ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)

Jñānābha ज्ञानाभ

(HALF YEARLY JOURNAL OF MATHEMATICAL SCIENCES)
[Included : UGC-CARE List]

VOLUME 53

NUMBER 1

JUNE 2023

Dedicated to
Professor G. C. Sharma
on His 85th Birth Anniversary Celebrations

Published by :

The Vijñāna Parishad of India
विज्ञान परिषद ऑफ इण्डिया
[Society for Applications of Mathematics]
DAYANAND VEDIC POSTGRADUATE COLLEGE
(Bundelkhand University)
ORAI-285001, U. P., INDIA
www.vijnanaparishadofindia.org/jnanabha

ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)

Jñānābha

This Special Volume of

JÑĀNĀBHA

is Being Dedicated to

PROFESSOR G. C. SHARMA

on His 85th Birth Anniversary Celebrations

PROFESSOR G. C. SHARMA

(Born : July 15, 1938)

ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 1-2**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

PROFESSOR G. C. SHARMA (GOKUL CHANDRA SHARMA): A LEADING MATHEMATICIAN

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DOI: https://doi.org/10.58250/jnanabha.2023.53100

Professor G.C. Sharma is well recognized leading mathematician well associated with *VIJÑĀNA PARISHAD OF INDIA* since 1990. He was honored by *DISTINGUISHED SERVICE AWARD* of VPI in 1996 during 6th Annual Conference of VPI held at Bundelkhand Institute of Engineering and Technology, Jhansi, Uttar Pradesh, India [December 26-28, 1996]. Professor G.C. Sharma has credit to grace the chair of President of VPI [April 2008- March 2011]. He was elected as Honorary Fellow of VPI *(FVPI)* in 2007 during 12th Annual Conference of VPI held at J.N.V. University, Jodhpur, Rajasthan, India [October 25-27,2007]. Professor Sharma was also honored by Highest Prestigius VPI Award *"LIFE-LONG ACHIEVEMENT AWARD"* during 20th Annual Conference of VPI held at Manipal University, Jaipur, Rajasthan, India [November 24-26, 2017]. Professor Sharma was recently honored by *"VPI GOLDEN JUBILEE AWARD"* during 5th International Conference and Goden Jubilee Celebrations of VPI held at J. N. U., New Delhi, India [June 16-18,2022]. Professor G. C. Sharma is also active member on Editorial Board of *JÑĀNĀBHA* .

All members of *JÑĀNĀBHA* family feel immense pleasure with heartiest congratulations and best wishes to publish current issue of *Jñānābha* , Vol.53(1) (June 2023)

(Dedicated to Professor G. C. Sharma on His 85^{th}Birth Anniversary Celebrations).

Professor G. C. Sharma: A Bibliographic Skatch

Professor G. C. Sharma was born on July 15, 1938 at Chindauli, District Mathura. His father Late Bhavani Shankar Sharma was a freedom fighter. Professor Sharma completed his graduation from K. R. College, Mathura and Post-Graduation in Mathematics from Agra College, Agra, Uttar Pradesh. As a student he had a brilliant record and secured throughout Ist division. He started his teaching career as Lecturer in Mathematics from St John's College, Agra in 1961 and after one year joined Agra College, Agra. He received doctorate degree in 1972 from Agra University, Agra. He was selected in 1981 as Reader in the Department of Mathematics, Institute of Basic Sciences, Agra University, Agra and became Professor in 1985.

Professor Sharma held the following academic posts (i) Ex Pro- Vice Chancellor, Dr. B. R. Ambedkar University, Agra (ii) Former Professor and Head, Department of Mathematics and Computer Science, Institute of Basic Science, Agra (iii) Ex Principal, Agra College, Agra, and (iv) Ex Director of-

o Institute of Basic Science o Institute of Management
o Institute of Vocational Education o Institute of Engineering and Technology

at one time of the Institutes of Khandari Campus, Dr B.R. Ambedkar University, Agra, Uttar Pradesh, India.

In addition to teaching experience of 38 years in Degree, P.G. and M.Phil., Professor Sharma is an active researcher and is on the frontline of Bio-mathematicians and Applied OR Specialists in India and abroad.

Research Experience: more than 50 years

- Guided 72 candidates for Ph.D. degree in Mathematics, Computer Science and Management.
- Published more than 220 research papers
- Conducted 7 Research Projects of CSIR and UGC
- Author of 24 Textbooks (from High school to P.G. classes) and two reference texts
- Organized 7 National/ International Conferences and 3 workshops
- Professor G. C. Sharma popularized blood flow models and their applications in medical sciences, queueing models of complex machining systems, and other topics by imparting plenary lectures and keynote addresses in more than 150 National/International Conferences/Seminars/symposium. His scientific contributions in the following areas are worth-noting:

- Development of fluid dynamic models in different frameworks and their applications.
- Study of blood concentration affecting erythrocyte sedimentation.
- Mathematical modeling of conjugate and its localization in cancer chemotherapy.
- Studies on population dynamic of infectious diseases and population genetics.
- Development of diffusion approximations models for manufacturing systems.
- Traffic models with customers' behavior and service interruption.
- Inventory system having stock dependent demand rates under an inflationary environment.

- **Life Member of following Societies**
  - Indian Science Congress
  - Indian Mathematical Society
  - OR Society of India (Ex Vice President)
  - Ramanujan Mathematical Society
  - Vijñāna Parishad of India (Ex President)
  - Bharat Ganita Parishad (Ex Vice President)
  - Global Society of Mathematical & Allied Sciences (Ex. President)
  - National Academy of Sciences, India
  - Indian Society of Biomechanics
  - Indian Society for Industrial and Applicable Mathematics (Founder member)
- He is on the Editorial Board of various National and International Journals and served as Reviewer of Research papers of various National & International Journals and research projects of CSIR and DST.
- Professor Sharma has visited many countries as Visiting Faculty and delivered Invited, plenary and Keynote Lectures in the conferences/seminars. He chaired many technical sessions and delivered lectures in educational Institute of repute in U.S.A. (Visiting faculty in Maryland University), Canada, Germany, U.K., Netherlands, France, Belgium, Taiwan, Italy, Mauritius, Thailand, etc.
- He was member of various Committees in the University namely, Research Degree Committees, Selection Committees, Advisory Boards and Inspection Committees.
- **Administrative Experience**
  - Attended more than a dozen, Summers and Winter Schools.
  - Faculty Member in Summer School held at Agra College, Agra.
  - Member of Mathematical Section, Indian Science Congress.
  - Expert in U.G.C. for various projects (Minor) in 1984-85.
  - External expert for selection committees.
  - External member for faculty of science in various University.
  - Member of panel for affiliation in various University.
  - Member of Court in the University.
  - Member of Academic Council.
  - Member of Executive Council (1991-92, 92-93, 96-98)
  - Member of Research Degree Committee.
  - Member of committees in the University and a special invitee to college development council.
  - Dean, Faculty of Science (1991-94).
  - Member, Academic Advisory Committee Academic Staff College, AMU Aligarh.
  - Director, Coaching Schemes of U. G. C., Govt. of India and Agra Universities.
  - Organiser, Model Parliament which stood II at National Level.
  - Organising Chairman, Literary Events North Zone, Universities Youth Festival 1996.
  - Organising Chairman, Universities Youth Festival 1987.
  - Co-ordinating Director Job Oriented Courses, Agra University.
  - Director, Computer Centre; Director, University Science Instrumentation Centre (upto 1996).

In addition to his academic bright career, Professor Sharma is very popular among his students, colleagues and fellow workers for his cordial support and personnel generosity. As a man, Professor Sharma is a great human being, excellent teacher, an eminent researcher, perfectionist and a disciplined man.

ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 3-21**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

UNSTEADY *MHD* SLIP FLOW OF RADIATING NANOFLUID THROUGH A POROUS MEDIA DUE TO A SHRINKING SHEET WITH CHEMICAL REACTION AND SORET EFFECT

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(*Received: May* 09, 2022; *In format: May* 12, 2022; *Revised: January* 01, 2023; *Accepted: January* 31, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53101

Abstract

Numerical investigations are performed to analyze heat and mass transfer in *MHD* (magnetohydrodynamic) nanofluid flow over a shrinking sheet in the presence of thermal radiation and chemical reaction with Soret effect. The transport equations involve the effect of Brownian motion,Thermophoresis, viscous dissipation, suction/injection, partial slip velocity and thermal slip. Using self suitable transformations, the governing equations are reduced to ordinary differential equations, further these equations are solved numerically using Runge-Kutta fourth order method with shooting technique. This study reveals that the governing parameters, namely, magnetic field parameter, thermal radiation parameter, chemical reaction parameter, velocity slip parameter etc., have major effects on velocity, temperature, concentration, skin friction coefficient and Nusselt number. The study admits that concentration rises with an increase in the Soret number. Numerical results are discussed with the assistance of graphs. The present problem has multiple applications in polymer, chemical and metallurgical industries such as formation of metallic and glass sheets.
2020 Mathematical Sciences Classification : 76D05 ,76D10, 76S05, 76W05, 80A21, 80A32.

Keywords: Nanofluid; *MHD* flow; Thermal radiation; Viscous dissipation; Chemical reaction; Slip and Convective boundary conditions; Soret effect.

1 Introduction

Nanofluid is mix suspension of nanometer sized solid particle (*Cu*, *Al*, *Ag*, etc) in base fluid such as water, oil and ethylene glycol, which is first introduced by Choi [6]. Nanofluids have different properties that make them potentially useful for many applications in heat transfer including microelectronics, fuel cells, cancer therapy, domestic refrigerator, pharmaceutical processes and hybrid-power engines. Nanofluids also have some biomedical applications, like in antibacterial and drug delivery. It also has biotechnological applications like nano fibers, nanoparticles, nanowires and nanostructures. Crane [5] initially examined fluid flow past a stretching sheet. An investigation is carried out by Makinde and Aziz [17] on "boundary layer flow of a nanofluid past a stretching sheet with a convective boundary condition". Electric field effect on nanofluid in an enclosure with sinusoidal wall under convective heat transfer was investigated by Sheikholeslami et al. [33]. Various researchers [3, 9, 15, 16, 24, 28] have presented the study of fluid moving over a nonlinear stretching sheet. A non-homogeneous model was presented by Buongiorno [4] to understand the convective transport phenomena in nanofluid. Brownian motion and thermophoresis are found most important in these studied. To analyze the natural convection boundary-layer flow of a nanofluid past a vertical plate, a revised model was presented by Kuznetsov and Nield [14].

The study of magnetic field effect has various impotrant applications in engineering and industry, for instance streamlined expulsion of plastic sheet, glass blowing, metal turning and condensation process of metallic plate in a cooling bath. Magnetic field plays an important role in geophysics, for example controlling heat transfer of different nanofluids and paper production also. Exact solution for *MHD* flow equation of fluid over a shrinking sheet was given by Fang and Zhang [11]. Prasad et al. [20] studied the influence of temperature-dependent fluid properties on the hydro-magnetic flow and heat transfer over a stretching sheet. Free convection flow of magnetohydrodynamics (*MHD*) nanofluid over an infinite flat plate was studied by Hamad et al. [13]. The *MHD* slip flow of Maxwell nanofluid over an exponentially stretching plane was analyzed by Reddy et al. [26]. Sheikholeslami and Rashidi [32], Sheikholeslami and Ganji [31] studied magnetic field effect on Ferro fluids flow and heat transfer. Daniel et al. [8] carried out an investigation on the electrical *MHD* nanofluid flow over a nonlinear stretching/shrinking sheet. "Finite element simulation of unsteady *MHD* transport phenomena on a stretching sheet in a rotating nanofluid" was discussed by Rana et al. [27]. At present, much attention has been paid to working in the presence of magnetic field effect.

When high temperature are encountered, the study of radiation heat transfer plays an important role in the field of equipment designing [30]. At extremely high-temperature levels, thermal radiation plays an important role in operating the devices in the space technology. Hady et al. [12] and Pal et al. [19] investigated viscous flow of a radiative nanofluid and heat transfer over a nonlinearly stretching/shrinking sheet. In-compressible water based nanofluid flow in the presence of transverse magnetic field with thermal radiation and buoyancy effect was investigated by Rashidi et al. [25]. In recent years, combined heat and mass transfer problem with chemical reaction received significant attention in many processes of interst in Engineering like as drying, evaporation at the surface of a water body, flow in a desert cooler and energy transfer in a wet cooling tower. Sandeep and Sulochana [34] investigated *MHD* flow of nanofluid over a permeable stretching/shrinking sheet with suction/injection. Combined effect of chemical reaction and magnetic field over non-linear stretching sheet was investigated by Awang [2] using Adomain decomposition method (ADM). Anwar et al. [1] studied heat generation/absorption effect on *MHD* flow of a nanofluid over porous stretching sheet with chemical reaction. Effect of chemical reaction and thermal radiation on *MHD* nanofluid flow over non-linear stretching sheet was discussed by Ramya et al. [22].

It is assumed that the velocity of the fluid particles reltive to the solid boundary is zero but the characteristics are different in case of micro and nano-scale fluid flow. Navier [18] first discussed the importance of slip boundary condition, which state that fluid velocity is proportional to shear stress at boundary. *MHD* nanofluid boundary layer slip flow over vertical stretching sheet with non-uniform heat generation/absorption was investigated by Das et al. [7]. *MHD* viscous nanofluid flow and heat transfer over a non-linearly slippery stretching sheet with heat generation/absorption and suction/injection was studied by Ramya et al. [23].

In a system with flowing fluid , dissimilar particles react in different ways to alter temperature, then this thermodynamics trend/phenomena is called the Soret effect. The mass flux can be generated both by the temperature and concentration gradients. Mass fluxes generated by temperature gradients are called the Soret effect. This effect is very important in the operation of solar ponds, the transportation across biological membranes induced by small thermal gradients in living creatures and micro-structure of seas and oceans. This effect is also useful in design and operation of dryers. The Soret effect related to the parting of isotopes and in the combination of gases with small molecular weight (H_2, He) and the average molecular weight (N_2, air) was highlighted by Eckert and Drake [10]. The impact of the Soret effect on stagnation-point flow past a sheet in a nanofluid with non-Darcy porous medium was studied by Reddy et al. [29]. Suneetha et al. [35] presented the Navier slip condition on time-dependent radiating nanofluid with the Soret effect.

In view of above literature survey and development of research in nanofluids, it is revealed that *MHD* slip flow of radiating nanofluids through a porous media with thermal slip in the presence of chemical reaction and Soret effect has not been studied yet, therefore our main aim in the present work is to investigate this aforesaid problem. The governing boundary layer equations are transformed as ordinary differential equations (*ODE's*) using similarity variables which are then solved numerically using fourth order Runge-Kutta method with shooting technique. The influence of various parameters on heat transfer characteristics and the flow field are explored and depicted through graphs or tables.

Nomenclature

A	Unsteadiness parameter	T	Temperature of the nanofluid (K)
B	Magnetic field strength	T_w	Wall temperature(K)
C	Concentration of the nanofluid	T_∞	Ambient temperature (K)
Cf_x	Local skin friction coefficient	T_m	Mean nanofluid temperature (K)
Cr^*	Chemical reaction parameter	u, v	Velocity components (m/s)
C_∞	Ambient Concentration	v_0	Suction velocity (m/s)
C_p	Specific heat at constant pressure	x, y	Cartesian coordinates (m)
D_B	Brownian diffusion coefficient (kg/ms)		
D_m	Mass diffusivity		
D_T	Thermophoretic diffusion coefficient (kg/msK)		*Greek symbols*
Ec	Eckert number	ρ	Density (kg/m^3)
f	Dimensionless stream function	α	Thermal diffusivity (m^2/s)
Ha^2	Magnetic field parameter	κ	Thermal conductivity (m^2/s)

K^*	Permeability parameter	μ	Dynamic viscosity (kg/ms)
K_T	Thermal diffusion ratio	ϕ	Dimentionless concentration
K_0	Permeability of the porous medium (m^2)	δ	Thermal slip parameter
N_1	Thermal slip factor (m)	λ^*	Velocity slip parameter
Le	Lewis number	χ	Shrinking parameter
L_1	Velocity slip factor (m)	ν	Kinematic viscosity (m^2/s)
Nb	Brownian motion parameter	σ	Electric conductivity($Kg^{-1}m^{-3}t^3A^2$)
Nt	Thermophoresis parameter	ψ	Stream function
Nu_x	Local Nusselt number	η	Similarity variable
Pr	prandlt number	θ	Dimentionless temperature
Q_0	Heat generation coefficient (W/m^2K)		
Q^*	Heat generation parameter		
R	Radiation parameter		*Subscript*
S	Suction parameter	f	Base fluid
Sr	Soret number	np	Nanoparticle
t	Time (s)	∞	Condition at infinity

2 Mathematical Formulation

We consider an unsteady, laminar, incompressible, two-dimensional boundary layer flow of an electrically conducting nanofluid. The flow behavior is examined along nonlinear shrinking sheet under the influence of magnetic field, nonlinear thermal radiation, chemical reaction, Soret effect and thermal slip boundary conditions. Fig. 2.1 shows the flow configuration in which x-axis is taken along the sheet and y-axis perpendicular to the sheet.

The sheet expands/contracts in the x-direction with a velocity $u_w(x,t) = \frac{ax^m}{1-\lambda t}$, where a and λ are constants, $m(\geq 1)$ is a power index and velocity across the wall is $v_w(x,t) = \frac{v_0}{\sqrt{(1-\lambda t)}} x^{\frac{(m-1)}{2}}$. The time dependent magnetic field of variable intensity $B(x,t)$ is assumed to be applied upright to the sheet. Initially the wall temperature T_w is assumed constant at the shrinking sheet. The ambiant fluid temperature T_∞ is considered less than to sheet's temperature T_w.

