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(Dedicated to Professor V. P. Saxena on His 80^{th} Birth Anniversary Celebrations)

AN ANALYTICAL STUDY OF SPACE-TIME FRACTIONAL ORDER GAS DYNAMIC EQUATIONS

R. K. Bairwa and Karan Singh
Department of Mathematics, University of Rajasthan, Jaipur, Rajasthan-302004, India.
Email: dr.rajendra.maths@gmail.com, karansinghmath@gmail.com
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Abstract

In this article, the Sumudu transform with iterative method is implemented to obtain approximate analytical solutions in series form to non-linear homogeneous and non-homogeneous space-time fractional gas dynamic equations. The fractional derivatives presented here are in the Caputo sense. Furthermore, the findings of this study are graphically represented and the solution graphs demonstrate a strong connection between the approximate and exact solutions.
2020 Mathematical Sciences Classification: 33E12, 26A33, 35A22, 35A24, 35G25.
Keywords and Phrases: Gas dynamic equations, Sumudu transform, iterative method, Mittag-Leffler functions, Caputo fractional derivatives, fractional differential equations.

1 Introduction

Fractional calculus is a branch of applied mathematics that is extremely useful in a variety of fields of research [13, 21]. The fractional differential equations have sparked the interest of a vast scope of researchers working on a variety of applications [2, 3, 10, 28, 31]. Many efforts have been made to develop analytical and numerical approaches for solving differential equations of fractional order, such as the homotopy analysis method (*HAM*) [17], the q-homotopy analysis method (q-*HAM*) [14], the optimal q-homotopy analysis method (Oq-*HAM*) [32], the homotopy analysis transform method (*HATM*) [30], the adomian decomposition method (*ADM*) [12], the Laplace decomposition method (*LDM*) [18], the homotopy perturbation method (*HPM*) [19], the homotopy perturbation transform method (*HPTM*) [20, 23], and so on.

In 2006, Daftardar-Gejji and Jafari [8, 15] proposed an iterative method for numerically solving nonlinear functional equations. Since then, the iterative technique has been used to solve a wide variety of nonlinear differential equations of integer and fractional order [5] as well as fractional boundary value problems [7]. Recently, Wang and Liu [33] introduced the Sumudu transform iterative method (*STIM*) by combining the Sumudu transform with an iterative technique to determine approximate analytical solutions of time-fractional Cauchy reaction diffusion equations. The Sumudu transform iterative technique has been used successfully to solve a variety of time and space fractional partial differential equations and related systems [22], as well as the random component time-fractional Klein-Gordon equation [27].

In this work, we consider the non-linear homogeneous and non-homogeneous fractional gas dynamic equations with space and time fractional derivatives as follows
(i) The non-linear homogeneous space-time-fractional gas dynamic partial differential equation of the form

$$D_t^{\alpha}u(x,t)+\frac{1}{2}D_x^{\beta}u^2(x,t)-u(x,t)\big(1-u(x,t)\big)=0, \quad 0<\alpha,\beta\leq 1, \tag{1.1}$$

$$u(x,0)=g(x), \tag{1.2}$$

(ii) The non-linear non-homogeneous space-time-fractional gas dynamic partial differential equation of the form

$$D_t^{\alpha}u(x,t)+\frac{1}{2}D_x^{\beta}u^2(x,t)-u(x,t)\big(1-u(x,t)\big)=f(x,t), \quad 0<\alpha,\beta\leq 1, \tag{1.3}$$

$$u(x,0)=g(x), \tag{1.4}$$

where α and β are the parameters that describe the order of the time-fractional and space-fractional derivatives, respectively. Also, $u(x,t)$ is the probability density function and f is a known analytic function.

2 Preliminaries and Basic Definitions

This section introduces some fundamental definitions, notations, and properties of fractional calculus utilizing Sumudu transform theory, which will be applied later in this paper.

Definition 2.1. *The Caputo fractional derivative of a function $u(x,t)$ is defined as* [28, 31]

$$D_t^{\alpha}u(x,t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\eta)^{m-\alpha-1}u^{(m)}(x,\eta)d\eta,\ m-1<\alpha\le m, m\in\mathbb{N}. \tag{2.1}$$

Definition 2.2. *The Sumudu transform is defined over the set of functions*

$$\Big\{f(t)|\exists M,\rho_1>0,\rho_2>0,|f(t)|<Me^{|t|/\rho_j}\ \text{if}\ t\in(-1)^j\times[0,\infty),\ j=1,2\Big\}$$

by the following formula [4, 34]

$$S[f(t)] = F(\omega) = \int_0^{\infty} e^{-t}f(\omega t)dt\ ,\ \omega\in(-\rho_1,\rho_2). \tag{2.2}$$

Definition 2.3. *The Sumudu transform of Caputo fractional derivative is defined in the following manner* [9, 33]

$$S[D_t^{\alpha}u(x,t)] = \omega^{-\alpha}S[u(x,t)] - \sum_{k=0}^{m-1}\omega^{-\alpha+k}u^{(k)}(x,0),\ m-1<\alpha\le m\ ,\ m\in\mathbb{N}, \tag{2.3}$$

where $u^{(k)}(x,0)$ is the k-order derivative of $u(x,t)$ with respect to t at $t=0$.

