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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

IDENTITIES OF A GENERAL MULTIPLE HURWITZ-LERCH ZETA FUNCTION AND APPLICATIONS

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Abstract

In this article we introduce a general multiple Hurwitz-Lerch Zeta function. Then its convergence conditions and identities are obtained under certain conditions. We also derive some of connections to the multiple Hurwitz-Lerch Zeta function based upon Srivastava-Daoust hypergeometric series in several variables and other related functions of one and more variables found in the literature. Further, we study its integral representations and find their applications for deriving generating relations and solving the non-homogeneous fractional differential equation.
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1 Introduction

Recently, Srivastava et al. [16] extended the double Hurwitz-Lerch Zeta function due to (1.1) - (1.3) into the multiple Hurwitz-Lerch Zeta function [16, Eqn. (4.1)] in terms of Srivastava-Daoust hypergeometric series in several variables [20, p.37] and defined in the form

$$(1.1)\quad {}_{\omega}^{\sigma}F\begin{matrix} A:B^{(1)};...;B^{(n)} \\ C:D^{(1)};...;D^{(n)}\end{matrix}\left(\begin{matrix}[(a);\theta^{(1)},...,\theta^{(n)}]:[(b^{(1)}:\psi^{(1)})];...;[(b^{(n)}):\psi^{(n)}]; \\ [(c);\delta^{(1)},...,\delta^{(n)}]:[(d^{(1)}:\phi^{(1)})];...;[(d^{(n)}):\phi^{(n)}];\end{matrix} z_1,...,z_n\right)$$
$$= \sum_{m_1,...,m_n=0}^{\infty} \mathcal{H}\begin{matrix} A:B^{(1)};\dots;B^{(n)} \\ C:D^{(1)};...;D^{(n)}\end{matrix}(m_1,...,m_n)\frac{z_1^{m_1}\dots z_n^{m_n}}{m_1!\dots m_n!\,(m_1+\dots+m_n+\omega)^{\sigma}},$$

where, $\sigma\in\mathbb{C},\omega\in\mathbb{C}\backslash\mathbb{Z}_0,\mathbb{C}=\{z:z=x+iy,i=\sqrt{(-1)},x,y\in\mathbb{R}\},\mathbb{R}=(-\infty,\infty),\mathbb{Z}_0=\{0,-1,-2,-3,\dots\}$. Additionally, as usual $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}^+$denotes the set of positive real numbers and for convenience
(1.2)
$$\mathcal{H}\begin{matrix} A:B^{(1)};\dots;B^{(n)} \\ C:D^{(1)};\dots;D^{(n)}\end{matrix}(m_1,...,m_n)=\frac{\prod_{j=1}^{A}(a_j)_{\theta_j^{(1)}m_1+\dots+\theta_j^{(n)}m_n}\prod_{j=1}^{B^{(1)}}\left(b_j^{(1)}\right)_{\psi_j^{(1)}m_1}\cdots\prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}\right)_{\psi_j^{(n)}m_n}}{\prod_{j=1}^{C}(c_j)_{\delta_j^{(1)}m_1+\dots+\delta_j^{(n)}m_n}\prod_{j=1}^{D^{(1)}}\left(d_j^{(1)}\right)_{\phi_j^{(1)}m_1}\cdots\prod_{j=1}^{D^{(n)}}\left(d_j^{(n)}\right)_{\phi_j^{(n)}m_n}}.$$

The coefficients

$$(1.3)\quad \begin{cases}\theta_j^{(l)},j=1,\dots,\ A;\ \psi_j^{(l)},j=1,\dots,\ B^{(l)};\delta_j^{(l)},j=1,\dots,C; \\ \phi_j^{(l)},j=1,\dots,D^{(l)};\ \forall\, l\in\{1,2,3,\dots,n\},\end{cases}$$

are the members of the set $\mathbb{R}^+$ and (a) abbreviates the array of A parameters $a_1,\dots,a_A$; $\left(b^{(l)}\right)$ abbreviates the array of $B^{(l)}$ parameters $b_1^{(l)},\dots,b_{B^{(l)}}^{(l)},\ \ \forall\, l\in\{1,2,3,\dots,n\}$, with similar interpretations for (c) and $\left(d^{(l)}\right),\ \ l=1,2,\dots,n$; are complex numbers *et cetera*.

The multiple series (1.1) with (1.2) and (1.3) converges due to [3, 7] for

$$(1.4)\quad |z_1|<\infty,\dots,|z_n|<\infty,$$

if

$$(1.5) \qquad \sum_{j=1}^{C}\delta_j^{(l)}+\sum_{j=1}^{D^{(l)}}\phi_j^{(l)}-\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{B^{(l)}}\psi_j^{(l)}+1>0 \quad (\ \forall\ l=1,2,3,\dots,n).$$

Further if $\sum_{j=1}^{C}\delta_j^{(l)}+\sum_{j=1}^{D^{(l)}}\phi_j^{(l)}-\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{B^{(l)}}\psi_j^{(l)}+1=0 \quad (\ \forall\ l=1,2,3,\dots,n)$, then the multiple series (1.1) with (1.2) and (1.3) converges for

$$(1.6) \quad (|z_1|)^{\frac{1}{\mathfrak{H}_1}}+\dots+(|z_n|)^{\frac{1}{\mathfrak{H}_n}}<1,\ \text{when}\ \mathfrak{H}_l=\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{C}\delta_j^{(l)}\ (\ \forall\ l=1,2,3,\dots,n),\ \mathfrak{H}_l>0;\ \text{and for}$$

$$(1.7) \qquad \{|z_1|\,(z_1\neq 1)\,,\dots,|z_n|\,(z_n\neq 1)\}<1,\ \text{when}\ \mathfrak{H}_l<0\ (\ \forall\ l=1,2,3,\dots,n).$$