Bases on the boundary layer approximation, the governing equations of momentum, thermal energy and concentration [21] can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu_f \frac{\partial^2 u}{\partial y^2} - \frac{\sigma}{\rho_f} B^2 u - \frac{\mu_f}{\rho_f K} u, \tag{2.2}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_f \frac{\partial^2 T}{\partial y^2} - \frac{\partial q_r}{\partial y} + \quad \tau \left[D_B \left(\frac{\partial C}{\partial y}\right)\left(\frac{\partial T}{\partial y}\right) + \left(\frac{D_T}{T_\infty}\right)\left(\frac{\partial T}{\partial y}\right)^2 \right] +$$

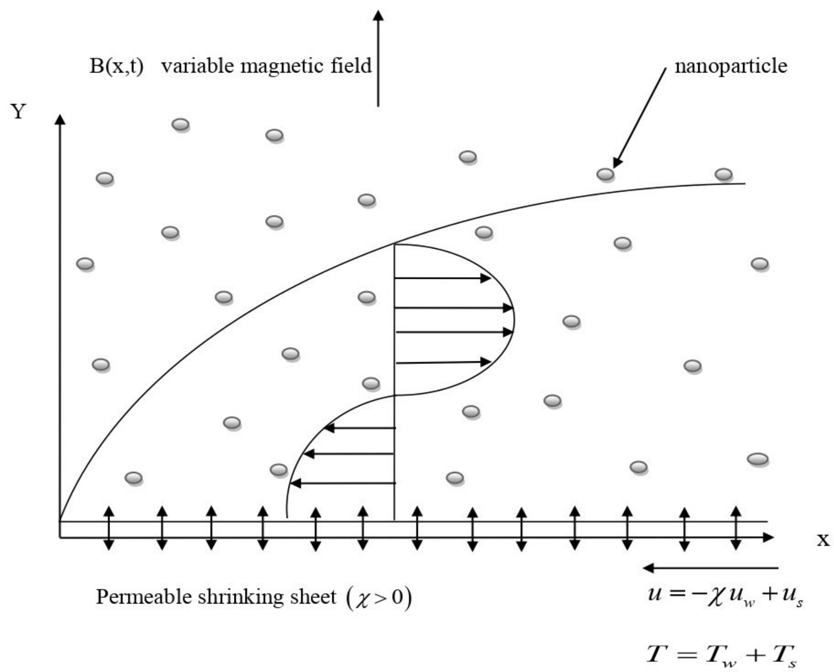

***Figure* 2.1:** Physical model and the coordinate system.

$$\frac{\mu_f}{(\rho C)_f}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{Q(T-T_\infty)}{(\rho C)_f}, \tag{2.3}$$

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} - Cr\left(C - C_\infty\right) + \frac{D_m K_T}{T_m}\frac{\partial^2 T}{\partial y^2}, \tag{2.4}$$

Subject to the following boundary conditions:

$$t \leq 0 : u = 0, v = 0, T = T_w; \tag{2.5}$$

u and v are function of x and t

$$t > 0 : u = -\chi u_w\left(x,t\right) + u_s, v = -v_w\left(x,t\right), T = T_w + T_s, D_B\frac{\partial C}{\partial y} + \frac{D_T}{T_\infty}\frac{\partial T}{\partial y} = 0$$

$$at \quad y = 0;$$

$$u \to 0, v \to 0, \quad T \to T_\infty, \quad C \to C_\infty \quad as \quad y \to \infty, \tag{2.6}$$

here $\tau = \frac{(\rho C_p)_{np}}{(\rho C_p)_f}$ is ratio of nanoparticle heat capacity and the base fluid heat capacity, $\alpha_f = \frac{\kappa_f}{(\rho C_p)_f}$ is thermal diffusivity of the fluid, $K = K_0\left(\frac{x^{m-1}}{1-\lambda t}\right)^{-1}$ is permeability of the porous medium, $Q = Q_0\left(\frac{x^{m-1}}{1-\lambda t}\right)$ is heat generation, $Cr = Cr_0\frac{x^{m-1}}{1-\lambda t}$ is rate of chemical reaction. The variable magnetic field is $B\left(x,t\right) = B_0\left(\frac{x^{m-1}}{1-\lambda t}\right)^{\frac{1}{2}}$, where B_0 is constant. $u_s = L\frac{\partial u}{\partial y}$ is slip velocity, where $L\left(x,t\right) = L_1\left(\frac{x^{m-1}}{1-\lambda t}\right)^{-\frac{1}{2}}$ is velocity slip factor and $T_s = N\frac{\partial T}{\partial y}$ is thermal slip, where $N\left(x,t\right) = N_1\left(\frac{x^{m-1}}{1-\lambda t}\right)^{-\frac{1}{2}}$ is velocity slip factor.

By using Rosselands approximation, the radiative heat flux is

$$q_r = -\left(\frac{4}{3}\frac{\sigma^*}{k^*}\right)\frac{\partial T^4}{\partial y}, \tag{2.7}$$

where k^* is the absorption coefficient, σ^* is the Stefan-Boltzmann constant.The temperature difference is assuming such that T^4 may be expended in a Taylor series about T_∞ and neglecting higher order terms, we get

$$T^4 = 4T_\infty^3 T - 3{T_\infty}^4. \tag{2.8}$$

Using eq.(2.8) in eq.(2.7), we get

$$q_r = -\left(\frac{4}{3}\frac{\sigma^*}{k^*}\right)\frac{\partial}{\partial y}\left(4T_\infty^3 T - 3{T_\infty}^4\right) = -\frac{16}{3}\frac{\sigma^* T_\infty^3}{k^*}\frac{\partial T}{\partial y}$$

and hence

$$\frac{\partial q_r}{\partial y} = -\frac{16}{3}\frac{\sigma^* T_\infty^3}{k^*}\frac{\partial^2 T}{\partial y^2}. \tag{2.9}$$

In terms of the stream function the velocity components are:

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}. \tag{2.10}$$

Now, using the similarity transformations

$$\eta = y\sqrt{\left(\tfrac{u_w(x,t)}{x\nu_f}\right)}, \psi = \sqrt{\nu_f x u_w\left(x,t\right)} f\left(\eta\right),$$
$$\theta\left(\eta\right) = \tfrac{T-T_\infty}{T_w-T_\infty}, \phi\left(\eta\right) = \tfrac{C-C_\infty}{C_\infty}, \tag{2.11}$$

equations (2.1) to (2.6) reduce to

$$f''' + \left(\frac{m+1}{2}\right) ff'' - mf'^2 - A\left(f' + \frac{\eta}{2}f''\right) - \left(Ha^2 + K^*\right) f' = 0, \tag{2.12}$$

$$\frac{1}{Pr_{eff}}\theta'' + \left(\frac{m+1}{2}\right) f\theta' + Nb\theta'\phi' + Nt\theta'^2 - \frac{A}{2}\eta\theta' + Ecf''^2 + Q^*\theta = 0, \tag{2.13}$$

$$\phi'' + \left(\frac{m+1}{2}\right) Lef\phi' + \frac{Nt}{Nb}\theta'' - LeCr^*\phi - \frac{\eta}{2}LeA\phi' + LeSr\theta'' = 0, \tag{2.14}$$

with the boundary conditions

$$\eta = 0 : f\left(\eta\right) = S, f'\left(\eta\right) = -\chi + \lambda^* f''\left(\eta\right), \theta\left(\eta\right) = 1 + \delta\theta'\left(\eta\right), Nb\phi'\left(\eta\right) + Nt\theta'\left(\eta\right) = 0;$$

$$\eta \to \infty : f'\left(\eta\right) \to 0, \theta\left(\eta\right) \to 0, \phi\left(\eta\right) \to 0. \tag{2.15}$$

The parameters used in equations (2.12) to (2.15) are as follows:

$$Pr_{eff} = \frac{Pr}{(1+R)}, Pr = \frac{\nu_f}{\alpha_f}, Le = \frac{\nu_f}{D_B}, Nb = \frac{\tau D_B C_\infty}{\nu_f}, R = \frac{16}{3}\frac{\sigma^* T_\infty^3}{\kappa_f k^*}, Nt = \frac{\tau D_T\left(T_w - T_\infty\right)}{\nu_f T_\infty},$$
$$Ha^2 = \frac{\sigma B_0^2}{\rho_f a}, K^* = \frac{\nu_f}{K_0 a}, Q^* = \frac{Q_0 \alpha_f}{\kappa_f a}, Cr^* = \frac{Cr_0}{a}, A = \frac{\lambda}{a x^{m-1}}, Ec = \frac{u_w^2}{\left(C_p\right)_f\left(T_w - T_\infty\right)},$$
$$Sr = \frac{D_m K_T\left(T_w - T_\infty\right)}{\nu_f T_m C_\infty}, \lambda^* = L_1\sqrt{\frac{a}{\nu_f}}, S = \frac{2v_0}{\sqrt{a\nu_f}\left(m+1\right)}, \delta = N_1\sqrt{\frac{a}{\nu_f}}.$$

Here Pr_{eff} denote effective Prandlt number.
In this study, the important physical quantities are the skin friction coefficient and the local Nusselt number, which are defined as

$$C_{fx} = \frac{\tau_w}{\rho_f u_w^2\left(x,t\right)}, Nu_x = \frac{x q_w}{\kappa_f\left(T_w - T_\infty\right)}, \tag{2.16}$$

$$where \quad \tau_w = \mu_f\left(\frac{\partial u}{\partial y}\right)_{y=0} \quad and \quad q_w = -\left[\left(\kappa_f + \frac{16}{3}\frac{\sigma^* T_\infty^3}{k^*}\right)\frac{\partial T}{\partial y}\right]_{y=0}, \tag{2.17}$$

are the wall shear stress and the wall heat flux, respectively.
Using equations (2.11) and (2.17) into equation (2.16), we have

$$C_{fx} Re_x^{\frac{1}{2}} = f''\left(0\right) \quad and \quad Nu_x Re_x^{-\frac{1}{2}} = -\left(1+R\right)\theta'\left(0\right), \tag{2.18}$$

where $Re_x = \frac{u_w x}{\nu_f}$ is the local Reynolds number.

3 Numerical Solution

In this study, similarity transformations are used to reduce the governing equations (2.1) - (2.4) into a system of coupled non-linear ordinary differential equations (2.12)-(2.14) with boundary conditions. The coupled non-linear ordinary differential equations (2.12)-(2.14) are third order in f and second order in both θ and ϕ. These coupled equations are reduced to a system of seven simultaneous differential equations of first order with seven unknowns. To solve this system of equations using the Runge-kutta fourth order method, we need seven initial conditions. We already know two initial conditions in f and one initial condition in each of θ and ϕ. Also, the value of f', θ and ϕ are known at $\eta \to \infty$. Thus, these three end conditions can be utilize to produce three unknown initial conditions at $\eta = 0$ by using shooting technique. After knowing all the seven initial conditions, we solve this system of equations using fourth order Runge-Kutta integration scheme with the help of *MATLAB* software.

The equations (2.12)-(2.14) can be expressed as

$$f = f_1,$$

$$f_1{}' = f_2,$$

$$f_2{}' = f_3,$$

$$f_3{}' = -\left(\frac{m+1}{2}\right) f_1 f_3 + m f_2^2 + A\left(f_2 + \frac{\eta}{2} f_3\right) + \left(Ha^2 + K^*\right) f_2,$$

$$\theta = f_4,$$

$$f_4{}' = f_5,$$

$$f_5{}' = -Pr_{eff}\left[\left(\frac{m+1}{2}\right) f_1 f_5 + Nb f_5 f_7 + Nt f_5^2 - \frac{A}{2}\eta f_5 + Ec f_3^2 + Q^* f_4\right],$$

$$\phi = f_6,$$

$$f_6{}' = f_7,$$

$$f_7{}' = \left(\frac{Nt}{Nb} + LeSr\right)\left[Pr_{eff}\left(\left(\frac{m+1}{2}\right) f_1 f_5 + Nb f_5 f_7 + Nt f_5^2 - \frac{A}{2}\eta f_5 + Ec f_3^2 + Q^* f_4\right)\right] - \left(\frac{m+1}{2}\right) Le f_1 f_7 + LeCr^* f_6 + \frac{\eta}{2} LeA f_7,$$

with reduced boundary conditions

$$\eta = 0 : f_1 = S, f_2 = -\chi + \lambda^* f_3, f_4 = 1 + \delta f_5, f_7 = -\frac{Nt}{Nb} f_5;$$

$$\eta \to \infty : f_2 \to 0, f_4 \to 0, f_6 \to 0.$$

In this study, the boundary value problem is first converted into an initial value problem (IVP). Then, the IVP is solved by appropriate guessing of the missing initial value by the shooting method for several sets of parameters. The obtained results have been discussed and shown graphically and in tables.

4 Results and discussion

The non-dimensional equations (2.12)-(2.14) with boundary conditions (2.15) are solved numerically by Runge-kutta fourth order method with shooting technique. For numerical computation, we consider the non-dimensional parameter's values as $0 \le K^* \le 4.0$, $0 \le Ha^2 \le 1.5$, $0.01 \le Nt \le 1.0$, $0.1 \le Cr^* \le 7.0$, $0.5 \le Nb \le 10.0$, $0.1 \le Ec \le 1.0$, $4.0 \le Pr_{eff} \le 15.0$, $0.0 \le A \le 1.2$, $3.0 \le S \le 5.0$, $0.0 \le \lambda^* \le 0.2$, $5.0 \le Le \le 30.0$, $1.0 \le m \le 3.0$, $0.1 \le Q^* \le 10.0$, $0.0 \le R \le 1.04$, $0.05 \le Sr \le 0.20$, $0.1 \le \delta \le 0.2$, and $1.0 \le \chi \le 1.5$. The numerical values of parameters are fixed as given below: $K^* = 1.0$, $Ha^2 = 0.5$, $Nt = Nb = 0.5$, $Ec = Cr^* = 0.1$, $Pr = 8.16$, $Pr_{eff} = 6.8$, $Q^* = 0.1$, $A = 0.5$, $S = 3.0$, $Le = 10$, $Sr = 0.05$, $\delta = 0.1$, $\chi = 1.0$, $R = 0.2$ and $\lambda^* = 0.1$, unless stated separately. In Figure 4.1, consequence of permeability parameter (K^*) on velocity profile is sketched. It is comprehended that rise in the value of permeability parameter increases the velocity profile. Figure 4.2 illustrates the effect of magnetic field parameter(Ha^2) on the nanofluid velocity profile. It is observed that increase in magnetic field parameter increases the velocity profile. Figures 4.3-4.4 and Figures 4.5-4.6 show the influence of the change of the thermophoresis parameter (Nt) and Brownian motion parameter (Nb) on the temperature and concentration profiles, respectively. It is noticed that, as the thermophoresis parameter and Brownian motion parameter increases, the thermal boundary layer thickness increases. Here the thermal rise is reported since higher Brownian motion includes the random acceleration of the fluid particles which generates extra energy. Nanofluid concentration increases with an increase in thermophoresis parameter and decreases near the surface with an increase in Brownian motion parameter. Figure 4.7 shows the effect of chemical reaction parameter (Cr^*) on the concentration of the nanofluid. It is observed that concentration of the nanofluid is on decline for higher estimation of Cr^*. Figure 4.8 and Figure 4.9 are graphed to comprehend the effect on temperature and concentration profiles for various values of effective Prandtl number (Pr_{eff}). The numerical results show that the impact of increasing values of effective Prandtl number leads to decrease in temperature and nanofluid concentration profiles. An enhancement in Eckert number causes increase in temperature and nanofluid concentration profiles. This effect is shown in Figure 4.10 and figure 4.11. Figure 4.12 shows the variation of velocity profile in response to a change in the values of unsteadiness parameter (A). It is seen that, as unsteadiness parameter increases, velocity increases. Figures 4.13-4.15 represent the effect of the suction parameter(S) on velocity, temperature and nanofluid concentration profiles, respectively. These plots show that velocity profile increases while temperature and nanofluid concentration profiles decrease with increasing value of suction parameter. Figures 4.16-4.18 illustrate the effect of the velocity slip parameter (λ^*) on velocity, temperature and concentration profiles, respectively. It is observed by Figure 4.16 that the velocity profile increases with increasing value of λ^* while Figure 4.17 and Figure 4.18 show that the velocity slip parameter affects the temperature and nanofluid concentration in an opposite manner. Figure 4.19 and Figure 4.20 show the impact of Lewis number (Le) on temperature and concentration distribution. It is seen by Figure 4.19 that temperature profile increases near the surface and reverses far from the surface with the increasing values of Le. From Figure 4.20, it is found that the nanoparticle concentration is a decreasing function of Le. Thinner concentration boundary layer and weaker mass diffusivity are proportional to increase in Le. Figures 4.21 and 4.22 show the effect of heat generation parameter (Q^*) on temperature and nanofluid concentration profiles and declare that, the temperature and concentration profiles increase with increasing value of heat generation parameter. Figure 4.23 and Figure 4.24 illustrate the effect of thermal slip parameter (δ) on the nanofluid temperature and concentration profiles. It can be seen that both nanofluid temperature and concentration profiles decrease with increasing values of thermal slip parameter. Figures 4.25-4.27 are plotted to analyze the behavior of $f^{'}(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ for various values of χ. It is observed by Figure 4.25 that velocity has decreasing tendency with decreasing value of χ. It can also be noted by Figure 4.26 and Figure 4.27 that temperature and nanofluid concentration have increasing tendency with decreasing values of χ. The effect of power-law index (m) on velocity, temperature and nanoparticle volume fraction is drawn in Figures 4.28-4.30, respectively. It is observed that temperature and nanofluid concentration decrease while velocity profile increases with the increasing values of m. This is due to the fact that increment in power-law index (m) enhances the intensity of the cold fluid at the ambient towards the hot fluid near the sheet. This decreases the fluid temperature near the shrinking sheet. Figure 4.31 and Figure 4.32 outline the temperature and nanofluid concentration in the presence of the Soret number Sr. An increase in Sr increases the temperature and concentration profiles within the boundary layer. Figure 4.33 and Figure 4.34 show the effect of radiation absorption parameter (R). Temperature and concentration profiles increse with increase in radiation absorption parameter (R). The effect of various governing parameters on Skin friction and Nusselt number are calculated numerically and also presented in Table 4.1,4.2 and 4.3. From Table 4.1, it is clear that Skin friction increases for increasing value of parameters K^*, Ha^2, A, S and m while it is a decreasing function of parameters λ^* and χ. Also from Table 4.2 and 4.3, it is noted that the Nusselt number increases for increasing value of parameters Pr_{eff}, Cr^*, S, m and λ^* while it is a decreasing function of parameters K^*, Ha^2, Nt, Nb, Le, Ec, A, Sr, Q^*, χ, R and δ.