Definition 2.4. *The Mittag-Leffler function, a generalization of the exponential function, is defined as follows* [28, 31]

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}\ ,\alpha\in\mathbb{C}, Re(\alpha)>0. \tag{2.4}$$

A further generalization of equation (2.4) is as follows [35]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}\ ,\alpha,\beta\in\mathbb{C}, Re(\alpha)>0, Re(\beta)>0, \tag{2.5}$$

where $\Gamma(.)$ is the well-known Gamma function.

3 Basic Idea of Sumudu Transform Iterative Method

To explain the basic idea of the Sumudu transform iterative technique [33], we take the following space and time general fractional partial differential equation having the prescribed initial conditions may be written in the form of an operator as

$$\begin{aligned} D_t^{\alpha}u(x,t) =& F\big[x,u(x,t),D_x^{\beta}u(x,t),...,D_x^{l\beta}u(x,t)\big], \\ & l-1<\alpha\le l,\ m-1<\beta\le m;\ l,m\in\mathbb{N} \end{aligned} \tag{3.1}$$

$$u^{(k)}(x,0) = h_k(x),\ k=0,1,2,...,n-1, \tag{3.2}$$

where $D_t^{\alpha}u(x,t)$ and $D_x^{\beta}u(x,t)$ are the Caputo fractional derivatives of order α, $l-1<\alpha\le l$ and β, $m-1<\beta\le m$, respectively, defined by the equation (2.1), $F\big[x,u,D_x^{\beta}u,...,D_x^{l\beta}u\big]$ is a linear/non-linear operator and $u=u(x,t)$ is the unknown function and fractional derivative $D_x^{l\beta}u(x,t), l\in\mathbb{N}$ is taken as the sequential fractional derivative [28] that is

$$D_x^{l\beta}u = D_x^{\beta}D_x^{\beta},...,D_x^{\beta}u\ \ (l\ times). \tag{3.3}$$

Applying the Sumudu transform on both sides of equation (3.1), we have

$$S\Big[D_t^{\alpha}u(x,t)\Big] = S\Big[F\big(x,u(x,t),D_x^{\beta}u(x,t),...,D_x^{l\beta}u(x,t)\big)\Big]. \tag{3.4}$$

Using the differentiation property of the Sumudu transform, we get

$$S[u(x,t)] = \omega^{\alpha}\sum_{k=0}^{m-1}\Big[\omega^{-\alpha+k}u^{(k)}(x,0)\Big] + \omega^{\alpha}S\Big[F\big(x,u,D_x^{\beta}u,...,D_x^{l\beta}u\big)\Big]. \tag{3.5}$$

On taking inverse Sumudu transform of equation (3.5), we have

$$u(x,t) = S^{-1}\Big[\omega^{\alpha}\sum_{k=0}^{m-1}\Big[\omega^{-\alpha+k}u^{(k)}(x,0)\Big]\Big] + S^{-1}\Big[\omega^{\alpha}S\Big[F\big(x,u,D_x^{\beta}u,...,D_x^{l\beta}u\big)\Big]\Big]. \tag{3.6}$$

Equation (3.6) may be written as

$$u(x,t) = f(x,t) + N\Big(x, u, D_x^{\beta}u, ..., D_x^{l\beta}u\Big), \tag{3.7}$$

where

$$f(x,t) = S^{-1}\Big[\omega^{\alpha}\sum_{k=0}^{m-1}\Big[\omega^{-\alpha+k}u^{(k)}(x,0)\Big]\Big], \tag{3.8}$$

$$N\Big(x, u, D_x^{\beta}u, ..., D_x^{l\beta}u\Big) = S^{-1}\Big[\omega^{\alpha}S\Big[F\big(x, u, D_x^{\beta}u, ..., D_x^{l\beta}u\big)\Big]\Big]. \tag{3.9}$$

Here N is a linear/nonlinear operator and f is a known function .
Furthermore, we employ the iterative method proposed by Daftardar-Gejji and Jafari [8], which represents a solution in an infinite series of components as

