty$ and using (3.16) (3.18) (3.19), we get $r_\theta(u,v) = 0$ and we can say that $u = v$, which is a contradiction. Hence, proof is complete. □

The following example shows the importance of a T-directed condition in the Theorem 3.2 for the uniqueness of a fixed point.

Example 3.3. In Example 3.2, we take $\preceq = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$ and a mapping $T : U \to U$ defined by:

$$T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 3 & 1 & 3 \end{pmatrix}.$$

By choosing $u, v \in U$ such that $u \preceq v$ and $Tu = Tv = 1$ or 3. Thus, the contraction condition (3.1) is trivially hold. Therefore, all the conditions of Theorem 3.1 are satisfied except that $(U, \preceq)$ is not $T-$directed. Observe that the mapping T has two fixed points namely $u = 1$ and $u = 3$.

Theorem 3.3. *In Theorems* 3.1 *and* 3.2*, if we replace some conditions namely: increasing mapping T to decreasing(or monotone) mapping, $\overline{O}-$complete to $\underline{O}-$complete(or O-complete), $\overline{O}-$continuos to $\underline{O}-$continuous(or O-continuous) and ICC-property to DCC-property(or MCC-property) also replace $u_0 \preceq T(u_0)$ by $u_0 \succeq T(u_0)$(or $u_0 \prec\succ T(u_0)$). Then the result of both the remains true.*

4 Conclusion

We use geraghty-weak contraction in ordered extended rectangular b-metric space to get fixed point results, with that we have given examples to exhibit the utility of the result.

Authors Contributions

Both author contributed equally to this work. The final draft was approved by both authors. **Acknowledgement.**
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