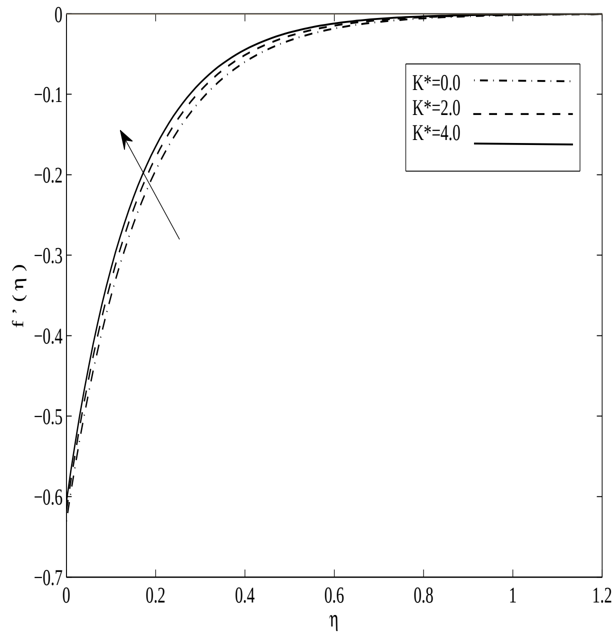

***Figure* 4.1:** Velocity behaviour for various values of K^*.

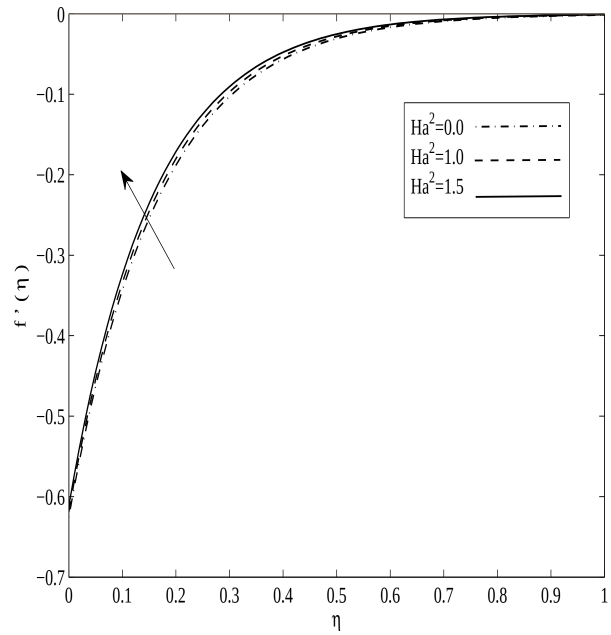

***Figure* 4.2:** Velocity behaviour for various values of Ha^2.

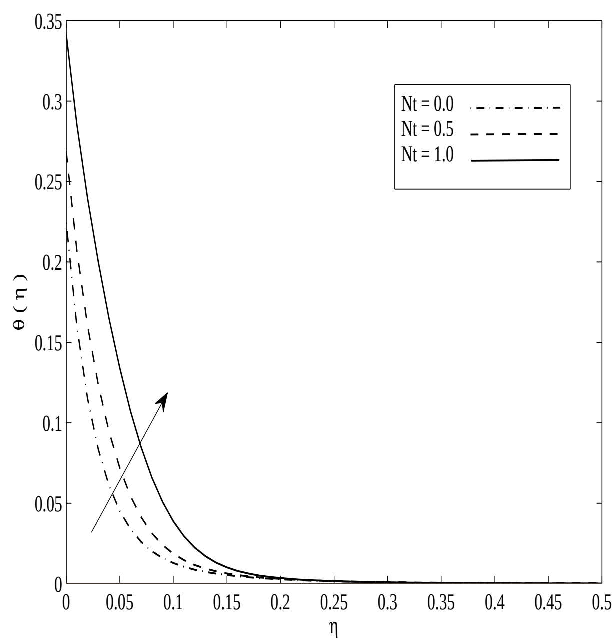

***Figure* 4.3:** Temperature behaviour for various values of Nt.

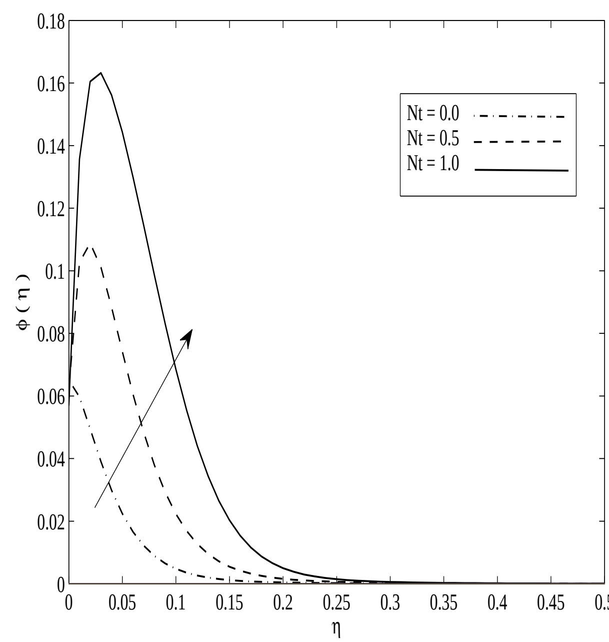

***Figure* 4.4:** Concentration behaviour for various values of Nt.

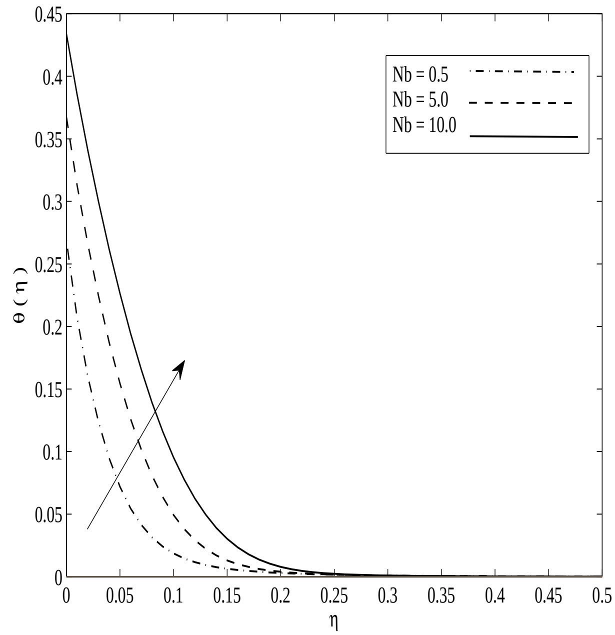

***Figure* 4.5:** Temperature behaviour for various values of Nb.

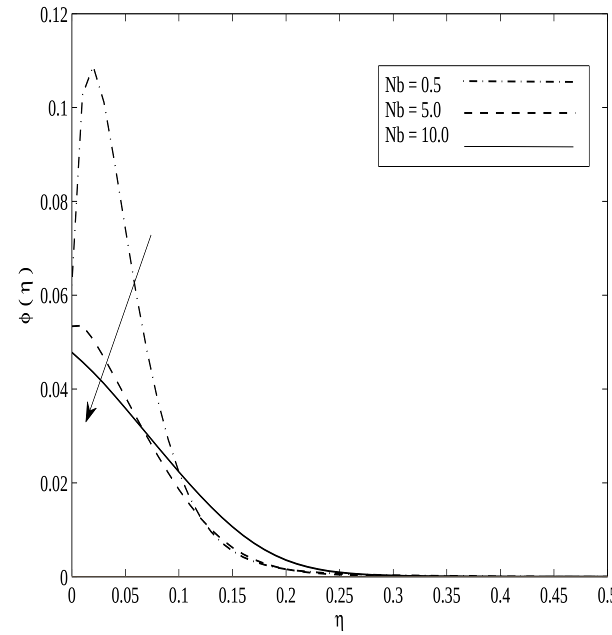

***Figure* 4.6:** Concentration behaviour for various values of Nb.

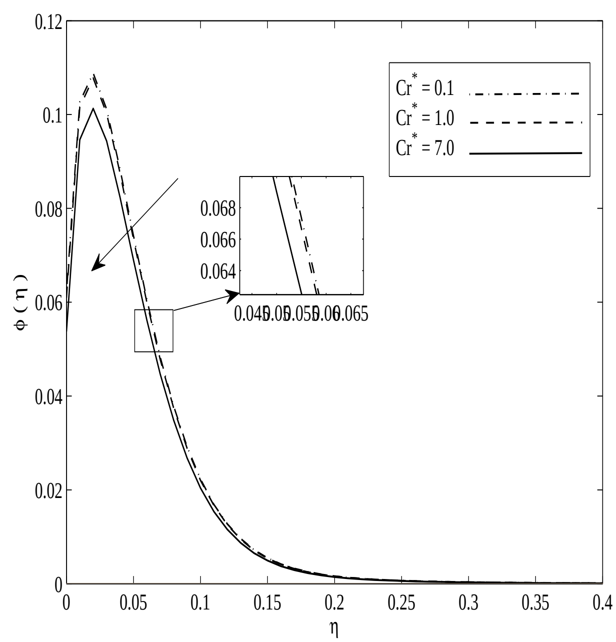

***Figure* 4.7:** Concentration behaviour for various values of Cr^*.

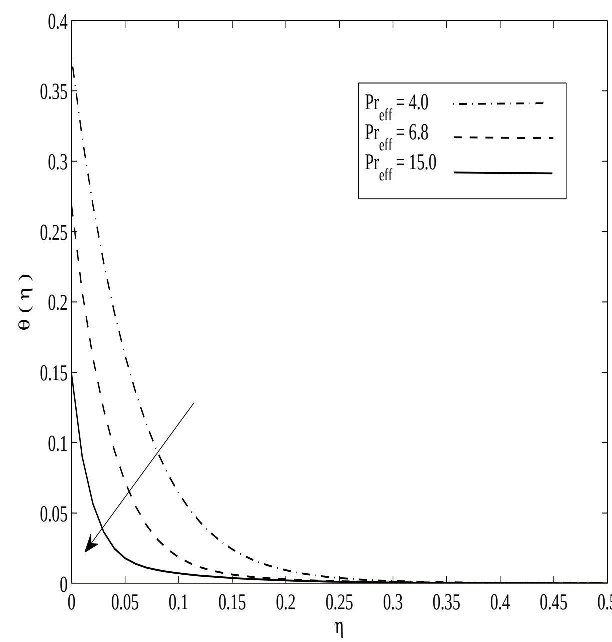

***Figure* 4.8:** Temperature behaviour for various values of Pr_{eff}.

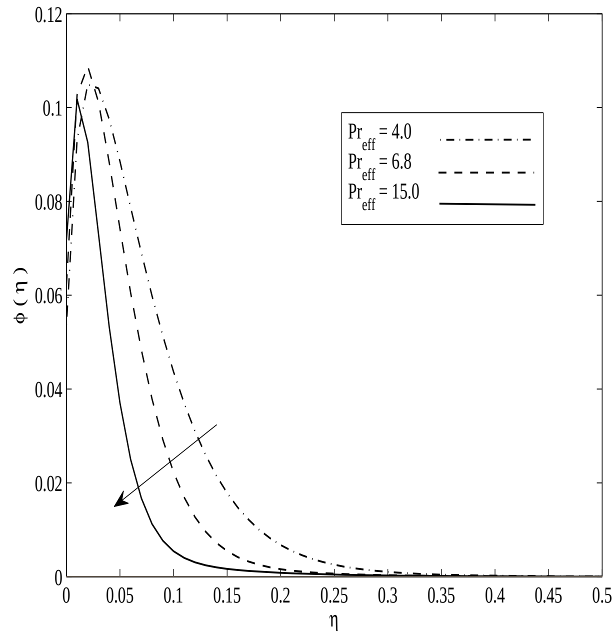

***Figure* 4.9:** Concentration behaviour for various values of Pr_{eff}.

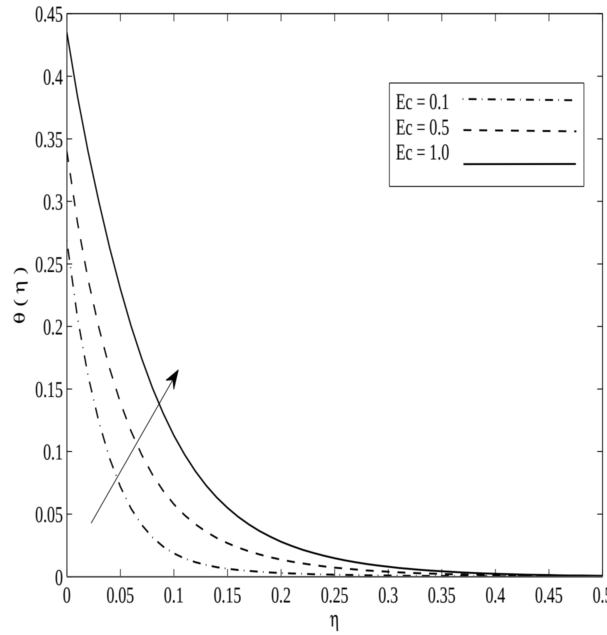

***Figure* 4.10:** Temperature behaviour for various values of Ec.

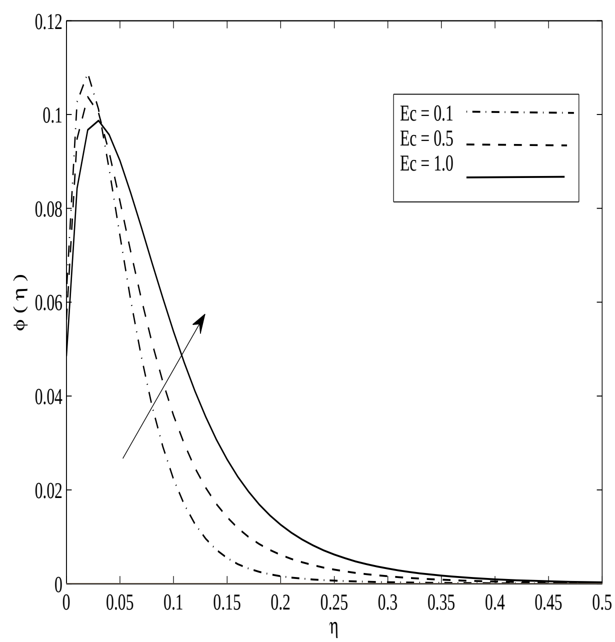

***Figure* 4.11:** Concentration behaviour for various values of Ec.

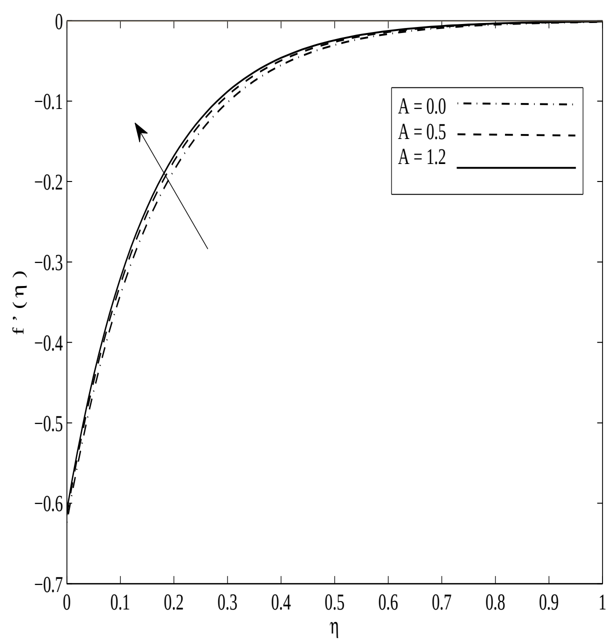

***Figure* 4.12:** Velocity behaviour for various values of A.

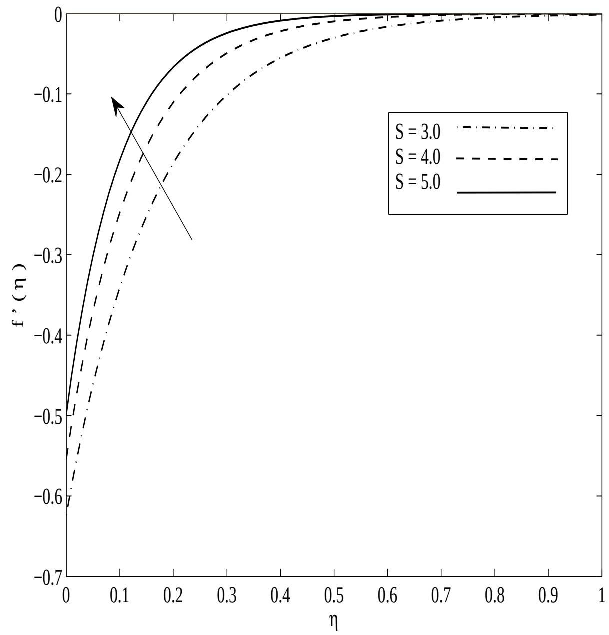

***Figure* 4.13:** Velocity behaviour for various values of S.

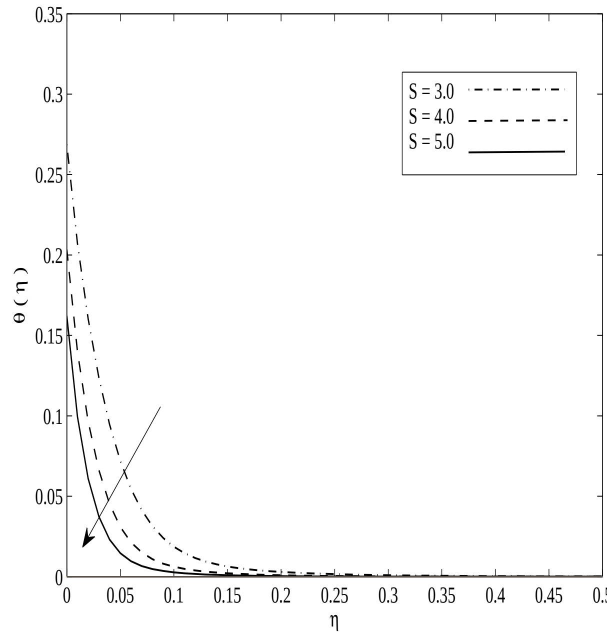

***Figure* 4.14:** Temperature behaviour for various values of S.

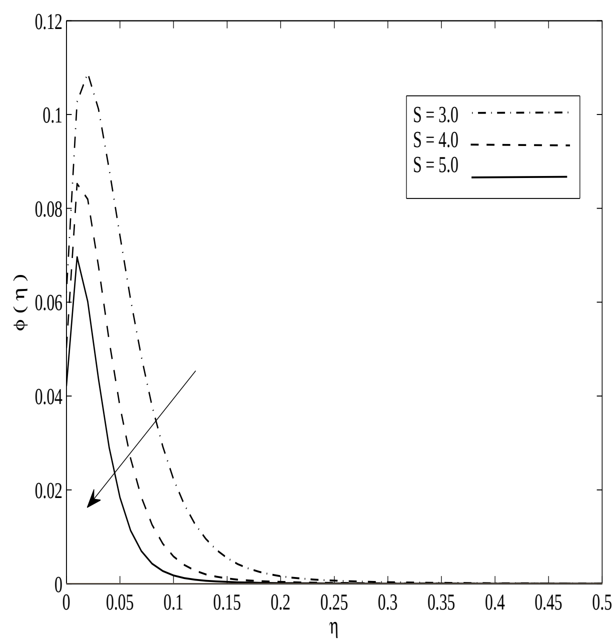

***Figure* 4.15:** Concentration behaviour for various values of S.

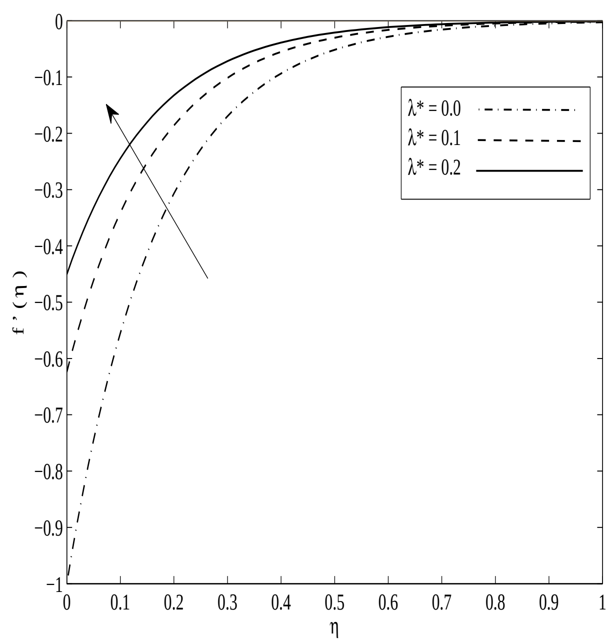

***Figure* 4.16:** Velocity behaviour for various values of λ^*.

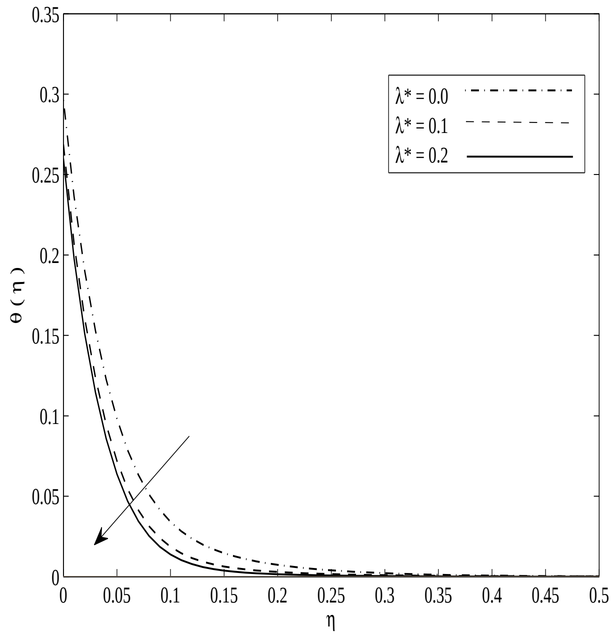

***Figure* 4.17:** Temperature behaviour for various values of λ^*.

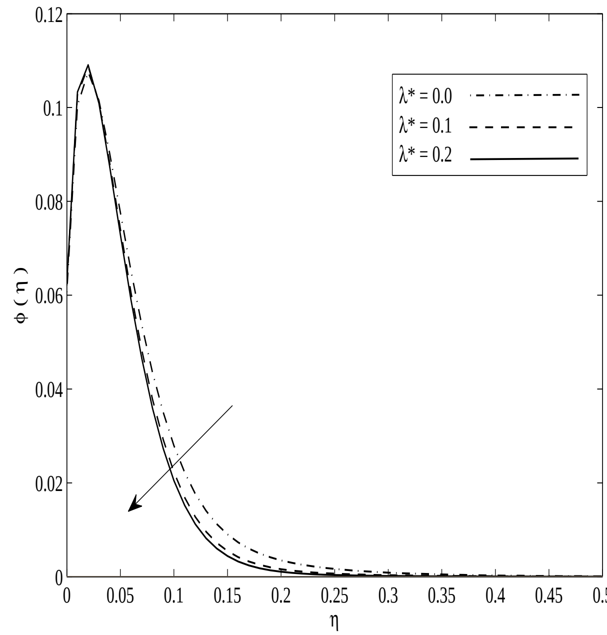

***Figure* 4.18:** Concentration behaviour for various values of λ^*.

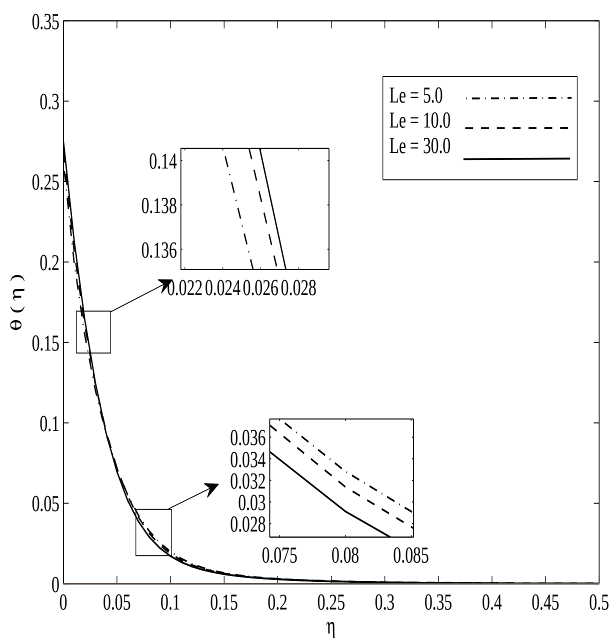

***Figure* 4.19:** Temperature behaviour for various values of Le.

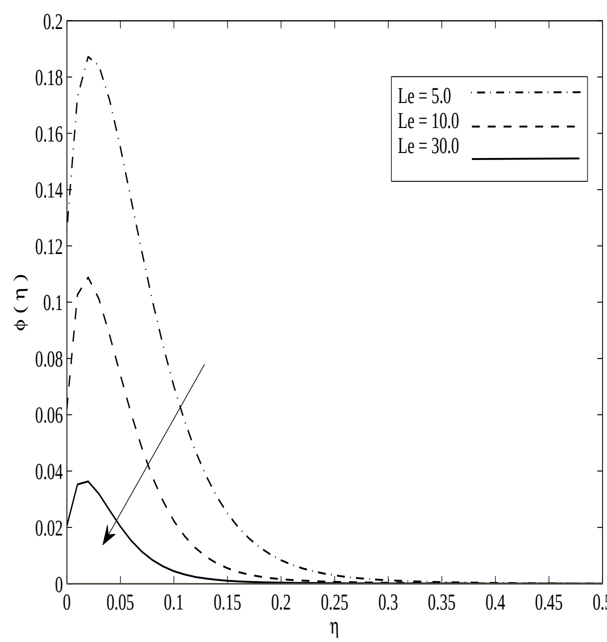