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t). \tag{3.10}$$

The operator N is decomposed as follows

$$\begin{aligned} N\Big(x, \sum_{i=0}^{\infty}u_i, D_x^{\beta}\Big(\sum_{i=0}^{\infty}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{\infty}u_i\Big)\Big) &= N\Big(x, u_0, D_x^{\beta}u_0, ..., D_x^{l\beta}u_0\Big) \\ &+ \sum_{j=1}^{\infty}\Big[N\Big(x, \sum_{i=0}^{j}u_i, D_x^{\beta}\Big(\sum_{i=0}^{j}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{j}u_i\Big)\Big)\Big] \\ &- \sum_{j=1}^{\infty}\Big[N\Big(x, \sum_{i=0}^{j-1}u_i, D_x^{\beta}\Big(\sum_{i=0}^{j-1}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{j-1}u_i\Big)\Big)\Big], \end{aligned} \tag{3.11}$$

$$\begin{aligned} &S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, \sum_{i=0}^{\infty}u_i, D_x^{\beta}\Big(\sum_{i=0}^{\infty}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{\infty}u_i\Big)\Big)\Big]\Big] \\ &= S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, u_0, D_x^{\beta}u_0, ..., D_x^{l\beta}u_0\Big)\Big]\Big] \\ &+ \sum_{j=0}^{\infty}\Big[S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, \sum_{i=0}^{j}u_i, D_x^{\beta}\Big(\sum_{i=0}^{j}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{j}u_i\Big)\Big)\Big]\Big]\Big] \\ &- \sum_{j=0}^{\infty}\Big[S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, \sum_{i=0}^{j-1}u_i, D_x^{\beta}\Big(\sum_{i=0}^{j-1}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{j-1}u_i\Big)\Big)\Big]\Big]\Big]. \end{aligned} \tag{3.12}$$

Using equations (3.10) to (3.12) in equation (3.7), we obtain

$$\begin{aligned} \sum_{i=0}^{\infty}u_i(x,t) &= S^{-1}\Big[\omega^{\alpha}\sum_{k=0}^{m-1}\Big(\omega^{-\alpha+k}u^{(k)}(x,0)\Big)\Big] \\ &+ S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, u_0, D_x^{\beta}u_0, ..., D_x^{l\beta}u_0\Big)\Big]\Big] \\ &+ \sum_{j=0}^{\infty}\Big[S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, \sum_{i=0}^{j}u_i, D_x^{\beta}\Big(\sum_{i=0}^{j}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{j}u_i\Big)\Big)\Big]\Big]\Big] \\ &- \sum_{j=0}^{\infty}\Big[S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, \sum_{i=0}^{j-1}u_i, D_x^{\beta}\Big(\sum_{i=0}^{j-1}u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{j-1}u_i\Big)\Big)\Big]\Big]\Big]. \end{aligned} \tag{3.13}$$

The recurrence relations have been defined as follows

$$u_0(x,t) = S^{-1}\Big[\omega^{\alpha}\sum_{k=0}^{m-1}\Big(\omega^{-\alpha+k}u^{(k)}(x,0)\Big)\Big], \tag{3.14}$$

$$u_1(x,t) = S^{-1}\Big[\omega^{\alpha}S\Big[F\Big(x, u_0, D_x^{\beta}u_0, ..., D_x^{l\beta}u_0\Big)\Big]\Big], \tag{3.15}$$

$$u_{r+1}(x,t) = S^{-1}\Big[\omega^\alpha S\Big[F\Big(x, \sum_{i=0}^{r} u_i, D_x^\beta\Big(\sum_{i=0}^{r} u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{r} u_i\Big)\Big)\Big]\Big] - S^{-1}\Big[\omega^\alpha S\Big[F\Big(x, \sum_{i=0}^{r-1} u_i, D_x^\beta\Big(\sum_{i=0}^{r-1} u_i\Big), ..., D_x^{l\beta}\Big(\sum_{i=0}^{r-1} u_i\Big)\Big)\Big]\Big], r \geq 1. \tag{3.16}$$

Therefore, the approximate analytical solution of equations (3.1) and (3.2) in truncated series form is given by

$$u(x,t) \cong \lim_{\mathbb{N}\to\infty} \sum_{m=0}^{\mathbb{N}} u_m(x,t). \tag{3.17}$$

In general, the solutions in the above series converge quickly. The classical approach to the convergence of this type of series has been presented by Bhalekar and Daftardar-Gejji [6] and Daftardar-Gejji and Jafari [8].

4 Solution of the Space-Time Fractional Gas Dynamic Equations

In this section, we make an attempt to solve non-linear homogeneous and non-homogeneous space-time fractional gas dynamic equations by means of the Sumudu transform iterative method.