But when $z_1=1,\dots,z_n=1$, the multiple Srivastava-Daoust-Hurwitz-Lerch Zeta function (1.1) - (1.3), with the aid of the formulae, $\lim_{n\to\infty}\Gamma(z+n+1)=\lim_{n\to\infty}\Gamma(n+1)n^z$, is written by (see in [5, 6])

$$(1.8) \qquad {}^{\sigma}_{\omega}F^{A:B^{(1)};\dots;B^{(n)}}_{C:D^{(1)};\dots;D^{(n)}}\left(\begin{matrix}\left[(a):\theta^{(1)},\dots,\theta^{(n)}\right]:\left[\left(b^{(1)}\right):\psi^{(1)}\right];\dots;\left[\left(b^{(n)}\right):\psi^{(n)}\right];\\ \left[(c):\delta^{(1)},\dots,\delta^{(n)}\right]:\left[\left(d^{(1)}\right):\phi^{(1)}\right];\dots;\left[\left(d^{(n)}\right):\phi^{(n)}\right];\end{matrix}1,\dots,1\right)$$

$$=\sum_{m_1,\dots,m_n=0}^{m_1=M_1-1,\dots,m_n=M_n-1}\mathcal{H}^{A:B^{(1)};\dots;B^{(n)}}_{C:D^{(1)};\dots;D^{(n)}}(m_1,\dots,m_n)\frac{1}{m_1!\dots m_n!\,(m_1+\dots+m_n+\omega)^{\sigma}}$$

$$+\sum_{m_1,\dots,m_n=0}^{\infty}\mathcal{H}^{A:B^{(1)};\dots;B^{(n)}}_{C:D^{(1)};\dots;D^{(n)}}(m_1+M_1,\dots,m_n+M_n)$$

$$\times\frac{\Gamma(m_1+M_1+\dots+m_n+M_n+\omega+1)}{(m_1+M_1)!\dots(m_n+M_n)!\Gamma(m_1+M_1+\dots+m_n+M_n+\sigma+\omega+1)}.$$

Here in (1.8), the first series which is finite and the second infinite series converges for

$$(1.9) \qquad \sum_{j=1}^{C}\delta_j^{(l)}+\sum_{j=1}^{D^{(l)}}\phi_j^{(l)}-\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{B^{(l)}}\psi_j^{(l)}+1=0 \quad (\ \forall\ l=1,2,3,\dots,n)$$

and if $\mathfrak{H}_l=0$, where, $\mathfrak{H}_l=\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{C}\delta_j^{(l)}\ (\ \forall\ l=1,2,3,\dots,n)$, then there exists

$$\Re\left(\sum_{j=1}^{C}c_j+\sum_{j=1}^{D^{(l)}}d_j^{(l)}+\sigma-\sum_{j=1}^{A}a_j-\sum_{j=1}^{B^{(l)}}b_j^{(l)}\right)>0\ \ (\ \forall\ l=1,2,3,\dots,n).$$

It is remarked that to obtain the condition (1.9), we make an appeal to the formula $\frac{\Gamma(n+a)}{\Gamma(n+b)}\sim n^{(a-b)}\ (n\to\infty)$, and apply the techniques due to Hái, Marichev and Srivastava [3].

Here in this research article, in order to explore new ideas for studying various known and unknown Hurwitz-Lerch Zeta functions of one, two and multiple Hurwitz-Lerch Zeta functions (see in [5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19]), we introduce a general multiple Hurwitz-Lerch Zeta function given by

$$(1.10) \qquad {}^{s}_{\eta}K(\chi;\rho_1,\dots,\rho_n;z_1,\dots,z_n)=\sum_{m_1=0,\dots,m_n=0}^{\infty}\frac{\chi(m_1,\dots,m_n)(\rho_1)_{m_1}\dots(\rho_n)_{m_n}}{m_1!\dots m_n!}\frac{(z_1)^{m_1}\dots(z_n)^{m_n}}{(m_1+\dots+m_n+\eta)^{s}},$$

provided that the multiple sequence of function $\chi(m_1,\dots,m_n)\ \ (\forall\ \ m_1\geq 0,\dots,m_n\geq 0)$ is convergent under certain restrictions, all $\rho_1,\dots,\rho_n;x_1,\dots,x_n;\ \ s\in\mathbb{C}$ and $\eta\in\mathbb{C}\backslash\mathbb{Z}_0$.

2 Identities of the general multiple Hurwitz-Lerch Zeta function (1.10)

In this section, we show that the general multiple Hurwitz-Lerch Zeta function (1.10) gives some identities under certain conditions.

Theorem 2.1. *If in* (1.10), $\chi(m_1,\dots,m_n)=\chi(m_1+\dots+m_n)$ *and* $z_1=\dots=z_n=z$, *then there exists an identity*

$$(2.1) \qquad {}^{s}_{\eta}K(\chi;\rho_1,\dots,\rho_n;z,\dots,z)=\sum_{k=0}^{\infty}\frac{\chi(k)(\rho_1+\dots+\rho_n)_k}{k!}\frac{z^k}{(k+\eta)^s},$$

provided that the series involved converges absolutely.