***Figure* 4.20:** Concentration behaviour for various values of Le.

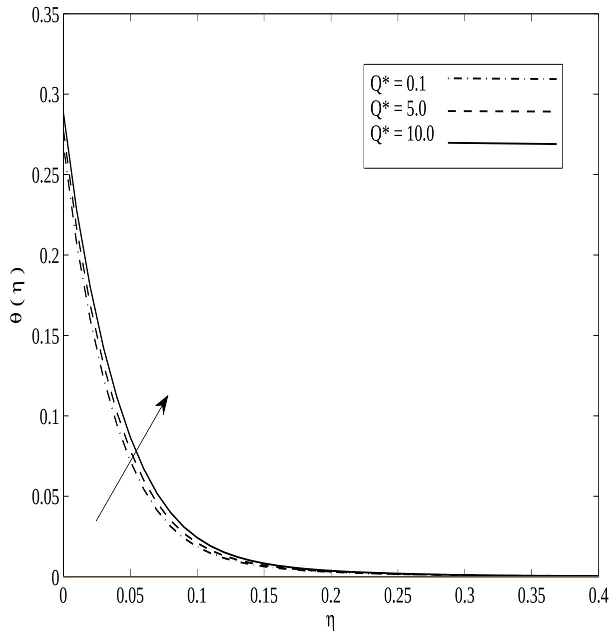

Figure **4.21:** Temperature behaviour for various values of Q^*.

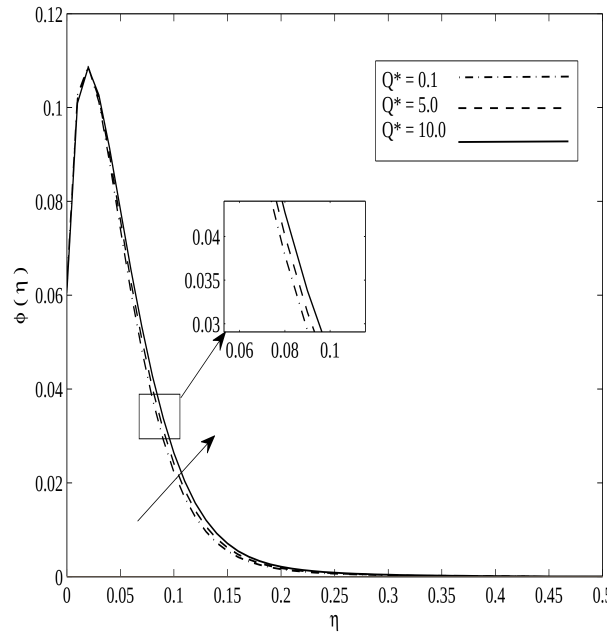

Figure **4.22:** Concentration behaviour for various values of Q^*.

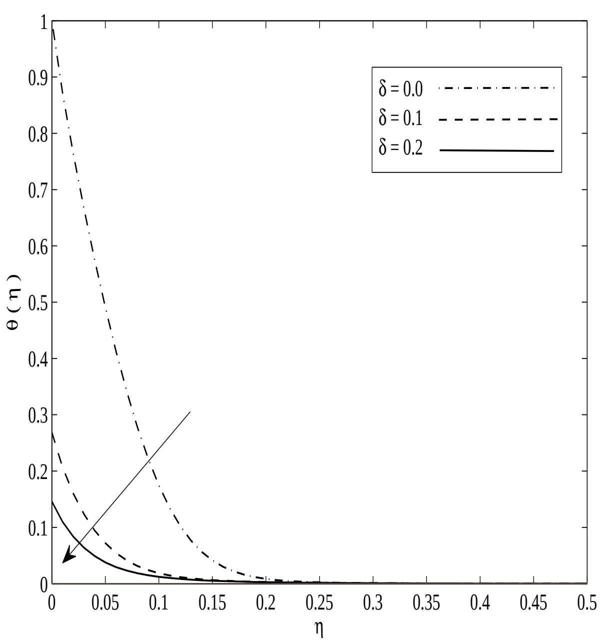

Figure **4.23:** Temperature behaviour for various values of δ.

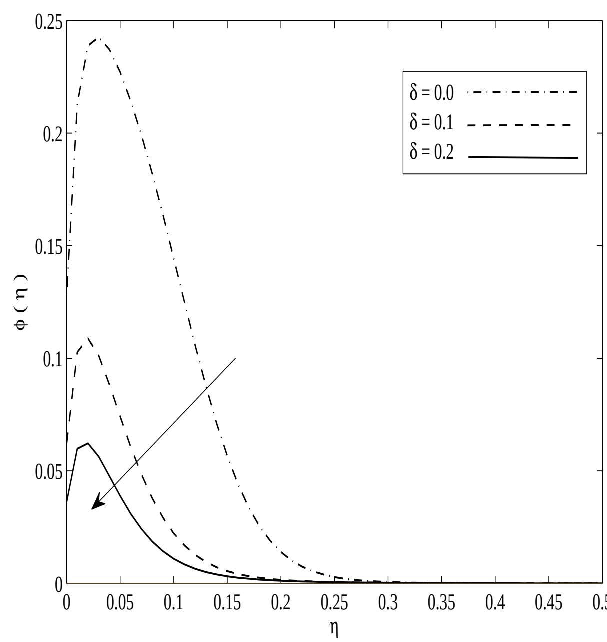

Figure **4.24:** Concentration behaviour for various values of δ.

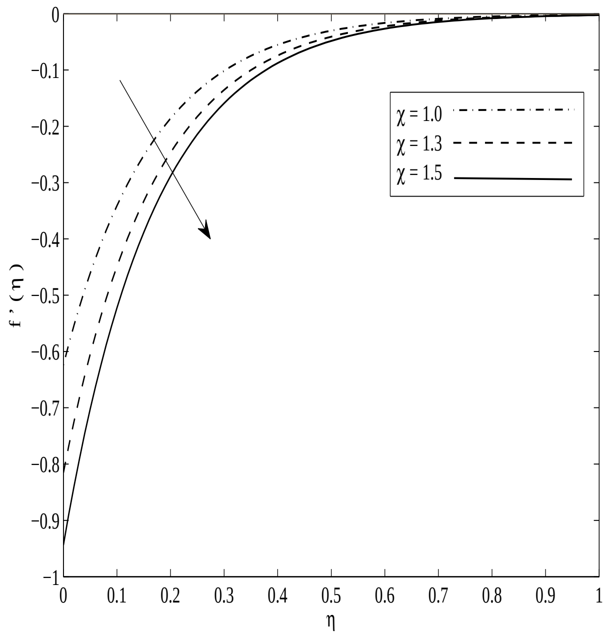

Figure **4.25:** Velocity behaviour for various values of χ.

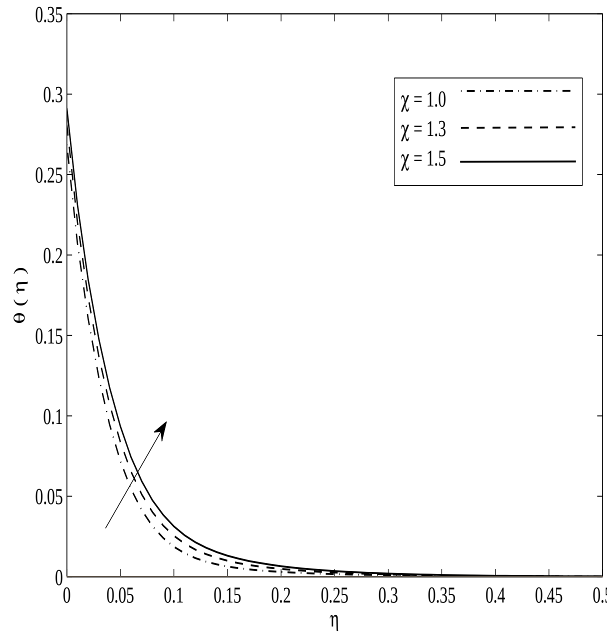

Figure **4.26:** Temperature behaviour for various values of χ.

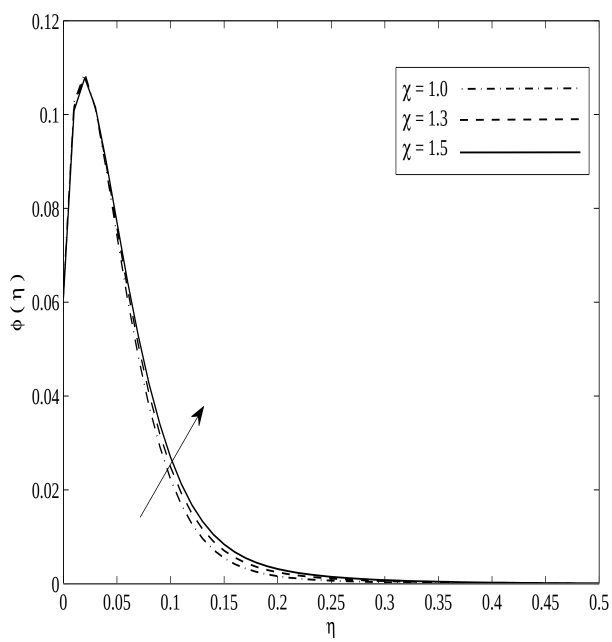

Figure **4.27:** Concentration behaviour for various values of χ.

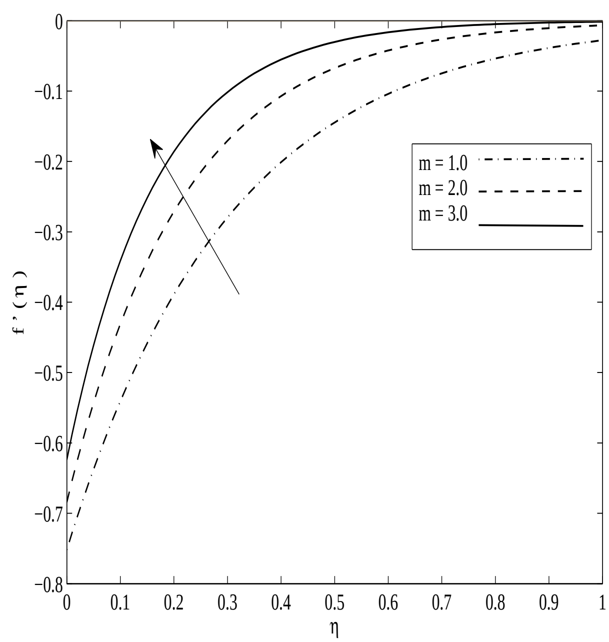

Figure **4.28:** Velocity behaviour for various values of m.

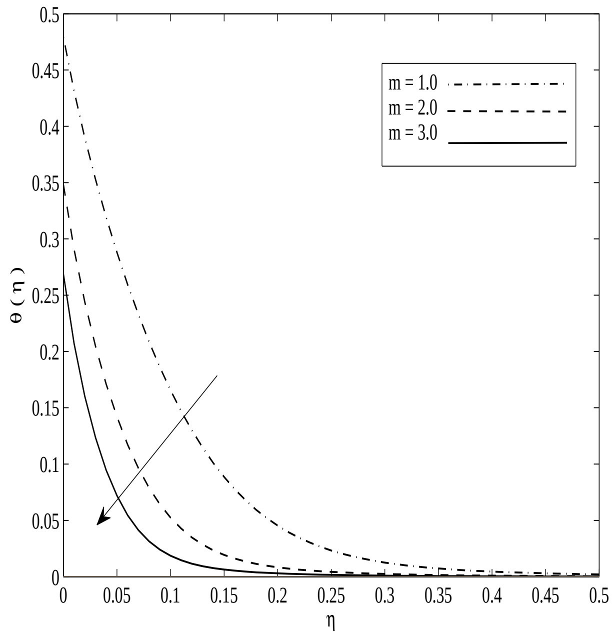

***Figure* 4.29:** Temperature behaviour for various values of m.

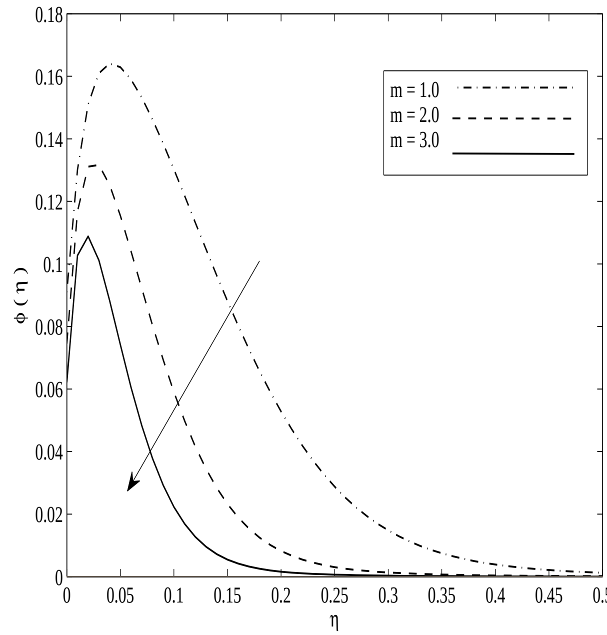

***Figure* 4.30:** Concentration behaviour for various values of m.

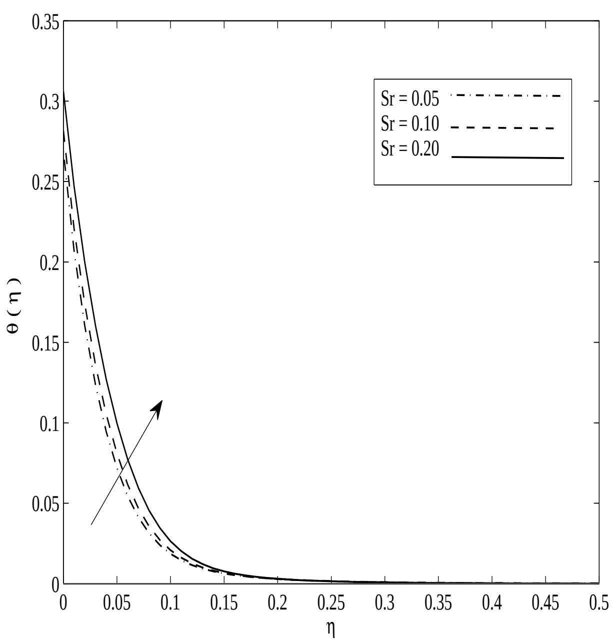

***Figure* 4.31:** Temperature behaviour for various values of Sr.

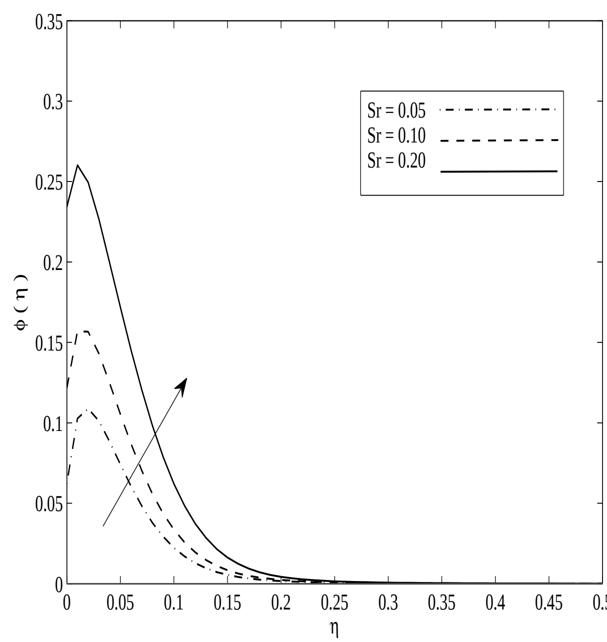

***Figure* 4.32:** Concentration behaviour for various values of Sr.

***Table* 4.1:** Numerical values of $C_{fx}Re_x^{1/2}$

K^*	Ha^2	A	S	λ^*	m	χ	$C_{fx}\ Re_x^{1/2}$
0.00	0.50	0.50	3.00	0.10	3.00	1.00	3.676558
1.00	**0.50**	**0.50**	**3.00**	**0.10**	**3.00**	**1.00**	**3.746045**
2.00							3.810179
4.00							3.925633
1.00	**0.00**	0.50	3.00	0.10	3.00	1.00	3.729214
	1.00						3.794596
	1.50						3.869821
1.00	0.50	**0.00**	3.00	0.10	3.00	1.00	3.728988
		1.20					3.769732
1.00	0.50	0.50	**4.00**	0.10	3.00	1.00	4.448826
			5.00				5.006289
1.00	0.50	0.50	3.00	**0.00**	3.00	1.00	5.828953
				0.20			2.739946
1.00	0.50	0.50	3.00	0.10	**1.00**		2.445871
					2.00		3.135196
1.00	0.50	0.50	3.00	0.10	3.00	**1.30**	4.828288
						1.50	5.537819