Example 4.1. Consider the following non-linear homogeneous space-time fractional gas dynamic equation [30, 32]

$$D_t^\alpha u(x,t) + \frac{1}{2} D_x^\beta u^2(x,t) - u(x,t)(1-u(x,t)) = 0, \quad t>0, \ 0<\alpha,\beta \leq 1, \tag{4.1}$$

with the initial condition

$$u(x,0) = e^{-x}. \tag{4.2}$$

Taking the Sumudu transform on the both sides of equation (4.1), and making use of the result given by equation (4.2), we have

$$S[u(x,t)] = e^{-x} + \omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta u^2(x,t)}{\partial x^\beta} + u(x,t)(1-u(x,t))\Big]. \tag{4.3}$$

On taking inverse Sumudu transform of equation (4.3), we get

$$u(x,t) = e^{-x} + S^{-1}\Big[\omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta u^2(x,t)}{\partial x^\beta} + u(x,t)(1-u(x,t))\Big]\Big]. \tag{4.4}$$

Substituting the results from equations (3.10) to (3.12) in the equation (4.4) and applying the equations (3.14) to (3.16), we determine the components of the solution as follows

$$u_0(x,t) = u(x,0) = e^{-x}, \tag{4.5}$$

$$\begin{aligned} u_1(x,t) &= S^{-1}\Big[\omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta u_0^2}{\partial x^\beta} + u_0(1-u_0)\Big]\Big] \\ &= -e^{-x}\Big[\frac{t^\alpha}{\Gamma(\alpha+1)}\Big]\Big(2^{-1+\beta}e^{-x+i\beta\pi} - 1 + e^{-x}\Big), \end{aligned} \tag{4.6}$$

$$\begin{aligned} u_2(x,t) &= S^{-1}\Big[\omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta (u_0+u_1)^2}{\partial x^\beta} + (u_0+u_1)\big(1-(u_0+u_1)\big)\Big]\Big] \\ &\quad - S^{-1}\Big[\omega^\alpha S\Big[-\frac{1}{2}\frac{\partial^\beta u_0^2}{\partial x^\beta} + u_0(1-u_0)\Big]\Big] \\ &= \Big[-e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 2e^{-2x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big](2^{-1+\beta}e^{-x+i\beta\pi} - 1 + e^{-x}) \\ &\quad + \Big[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big](3^\beta e^{-3x+i\beta\pi} - 2^\beta e^{-2x+i\beta\pi} + 2^{-1+\beta}3^\beta e^{-3x+2i\beta\pi}), \end{aligned} \tag{4.7}$$

and so on. The remaining components may be obtained in the same way.

Thus, the approximate analytical solution in the series form can be obtained as

$$u(x,t) \cong \lim_{\mathbb{N}\to\infty} \sum_{m=0}^{\mathbb{N}} u_m(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t)+, ..., \tag{4.8}$$

$$
\begin{aligned}
&= e^{-x} - e^{-x}\Big[\frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big]\Big(2^{-1+\beta}e^{-x+i\beta\pi} - 1 + e^{-x}\Big)\\
&+ \Big[-e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 2e^{-2x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big](2^{-1+\beta}e^{-x+i\beta\pi} - 1 + e^{-x})\\
&+ \Big[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big](3^{\beta}e^{-3x+i\beta\pi} - 2^{\beta}e^{-2x+i\beta\pi} + 2^{-1+\beta}3^{\beta}e^{-3x+2i\beta\pi}) +, \ldots, \ .
\end{aligned}
$$

The same result was obtained by Saad *et al.* [32] by using the method of optimal *q-HAM.*
If we put $\alpha = \beta = 1$, in equation (4.8), we have the result in simple form

$$u(x,t) = e^{t-x}, \tag{4.9}$$

which is the exactly the same solution obtained by earlier by Jafari *et al.* [19] by using *HPM* method.