Proof. In the formula (1.10), setting $\chi(m_1,\ldots,m_n)=\chi(m_1+\ldots+m_n)$ and $z_1=\ldots=z_n=z$, we find (2.2)

$$ {}^s_\eta K(\chi;\rho_1,\ldots,\rho_n;z,\ldots,z)=\sum_{m_1=0,\ldots,m_n=0}^{\infty}\frac{\chi(m_1+\ldots+m_n)(\rho_1)_{m_1}\ldots(\rho_n)_{m_n}}{m_1!\ldots m_n!}\frac{(z)^{m_1+\ldots+m_n}}{(m_1+\ldots+m_n+\eta)^s}. $$

Now in the equality (2.1) applying the formula [21, pp. 61-62]

$$ \sum_{m_1=0,\ldots,m_n=0}^{\infty}F(m_1+\ldots+m_n)(\rho_1)_{m_1}\ldots(\rho_n)_{m_n}\frac{x^{m_1+\ldots+m_n}}{m_1!\ldots m_n!}=\sum_{k=0}^{\infty}F(k)(\rho_1+\ldots+\rho_n)_k\frac{x^k}{k!}, \tag{2.3} $$

provided that the series involved in (2.3) converges absolutely, we get the result (2.1). □

Corollary 2.1. *If in the Theorem* 2.1*, for all* $p,q\in\mathbb{N}_0$*, where* $\mathbb{N}_0=\mathbb{N}\cup\{0\}$; $a_j\in\mathbb{C}$ $(j=1,\ldots,p)$, $b_j\in\mathbb{C}\backslash\mathbb{Z}_0$ $(j=1,..,q)$; $\lambda_j\in\mathbb{R}^+$ $(j=1,\ldots,p)$, $\mu_j\in\mathbb{R}^+$ $(j=1,\ldots,q)$*; set*

$$ \chi(m_1,\ldots,m_n)=\frac{\prod_{j=1}^{p}(a_j)_{(m_1+\ldots+m_n)\lambda_j}}{\prod_{j=1}^{q}(b_j)_{(m_1+\ldots+m_n)\mu_j}}, $$

and make an appeal to the Zeta function due to [15, Eqn. (16)]*, then there exists following identities*

$$ \begin{aligned} {}^s_\eta K(\chi;\rho_1,\ldots,\rho_n;z,\ldots,z)&=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{k\lambda_j}(\rho_1+\ldots+\rho_n)_k}{\prod_{j=1}^{q}(b_j)_{k\mu_j}\,k!}\frac{z^k}{(k+\eta)^s}\\ &=\Phi^{(\lambda_1,\ldots\lambda_p,1;\mu_1,\ldots,\mu_q)}_{(a_1,\ldots a_p,\rho_1+\ldots+\rho_n;b_1,\ldots,b_q)}(z,s,\eta). \end{aligned} \tag{2.4} $$

Now make an appeal to the conditions given in (1.4)-(1.7) *and* (1.9)*, the series in* (2.4) *converges due to the conditions*
$\sum_{j=1}^{q}\mu_j-\sum_{j=1}^{p}\lambda_j>0$, *if* $|z|<\infty$; $\sum_{j=1}^{q}\mu_j-\sum_{j=1}^{p}\lambda_j=0$, *if* $|z|<1$; *again for* $\sum_{j=1}^{q}\mu_j-\sum_{j=1}^{p}\lambda_j=0$, *along with* $z=1$, *if* $\sum_{j=1}^{q}b_j+s-\sum_{j=1}^{p}a_j-\rho_1-\ldots-\rho_n>0$.

3 Various connected known and unknown multiple Hurwitz-Lerch Zeta functions to the general multiple Hurwitz-Lerch Zeta function (1.10)

In this section, we derive various known and unknown multiple Hurwitz-Lerch Zeta functions as on manipulation of the multiple sequence of function $\chi(m_1,\ldots,m_n)$ $\forall$ $m_1\geq 0,\ldots,m_n\geq 0$ and on application of the Theorem 2.1.

Theorem 3.1. *In Eqn.* (1.10)*, if* $s\in\mathbb{C}$, $\eta\in\mathbb{C}\backslash\mathbb{Z}_0$, *max* $\{|z_1|(z_1\neq 1),\ldots,|z_n|(z_n\neq 1)\}<1$, *and*

$$ \sum_{j=1}^{C}\delta_j^{(l)}+\sum_{j=1}^{D^{(l)}}\phi_j^{(l)}-\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{B^{(l)}}\psi_j^{(l)}=0,\mathfrak{H}_l=\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{C}\delta_j^{(l)}\leq 0. \tag{3.1} $$

Again when $z_1=1,\ldots,$ $z_n=1$, $(\ \forall\ l=1,2,3,\ldots,n)$ *and there are*

$$ \sum_{j=1}^{C}\delta_j^{(l)}+\sum_{j=1}^{D^{(l)}}\phi_j^{(l)}-\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{B^{(l)}}\psi_j^{(l)}=0,\mathfrak{H}_l=\sum_{j=1}^{A}\theta_j^{(l)}-\sum_{j=1}^{C}\delta_j^{(l)}=0, $$

along with

$$ \Re\left(\sum_{j=1}^{C}c_j+\sum_{j=1}^{D^{(l)}}d_j^{(l)}+s-\sum_{j=1}^{A}a_j-\sum_{j=1}^{B^{(l)}}b_j^{(l)}-\rho_l\right)>0\quad(\ \forall\ l=1,2,3,\ldots,n). $$

Then by the coefficient

$$ \chi(m_1,\ldots,m_n)=\mathcal{H}^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}}(m_1,\ldots,m_n), \tag{3.2} $$

the formula (1.10) *is connected by a multiple Hurwitz-Lerch Zeta function* [16, Eqn.(4.1)] *based upon the Srivastava-Daoust hypergeometric series in several variables* [20, P. 37] *as*

$$ \begin{aligned} {}^s_\eta K\left(\mathcal{H}^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}};\rho_1,\ldots,\rho_n;z_1,\ldots,z_n\right)&={}^s_\eta F^{A:B^{(1)}+1;\ldots;B^{(n)}+1}_{C:D^{(1)};\ldots;D^{(n)}}\\ &\times\left(\begin{matrix}[(a):\theta^{(1)},\ldots,\theta^{(n)}]:[(b^{(1)}):\psi^{(1)}],\ [\rho_1:1];\ldots;[(b^{(n)}):\psi^{(n)}],[\rho_n:1];\\ [(c):\delta^{(1)},\ldots,\delta^{(n)}]:[(d^{(1)}):\phi^{(1)}];\ldots;[(d^{(n)}):\phi^{(n)}];\end{matrix}z_1,\ldots,z_n\right). \end{aligned} \tag{3.3} $$