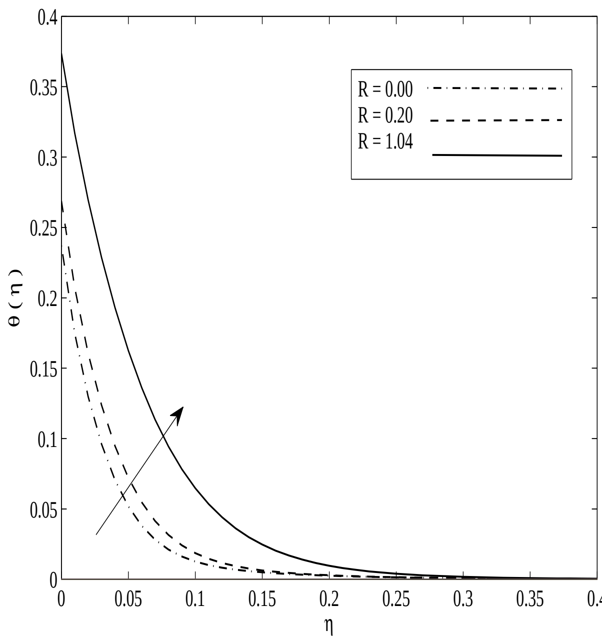

***Figure* 4.33:** Temperature behaviour for various values of R.

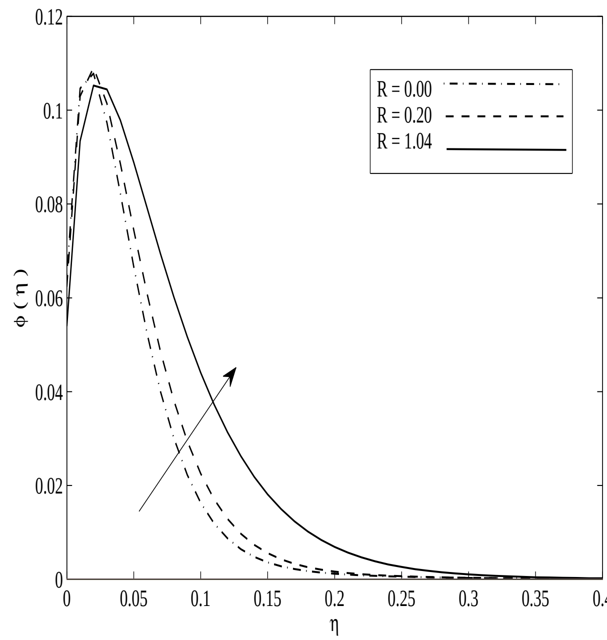

***Figure* 4.34:** Concentration behaviour for various values of R.

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5 Conclusions

The numerical investigation has been carried out to analyze the Soret effect on *MHD* nanofluid flow past a nonlinear shrinking sheet in the presence of thermal radiation, viscous dissipation and chemical reaction under

***Table* 4.2:** Numerical values of $Nu_xRe_x^{-1/2}$

K^*	Ha^2	Nt	Nb	Ec	Cr^*	Pr_{eff}	Le	$Nu_xRe_x^{-1/2}$
0.00	0.50	0.50	0.50	0.10	0.10	6.80	10.0	8.776111
1.00	**0.50**	**0.50**	**0.50**	**0.10**	**0.10**	**6.80**	**10.0**	**8.773807**
2.00								8.772007
4.00								8.768707
1.00	**0.00**	0.50	0.50	0.10	0.10	6.80	10.0	8.774323
	1.00							8.772523
	1.50							8.770087
1.00	0.50	**0.00**	0.50	0.10	0.10	6.80	10.0	9.306258
		1.00						7.895671
1.00	0.50	0.50	**5.00**	0.10	0.10	6.80	10.0	7.586038
			10.00					6.785098
1.00	0.50	0.50	0.50	**0.50**	0.10	6.80	10.0	7.914547
				1.00				6.778293
1.00	0.50	0.50	0.50	0.10	0.10	**4.00**		7.516687
						15.00		10.216327
1.00	0.50	0.50	0.50	0.10	0.10	6.80	**5.00**	8.854783
							30.0	8.696666
1.00	0.50	0.50	0.50	0.10	**1.00**	6.80	10.0	8.776197
					7.00			8.788108

***Table* 4.3:** Numerical values of $Nu_xRe_x^{-1/2}$

S	λ^*	m	Sr	A	R	Q^*	δ	χ	$Nu_xRe_x^{-1/2}$
3.00	**0.10**	**3.00**	**0.05**	**0.50**	**0.20**	**0.10**	**0.10**	**1.00**	**8.773807**
4.00	0.10	3.00	0.05	0.50	0.20	0.10	0.10	1.00	9.558962
5.00									10.053549
3.00	**0.00**	3.00	0.05	0.50	0.20	0.10	0.10	1.00	8.445462
	0.20								8.877429
3.00	0.10	**1.00**	0.05	0.50	0.20	0.10	0.10	1.00	6.230143
		2.00							7.816951
3.00	0.10	3.00	**0.10**	0.50	0.20	0.10	0.10	1.00	8.617619
			0.20						8.321183
3.00	0.10	3.00	0.05	0.50	0.20	**5.00**	0.10	1.00	8.662975
						10.00			8.256715
3.00	0.10	3.00	0.05	0.50	0.20	0.10	**0.00**	1.00	17.873911
							0.20		5.124427
3.00	0.10	3.00	0.05	0.50	0.20	0.10	0.10	**1.30**	8.623513
								1.50	8.501317
3.00	0.10	3.00	0.05	**0.00**	0.20	0.10	1.10	1.00	8.778535
				1.20					8.767389
3.00	0.10	3.00	0.05	0.50	**0.00**	0.10	0.10	1.00	7.629306
					1.04				12.778368

velocity and thermal slip conditions. The influence of the governing parameters on velocity, temperature and concentration profiles has been numerically evaluated using Runge- Kutta fourth order method with shooting technique in *MATLAB* software. Some of the important results are as follows:

- Velocity profiles increase for permeability parameter (K^*) and velocity slip parameter (λ^*) while decrease for shrinking parameter (χ).
- The surface temperature increases with an increase in the values of the governing parameters, such as Brownian motion parameter (Nb), thermophoresis parameter (Nt), heat generation parameter (Q^*), shrinking parameter (χ) and Lewis number (Le).
- The enhancement in chemical reaction parameter (Cr^*) declines the nanofluid concentration profile while the nanofluid concentration profile rises up with Soret number (Sr).
- An increase in suction parameter (S) increases both the Skin friction and local Nusselt number.
- Skin friction decreases whereas Nusselt number increases with increasing value of velocity slip parameter (λ^*) whereas reverse effect obtained with the permeability parameter (K^*), unsteadiness parameter (A), magnetic field parameter (Ha^2) and shrinking parameter (χ).
- The Nusselt number decreases with an increase of Brownian motion parameter (Nb), thermophoresis parameter (Nt), heat generation parameter (Q^*), thermal slip parameter (δ), Eckert number (Ec) and Soret number (Sr).

Acknowledgement: Authors are very much grateful to the Editors and Reviewers for their fruitful suggations to bring the paper in present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 22-27**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

FEKETE-SZEGÖ INEQUALITY AND ZALCMAN FUNCTIONAL FOR CERTAIN SUBCLASS OF ALPHA-CONVEX FUNCTIONS

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(Received: June 28, 2022; In format: July 17, 2022; Finally revised: January 06, 2023; Accepted: January 31, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53102

Abstract

In the present investigation, we introduce a subclass of α-convex functions defined with subordination and associated with Cardioid domain in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. We establish the bounds for $|a_2|$, $|a_3|$ and $|a_4|$, Fekete-Szegö inequality and bound for the Zalcman functional for this class. The results proved earlier will follow as special cases.
2020 Mathematical Sciences Classification: 30C45, 30C50.
Keywords and Phrases: Analytic functions, Alpha-convex functions, Subordination, Cardioid domain, Coefficient bounds, Fekete-Szegö inequality, Zalcman functional.

1 Introduction

By $\mathcal{A}$, we denote the class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, defined in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. The subclass of $\mathcal{A}$, which consists of univalent functions in E, is denoted by $\mathcal{S}$.

In the theory of univalent functions, a very noted result was Bieberbach's conjecture which was established by Bieberbach [2]. It states that, for $f \in \mathcal{S}$, $|a_n| \leq n$, $n = 2, 3, \ldots$ and it remained as a challenge for the mathematicians for a long time. Finally, L. De-Branges [4], proved this conjecture in 1985. During the course of proving this conjecture, various results related to the coefficients were established and some new subclasses of $\mathcal{S}$ were developed.

For two analytic functions f and g in E, f is said to be subordinate to g (symbolically $f \prec g$) if there exists a function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$ such that $f(z) = g(w(z))$. Further, if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

Before defining our main classes, firstly we review some basic and relevant classes mentioned below:

$\mathcal{S}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}$, the class of starlike functions.

$\mathcal{K} = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}$, the class of convex functions.

Mocanu [11] introduced a unifying class $\mathcal{M}(\alpha)$ as below:

$$\mathcal{M}(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}.$$

The functions in the class $\mathcal{M}(\alpha)$ are known as alpha-convex functions. In particular, $\mathcal{M}(0) \equiv \mathcal{S}^*$ and $\mathcal{M}(1) \equiv \mathcal{K}$.

For $f \in \mathcal{A}$, the relation $f \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2$ means that f lies in the region bounded by the cardioid given by

$$(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0.$$

Sharma et al. [16] introduced the classes $\mathcal{S}^*_{car}$ and $\mathcal{K}_{car}$ defined as follow:

$$\mathcal{S}^*_{car} = \left\{ f : f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, z \in E \right\}$$

and

$$\mathcal{K}_{car} = \left\{ f : f \in \mathcal{A}, \frac{(zf'(z))'}{f'(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, z \in E \right\}.$$

Obviously, $\mathcal{S}^*_{car}$ and $\mathcal{K}_{car}$ are the subclasses of starlike and convex functions associated with cardioid domain, respectively. Various subclasses of analytic functions were studied by subordinating to different kind of functions. Malik et al. [9, 10], Sharma et al. [16], Zainab et al. [18], Shi et al. [17] and Raza et al. [15] studied certain classes af analytic functions associated with cardioid domain.

Getting inspired from the above works, now we define the following subclass of α-convex functions by subordinating to $1 + \frac{4}{3}z + \frac{2}{3}z^2$.

Definition 1.1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}^\alpha_{car}$ if it satisfying the condition*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2.$$

The class $\mathcal{M}^\alpha_{car}$ is the unification of the classes $\mathcal{S}^*_{car}$ and $\mathcal{K}_{car}$ and for particular values of α, the results for the these classes can be obtained. In particular, we have the following observations:
(i) $\mathcal{M}^0_{car} \equiv \mathcal{S}^*_{car}$.
(ii) $\mathcal{M}^1_{car} \equiv \mathcal{K}_{car}$.

Fekete and Szegö [5] established the estimate $|a_3 - \mu a_2^2|$, where μ is real and $f \in \mathcal{S}$. Further, the upper bound of $|a_3 - \mu a_2^2|$ for various classes of analytic functions were extensively studied by several authors. There is another very useful functional $J_{n,m}(f) = a_n a_m - a_{m+n-1}$, $n, m \in \mathbb{N} - \{1\}$, which was investigated by Ma [8] and it is known as generalized Zalcman functional. The functional $J_{2,3}(f) = a_2 a_3 - a_4$ is a specific case of the generalized Zalcman functional. Various authors including Khan et al. [7], Mohamad and Wahid [12] and Cho et al. [3], computed the upper bound for the functional $J_{2,3}(f)$ over different subclasses of analytic functions as it plays very important role in finding the bounds for the third Hankel determinant.

In the present paper, we establish the upper bounds for the initial coefficients, Fekete-Szegö inequality and bound for the Zalcman functional for the class $\mathcal{M}^\alpha_{car}$. Also various known results follow as particular cases.

2 Preliminary Results

By $\mathcal{P}$, we denote the class of analytic functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

whose real parts are positive in E.

To prove our main results, we shall make use of the following lemmas:

Lemma 2.1. [2] ([14, 6]) *If $p \in \mathcal{P}$, then*

$$|p_k| \leq 2, k \in \mathbb{N},$$

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2},$$

$$|p_{i+j} - \mu p_i p_j| \leq 2, 0 \leq \mu \leq 1,$$

and for complex number ρ, we have

$$|p_2 - \rho p_1^2| \leq 2\max\{1, |2\rho - 1|\}.$$

Lemma 2.2. ([1]). *Let $p \in \mathcal{P}$, then*

$$|Jp_1^3 - Kp_1p_2 + Lp_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|.$$

In particular, it is proved in [14] *that*

$$|p_1^3 - 2p_1p_2 + p_3| \leq 2.$$

3 Main Results

Theorem 3.1. *If $f \in \mathcal{M}_{car}^{\alpha}$, then*

$$|a_2| \leq \frac{4}{3(1+\alpha)}, \tag{3.1}$$

$$|a_3| \leq \frac{3\alpha^2+30\alpha+11}{9(1+2\alpha)(1+\alpha)^2}, \tag{3.2}$$

and

$$|a_4| \leq \frac{180\alpha^3+940\alpha^2+444\alpha+68}{81(1+2\alpha)(1+3\alpha)(1+\alpha)^3}. \tag{3.3}$$

The bounds are sharp.

Proof. As $f \in \mathcal{M}_{car}^{\alpha}$, by the principle of subordination, we have

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = 1 + \frac{4}{3}w(z) + \frac{2}{3}(w(z))^2. \tag{3.4}$$

Define $p(z) = \frac{1+w(z)}{1-w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$, which implies $w(z) = \frac{p(z)-1}{p(z)+1}$.

On expanding, we have

$$(1-\alpha)\tfrac{zf'(z)}{f(z)} + \alpha\tfrac{(zf'(z))'}{f'(z)} = 1 + (1+\alpha)a_2 z + \left[2(1+2\alpha)a_3 - (1+3\alpha)a_2^2\right] z^2$$
$$+ \left[3(1+3\alpha)a_4 - 3(1+5\alpha)a_2 a_3 + (1+7\alpha)a_2^3\right] z^3 + ... \tag{3.5}$$

Also

$$1 + \frac{4}{3}w(z) + \frac{2}{3}(w(z))^2 = 1 + \frac{2}{3}p_1 z + \left(\frac{2}{3}p_2 - \frac{p_1^2}{6}\right) z^2 + \left(\frac{2}{3}p_3 - \frac{1}{3}p_1 p_2\right) z^3 + ... \tag{3.6}$$

Using (3.5) and (3.6), (3.4) yields

$$1 + (1+\alpha)a_2 z + \left[2(1+2\alpha)a_3 - (1+3\alpha)a_2^2\right] z^2$$
$$+ \left[3(1+3\alpha)a_4 - 3(1+5\alpha)a_2 a_3 + (1+7\alpha)a_2^3\right] z^3 + ...$$
$$= 1 + \frac{2}{3}p_1 z + \left(\frac{2}{3}p_2 - \frac{p_1^2}{6}\right) z^2 + \left(\frac{2}{3}p_3 - \frac{1}{3}p_1 p_2\right) z^3 + ... \tag{3.7}$$

On equating the coefficients of z, z^2 and z^3 in (3.7) and on simplification, we obtain

$$a_2 = \frac{2}{3(1+\alpha)}p_1, \tag{3.8}$$

$$a_3 = \frac{1}{2(1+2\alpha)}\left[\frac{2}{3}p_2 + \left(\frac{5+18\alpha-3\alpha^2}{18(1+\alpha)^2}\right)p_1^2\right], \tag{3.9}$$

and

$$a_4 = \frac{1}{9(1+3\alpha)}\left[2p_3 + \frac{1+7\alpha-2\alpha^2}{(1+\alpha)(1+2\alpha)}p_1 p_2 + \frac{-45\alpha^3+37\alpha^2-15\alpha-1}{18(1+2\alpha)(1+\alpha)^3}p_1^3\right]. \tag{3.10}$$

Using first inequality of Lemma 2.1 in (3.8), the result (3.1) is obvious.

From (3.9), we have

$$|a_3| = \frac{1}{3(1+2\alpha)}\left|p_2 - \frac{3}{2}\left(\frac{3\alpha^2-18\alpha-5}{18(1+\alpha)^2}\right)p_1^2\right|. \tag{3.11}$$

Using fourth inequality of Lemma 2.1 in (3.11), the result (3.2) can be easily obtained.

Furthermore, on applying Lemma 2.2 in (3.10), the result (3.3) is obvious. □

Remark 3.1. The results of Theorem 3.1 are sharp and the equality is attained in (3.1), (3.2) and (3.3) for the function f given by

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = 1 + \frac{4}{3}z + \frac{2}{3}z^2.$$

Proof. The expansion of $(1-\alpha)\dfrac{zf'(z)}{f(z)}+\alpha\dfrac{(zf'(z))'}{f'(z)}=1+\frac{4}{3}z+\frac{2}{3}z^2$, yields

$1+(1+\alpha)a_2z+\left[2(1+2\alpha)a_3-(1+3\alpha)a_2^2\right]z^2+\left[3(1+3\alpha)a_4-3(1+5\alpha)a_2a_3+(1+7\alpha)a_2^3\right]z^3+...=1+\frac{4}{3}z+\frac{2}{3}z^2.$

On equating the coefficients of z, it gives

$(1+\alpha)a_2=\frac{4}{3}$, which implies $|a_2|=\dfrac{4}{3(1+\alpha)}$ and it gives equality in (3.1).

Equating the coefficients of z^2, we obtain

$$2(1+2\alpha)a_3-(1+3\alpha)a_2^2=\frac{2}{3}.$$

On sustituting the value of a_2, we can easily obtain

$$|a_3|=\frac{3\alpha^2+30\alpha+11}{9(1+2\alpha)(1+\alpha)^2},$$

which shows equality in (3.2).

Further equating the coefficients of z^3, we get

$$3(1+3\alpha)a_4-3(1+5\alpha)a_2a_3+(1+7\alpha)a_2^3=0.$$

On substituting the values of a_2 and a_3 and after simplification, it is obvious to get

$$|a_4|=\frac{180\alpha^3+940\alpha^2+444\alpha+68}{81(1+2\alpha)(1+3\alpha)(1+\alpha)^3},$$

which shows equality in (3.3).

For $\alpha=0$, Theorem 3.1 yields the following result proved by Shi et al. [17]: □

Corollary 3.1. *If $f\in\mathcal{S}^*_{car}$, then*

$$|a_2|\leq\frac{4}{3},|a_3|\leq\frac{11}{9},|a_4|\leq\frac{68}{81}.$$

On putting $\alpha=1$ in Theorem 3.1, the following result due to Shi et al. [17] can be easily obtained:

Corollary 3.2. *If $f\in\mathcal{K}_{car}$, then*

$$|a_2|\leq\frac{2}{3},|a_3|\leq\frac{11}{27},|a_4|\leq\frac{17}{81}.$$

Theorem 3.2. *If $f\in\mathcal{M}^{\alpha}_{car}$, then*

$$|a_3-\mu a_2^2|\leq\frac{2}{3(1+2\alpha)}max\left\{1,\frac{16\mu(1+2\alpha)-3\alpha^2-30\alpha-11}{6(1+\alpha)^2}\right\}. \tag{3.12}$$

Proof. From (3.8) and (3.9), we have

$$|a_3-\mu a_2^2|=\frac{1}{3(1+2\alpha)}\left|p_2-\frac{16\mu(1+2\alpha)+3\alpha^2-18\alpha-5}{12(1+\alpha)^2}p_1^2\right|. \tag{3.13}$$

Using fourth inequality of Lemma 2.1, (3.13) yields

$$|a_3-\mu a_2^2|\leq\frac{2}{3(1+2\alpha)}max\left\{1,\frac{16\mu(1+2\alpha)-3\alpha^2-30\alpha-11}{6(1+\alpha)^2}\right\}. \tag{3.14}$$

Hence, the result (3.12) is obvious from (3.14).