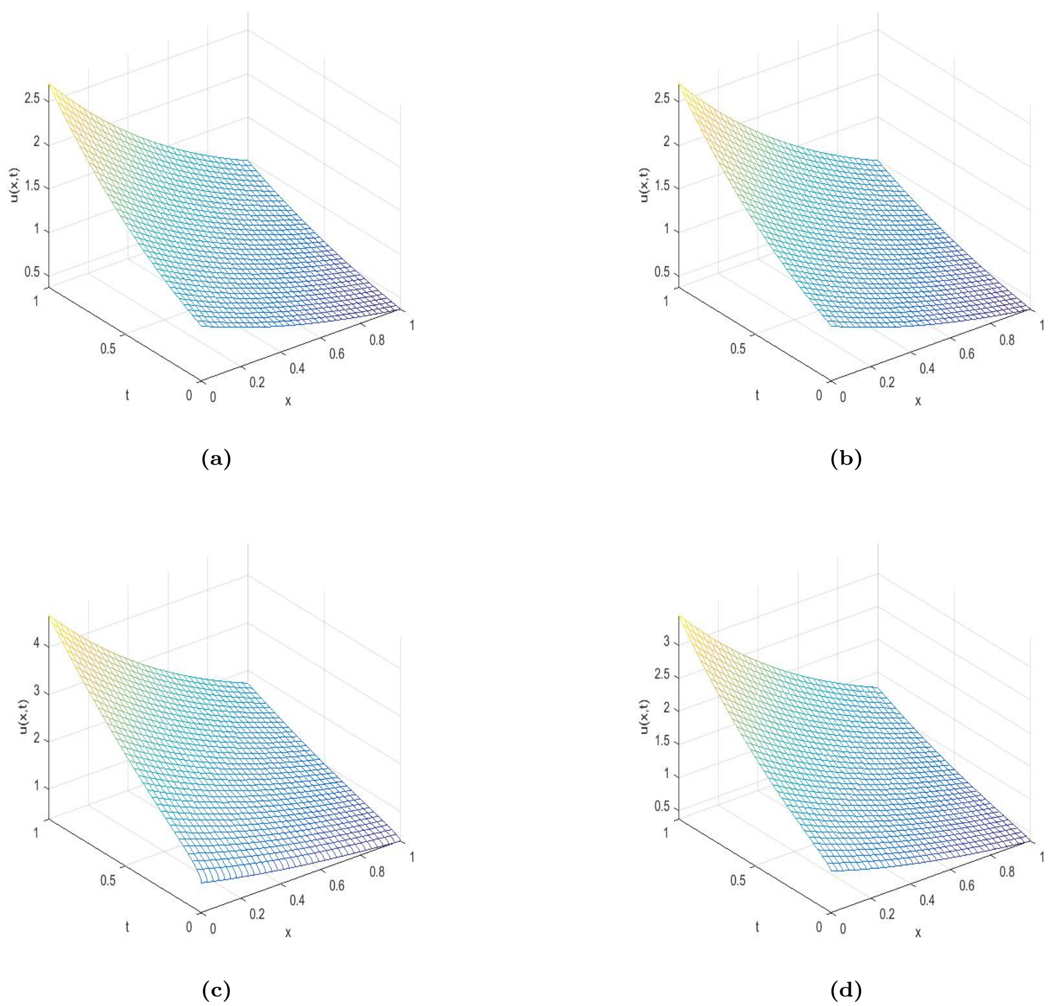


***Figure* 4.1:** Graph of the $u(x,t)$ for Example 4.1, when $\beta = 1$: (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.75$.

Example 4.2. Consider the following non-linear non-homogeneous space-time fractional gas dynamic equation [32]

$$D_t^{\alpha}u(x,t)+\frac{1}{2}D_x^{\beta}u^2(x,t)-u(x,t)(1-u(x,t))=-e^{t-x},\quad t>0,\ 0<\alpha,\beta\le 1, \tag{4.10}$$

with the initial condition

$$u(x,0)=1-e^{-x}. \tag{4.11}$$

Taking the Sumudu transform on the both sides of equation (4.10), and making use of the result given by equation (4.11), we have

$$S[u(x,t)]=1-e^{-x}+\omega^{\alpha}S\Big[-\frac{1}{2}\frac{\partial^{\beta}u^2(x,t)}{\partial x^{\beta}}+u(x,t)(1-u(x,t))-e^{t-x}\Big]. \tag{4.12}$$

On taking inverse Sumudu transform of equation (4.12), we get

$$u(x,t)=1-e^{-x}+S^{-1}\Big[\omega^{\alpha}S\Big[-\frac{1}{2}\frac{\partial^{\beta}u^2(x,t)}{\partial x^{\beta}}+u(x,t)(1-u(x,t))-e^{t-x}\Big]\Big]. \tag{4.13}$$