Proof. Making an appeal to the formulae (1.10) and ((3.2), we get the series

$$(3.4)\quad {}_{\eta}^{s}K\left(\mathcal{H}_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}};\rho_1,\ldots,\rho_n;z_1,\ldots,z_n\right)$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\mathcal{H}_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}}(m_1,\ldots,m_n)\frac{(\rho_1)_{m_1}\ldots(\rho_n)_{m_n}\ z_1^{m_1}\ldots z_n^{m_n}}{m_1!\ldots m_n!\,(m_1+\ldots+m_n+\omega)^{\sigma}}.$$

Then in the series of (3.4), define (1.2) and the multiple hypergeometric function under the conditions given in the Theorem 3.1, we get the multiple Hurwitz-Lerch Zeta function (1.1)-(1.3) based upon the Srivastava-Daoust hypergeometric series in several variables [20, p.37] as given in (3.3). □

Special cases of the multiple Hurwitz-Lerch Zeta function based upon the Srivastava-Daoust hypergeometric series in several variables (3.3)

In (3.3) set $\delta_j^{(l)}=1,\ \phi_j^{(l)}=1,\ \theta_j^{(l)}=1,\ \psi_j^{(l)}=1,\ (\ \forall\ l=1,2,3,\ldots,n)$ and then for $s\in\mathbb{C},\ \eta\in\mathbb{C}\backslash\mathbb{Z}_0$, we find $C+D^{(l)}-A-B^{(l)}=0, A-C\leq 0\ \forall\ l=1,2,3,\ldots,n, \max\{|z_1|\,(z_1\neq 1),\ldots,|z_n|\,(z_n\neq 1)\}<1$.

But when $z_1=1,\ldots,\ z_n=1$, there exists $C+D^{(l)}-A-B^{(l)}=0,\ A-C=0$, along with

$$\Re\left(\sum_{j=1}^{C}c_j+\sum_{j=1}^{D^{(l)}}d_j^{(l)}+s-\sum_{j=1}^{A}a_j-\sum_{j=1}^{B^{(l)}}b_j^{(l)}-\rho_l\right)>0\quad(\ \forall\ l=1,2,3,\ldots,n),$$

then we get the multiple Hurwitz-Lerch Zeta function based upon the Srivastava-Panda hypergeometric series in several variables [18] as

$$(3.5)\quad {}_{\eta}^{s}K\left(\mathcal{H}_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}};\rho_1,\ldots,\rho_n;z_1,\ldots,z_n\right)$$

$$={}_{\eta}^{s}F_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)}+1;\ldots;B^{(n)}+1}\left(\begin{matrix}(a):\left(b^{(1)}\right),\ \rho_1;\ldots;\left(b^{(n)}\right),\rho_n;\\ (c):\left(d^{(1)}\right);\ldots;\left(d^{(n)}\right);\end{matrix}z_1,\ldots,z_n\right).$$

Further in cite (3.5), set $A=1,\ C=1,\ \ B^{(l)}=0,\ D^{(l)}=0,\ \left(b^{(l)}\right)=\left(d^{(l)}\right)\ (\ \forall\ \ l=1,2,3,\ldots,n),\ a,s\in\mathbb{C}$ and $\eta,\quad c\in\mathbb{C}\backslash\mathbb{Z}_0$, we find a multiple Hurwitz-Lerch Zeta function based upon the Lauricella's hypergeometric series in several variables [9] as

$$(3.6)\quad {}_{\eta}^{s}K\left(\mathcal{H}_{1:0;\ldots;0}^{1:0;\ldots;0};\rho_1,\ldots,\rho_n;z_1,\ldots,z_n\right)={}_{\eta}^{s}F_{1:0;\ldots;0}^{1:1;\ldots;1}\left(\begin{matrix}a:\ \rho_1;\ldots;\rho_n;\\ c:0;\ldots;0;\end{matrix}z_1,\ldots,z_n\right)$$

$$=F_D^{(n)}\left(a,\rho_1,\ldots,\rho_n;c;z_1,\ldots,z_n\right),$$

provided that $\max\left\{|z_1|\,(z_1\neq 1),\ldots,|z_n|\,(z_n\neq 1)\right\}<1$; as well as with aid of Theorem 2.1, for $z_1=\ldots=z_n=z$, it converges for $|z|<1\ (z\neq 1)$, and $\Re\,(c+s-a-\rho_1-\ldots-\rho_n)>0$ with $z=1$.

Again, in (3.5) put $n=2$, we find the double Hurwitz-Lerch Zeta function based upon the Kampé de Fériet hypergeometric series in two variables [21, p.63, Eqn.(16)] as

$$(3.7)\quad {}_{\eta}^{s}K\left(\mathcal{H}_{C:D^{(1)};D^{(2)}}^{A:B^{(1)};B^{(2)}};\rho_1,\rho_2;z_1,z_2\right)={}_{\eta}^{s}F_{C:D^{(1)};D^{(2)}}^{A:B^{(1)}+1;B^{(2)}+1}\left(\begin{matrix}(a):\left(b^{(1)}\right),\ \rho_1;\left(b^{(2)}\right),\rho_2;\\ (c):\left(d^{(1)}\right);\left(d^{(2)}\right);\end{matrix}z_1,z_2\right).$$

It is provided that $s\in\mathbb{C},\ \eta\in\mathbb{C}\backslash\mathbb{Z}_0$, we find $C+D^{(l)}-A-B^{(l)}=0$ and $\mathfrak{H}_l=A-C\leq 0,\ \ \forall\ \ l=1,2$; $\max\left\{|z_1|\,(z_1\neq 1),|z_2|\,(z_2\neq 1)\right\}<1$.