For $\mu=1$, the result (3.12) yields

$$|a_3-a_2^2|\leq\frac{2}{3(1+2\alpha)}max\left\{1,\frac{5+2\alpha-3\alpha^2}{6(1+\alpha)^2}\right\}.$$

But $\dfrac{5+2\alpha-3\alpha^2}{6(1+\alpha)^2}\leq 1$, for $0\leq\alpha\leq 1$.

Hence, we have

$$|a_3-a_2^2|\leq\frac{2}{3(1+2\alpha)}. \tag{3.15}$$

For $\alpha=0$, the following result is obvious from Theorem 3.2: □

Corollary 3.3. *If $f \in \mathcal{S}^*_{car}$, then*

$$|a_3 - \mu a_2^2| \leq \frac{2}{3} max \left\{1, \frac{16\mu - 11}{6}\right\}.$$

For $\alpha = 1$, Theorem 3.2 agrees with the following result:

Corollary 3.4. *If $f \in \mathcal{K}_{car}$, then*

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} max \left\{1, \frac{12\mu - 11}{6}\right\}.$$

For $\mu = 1, \alpha = 0$, Theorem 3.2 yields the following result:

Corollary 3.5. *If $f \in \mathcal{S}^*_{car}$, then*

$$|a_3 - a_2^2| \leq \frac{2}{3}.$$

For $\mu = 1, \alpha = 1$, Theorem 3.2 gives the following result:

Corollary 3.6. *If $f \in \mathcal{K}_{car}$, then*

$$|a_3 - a_2^2| \leq \frac{2}{9}.$$

Theorem 3.3. *If $f \in \mathcal{M}^{\alpha}_{car}$, then*

$$|a_2 a_3 - a_4| \leq \frac{72\alpha^3 + 216\alpha^2 + 340\alpha + 68}{81(1+\alpha)^2(1+2\alpha)(1+3\alpha)}. \tag{3.16}$$

Proof. Using (3.8), (3.9) and (3.10), we have $a_2 a_3 - a_4$

$$= \frac{1}{81(1+\alpha)^2(1+2\alpha)(1+3\alpha)} \left[(9\alpha^2 + 49\alpha + 8)p_1^3 - 9(1+\alpha)(-1+\alpha-2\alpha^2)p_1 p_2 + 18(1+\alpha)^2(-1-2\alpha)p_3\right]. \tag{3.17}$$

Taking modulus and on applying Lemma 2.2, the result (3.16) is obvious from (3.17).

For $\alpha = 0$, Theorem 3.3 yields the following result: □

Corollary 3.7. *If $f \in \mathcal{S}^*_{car}$, then*

$$|a_2 a_3 - a_4| \leq \frac{68}{81}.$$

For $\alpha = 1$, Theorem 3.3 yields the following result:

Corollary 3.8. *If $f \in \mathcal{K}_{car}$, then*

$$|a_2 a_3 - a_4| \leq \frac{29}{162}.$$

4 Conclusion and Open Problems

Till now, many researchers have studied the coefficient problems for various fundamental subclasses of analytic functions, but not much work has been done on the coefficients of subclasses of alpha-convex functions as it involves some lengthy and complicated calculations. In the present investigation, a new subclass of alpha-convex functions is introduced by subordinating to the cardioid domain. We establish the bounds for the first three coefficients, Fekete-Szegö inequality and Zalcman functional for the class $\mathcal{M}^{\alpha}_{car}$. The results obtained here, generalize the results of various authors. The results of this paper can be extended towards the estimation of third and fourth Hankel determinants and also this work will motivate the other researchers to study some more generalized classes of functions.

Acknowledgement. The authors are very much grateful to the Editor and referees for their valuable suggestions to revise the paper.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 28-37**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

RELATIONS AND IDENTITIES DUE TO DOUBLE SERIES ASSOCIATED WITH GENERAL HURWITZ-LERCH TYPE ZETA FUNCTIONS

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(Received: December 20, 2022; In format: December 29, 2022; Revised: January 06, 2023; Accepted: February 02, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53103

Abstract

In this paper, we introduce certain families of double series associated with general Hurwitz-Lerch type Zeta functions and then derive their summation formulae, series and integral identities. Again then using these identities, we obtain various known and unknown results and hypergeometric generating relations.

2020 Mathematical Sciences Classification: 11M35, 33C65.
Keywords and Phrases: Double series associated with general HurwitzLerch type Zeta functions, summation formulae, series and integral identities, hypergeometric generating relations.

1 Introduction and preliminaries

In the entire paper, in the standard notation it is provided that
$\mathbb{C}=\left\{z: z=x+iy: x, y\in \mathbb{R}, i=\sqrt{(-1)}\right\}, \mathbb{Z}_0^-=\{0,-1,-2,\ldots\}, \mathbb{R}=(-\infty,\infty)$ and
$\mathbb{N}_0=\mathbb{N}\cup\{0\}=\{0,1,2,3,\ldots\}$.

The generalized Gaussian hypergeometric function has been studied and applied in computation of various problems occurring in different fields of science and technology (for example [5], [10], [11] and others) as defined by (see [13, pp. 73-74], [18, pp. 42-43])

$$ {}_pF_q\left(\begin{matrix}(\alpha)_{1,p};\\(\gamma)_{1,q};\end{matrix} z\right)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\gamma_i)_n}\frac{z^n}{n!}, \tag{1.1}$$

where $p,q\in\mathbb{N}_0, \alpha_i\in\mathbb{C}, (i=1,2,3,\ldots,p); \gamma_i\in\mathbb{C}\backslash\mathbb{Z}_0^-, (i=1,2,3,\ldots,q);\ z\in\mathbb{C}$.

The series in (1.1) (i) converges for $|z|<\infty$, if $p\le q$; (ii) converges for $|z|<1$, if $p=q+1$; (iii) diverges for all $z,\ z\neq 0$, if $p>q+1$; (iv) converges absolutely for $|z|=1$, if $p=q+1$, and $\Re(\omega)>0, \omega=\sum_{i=1}^{q}\gamma_i-\sum_{i=1}^{p}\alpha_i$; (v) converges conditionally for $|z|=1, z\neq 1$, if $p=q+1$, and $-1<\Re(\omega)\le 0$; (vi) diverges for $|z|=1$, if $p=q+1$, and $\Re(\omega)<-1$.

In reference of (1.1), when $p=2,\ q=1$, following extended Hurwitz-Lerch type hypergeometric Zeta function is studied in [2], written by

$$\phi_{\alpha,\beta;\gamma}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}\frac{z^n}{(n+a)^s}=\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-at}t^{s-1}{}_2F_1\left(\begin{matrix}\alpha,\beta;\\\gamma;\end{matrix} ze^{-t}\right)dt,$$
$$\forall\ a,\ \alpha,\beta,s,z\in\mathbb{C}, \Re(a)>0, \Re(s)>0 \text{ and } \gamma\in\mathbb{C}\backslash\mathbb{Z}_0^-. \tag{1.2}$$

It is remarked that the series of the extended Hurwitz-Lerch type hypergeometric Zeta function (1.2) converges if we have $\Re(s)>0$, when $|z|<1,\ (z\neq 1)$.

But when $z=1$, we apply the techniques of Gaussian gamma function ([6], [7], [12]) and then Watson's theorem (see [14, p.54, Eqn. (2.3.3.13)], [18, p. 95, Problem 26]) and it is provided $\Re(\gamma)>\frac{1}{2}\Re(\alpha+\beta+1)>0$, then the series given in (1.2) converges if

$$\Re(s)>\frac{1}{2}\Re(\alpha+\beta)-\frac{1}{2}. \tag{1.3}$$

Also in Eqns. (1.1)-(1.2), for $a \neq 0$ the Pochhammer symbol [18, p. 22] is used and defined as factorial function given by

$$(a)_n = \begin{cases} a(a+1)(a+2)\dots(a+n-1); n \geq 1, \\ 1; n = 0, \end{cases}$$

and is related with the gamma function as

$$(a)_\nu = \frac{\Gamma(a+\nu)}{\Gamma(a)}, \forall \nu \in \mathbb{R}. \tag{1.4}$$

Clearly, a relation of (1.2) with the Hurwitz-Lerch Zeta function (see in [17]) is given as

$$\phi_{\alpha,1;\alpha}(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} = \phi(z,s,a), \tag{1.5}$$

converges for all $s, z \in \mathbb{C}$, and $a \in \mathbb{C} \backslash \mathbb{Z}_0^-$, $\Re(s) > 0$, when $|z| < 1$, $(z \neq 1)$, and when $z = 1$, the series in (1.2) is convergent for $\Re(s) > 1$.

Further by (1.5) at $z = 1$, we have a relation with shifted Hurwitz Zeta function ([3], see in also [8])

$$\phi_{\alpha,1,\alpha}(1,s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \zeta(s;a), \text{ where, } a \in \mathbb{C} \backslash \mathbb{Z}_0^- \text{ and } \Re(s) > 1. \tag{1.6}$$

Generalized Kobayashi-Stieltjes type operators [9] seem identical to extended Hurwitz-Lerch type Zeta functions, Srivastava-Daoust Double series used in initial value problems [4], Hurwitz-Lerch Zeta functions associated with double series of the Appell, Kampé de Fériet and Srivastava-Daoust functions studied in ([17], [12] and others). Srivastava [16] obtained various generating relations associated with Hurwitz-Lerch Zeta functions. In this motivation, here in our researches for exploring new ideas in the theory of Hurwitz-Lerch Zeta functions and for obtaining of generating relations, series and integral identities, we consider the parameters $x, y, s, d \in \mathbb{C}, a \in \mathbb{C} \backslash \mathbb{Z}_0^-, |x^2| < 1$ and A_n be bounded real or complex sequences $\forall n \in \mathbb{N}_0$ and $A_0 \neq 0$. Then we present following the families of double series associated with general Hurwitz-Lerch type Zeta functions defined as

$$R_1\left(A, \frac{d}{2}+\frac{1}{2}, \frac{d}{2}+1; \frac{3}{2}; x, y; s, a\right) = \sum_{m,n=0}^{\infty} \frac{A_n \left(\frac{d}{2}+\frac{1}{2}\right)_{m+n} \left(\frac{d}{2}+1\right)_{m+n}}{\left(\frac{3}{2}\right)_m} \frac{x^{2m+2n} y^n}{(n+a)^s m! n!}, \tag{1.7}$$

$$R_2\left(A, \frac{d}{2}, \frac{d}{2}+\frac{1}{2}; \frac{1}{2}; x, y; s, a\right) = \sum_{m,n=0}^{\infty} \frac{A_n \left(\frac{d}{2}\right)_{m+n} \left(\frac{d}{2}+\frac{1}{2}\right)_{m+n}}{\left(\frac{1}{2}\right)_m} \frac{x^{2m+2n} y^n}{(n+a)^s m! n!}. \tag{1.8}$$

For these double series (1.7) and (1.8), we evaluate their summation formulae and derive various interesting series and integral identities. Further applying these identities, we obtain various known and unknown results involving the Hurwitz-Lerch type Zeta functions and hypergeometric generating relations.

2 Summation Formulae

In this section, we obtain summation formulae of the families of double series associated with general Hurwitz-Lerch type Zeta functions $\forall s \in \mathbb{C}$ and $\Re(s) > 1$, defined in the Eqns. (1.7) and (1.8) in form of the generalized Dirichlet type L-functions below in Eqn. (2.2) studied in [8].

For a bounded sequence A_n, an extended Dirichlet type L-function [8] is defined by

$$L(s, A; z) = \sum_{n=1}^{\infty} \frac{A_n z^n}{n^s}, \forall s \in \mathbb{C}, |z| < 1 (z \neq 1) \tag{2.1}$$

and

$$L(s, A) = \sum_{n=1}^{\infty} \frac{A_n}{n^s} \forall s \in \mathbb{C} \text{ and } \Re(s) > 1. \tag{2.2}$$

Further we extend (2.1) and (2.2) $\forall s, z \in \mathbb{C}$, and $a \in \mathbb{C} \backslash \mathbb{Z}_0^-$, A_n be bounded sequence, in general Hurwitz-Lerch type Zeta functions as

$$\phi(s, A, a; z) = \sum_{n=0}^{\infty} \frac{A_n z^n}{(n+a)^s}, \quad |z| < 1 (z \neq 1), a \in \mathbb{C} \backslash \mathbb{Z}_0^- \ \forall s \in \mathbb{C}, \tag{2.3}$$

and

$$\phi(s, A, a; 1) = \sum_{n=0}^{\infty} \frac{A_n}{(n+a)^s}, \quad a \in \mathbb{C} \backslash \mathbb{Z}_0^- \ \forall s \in \mathbb{C} \text{ and } \Re(s) > 1. \tag{2.4}$$

Lemma 2.1. *Let for all $s, z \in \mathbb{C}, a \in \mathbb{C}\backslash Z_0^-$ and A_n be bounded sequence, then summation formulas (2.3) and (2.4) for a general Hurwitz-Lerch type Zeta function exist in the form*

$$\phi(s, A, a; z) = \frac{A_0}{a^s} + \sum_{r=0}^{\infty} \begin{pmatrix} -s \\ r \end{pmatrix} L(s+r, A; z) a^r, \tag{2.5}$$

where $|z| < 1 (z \neq 1), a \in \mathbb{C}\backslash\mathbb{Z}_0^-, \forall s \in \mathbb{C}$,
and

$$\phi(s, A, a; 1) = \frac{A_0}{a^s} + \sum_{r=0}^{\infty} \begin{pmatrix} -s \\ r \end{pmatrix} L(s+r, A) a^r, \tag{2.6}$$

where $a \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ s \in C$ *and* $\Re(s) > 1$.

Proof. Under the conditions of Lemma 2.1, we write (2.3) as

$$\phi(s, A, a; z) = \frac{A_0}{a^s} + \sum_{n=1}^{\infty} \frac{A_n z^n}{n^s} \left(1 + \frac{a}{n}\right)^{-s}. \tag{2.7}$$

Now applying binomial theorem and (2.1) we obtain (2.6).
Similarly making an appeal to (2.2) and (2.4) we get (2.6).
Hence Lemma 2.1 is proved. □

It is remarked that the formula (2.6) is identical to the summation formula due to Murthy and Sinha [8], when $z = 1$.

Lemma 2.2. *Under the conditions $\alpha, \beta, s, z \in \mathbb{C},\ a,\ \gamma \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $|z| < 1 (z \neq 1)$, the function* (1.2) *follows following summation formula*

$$\phi_{\alpha,\beta;\gamma}(z, s, a) = \frac{1}{a^s} + \left(\frac{\alpha\beta}{\gamma} z\right) \sum_{r=0}^{\infty} \begin{pmatrix} -s \\ r \end{pmatrix} \phi_{\alpha+1,\beta+1;\gamma+1}(z, s+r+1, 1) a^r, \tag{2.8}$$

and for $\alpha, \beta, s, z \in \mathbb{C},\ a,\ \gamma \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ z = 1$, there exists the formula

$$\phi_{\alpha,\beta;\gamma}(1, s, a) = \frac{1}{a^s} + \left(\frac{\alpha\beta}{\gamma}\right) \sum_{r=0}^{\infty} \begin{pmatrix} -s \\ r \end{pmatrix} \phi_{\alpha+1,\beta+1;\gamma+1}(1, s+r+1, 1) a^r, \tag{2.9}$$

provided that

$$\Re(\gamma) > \frac{1}{2}\Re(\alpha+\beta+1) > 0.$$

Then the inner function in right hand side of (2.9) *converges for*

$$\Re(s) + r > \frac{1}{2}\Re(\alpha+\beta) - \frac{1}{2}, r = 0, 1, 2, \dots . \tag{2.10}$$

Proof. In Eqn. (2.3) setting $A_n = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}$ and making an appeal to the formulae (1.2) and (2.5), we get the summation formula for extended Hurwitz-Lerch type hypergeometric Zeta function (1.2) as

$$\begin{aligned} \phi_{\alpha,\beta;\gamma}(z, s, a) &= \frac{1}{a^s} + \sum_{r=0}^{\infty} \begin{pmatrix} -s \\ r \end{pmatrix} \left\{ \sum_{n=1}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} \frac{z^n}{n^{s+r}} \right\} a^r \\ &= \frac{1}{a^s} + \left(\frac{\alpha\beta}{\gamma} z\right) \sum_{r=0}^{\infty} \begin{pmatrix} -s \\ r \end{pmatrix} \left\{ \sum_{n=0}^{\infty} \frac{(\alpha+1)_n(\beta+1)_n}{(\gamma+1)_n n!} \frac{z^n}{(n+1)^{s+r+1}} \right\} a^r \\ &= \frac{1}{a^s} + \left(\frac{\alpha\beta}{\gamma} z\right) \sum_{r=0}^{\infty} \begin{pmatrix} -s \\ r \end{pmatrix} \phi_{\alpha+1,\beta+1;\gamma+1}(z, s+r+1, 1) a^r. \end{aligned} \tag{2.11}$$

For $z = 1$, by the second equality of the Eqn. (2.11) under the restrictions

$$\Re(\gamma) > \frac{1}{2}\Re(\alpha+\beta+1) > 0,$$

and also for large values of N, we write

$$\phi_{\alpha,\beta;\gamma}(1,s,a)=\frac{1}{a^s}+\left(\frac{\alpha\beta}{\gamma}\right)\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)\sum_{n=0}^{N-1}\frac{(\alpha+1)_n(\beta+1)_n}{(\gamma+1)_n n!}\frac{1}{(n+1)^{s+r+1}}a^r$$
$$+\left(\frac{\alpha\beta}{\gamma}\right)\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)\sum_{n=N}^{\infty}\frac{(\alpha+1)_n(\beta+1)_n}{(\gamma+1)_n n!}\frac{a^r}{(n+1)^{s+r+1}}$$
$$\Rightarrow \phi_{\alpha,\beta;\gamma}(1,s,a)=\frac{1}{a^s}+\left(\frac{\alpha\beta}{\gamma}\right)\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)\sum_{n=0}^{N-1}\frac{(\alpha+1)_n(\beta+1)_n}{(\gamma+1)_n n!}\frac{1}{(n+1)^{s+r+1}}a^r$$
$$+\left(\frac{\alpha\beta}{\gamma}\right)\frac{(\alpha+1)_N(\beta+1)_N}{(\gamma+1)_N\Gamma(N+s+r+2)}\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)\sum_{n=0}^{\infty}\frac{(\alpha+N+1)_n(\beta+N+1)_n(1)_n}{(\gamma+N+1)_n(N+s+r+2)_n n!}a^r.$$

Again if we suppose that $\alpha_1,\beta_1,\gamma_1,a_1,s_1$ are the real parts of α,β,γ,a,s respectively and $\gamma_1>\frac{1}{2}(\alpha_1+\beta_1+1)$, we get an inequality

$$|\phi_{\alpha_1,\beta_1;\gamma_1}(1,s_1,a_1)|<\frac{1}{(a_1)^{s_1}}$$
$$+\left(\frac{\alpha_1\beta_1}{\gamma_1}\right)\sum_{r=0}^{\infty}\left(\begin{array}{c}-s_1\\ r\end{array}\right)\sum_{n=0}^{N-1}\frac{(\alpha_1+1)_n(\beta_1+1)_n}{(\gamma_1+1)_n n!}\frac{1}{(n+1)^{s_1+r+1}}(a_1)^r$$
$$+\left(\frac{\alpha_1\beta_1}{\gamma_1}\right)\frac{(\alpha_1+1)_N(\beta_1+1)_N}{(\gamma_1+1)_N\Gamma(N+s_1+r+2)}\sum_{r=0}^{\infty}\left(\begin{array}{c}-s_1\\ r\end{array}\right)$$
$$\times\,{}_3F_2\left[\begin{array}{c}\alpha_1+N+1,\beta_1+N+1,1+\frac{N}{2}+\frac{S_1}{2}+\frac{r}{2};\\ \frac{1}{2}(\alpha_1+\beta_1+2N+3),2+N+s_1+r;\end{array}1\right](a_1)^r. \quad (2.12)$$

Now applying the Watson's theorem (see [14, p.54, Eqn. (2.3.3.13)], [18, p. 95, Problem 26]) in the function ${}_3F_2[\cdot]$ of right hand side of (2.12), we find the convergence conditions as

$$\Re(s)+r>\frac{1}{2}\Re(\alpha+\beta)-\frac{1}{2}\ \forall r=0,1,2,\dots .$$

Hence the Lemma 2.2 is proved. □

Making an appeal to theory of the Lemmas 2.1 and 2.2, we present following theorems:

Theorem 2.1. *For all $x,y,s,\ d\in\mathbb{C},\Re(s)>1,\ a\in\mathbb{C}\backslash\mathbb{Z}_0^-,\ |x^2|<1$ and A stands for a sequence A_n be bounded real or complex sequences $\forall n\in\mathbb{N}_0$ and $A_0\neq 0$, then by the double series associated with general Hurwitz-Lerch Zeta function* (1.7)*, following summation formula exists*

$$R_1\left(A,\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\frac{3}{2};x,y;s,a\right)=\frac{A_0}{(a)^s}\,{}_2F_1\left(\begin{array}{c}\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\\ \frac{3}{2};\end{array}x^2\right)$$
$$+x^2y\left(\frac{d^2}{4}+\frac{3d}{4}+\frac{1}{2}\right)\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)R_1\left(A^+,\frac{d}{2}+\frac{3}{2},\frac{d}{2}+2;\frac{3}{2};x,y;s+r+1,1\right)a^r \quad (2.13)$$

where, A^+stands for the sequence A_{n+1} $\forall n\in\mathbb{N}_0$.