Substituting the results from equations (3.10) to (3.12) in the equation (4.13) and applying the equations (3.14) to (3.16), we determine the components of the solution as follows

$$u_0(x,t)=u(x,0)=1-e^{-x}, \tag{4.14}$$

$$\begin{aligned} u_1(x,t)&=S^{-1}\Big[\omega^{\alpha}S\Big[-\frac{1}{2}\frac{\partial^{\beta}u_0^2}{\partial x^{\beta}}+u_0(1-u_0)-e^{t-x}\Big]\Big]\\ &=-e^{-x}\Big[\frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big]\Big(-e^{i\beta\pi}+2^{-1+\beta}e^{-x+i\beta\pi}-1+e^{-x}\Big)-e^{-x}t^{\alpha}E_{1,\alpha+1}(t), \end{aligned} \tag{4.15}$$

$$\begin{aligned} u_2(x,t)&=S^{-1}\Big[\omega^{\alpha}S\Big[-\frac{1}{2}\frac{\partial^{\beta}(u_0+u_1)^2}{\partial x^{\beta}}+(u_0+u_1)\big(1-(u_0+u_1)\big)-e^{t-x}\Big]\Big]\\ &-S^{-1}\Big[\omega^{\alpha}S\Big[-\frac{1}{2}\frac{\partial^{\beta}u_0^2}{\partial x^{\beta}}+u_0(1-u_0)-e^{t-x}\Big]\Big]\\ &=t^{2\alpha}(-e^{i\pi\beta}+2^{-1+\beta}e^{-x+i\pi\beta}-1+e^{-x})\Big(-\frac{2e^{-2x}}{\Gamma(2\alpha+1)}+\frac{e^{-x}t^{2\alpha}}{\Gamma(2\alpha+1)}\Big)\\ &-\frac{(3^{\beta}e^{-3x+i\pi\beta}-2^{\beta}e^{-2x+i\pi\beta}+2^{-1+\beta}3^{\beta}e^{-3x+2i\pi\beta}-2^{\beta}e^{-2x+2i\pi\beta})t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &+\frac{(2^{\beta}e^{-2x+i\pi\beta}-e^{-x+i\pi\beta}+2^{\beta}e^{-2x+2i\pi\beta}-e^{-x+2i\pi\beta})t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &+\big(-2e^{-2x}E_{1,2\alpha+1}(t)+e^{-x}E_{1,2\alpha+1}(t)\\ &\qquad+e^{-x+i\pi\beta}E_{1,2\alpha+1}(t)-2^{\beta}e^{-2x+i\pi\beta}E_{1,2\alpha+1}(t)\big)t^{2\alpha}, \end{aligned} \tag{4.16}$$

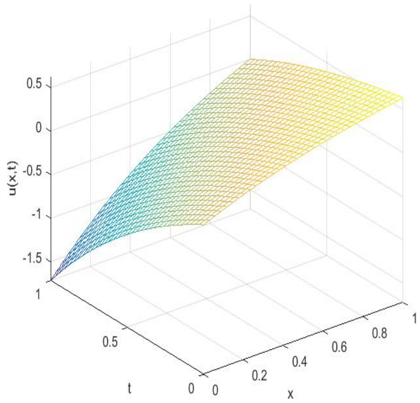
and so on. The remaining components may be obtained in the same way.
Thus, the approximate analytical solution in the series form can be obtained as

$$\begin{aligned} u(x,t)&\cong\lim_{\mathbb{N}\to\infty}\sum_{m=0}^{\mathbb{N}}u_m(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+,...,\\ &=1-e^{-x}-e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\big(-e^{i\pi\beta}+2^{-1+\beta}e^{-x+i\pi\beta}-1+e^{-x}\big)-e^{-x}\ t^{\alpha}E_{1,\alpha+1}(t)\\ &+t^{2\alpha}(-e^{i\pi\beta}+2^{-1+\beta}e^{-x+i\pi\beta}-1+e^{-x})\Big(-\frac{2e^{-2x}}{\Gamma(2\alpha+1)}+\frac{e^{-x}t^{2\alpha}}{\Gamma(2\alpha+1)}\Big)\\ &-\frac{(3^{\beta}e^{-3x+i\pi\beta}-2^{\beta}e^{-2x+i\pi\beta}+2^{-1+\beta}3^{\beta}e^{-3x+2i\pi\beta}-2^{\beta}e^{-2x+2i\pi\beta})t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &+\frac{(2^{\beta}e^{-2x+i\pi\beta}-e^{-x+i\pi\beta}+2^{\beta}e^{-2x+2i\pi\beta}-e^{-x+2i\pi\beta})t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &+\big(-2e^{-2x}E_{1,2\alpha+1}(t)+e^{-x}E_{1,2\alpha+1}(t)\\ &\qquad+e^{-x+i\pi\beta}E_{1,2\alpha+1}(t)-2^{\beta}e^{-2x+i\pi\beta}E_{1,2\alpha+1}(t)\big)t^{2\alpha}+,...,\quad. \end{aligned} \tag{4.17}$$