But when $z_1=1,\ z_2=1$, there exists $C+D^{(l)}-A-B^{(l)}=0$ and $\mathfrak{H}_l=A-C=0\quad\forall\ l=1,2$; along with

$$\Re\left(\sum_{j=1}^{C}c_j+\sum_{j=1}^{D^{(l)}}d_j^{(l)}+s-\sum_{j=1}^{A}a_j-\sum_{j=1}^{B^{(l)}}b_j^{(l)}-\rho_l\right)>0\quad(\forall\ l=1,2).$$

Obviously, by (3.7) we get a double zeta function due to Choi and Parmar [1]

$$(3.8)\quad {}_{\eta}^{s}K\left(\mathcal{H}_{1:0;0}^{1:0;0};\rho_1,\rho_2;z_1,z_2\right)={}_{\eta}^{s}F_{1:0;0}^{1:1;1}\left(\begin{matrix}a:\ \rho_1;\rho_2;\\ c:-;-;\end{matrix}z_1,z_2\right)=\phi_{a,\rho_1,\rho_2,c}\left(z_1,z_2,s,\eta\right).$$

provided that $\max\left\{|z_1|\,(z_1\neq 1),|z_2|\,(z_2\neq 1)\right\}<1,\ s\in\mathbb{C},\ \ \eta,c\in\mathbb{C}\backslash\mathbb{Z}_0$.

But when $(z_1=1),(z_2=1),\Re\,(c+s-a-\rho_1-\rho_2)>0,(\forall\ l=1,2)$ along with $\eta,c\in\mathbb{C}\backslash\mathbb{Z}_0$.

4 Integral representations of the multiple Hurwitz-Lerch Zeta function (1.10)

We use the Eulerian integral formula [8, 10, 11] and find an integral representation of the multiple Hurwitz-Lerch Zeta function (1.10) as

$$(4.1)\qquad {}_{\eta}^{s}K\left(\chi;\rho_1,\ldots,\rho_n;z_1,\ldots,z_n\right)$$
$$=\frac{1}{\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}\left\{\sum_{k_1=0,\ldots,k_n=0}^{\infty}\chi\left(k_1,\ldots,k_n\right)\prod_{j=1}^{n}\frac{(\rho_j)_{k_j}}{k_j!}\left(z_je^{-t}\right)^{k_j}\right\}dt,$$

provided that $\eta, s\in\mathbb{C}$ such that $\Re(s)>0$, $\Re(\eta)>0$.

Theorem 4.1. *If* $\zeta(u)=u\left(1+\zeta(u)\right)^{\beta+1}$, $\zeta(0)=0$ *and* $M=m_1k_1+\ldots+m_nk_n\leq N$*; and*

$$(4.2)\qquad \Delta_N^{(\alpha,\beta)}\left[\chi;m_1,\ldots,m_n;z_1,\ldots,z_n,t\right]=\sum_{k_1,\ldots,k_n=0}^{M\leq N}\frac{(-N)_M}{(\alpha+\beta N+1)_M}\chi\left(k_1,\ldots,k_n\right)\prod_{j=1}^{n}\frac{(\rho_j)_{k_j}}{k_j!}\left(z_je^{-t}\right)^{k_j},$$

then the exists an integral formula

$$(4.3)\qquad \frac{1}{\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}\left\{\sum_{N=0}^{\infty}u^N\binom{\alpha+(\beta+1)N}{N}\Delta_N^{(\alpha,\beta)}\left[\chi;m_1,\ldots,m_n;z_1,\ldots,z_n,t\right]\right\}dt$$
$$=\frac{\left(1+\zeta(u)\right)^{\alpha+1}}{\{1-\beta\zeta(u)\}}{}_{\eta}^{s}K\left(\chi;\rho_1,\ldots,\rho_n;z_1\{-\zeta\}^{m_1},\ldots,z_n\{-\zeta\}^{m_n}\right).$$

Proof. Consider left hand side of (4.3) and make an appeal to the formula (4.2) to get

$$(4.4)\qquad \frac{1}{\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}\sum_{N=0}^{\infty}u^N\binom{\alpha+(\beta+1)N}{N}\sum_{k_1,\ldots,k_n=0}^{m_1k_1+\ldots+m_nk_n\leq N}\frac{(-N)_{m_1k_1+\ldots+m_nk_n}}{(\alpha+\beta N+1)_{m_1k_1+\ldots+m_nk_n}}$$
$$\times\chi\left(k_1,\ldots,k_n\right)\frac{(\rho_1)_{k_1}}{k_1!}\cdots\frac{(\rho_n)_{k_n}}{k_n!}\left(z_1e^{-t}\right)^{k_1}\cdots\left(z_ne^{-t}\right)^{k_n}dt.$$

Now in the integrand of (4.4) use generalized formula due to Carlitz (see in Srivastava and Manocha [21, p.360, Eqn.(1)]), given by

$$(4.5)\qquad \sum_{N=0}^{\infty}u^N\binom{\alpha+(\beta+1)N}{N}\sum_{k=0}^{\frac{N}{m}}\frac{(-N)_{\mathrm{mk}}}{(\alpha+\beta N+1)_{\mathrm{mk}}}\frac{\gamma_kx^k}{k!}=\frac{\left(1+\zeta(u)\right)^{\alpha+1}}{\{1-\beta\zeta(u)\}}\sum_{k=0}^{\infty}\frac{\gamma_k}{k!}\left(x\{-\zeta(u)\}^m\right)^k,$$

and then we obtain

$$(4.6)\qquad \frac{\left(1+\zeta(u)\right)^{\alpha+1}}{\{1-\beta\zeta(u)\}\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}\sum_{k_1,\ldots,k_n=0}^{\infty}\chi\left(k_1,\ldots,k_n\right)\prod_{j=1}^{n}\frac{(\rho_j)_{k_j}}{k_j!}\left(z_je^{-t}\{-\zeta(u)\}^{m_j}\right)^{k_j}\mathrm{dt}.$$