Proof. We write the formula (1.7) as

$$R_1\left(A,\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\frac{3}{2};x,y;s,a\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_{m+n}\left(\frac{d}{2}+1\right)_{m+n}}{\left(\frac{3}{2}\right)_m}\frac{x^{2m}(yx^2)^n}{(n+a)^s m!n!}.$$

Then for $a\in\mathbb{C}\backslash\mathbb{Z}_0^-$, we derive

$$R_1\left(A,\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\frac{3}{2};x,y;s,a\right)$$
$$=\frac{A_0}{(a)^s}\sum_{m=0}^{\infty}\frac{\left(\frac{d}{2}+\frac{1}{2}\right)_m\left(\frac{d}{2}+1\right)_m}{\left(\frac{3}{2}\right)_m}\frac{x^{2m}}{m!}+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_{m+n}\left(\frac{d}{2}+1\right)_{m+n}}{\left(\frac{3}{2}\right)_m}\frac{x^{2m}(yx^2)^n}{(n+a)^s m!n!}$$
$$=\frac{A_0}{(a)^s}\sum_{m=0}^{\infty}\frac{\left(\frac{d}{2}+\frac{1}{2}\right)_m\left(\frac{d}{2}+1\right)_m}{\left(\frac{3}{2}\right)_m}\frac{(x^2)^m}{m!}$$

$$+ x^2y\left(\frac{d^2}{4}+\frac{3d}{4}+\frac{1}{2}\right)\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)a^r\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{A_{n+1}\left(\frac{d}{2}+\frac{3}{2}\right)_{m+n}\left(\frac{d}{2}+2\right)_{m+n}}{\left(\frac{3}{2}\right)_m(n+1)^{s+r+1}}\frac{x^{2m}(yx^2)^n}{m!n!}. \quad (2.14)$$

□

Theorem 2.2. *For all $x, y, s, d \in \mathbb{C}, \Re(s) > 1,\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ |x^2| < 1$ and A stands for the sequence A_n be bounded real or complex sequences $\forall n \in \mathbb{N}_0$ and $A_0 \neq 0$, then by the double series associated with general Hurwitz-Lerch Zeta function* (1.8), *following summation formula exists*

$$R_2\left(A,\frac{d}{2},\frac{d}{2}+\frac{1}{2};\frac{1}{2};x,y;s,a\right) = \frac{A_0}{a^s}\ {}_2F_1\left(\begin{array}{c}\frac{d}{2},\frac{d}{2}+\frac{1}{2};\\ \frac{1}{2};\end{array} x^2\right)$$

$$+\left(\frac{d^2}{4}+\frac{d}{4}\right)x^2y\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)R_2\left(A^+,\frac{d}{2}+1,\frac{d}{2}+\frac{3}{2};\frac{1}{2};x,y;s+r+1,1\right)a^r. \quad (2.15)$$

Proof. Considering the double series associated with general Hurwitz-Lerch Zeta function (1.8) and applying the same techniques as in the proof of the Theorem 2.1, we establish the required result (2.15). □

3 Series and integral identities

In this section, we derive series and integral identities associated with general Hurwitz-Lerch Zeta functions due to double series defined in the Eqns. (1.7) and (1.8).

Theorem 3.1. *If $|x^2| < 1$, then the double series associated with general Hurwitz-Lerch Zeta function* (1.7) *generates following series identity*

$$\sum_{n=0}^{\infty}\frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!}\sum_{m=0}^{n}\frac{A_m(-n)_m\left(-\frac{1}{2}-n\right)_m}{m!}\frac{y^m}{(m+a)^s}$$

$$=\frac{1}{2xd(1-x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2y}{(1-x)^2}\right)^n}{n!(n+a)^s}$$

$$-\frac{1}{2xd(1+x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2y}{(1+x)^2}\right)^n}{n!(n+a)^s}, \quad (3.1)$$

provided that all conditions of the Theorem 2.1 *are satisfied.*

Proof. Consider the double series associated with general Hurwitz-Lerch Zeta function (1.7) in the form

$$R_1\left(A,\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\frac{3}{2};x,y;s,a\right)$$

$$=\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}+1\right)_n x^{2n}y^n}{(n+a)^s n!}\sum_{m=0}^{\infty}\frac{\left(\frac{d}{2}+\frac{1}{2}+n\right)_m\left(\frac{d}{2}+1+n\right)_m}{\left(\frac{3}{2}\right)_m}\frac{x^{2m}}{m!}. \quad (3.2)$$

Now making an appeal to the result due to Sneddon [15, p. 42, Example II (1 (iii))] given by

$$\sum_{m=0}^{\infty}\frac{\left(\frac{d}{2}+\frac{1}{2}\right)_m\left(\frac{d}{2}+1\right)_m}{\left(\frac{3}{2}\right)_m}\frac{x^{2m}}{m!}=\frac{1}{2dx}(1-x)^{-d}-\frac{1}{2dx}(1+x)^{-d}, x\neq\pm1, \quad (3.3)$$

in right hand side of (3.2), we derive

$$R_1\left(A,\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\frac{3}{2};x,y;s,a\right)$$

$$=\frac{\Gamma\left(\frac{d}{2}\right)}{4x(1-x)^d\Gamma\left(\frac{d}{2}+1\right)}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2y}{(1-x)^2}\right)^n}{n!(n+a)^s}$$

$$-\frac{\Gamma\left(\frac{d}{2}\right)}{4x(1+x)^d\Gamma\left(\frac{d}{2}+1\right)}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2y}{(1+x)^2}\right)^n}{n!(n+a)^s}. \quad (3.4)$$

Further in the double series (1.7), making and appeal to the series rearrangement techniques [18, p. 100], we obtain

$$R_1\left(A, \frac{d}{2}+\frac{1}{2}, \frac{d}{2}+1; \frac{3}{2}; x, y; s, a\right)$$
$$= \sum_{m=0}^{\infty} \frac{\left(\frac{d}{2}+\frac{1}{2}\right)_m \left(\frac{d}{2}+1\right)_m x^{2m}}{m!} \sum_{n=0}^{m} \frac{A_n}{\left(\frac{3}{2}\right)_{m-n}} \frac{y^n}{(n+a)^s (m-n)! n!}$$
$$= \sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!} \sum_{m=0}^{n} \frac{A_m (-n)_m \left(-\frac{1}{2}-n\right)_m}{m!} \frac{y^m}{(m+a)^s}. \tag{3.5}$$

Finally, employing (3.4) and (3.5) we establish the identity (3.1). □

Srivastava [16] obtained various generating relations associated with some families of the extended Hurwitz-Lerch Zeta functions, then to make extension in this area we derive following generating relations for our defined families of the extended Hurwitz-Lerch Zeta functions (1.7) and (1.8), given by

Corollary 3.1. *In the Theorem* 3.1 *set* $A_n = \frac{\prod_{i=1}^{p} (\alpha_i)_n}{\prod_{i=1}^{q} (\gamma_i)_n}, \forall n = 0, 1, 2, 3, \ldots,$ *and define an extended semi-hypergeometric Hurwitz-Lerch Zeta function*

$${}_{p+2}H_q\left(\begin{matrix}(\alpha)_{1,p}, -n, -\frac{1}{2}-n; \\ (\gamma)_{1,q};\end{matrix}\ y, s, a\right) = \sum_{m=0}^{n} \frac{\prod_{i=1}^{p} (\alpha_i)_n (-n)_m \left(-\frac{1}{2}-n\right)_m}{\prod_{i=1}^{q} (\gamma_i)_n m!} \frac{y^m}{(m+a)^s}, \tag{3.6}$$

and then make an appeal to equality (3.1) *for* $|x^2| < 1$, *there exists a generating relation of extended generalized hypergeometric Hurwitz-Lerch Zeta function due to the formula* (1.7) *given by*

$$\sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!}\ {}_{p+2}H_q\left(\begin{matrix}(\alpha)_{1,p}, -n, -\frac{1}{2}\ n; \\ (\gamma)_{1,q};\end{matrix} y, s, a\right)$$
$$= \frac{1}{2xd(1-x)^d}\ {}_{p+2}H_q\left(\begin{matrix}(\alpha)_{1,p}, \frac{d}{2}+\frac{1}{2}, \frac{d}{2}; \\ (\gamma)_{1,q};\end{matrix} \frac{x^2 y}{(1-x)^2}, s, a\right)$$
$$- \frac{1}{2xd(1+x)^d}\ {}_{p+2}H_q\left(\begin{matrix}(\alpha)_{1,p}, \frac{d}{2}+\frac{1}{2}, \frac{d}{2}; \\ (\gamma)_{1,q};\end{matrix} \frac{x^2 y}{(1+x)^2}, s, a\right). \tag{3.7}$$

Theorem 3.2. *The double series associated with general Hurwitz-Lerch Zeta function* (1.7), *generates the following integral identity*

$$\int_0^{\infty} e^{-at} t^{s-1} \left\{\sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!} \sum_{m=0}^{n} \frac{A_m (-n)_m \left(-\frac{1}{2}-n\right)_m (ye^{-t})^m}{m!}\right\} dt$$
$$= \frac{1}{2xd} \int_0^{\infty} e^{-at} t^{s-1} \left\{\frac{1}{(1-x)^d} \sum_{n=0}^{\infty} \frac{A_n \left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{d}{2}\right)_n \left(\frac{x^2 y e^{-t}}{(1-x)^2}\right)^n}{n!}\right.$$
$$\left. - \frac{1}{(1+x)^d} \sum_{n=0}^{\infty} \frac{A_n \left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{d}{2}\right)_n \left(\frac{x^2 y e^{-t}}{(1+x)^2}\right)^n}{n!}\right\} dt, \tag{3.8}$$

where, $\Re(a) > 0, \Re(s) > 0, |x^2| < 1.$

Proof. Making an appeal to the equality (3.5) and to the Euler integral formula ([6], [7], [17]), we find an integral representation for $\Re(a) > 0, \Re(s) > 0$ as

$$R_1\left(A, \frac{d}{2}+\frac{1}{2}, \frac{d}{2}+1; \frac{3}{2}; x, y; s, a\right)$$
$$= \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-at} t^{s-1} \left\{\sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!} \sum_{m=0}^{n} \frac{A_m (-n)_m \left(-\frac{1}{2}-n\right)_m (ye^{-t})^m}{m!}\right\} dt. \tag{3.9}$$

Again starting with the equality (3.4) and applying the same techniques as in (3.9), we obtain the result

$$R_1\left(A,\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\frac{3}{2};x,y;s,a\right) = \frac{1}{\Gamma(s)}\int_0^\infty e^{-at}t^{s-1}\left\{\frac{1}{2xd(1-x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2ye^{-t}}{(1-x)^2}\right)^n}{n!} - \frac{1}{2xd(1+x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2ye^{-t}}{(1+x)^2}\right)^n}{n!}\right\}dt. \quad (3.10)$$

The relations (3.9) and (3.10) immediately give the integral equality (3.8). □

Theorem 3.3. *The double series associated with general Hurwitz-Lerch Zeta function* (1.7) *generates the following general generating relation for* $|x^2| < 1$,

$$\sum_{n=0}^{\infty}\frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!}\sum_{m=0}^{n}\frac{A_m(-n)_m\left(-\frac{1}{2}-n\right)_m(ye^{-t})^m}{m!} = \frac{1}{2xd(1-x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2ye^{-t}}{(1-x)^2}\right)^n}{n!} - \frac{1}{2xd(1+x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2ye^{-t}}{(1+x)^2}\right)^n}{n!}. \quad (3.11)$$

Proof. Making an appeal to the result (3.8) of the Theorem 3.2 we get an identity. This identity gives us the general generating relation (3.11). □

In the similar manner by the double series associated with general Hurwitz-Lerch Zeta function (1.8), we derive:

Theorem 3.4. *Double series associated with general Hurwitz-Lerch Zeta function* (1.8) *generates following series identity for* $|x^2| < 1,$ *as*

$$\sum_{n=0}^{\infty}\frac{\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n(x^2)^n}{\left(\frac{1}{2}\right)_n n!}\sum_{m=0}^{n}\frac{A_m(-n)_m\left(\frac{1}{2}-n\right)_m y^m}{(m+a)^s m!} = \frac{1}{2(1-x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{x^2y}{(1-x)^2}\right)^n}{(n+a)^s n!} + \frac{1}{2(1+x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{x^2y}{(1+x)^2}\right)^n}{(n+a)^s n!}. \quad (3.12)$$

provided that all conditions of the Theorem 2.2 are satisfied.

Proof. Considering the formula (1.8) and making an appeal to the revised result due to Sneddon [15, p. 42, Example II (1 (ii))]

$$\sum_{m=0}^{\infty}\frac{\left(\frac{d}{2}\right)_m\left(\frac{d}{2}+\frac{1}{2}\right)_m}{\left(\frac{1}{2}\right)_m}\frac{(x^2)^m}{m!} = \frac{1}{2}\left\{(1-x)^{-d}+(1+x)^{-d}\right\},$$

we arrive at

$$R_2\left(A,\frac{d}{2},\frac{d}{2}+\frac{1}{2};\frac{1}{2};x,y;s,a\right) = \frac{1}{2(1-x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{x^2y}{(1-x)^2}\right)^n}{(n+a)^s n!} + \frac{1}{2(1+x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{x^2y}{(1+x)^2}\right)^n}{(n+a)^s n!}. \quad (3.13)$$

provided that all conditions of the Theorem 2.2 are satisfied.

Further for the same conditions of (3.13). making an appeal to formula (1.8) and series rearrangement techniques, we obtain

$$R_2\left(A,\frac{d}{2},\frac{d}{2}+\frac{1}{2};\frac{1}{2};x,y;s,a\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{A_n\left(\frac{d}{2}\right)_m\left(\frac{d}{2}+\frac{1}{2}\right)_m}{\left(\frac{1}{2}\right)_{m-n}}\frac{\left(x^2\right)^m y^n}{(n+a)^s(m-n)!n!}. \tag{3.14}$$

But for all n such that $0\leq n\leq m$, we have

$$\left(\frac{1}{2}\right)_{m-n}=\frac{(-1)^n\left(\frac{1}{2}\right)_m}{\left(\frac{1}{2}-m\right)_n}\text{ and }\frac{1}{(m-n)!}=\frac{(-1)^n(-m)_n}{m!}.$$

Therefore,

$$R_2\left(A,\frac{d}{2},\frac{d}{2}+\frac{1}{2};\frac{1}{2};x,y;s,a\right)=\sum_{n=0}^{\infty}\frac{\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(x^2\right)^n}{\left(\frac{1}{2}\right)_n n!}\sum_{m=0}^{n}\frac{A_m(-n)_m\left(\frac{1}{2}-n\right)_m y^m}{(m+a)^s m!}. \tag{3.15}$$

Finally, making an appeal to the results (3.13) and (3.15), we establish the formula (3.12). □

Corollary 3.2. *In the Theorem* 3.4 *setting* $A_n=\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\gamma_i)_n}, \forall n=0,1,2,3,\ldots,$ *and defining an extended semi-hypergeometric Hurwitz-Lerch Zeta function*

$${}_{p+2}H_q\left(\begin{matrix}(\alpha)_{1,p},-n,\frac{1}{2}-n;\\(\gamma)_{1,q};\end{matrix}\ y,s,a\right)=\sum_{m=0}^{n}\frac{\prod_{i=1}^{p}(\alpha_i)_n(-n)_m\left(\frac{1}{2}-n\right)_m}{\prod_{i=1}^{q}(\gamma_i)_n m!}\frac{y^m}{(m+a)^s}, \tag{3.16}$$

and then making an appeal to equality (3.12)*, we obtain the generating relation of extended generalized hypergeometric Hurwitz-Lerch Zeta function defined by* (1.8)

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n x^{2n}}{\left(\frac{1}{2}\right)_n n!}\ {}_{p+2}H_q\left(\begin{matrix}(\alpha)_{1,p},-n,\frac{1}{2}-n;)\\(\gamma)_{1,q};\end{matrix}\ y,s,a\right)\\=\frac{1}{2(1-x)^d}\ {}_{p+2}H_q\left(\begin{matrix}(\alpha)_{1,p},\frac{d}{2},\frac{d}{2}+\frac{1}{2};\\(\gamma)_{1,q};\end{matrix}\ \frac{x^2y}{(1-x)^2},s,a\right)\\+\frac{1}{2(1-x)^d}\ {}_{p+2}H_q\left(\begin{matrix}(\alpha)_{1,p},\frac{d}{2},\frac{d}{2}+\frac{1}{2};\\(\gamma)_{1,q};\end{matrix}\ \frac{x^2y}{(1+x)^2},s,a\right).\end{aligned} \tag{3.17}$$

Theorem 3.5. *The double series associated with general Hurwitz-Lerch Zeta function* (1.8) *generates following integral identity*

$$\begin{aligned}\int_0^{\infty}e^{-at}t^{s-1}\sum_{n=0}^{\infty}\frac{\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(x^2\right)^n}{\left(\frac{1}{2}\right)_n n!}\sum_{m=0}^{n}\frac{A_m(-n)_m\left(\frac{1}{2}-n\right)_m\left(ye^{-t}\right)^m}{m!}dt\\=\frac{1}{2}\int_0^{\infty}e^{-at}t^{s-1}\Bigg\{\frac{1}{(1-x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{x^2ye^{-t}}{(1-x)^2}\right)^n}{n!}\\+\frac{1}{(1+x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{x^2ye^{-t}}{(1+x)^2}\right)^n}{n!}\Bigg\}dt,\end{aligned} \tag{3.18}$$

where $\Re(a)>0,\ \Re(s)>0.$

Proof. By equation (3.15) we immediately obtain the result (3.18) on applying Euler integral formule. □

Theorem 3.6. *The double series associated with general Hurwitz-Lerch Zeta function* (1.8) *generates following general generating relation*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(x^2\right)^n}{\left(\frac{1}{2}\right)_n n!}\sum_{m=0}^{n}\frac{A_m(-n)_m\left(\frac{1}{2}-n\right)_m\left(ye^{-t}\right)^m}{m!}\\=\frac{1}{2(1-x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{x^2ye^{-t}}{(1-x)^2}\right)^n}{n!}+\frac{1}{2(1+x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{x^2ye^{-t}}{(1+x)^2}\right)^n}{n!}.\end{aligned} \tag{3.19}$$

Proof. Make an appeal to the Theorem 3.5 and by the identity of Eqn. (3.18) we establish the result (3.19). □

This result (3.19) is identical to the generating relation due to H. Exton [1, (1999)].