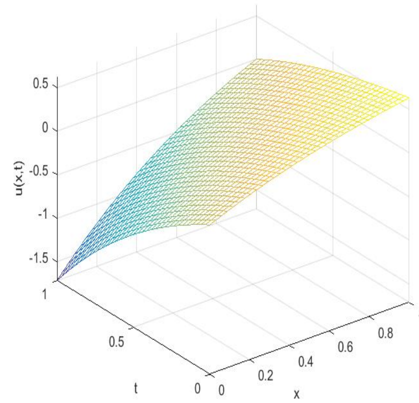
The same result was obtained by Saad *et al.* [32] by using the method of optimal *q-HAM*.
If we put $\alpha = \beta = 1$, in equation (4.17), we have the result in simple form

$$u(x,t) = 1 - e^{t-x}, \tag{4.18}$$

which is the exactly the same solution obtained by earlier by Jafari *et al.* [16] by using two-dimensional *DTM* method.



(a)



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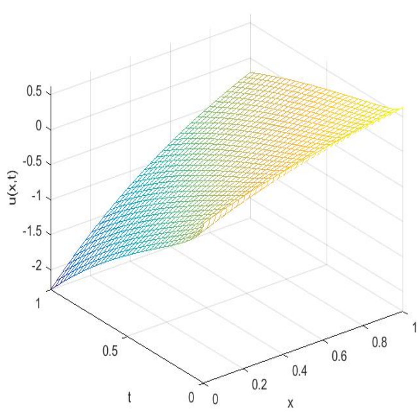

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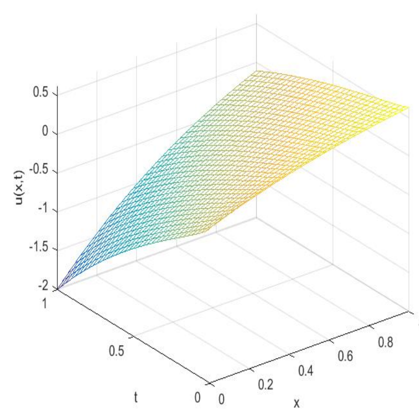

(d)

***Figure* 4.2:** Graph of the $u(x,t)$ for Example 4.2, when $\beta = 1$: (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.75$.

5 Conclusion

In this paper, we have successfully and efficiently applied the Sumudu transform iterative method (*STIM*) to derive the approximate analytical solutions of the non-linear homogeneous and non-homogeneous space-time fractional gas dynamic equations with Caputo fractional derivatives. *STIM* is a hybrid approach of the Sumudu transform and the iterative method. The graphical representation of the obtained solutions was completed successfully by the *MATLAB* software. The analytical results derived from the proposed approach indicate that the method is simple to use and precise.

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References

[1] A. Akgl, A. Cordero and J.R. Torregrosa, Solutions of fractional gas dynamics equation by a new technique, *Math. Methods Appl. Sci.*, **43**(3)(2020), 1349-1358.

[2] T.M. Atanackovic, S. Pilipovic, B.Stankovic and D. Zorica, *Fractional calculus with applications in mechanics : wave propagation, impact and variational principles*, John Wiley and Sons, 2014.

[3] D. Baleanu, K. Diethelm, E. Scalas and J.J. Trujillo, *Fractional Calculus,* **3** *Series on Complexity, Nonlinearity and Chaos*, World Scientific, Singapore, 2012.

[4] F.B.M. Belgacem and A.A. Karaballi, Sumudu transform fundamental properties investigations and applications, *Int. J. Stoch. Anal.*, **2006**(2006), Article ID 91083, 1-23.

[5] S. Bhalekar and V. Daftardar-Gejji, Solving evolution equations using a new iterative method, *Numer. Methods Partial Diff. Equ. : An Int. J.*, **26**(4)(2010), 906-916.

[6] S. Bhalekar and V. Daftardar-Gejji, Convergence of the new iterative method, *Int. J. Diff. Equ.*, **2011**(2011), Article ID 989065, 10 pages.

[7] V. Daftardar-Gejji and S.Bhalekar, Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method, *Comp. Math. Appli.*, **59**(5)(2010), 1801-1809.

[8] V. Daftardar-Gejji and H. Jafari, An iterative method for solving nonlinear functional equations, *J. Math. Anal. Appli.*, **316**(2)(2006), 753-763.

[9] S.T. Demiray, H. Bulut and F.B.M. Belgacem, Sumudu transform method for analytical solutions of fractional type ordinary differential equations, *Math. Prob. Eng.*, **2015**(2015), Article ID 131690, 6 pages.

[10] K. Diethelm, *Analysis of Fractional Differential Equations, Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2010.

[11] A. Erdlyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher transcendental functions*, **I, II & III**, McGraw-Hill, New York-Toronto-London, 1953-1955.