In Eqn. (4.6) we use the formula (4.1) and derive right hand side of (4.3). □

Corollary 4.1. *If all conditions of the Theorem* 4.1 *are satisfied and if*

$$(4.7)\qquad \chi=\mathcal{H}_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}},\ \text{(given in (1.2))},$$

then by the definition *of Srivastava-Panda hypergeometric series in several variables* [18]*, there exists*

$$(4.8)\qquad \Delta_N^{(\alpha,\beta)}\left[\mathcal{H}_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}};m_1,\ldots,m_n;z_1,\ldots,z_n,t\right]$$
$$=F_{C+1:D^{(1)};\ldots;D^{(n)}}^{A+1:B^{(1)}+1;\ldots;B^{(n)}+1}\left(\begin{matrix}\left[(a):\theta^{(1)},\ldots,\theta^{(n)}\right],\ [-N:m_1,\ldots,m_n]:\\ \left[(c):\delta^{(1)},\ldots,\delta^{(n)}\right],[\alpha+\beta N+1:m_1,\ldots,m_n]:\end{matrix}\right.$$
$$\left.\begin{matrix}\left[\left(b^{(1)}\right):\psi^{(1)}\right],\ [\rho_1:1];\ldots;\left[\left(b^{(n)}\right):\psi^{(n)}\right],[\rho_n:1];\\ \left[\left(d^{(1)}\right):\phi^{(1)}\right];\ldots;\left[\left(d^{(n)}\right):\phi^{(n)}\right];\end{matrix}z_1e^{-t},\ldots,z_ne^{-t}\right).$$

and the integral representation

$$(4.9)\qquad \frac{\left(1+\zeta(u)\right)^{\alpha+1}}{\{1-\beta\zeta(u)\}}{}_{\eta}^{s}K\left(\mathcal{H}_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}};\rho_1,\ldots,\rho_n;z_1\{-\zeta(u)\}^{m_1},\ldots,z_n\{-\zeta(u)\}^{m_n}\right)$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty e^{-\eta t} t^{s-1} \left\{ \sum_{N=0}^{\infty} u^N \binom{\alpha+(\beta+1)N}{N} \right.$$

$$\times F^{A+1:B^{(1)}+1;\ldots;B^{(n)}+1}_{C+1:D^{(1)};\ldots;D^{(n)}} \left(\begin{matrix} \left[(a):\theta^{(1)},\ldots,\theta^{(n)}\right],\ [-N:m_1,\ldots,m_n]: \\ \left[(c):\delta^{(1)},\ldots,\delta^{(n)}\right],[\alpha+\beta N+1:m_1,\ldots,m_n]: \end{matrix} \right.$$

$$\left. \begin{matrix} \left[\left(b^{(1)}\right):\psi^{(1)}\right],\ [\rho_1:1];\ldots;\left[\left(b^{(n)}\right):\psi^{(n)}\right],[\rho_n:1]; \\ \left[\left(d^{(1)}\right):\phi^{(1)}\right];\ldots;\left[\left(d^{(n)}\right):\phi^{(n)}\right]; \end{matrix} z_1e^{-t},\ldots,z_ne^{-t} \right) dt$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty e^{-\eta t} t^{s-1} \left\{ \sum_{N=0}^{\infty} u^N \binom{\alpha+(\beta+1)N}{N} \quad \Delta_N^{(\alpha,\beta)} \left[\mathcal{H}^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}}; m_1,\ldots,m_n; z_1,\ldots,z_n,t \right] \right\} dt.$$

Corollary 4.2. *If all conditions of the Theorem* 4.1 *are satisfied and in the coefficients*

$$\chi = \mathcal{H}^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}},$$

we consider

(4.10) $$A=C=0; B^{(1)}=D^{(1)},\ldots,\ B^{(n)}=D^{(n)}; \theta^{(1)}=\ldots=\theta^{(n)}=1; \delta^{(1)}=\ldots=\delta^{(n)}=1;$$
$$\psi^{(1)}=\ldots=\psi^{(n)}=1; \phi^{(1)}=\ldots=\phi^{(n)}=1; \left(b^{(1)}\right)=\left(d^{(1)}\right);\ \ldots; \left(b^{(n)}\right)=\left(d^{(n)}\right).$$

Also let $m_1=\ldots=m_n=1$.

Then by the definition *of Lauricella's hypergeometric series in several variables* [9],*following results hold*

(4.11) $$\Delta_N^{(\alpha,\beta)} \left[\mathcal{H}^{0:B^{(1)};\ldots;B^{(n)}}_{0:B^{(1)};\ldots;B^{(n)}}; 1,\ldots,1; z_1,\ldots,z_n,t \right]$$
$$= F_D^{(n)}(-N,\rho_1,\ldots,\rho_n;\alpha+\beta N+1; z_1e^{-t},\ldots,z_ne^{-t}).$$

(4.12) $$\frac{(1+\zeta(u))^{\alpha+1}}{\{1-\beta\zeta(u)\}} {}^s_\eta K \left(\mathcal{H}^{0:B^{(1)};\ldots;B^{(n)}}_{0:B^{(1)};\ldots;B^{(n)}}; \rho_1,\ldots,\rho_n; z_1\{-\zeta(u)\},\ldots,z_n\{-\zeta(u)\} \right)$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty e^{-\eta t} t^{s-1} \left\{ \sum_{N=0}^{\infty} u^N \binom{\alpha+(\beta+1)N}{N} \quad \Delta_N^{(\alpha,\beta)} \left[\mathcal{H}^{0:B^{(1)};\ldots;B^{(n)}}_{0:B^{(1)};\ldots;B^{(n)}}; 1,\ldots,1; z_1,\ldots,z_n,t \right] \right\} dt.$$

In the Eqns. (4.11) *and* (4.12), *set* $z_1=\ldots=z_n=z$ *to get*

(4.13) $$\Delta_N^{(\alpha,\beta)} \left[\mathcal{H}^{0:B^{(1)};\ldots;B^{(n)}}_{0:B^{(1)};\ldots;B^{(n)}}; 1,\ldots,1; z,\ldots,z,t \right] = {}_2F_1(-N,\rho_1+\ldots+\rho_n;\alpha+\beta N+1; ze^{-t}).$$

(4.14) $${}^s_\eta K \left(\mathcal{H}^{0:B^{(1)};\ldots;B^{(n)}}_{0:B^{(1)};\ldots;B^{(n)}}; \rho_1,\ldots,\rho_n; z\{-\zeta(u)\},\ldots,z\{-\zeta(u)\} \right)$$
$$= \frac{(1+\zeta(u))^{\alpha+1}}{\{1-\beta\zeta(u)\}} \int_0^\infty \frac{e^{-\eta t}}{(1-z\{-\zeta(u)\}e^{-t})^{\rho_1+\ldots+\rho_n}} \frac{t^{s-1}}{\Gamma(s)} dt$$
$$= \frac{(1+\zeta(u))^{\alpha+1}}{\{1-\beta\zeta(u)\}} \int_0^\infty \frac{e^{-(\eta-\rho_1-\ldots-\rho_n)t}}{(e^t-z\{-\zeta(u)\})^{\rho_1+\ldots+\rho_n}} \frac{t^{s-1}}{\Gamma(s)} dt,$$

provided that $\Re(\eta-\rho_1-\ldots-\rho_n)>0$.

Again since we know that a relation of Lauricella's multiple hypergeometric function with the Appell's double hypergeometric function given by

(4.15) $$F_D^{(n)} = F_1,$$

and hence on setting $n=2$ *in* (4.11) *and* (4.12), *we get*

(4.16) $$\Delta_N^{(\alpha,\beta)} \left[\mathcal{H}^{0:B^{(1)};B^{(2)}}_{0:B^{(1)};B^{(2)}}; 1,1; z_1,z_2,t \right] = F_1(-N,\rho_1,\rho_2;\alpha+\beta N+1; z_1e^{-t}, z_2e^{-t}).$$

(4.17) $$\frac{(1+\zeta(u))^{\alpha+1}}{\{1-\beta\zeta(u)\}} {}^s_\eta K \left(\mathcal{H}^{0:B^{(1)};B^{(2)}}_{0:B^{(1)};B^{(2)}}; \rho_1,\rho_2; z_1\{-\zeta(u)\}, z_2\{-\zeta(u)\} \right)$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty e^{-\eta t} t^{s-1} \left\{ \sum_{N=0}^{\infty} u^N \binom{\alpha+(\beta+1)N}{N} \quad \Delta_N^{(\alpha,\beta)} \left[\mathcal{H}^{0:B^{(1)};B^{(2)}}_{0:B^{(1)};B^{(2)}}; 1,1; z_1,z_2,t \right] \right\} dt$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty e^{-\eta t} t^{s-1} \left\{ \sum_{N=0}^{\infty} u^N \binom{\alpha+(\beta+1)N}{N} \quad F_1(-N,\rho_1,\rho_2;\alpha+\beta N+1; z_1e^{-t}, z_2e^{-t}) \right\} dt.$$

5 Application in non-homogeneous initial value fractional differential equation

It is familiar that the Caputo fractional differential operator $\left({}^{C}D_{a+}^{\alpha}y\right)(x)$ is defined on a finite interval $[a,b]$ where $y(x)\in \mathrm{AC}^{n}[a,b]$, $\Re(\alpha)\geqq 0$ and for $n=[\Re(\alpha)]+1$ for $\alpha\notin\mathbb{N}_0$; $\mathbb{N}_0=\mathbb{N}\cup\{0\}$; $n=\alpha$ for $\alpha\in\mathbb{N}_0$. $\mathrm{AC}^{n}[a,b]=\left\{y:[a,b]\rightarrow\mathbb{C}\quad\text{and}\quad\left(D^{n-1}y\right)(x)\in AC[a,b]\ \left(D=\frac{d}{\mathrm{d}x}\right)\right\}$; again if $\varphi(t)\in L(a,b)$, then for $y(x)\in AC[a,b]\Leftrightarrow y(x)=c+\int_a^x\varphi(t)dt\Longrightarrow\frac{d}{\mathrm{d}x}y(x)=\varphi(x),\ \frac{d}{\mathrm{d}x}y(x)\Big|_{x\,=\,a}=c$; then there exists

$$\left({}^{C}D_{a+}^{\alpha}y\right)(x)=\frac{1}{\Gamma\left((n-\alpha)\right)}\int_0^x\frac{y^{(n)}(t)dt}{(x-t)^{\alpha+1-n}}\quad\left(x\in\mathbb{R}^{+}\right). \tag{5.1}$$

(See in [5, p.97]).

Again if $0<\alpha\leqq 1$, then by [4, p.98] where $(\mathcal{L}y)(\eta)=\frac{1}{\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}y(t)dt,\quad \eta\in\mathbb{C},\ \Re(\eta)>0$, there exists

$$\left(\mathcal{L}^{C}D_{0+}^{\alpha}y\right)(\eta)=\eta^{\alpha}\left(\mathcal{L}y\right)(\eta)-\eta^{\alpha-1}y(0). \tag{5.2}$$

Theorem 5.1. *In reference of* (5.1), *for* $t\in\mathbb{R}^{+}=(0,\infty)$, *if we introduce a non-homogeneous initial value fractional differential equation as*

$$\left({}^{C}D_{a+}^{\alpha}y\right)(t)+y(t)=\frac{t^{s-1}}{\Gamma(s)}\left\{\sum_{k_1=0,\ldots,k_n=0}^{\infty}\chi\left(k_1,\ldots,k_n\right)\prod_{j=1}^{n}\frac{(\rho_j)_{k_j}}{k_j!}\left(z_je^{-t}\right)^{k_j}\right\};y(0)=0; \tag{5.3}$$

provided that $s\in\mathbb{C}$, $\Re(s)>0$.