[12] D.J. Evans and H. Bulut, A new approach to the gas dynamics equation: an application of the decomposition method, *Int. J. Comp. Math.*, **79**(7)(2002), 817-822.

[13] R. Herrmann, *Fractional Calculus for Physicist*, World Scientific, Singapore, 2014.

[14] O.S. Iyiola, On the solutions of non-linear time-fractional gas dynamic equations: an analytical approach, *Int. J. Pure & Appl. Math.*, **98**(4)(2015), 491-502.

[15] H. Jafari, *Iterative methods for solving system of fractional differential equations* [Ph.D. thesis], Pune University, 2006.

[16] H. Jafari, M. Alipour and H. Tajadodi, Two-dimensional differential transform method for solving nonlinear partial differential equations, *Int. J. Res. Revi. Appl. Sci.*, **2**(1)(2010), 47-52.

[17] H. Jafari, C. Chun, S. Seifi and M. Saeidy, Analytical solution for nonlinear gas dynamic equation by homotopy analysis method, *Appli. Appl. Math. : An Int. J. (AAM)*, **4**(1)(2009), Article 12, 1-6.

[18] H. Jafari, C.M. Khalique and M.Nazari, Application of the Laplace decomposition method for solving linear and nonlinear fractional diffusion-wave equations, *Appl. Math. Lett.*, **24**(11)(2011), 1799-1805.

[19] H. Jafari, M. Zabihi and M. Saidy, Application of homotopy perturbation method for solving gas dynamics equation, *Appl. Math. Sci.*, **48**(2)(2008), 2393-2396.

[20] Y. Khan and Q. Wu, Homotopy perturbation transform method for nonlinear equations using Hes polynomials, *Comp. Math. Appli.*, **61**(8)(2011), 1963-1967.

[21] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations, North Holand Mathematics Studies*, **24**, *Elsevier*, Amsterdam, Netherlands, 2006.

[22] M. Kumar and V. Daftardar-Gejji, Exact solutions of fractional partial differential equations by Sumudu transform iterative method, *Fractional Calculus and Fractional Differential Equations*, Birkhuser, Singapore, (2018), 157-180.

[23] S. Kumar, H. Kocak and A.Yldrm, A fractional model of gas dynamics equations and its analytical approximate solution using Laplace transform, *Zeitschriftfr Naturforschung A*, **67**(6-7)(2012), 389-396.

[24] H. Kumar, M.A. Pathan and H. Srivastava, Application of two variable H-function for obtaining analytic sequence of solutions of three-variable space-and-time fractional diffusion problem, *Bull. Pure Appl. Math.* **5**(1)(2011), 151-160.

[25] H. Kumar, R.C.S. Chandel and H. Srivastava, On a fractional non linear biological model problem and its approximate solutions through Volterra integral equation, *Jñānābha* **47**(1)(June2017), 143-154.

[26] H. Kumar, Large values of approximate solution of nonlinear differential equations due to the Laplace transforms and their computations, *Proceedings of the 16th Annual Conf. SSFA,* **16** (2017), 96-112.
[27] M. Merdan, H. Anac and T. Kesemen, The New Sumudu Transform Iterative Method For Studying The Random Component Time-Fractional Klein-Gordon Equation, *Sigma*, **10**(3)(2019), 343-354.
[28] K.S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley and Sons New York, USA, 1993.
[29] G.M. Mittag-Leffler, Surla nouvelle fonction E(x), *CR Acad. Sci. Paris*, **137**(1903), 554-558.
[30] M.S. Mohamed, F. Al-Malki and M. Al-Humyani, Homotopy analysis transform method for time-space fractional gas dynamics equation, *Gen. Math. Notes*, **24**(1)(2014), 1-16.
[31] I. Podlubny, *Fractional differential equations, mathematics in science and engineering*, Academic Press, New York, USA, 1999.
[32] K.M. Saad, E.H. Al-Shareef, M.S. Mohamed and X.J. Yang, Optimal q-homotopy analysis method for time-space fractional gas dynamics equation, *The European Phy. J. Plus*, **132**(1)(2017), 1-11.
[33] K. Wang and S. Liu, A new Sumudu transform iterative method for time-fractional Cauchy reaction-diffusion equation, *Springer Plus*, **5**(1)(2016), 1-20.
[34] G. Watugala, Sumudu transform: a new integral transform to solve differential equations and control engineering problems, *Int. J. Math. Edu. Sci. Tech.*, **24**(1)(1993), 35-43.
[35] A. Wiman, Uber den Fundamentalsatz in der Teorie der Funktionen $E_a(x)$, *Acta Math.*, **29**(1905), 191-201.