Then for $s\in\mathbb{C}$, $\Re(s)>0,\quad 0<\alpha\leqq 1$, *there exists*

$$y(t)=\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}e^{t\eta}\frac{{}_{\eta}^{s}K(\chi;\rho_1,...,\rho_n;z_1,...,z_n)}{(\eta^{\alpha}+1)}d\eta\qquad(\tau=\Re(\eta)>0,\quad i=\sqrt{(-1)}). \tag{5.4}$$

Proof. Consider $\eta\in\mathbb{C}$,$\Re(\eta)>0$, and take *Laplace transformation* of both sides of the Eqn. (5.3) and then use formulae (4.1) and (5.2) with initial value given in (5.3), we get

$$\left(\eta^{\alpha}+1\right)\left(\mathcal{L}y\right)(\eta)={}_{\eta}^{s}K\left(\chi;\rho_1,\ldots,\rho_n;z_1,\ldots,z_n\right), \tag{5.5}$$

which on taking *inverse Laplace transformation* gives a solution

$$y(t)=\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}e^{t\eta}\frac{{}_{\eta}^{s}K(\chi;\rho_1,...,\rho_n;z_1,...,z_n)}{(\eta^{\alpha}+1)}d\eta\qquad(\tau=\Re(\eta)>0,\quad i=\sqrt{(-1)}) \tag{5.6}$$

provided that $s\in\mathbb{C}$, $\Re(s)>0,\quad 0<\alpha\leqq 1$. □

6 Concluding remarks

In this section, in the Corollary 2.2, set $p=1,\ q=1,\ a_1=\rho_1+\ldots+\rho_n-\frac{1}{2},\ b_1=2\left(\rho_1+\ldots+\rho_n\right)\in\mathbb{C}\backslash\mathbb{Z}_0$, $\lambda_1=1$, $\mu_1=1$ and then apply the result [12, p.70].Thus we find an interesting Zeta function for $\Re(s)>0,\Re(\eta)>0$ as

$$\begin{aligned}{}_{\eta}^{s}K\left(\chi;\rho_1,\ldots,\rho_n;z,\ldots,z\right)&=\sum_{k=0}^{\infty}\frac{(a_1)_k\ \ (\rho_1+\ldots+\rho_n)_k}{(b_1)_k\ \ \ k!}\frac{z^k}{(k+\eta)^s}=\Phi^{(1,1;1)}_{(a_1,\rho_1+\ldots+\rho_n;b_1)}(z,s,\eta)\\
&=\frac{1}{\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}\left\{{}_2F_1\left(\begin{matrix}\rho_1+\ldots+\rho_n-\ \frac{1}{2},\rho_1+\ldots+\rho_n;\\ 2\left(\rho_1+\ldots+\rho_n\right);\end{matrix}ze^{-t}\right)\right\}dt\\
&=\frac{1}{\Gamma(s)}\int_0^\infty e^{-\eta t}t^{s-1}\left\{\left(\frac{2}{1+\sqrt{(1-ze^{-t})}}\right)^{2(\rho_1+\ldots+\rho_n)\ -1}\right\}dt\\
&=\frac{1}{\Gamma(s)}\int_0^\infty e^{-(\eta\ +\ \ \frac{1}{2}\ -\rho_1-\ldots-\rho_n)t}t^{s-1}\left\{\left(\frac{2}{e^{\frac{t}{2}}\ +\sqrt{(e^t-z)}}\right)^{2(\rho_1+\ldots+\rho_n)\ -1}\right\}dt,\end{aligned} \tag{6.1}$$

provided $\Re\left(\eta\ +\ \ \frac{1}{2}\ -\rho_1-\ldots-\rho_n\right)>0$.

Again, in the Corollary 2.2, set $p=1,\ q=1,\ a_1=\rho_1+\ldots+\rho_n+\frac{1}{2},\ b_1=2\left(\rho_1+\ldots+\rho_n\right)\in\mathbb{C}\backslash\mathbb{Z}_0$,

$\lambda_1 = 1$, $\mu_1 = 1$, and then on application of the result [12, p.70], we find another interesting Zeta function for $\Re(s) > 0, \Re(\eta) > 0$ as

$$(6.2) \qquad {}^{s}_{\eta}K\left(\chi;\rho_1,\ldots,\rho_n;z,\ldots,z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ (\rho_1+\ldots+\rho_n)_k}{(b_1)_k \quad k!} \frac{z^k}{(k+\eta)^s} = \Phi^{(1,1;1)}_{(a_1,\rho_1+\ldots+\rho_n;b_1)}(z,s,\eta)$$

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-\eta t} t^{s-1} \left\{ {}_2F_1\left(\begin{matrix} \rho_1+\ldots+\rho_n+\ \frac{1}{2}, \rho_1+\ldots+\rho_n; \\ 2\left(\rho_1+\ldots+\rho_n\right); \end{matrix} ze^{-t} \right) \right\} dt$$

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-(\eta\ -\rho_1-\ldots-\rho_n)t} t^{s-1} \left(\frac{1}{\sqrt{(e^t-z)}} \right) \left\{ \left(\frac{2}{e^{\frac{t}{2}}\ +\sqrt{(e^t-z)}} \right)^{2(\rho_1+\ldots+\rho_n)\ -1} \right\} dt,$$

provided $\Re\left(\eta\ +\ \ \frac{1}{2}\ -\rho_1-\ldots-\rho_n\right) > 0$.

Various generating relations and integral representations may be found by applying results obtained in the *Sections* 3 to 5.

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