4 Applications

In this section, we present some known and unknown generating relations and summation formulae. Making an appeal to the Corollary 3.1 and the identity (3.8) of the Theorem 3.2 we derive

$$\sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!} \; {}_{p+2}F_q \left(\begin{array}{c} (\alpha)_{1,p}, -n, -\frac{1}{2}-n; \\ (\gamma)_{1,q}; \end{array} ye^{-t} \right)$$
$$= \frac{1}{2xd(1-x)^d} \; {}_{p+2}F_q \left(\begin{array}{c} (\alpha)_{1,p}, \frac{d}{2}+\frac{1}{2}, \frac{d}{2}; \\ (\gamma)_{1,q}; \end{array} \frac{x^2ye^{-t}}{(1-x)^2} \right)$$
$$- \frac{1}{2xd(1+x)^d} \; {}_{p+2}F_q \left(\begin{array}{c} (\alpha)_{1,p}, \frac{d}{2}+\frac{1}{2}, \frac{d}{2}; \\ (\gamma)_{1,q}; \end{array} \frac{x^2ye^{-t}}{(1+x)^2} \right). \quad (4.1)$$

Further making an appeal to the Corollary 3.2 and the identity (3.18) of the Theorem 3.5, we obtain another generating relation

$$\sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}\right)_n \left(\frac{d}{2}+\frac{1}{2}\right)_n x^{2n}}{\left(\frac{1}{2}\right)_n n!} \; {}_{p+2}F_q \left(\begin{array}{c} (\alpha)_{1,p}, -n, \frac{1}{2}-n; \\ (\gamma)_{1,q}; \end{array} ye^{-t} \right)$$
$$= \frac{1}{2(1-x)^d} \; {}_{p+2}F_q \left(\begin{array}{c} (\alpha)_{1,p}, \frac{d}{2}, \frac{d}{2}+\frac{1}{2}; \\ (\gamma)_{1,q}; \end{array} \frac{x^2ye^{-t}}{(1-x)^2} \right)$$
$$+ \frac{1}{2(1+x)^d} \; {}_{p+2}F_q \left(\begin{array}{c} (\alpha)_{1,p}, \frac{d}{2}, \frac{d}{2}+\frac{1}{2}; \\ (\gamma)_{1,q}; \end{array} \frac{x^2ye^{-t}}{(1+x)^2} \right). \quad (4.2)$$

Now in the results (4.1) and (4.2), setting $p = 0, q = 1, \gamma_1 = \frac{3}{2}$ and $d = 1$ so that $A_m = \frac{1}{\left(\frac{3}{2}\right)_m}$ and supposing that for all $n \in \mathbb{N}_0, x, y \in \mathbb{C}$ and $t \in [0^+, \infty)$, then for following sequences of functions defined by

$$H_n^{(1)}(y,t) = {}_2F_1 \left(\begin{array}{c} -n, -n-\frac{1}{2}; \\ \frac{3}{2}; \end{array} ye^{-t} \right) \text{ and } H_n^{(2)}(y,t) = {}_2F_1 \left(\begin{array}{c} -n, n-\frac{1}{2}; \\ \frac{1}{2}; \end{array} ye^{-t} \right), \quad (4.3)$$

there exist following summation formulae

$$\sum_{n=0}^{\infty} H_n^{(1)}(y,t) x^{2n} = \frac{y^{-1/2}e^{t/2}}{4x^2} \left[\log\left\{ \frac{1-x+xy^{1/2}e^{-t/2}}{1-x-xy^{1/2}e^{-t/2}} \right\} - \log\left\{ \frac{1+x+xy^{1/2}e^{-t/2}}{1+x-xy^{1/2}e^{-t/2}} \right\} \right], \quad (4.4)$$

and

$$\sum_{n=0}^{\infty} H_n^{(2)}(y,t) x^{2n} = \frac{y^{-1/2}e^{t/2}}{4x} \left[\log\left\{ \frac{1-x+xy^{1/2}e^{-t/2}}{1-x-xy^{1/2}e^{-t/2}} \right\} + \log\left\{ \frac{1+x+xy^{1/2}e^{-t/2}}{1+x-xy^{1/2}e^{-t/2}} \right\} \right], \quad (4.5)$$

respectively.

Further for all $n \in \mathbb{N}_0, \Re(a) > \frac{1}{2}, x, y \in \mathbb{C}$ and $\Re(s) > 0$, considering sequence of functions

$$H_n^{(3)}(a,y,s) = \int_0^{\infty} e^{-at} t^{s-1} \; {}_2F_1 \left(\begin{array}{c} -n, -n-\frac{1}{2}; \\ \frac{3}{2}; \end{array} ye^{-t} \right) dt \quad (4.6)$$

and

$$H_n^{(4)}(a,y,s) = \int_0^{\infty} e^{-at} t^{s-1} \; {}_2F_1 \left(\begin{array}{c} -n, n-\frac{1}{2}; \\ \frac{1}{2}; \end{array} ye^{-t} \right) dt \quad (4.7)$$

and making an appeal to the Theorems 3.2 and 3.5 in Eqns. (4.4) and (4.5), the following summation formulae are computed as

$$\sum_{n=0}^{\infty} H_n^{(3)}(a,y,s) x^{2n}$$
$$= \frac{y^{-1/2}}{4x^2} \int_0^{\infty} e^{-\left(a-\frac{1}{2}\right)t} t^{s-1} \left[\log\left\{ \frac{1-x+xy^{1/2}e^{-t/2}}{1-x-xy^{1/2}e^{-t/2}} \right\} - \log\left\{ \frac{1+x+xy^{1/2}e^{-t/2}}{1+x-xy^{1/2}e^{-t/2}} \right\} \right] dt. \quad (4.8)$$

and

$$\sum_{n=0}^{\infty} H_n^{(4)}(a,y,s) x^{2n}$$
$$= \frac{y^{-1/2}}{4x} \int_0^{\infty} e^{-\left(a-\frac{1}{2}\right)t} t^{s-1} \left[\log\left\{ \frac{1-x+xy^{1/2}e^{-t/2}}{1-x-xy^{1/2}e^{-t/2}} \right\} + \log\left\{ \frac{1+x+xy^{1/2}e^{-t/2}}{1+x-xy^{1/2}e^{-t/2}} \right\} \right] dt, \quad (4.9)$$

respectively.

Several other results, integral identities and generating relations may be derived on making an application of our formulae evaluated in previous sections, due to lack of space we omit them.

5 Conclusion

The summation formulae of the families of double series associated with general Hurwitz-Lerch type Zeta functions presented in the Section 2 may be useful in computational work. The identities found in the Section 3 applicable in evaluation of various generating relations of hypergeometric functions and the Zeta functions found in the literature. The sequence of functions given in the Section 4 may be useful in various problems of science and technology.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 38-43**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

A STUDY ON *Fuzzy* SEPARATION AXIOMS ($T_i, i = 0, 1, 2$) VIA *Fuzzy* gp^*-OPEN SETS

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(Received: March 20, 2022; In format: November 02, 2022; Revised: January 12, 2023; Accepted: January 24, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53104

Abstract

In this paper we have introduced *Fuzzy* gp^* closure, *Fuzzy* gp^*-interior and separation axioms via *Fuzzy* gp^*-open sets. Also we found out the relationship between *Fuzzy* separation axioms, *Fuzzy* gp^* separation axioms and *Fuzzy* pre separation axioms.

2020 Mathematical Sciences Classification: 54A40.
Keywords and Phrases: Fuzzy gp^*- closure, $Fgp^* - T_0$ spaces, $Fgp^* - T_1$ spaces, $Fgp^* - T_2$ spaces.

1 Introduction

In this paper we introduce and study separation properties of *Fuzzy* topological spaces via *Fuzzy* gp^* closed sets and draw a valid implication between the different axioms introduced earlier. *Fuzzy* separation axioms were introduced and studied by Ghanim et al. [3]. Similarly *Fuzzy* pre separation axioms were introduced and many of their properties were established by Singal et al. [11]. In 2011 Lee and Yun [9] introduced and studied *Fuzzy* delta separation axioms based on *Fuzzy* δ-open sets. They investigated the relationship between *Fuzzy* separation axioms and *Fuzzy* δ-separation axioms and showed *Fuzzy* δ-separation axioms are hereditary in *Fuzzy* regular open subspaces. In 2018 Paul et al. [10] studied and introduced separation axioms ($T_i, i = 0, 1, 2$) in the light of *Fuzzy* γ^*-open set via quasi-coincidence, quasi-neighborhood and also established relation between *Fuzzy* separation axioms, *Fuzzy* pre-separation axioms and *Fuzzy* γ^*-separation axioms.

In this paper, we introduce *Fuzzy* separation axioms via *Fuzzy* gp^*-open sets and find out there relation with *Fuzzy* separation axioms and *Fuzzy* pre separation axioms introduced earlier. We find out that every FT_i space [3] is Fgp^*T_i space for $i = 0, 1, 2$ and every FPT_i space [11] is Fgp^*T_i space for $i = 0, 1, 2$. But the converse is not true for both the cases, which we proved by counter examples.

2 Preliminaries

In this paper (Z, τ) always mean *Fuzzy* topological space on which no separation axioms are mentioned unless otherwise explicitly stated. A *Fuzzy* set in topological space (X, τ) is called a *Fuzzy* point iff it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is λ $(0 < \lambda \leq 1)$ we denote this *Fuzzy* point by x_λ, where the point x is called its support see [11]. From the previous literature, following definitions and remarks play a key role in establishing the main work of this paper.

Definition 2.1 ([5]). *Suppose (Y, τ) is a Fuzzy topological space. Then a subset λ of (Y, τ) is called Fuzzy generalized pre regular weakly closed (briefly Fuzzy gp^*-closed) if $pcl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is a Fuzzy regular semi open set in (Y, τ). Complement of Fuzzy generalized pre regular weakly closed set is called Fuzzy generalized pre regular weakly open (briefly Fuzzy gp^*-open).*

Definition 2.2 ([2]). *A Fuzzy set on X is called a Fuzzy singleton if it takes the value zero (0) for all points x in X except one point. The point at which a Fuzzy singleton takes the non-zero value is called the support and the corresponding element of $(0, 1]$ its value. A Fuzzy singleton with value 1 is called a Fuzzy crisp singleton.*

Definition 2.3 ([3]). *A Fuzzy topological space is said to be FT_0 iff for every pair of Fuzzy singletons P_1 and P_2 with different supports, there exists an open Fuzzy set O such that $p_1 \leq O \leq cop_2$ or $p_2 \leq O \leq cop_1$.*

Definition 2.4 ([3]). *A Fuzzy topological space is said to be FT_1 iff for every pair of Fuzzy singletons p_1 and p_2 with different supports, there exists open Fuzzy sets O_1 and O_2 such that $p_1 \leq O_1 \leq cop_2$ and $p_2 \leq O_2 \leq cop_1$.*

Definition 2.5 ([3]). *A Fuzzy topological space is said to be FT_2 (F-Hausdorff) iff for every pair of Fuzzy singletons p_1 and p_2 with different supports, there exists open Fuzzy sets O_1 and O_2 such that $p_1 \leq O_1 \leq cop_2$, $p_2 \leq O_2 \leq cop_1$ and $O_1 \leq coO_2$.*

Definition 2.6 ([11]). *A Fuzzy topological space is said to be Fuzzy pre-T_0 or in short FPT_0 if for every pair of Fuzzy singletons p_1 and p_2 with different supports, there exists a Fuzzy pre-open set u such that either $p_1 \leq u \leq cop_2$ or $p_2 \leq u \leq cop_1$.*

Definition 2.7 ([11]). *A Fuzzy topological space (X, τ) is said to be Fuzzy pre-T_1 or in short FPT_1 if for every pair of Fuzzy singletons p_1 and p_2 with different supports x_1 and x_2, ($x_1 \neq x_2$), there exists Fuzzy pre-open sets u and v such that $p_1 \leq u \leq cop_2$ and $p_2 \leq v \leq cop_1$.*

Definition 2.8 ([11]). *A Fuzzy topological space is said to be Fuzzy pre-Hausdorff or in short FPT_2 iff for every pair of Fuzzy singletons p_1 and p_2 with different supports, there exists two Fuzzy pre-open sets u and v such that $p_1 \leq u \leq cop_2$, $p_2 \leq v \leq cop_1$ and $u \leq cov$.*

Remark 2.1 ([5]). Suppose (Y, τ) is a *Fuzzy* topological space and $\lambda \leq Y$. Then we call λ *Fuzzy gp**-open if $(1-\lambda)$ is *Fuzzy gp** closed in (Y, τ).

Remark 2.2 ([11]). In *Fuzzy* topological space (Y, τ) every *Fuzzy* closed set is *Fuzzy* pre-closed.

Remark 2.3 ([5]). In *Fuzzy* topological space (Y, τ), every *Fuzzy* open set is *Fuzzy gp**-open.

3 Fuzzy gp^*-closure

Definition 3.1. *Suppose (Y, τ) is a Fuzzy topological space and $\alpha \leq Y$. Then Fuzzy gp^*-closure (briefly Fgp^*-cl) and Fuzzy gp^*-interior (briefly Fgp^*-int) of α are respectively defined as,*

$$\text{Fuzzy } gp^*\text{-}cl(\alpha) = \wedge \{\mu : \alpha \leq \mu, \mu \text{ is Fuzzy } gp^*\text{-closed set in } Y \},$$
$$\text{Fuzzy } gp^*\text{-}int(\alpha) = \vee \{\mu : \alpha \geq \mu, \mu \text{ is Fuzzy } gp^*\text{-open set in } Y\}.$$

Theorem 3.1. *In Fuzzy topological space (Y, τ) every Fuzzy pre-closed set is Fuzzy gp^*-closed.*

Proof. Suppose λ is a *Fuzzy* pre-closed set in (Y, τ) such that $\lambda \leq \mu$, where μ is *Fuzzy* generalized pre-open in (Y, τ). Now as λ is *Fuzzy* pre-closed implying that $pcl(\lambda) = \lambda$. Also by Remark 2.2 every *Fuzzy* closed set is *Fuzzy* pre-closed, implying $cl(\lambda) \leq pcl(\lambda) = \lambda \leq \mu$, whenever $\lambda \leq \mu$ and μ is *Fuzzy* generalized pre-open in (Y, τ). So λ is *Fuzzy gp**-closed. □

Theorem 3.2. *Suppose λ is a Fuzzy set in Fuzzy space (Y, τ). Then Fuzzy $gp^* - cl(1-\lambda)$ = 1- (Fuzzy $gp^* - int(\lambda)$) and Fuzzy $gp^* - int(1-\lambda)$ = 1-(Fuzzy $gp^* - cl(\lambda)$).*

Proof. From Remark 2.1, a *Fuzzy gp**-open set $p \leq \lambda$ is the complement of *Fuzzy gp**-closed set $q \geq 1-\lambda$. So

Fuzzy gp*-int(λ) $=\vee\{1-q : q$ is *fuzzy gp** closed and $q \geq 1-\lambda\}$,
Fuzzy $gp^* - int(\lambda) = 1 - \wedge\{q : q$ is *Fuzzy gp** closed and $q \geq 1-\lambda\}$,
Fuzzy $gp^* - int(\lambda)$ = 1- *Fuzzy* $gp^* - cl(1-\lambda)$
$\Longrightarrow gp^* - cl(1-\lambda)$ = 1-*Fuzzy* $gp^* - int(\lambda)$.

Now, suppose r is a *Fuzzy gp**-open set so for *fuzzy gp**-closed set s$\geq\lambda$, $r = 1-s \leq 1-\lambda$

Fuzzy $gp^* - cl(\lambda) = \wedge\{1-r : r$ is *fuzzy gp**-open and $r \leq 1-\lambda$ },
Fuzzy gp*-cl(λ)=1-$\vee\{r : r$ is *Fuzzy gp**-open and $r \leq 1-\lambda\}$,
Fuzzy $gp^* - cl(\lambda)$ = 1- *Fuzzy* $gp^* - int(1-\lambda)$
$\Longrightarrow$ *Fuzzy* $gp^* - int(1-\lambda)$ = 1-Fuzzy $gp^* - cl(\lambda)$.

□

Theorem 3.3. *Suppose* (Y, τ) *is a Fuzzy topological space and* α, μ *are Fuzzy subsets of* Y. *Then*
(a) *Fuzzy* $gp^* - cl(1_Y) = 1_Y$ *and Fuzzy* $gp^* - cl(0_Y) = 0_Y$,
(b) $\alpha \leq$ *Fuzzy* gp^**-cl*(α),
(c) *suppose* $\mu \leq \alpha$ *where* α *is Fuzzy* gp^**-closed set. Then Fuzzy* gp^**-cl*$(\mu) \leq \alpha$,
(d) *If* $\alpha \leq \mu$ *then Fuzzy* $gp^* - cl(\alpha) \leq$ *Fuzzy* $gp^* - cl(\mu)$.

Proof. (a) Since *Fuzzy* $gp^* - cl(1_Y)$ is the intersection i.e. minimum of all *Fuzzy* gp^*-closed sets in Y containing 1_Y and since 1_Y is the minimum *Fuzzy* gp^*-closed set containing 1_Y. So *Fuzzy* $gp^* - cl(1_Y) = 1_Y$. Now *Fuzzy* $gp^* - cl(0_Y)$ is the intersection i.e. minimum of all *Fuzzy* gp *-closed sets in Y containing 0_Y and since 0_Y is the minimum *Fuzzy* gp^*-closed set containing 0_Y, implying *Fuzzy* $gp^* - cl(0_Y) = 0_Y$.

(b) As *Fuzzy* $gp^* - cl(\alpha)$ is the intersection of all *Fuzzy* gp^*-closed sets containing α. So $\alpha \leq$ *Fuzzy* $gp^* - cl(\alpha)$ is obvious.

(c) Suppose $\mu \leq \alpha$, where α is *Fuzzy* gp^*-closed set. Now,

$$\textit{Fuzzy } gp^*\text{-cl}(\mu) = \wedge \{\pi : \mu \leq \pi, \pi \text{ is } \textit{Fuzzy } gp^*\text{-closed set in } Y \}$$

i.e. *Fuzzy* $gp^* - cl(\mu)$ is contained in all *Fuzzy* gp^*-closed sets, so in particular *Fuzzy* $gp^* - cl(\mu) \leq \alpha$.

(d) Suppose $\alpha \leq \mu$, also

$$\textit{Fuzzy } gp^*\text{-cl}(\mu) = \wedge \{\pi : \mu \leq \pi, \pi \text{ is } \textit{Fuzzy } gp^*\text{-closed set in Y} \} \quad \rightarrow (d.1).$$

Now if$\mu \leq \pi$, where π is *Fuzzy* gp^*-closed in Y, then by (c) of this theorem, *Fuzzy* $gp^* - cl(\mu) \leq \pi$. Now by (b) of this theorem $\mu \leq$ *Fuzzy* $gp^* - cl(\mu)$ implies $\alpha \leq \mu \leq \pi$ where π is *Fuzzy* gp^*-closed. So *Fuzzy* $gp^* - cl(\alpha) \leq \pi$ (by (c) of this theorem). Therefore

$$\textit{Fuzzy } gp^*\text{-cl}(\alpha) \leq \wedge \{\pi : \mu \leq \pi, \pi \text{ is } \textit{Fuzzy } gp^*\text{-closed set in Y} \}$$

$$\Longrightarrow \quad \textit{Fuzzy } gp^*\text{-cl}(\alpha) \leq \textit{Fuzzy} gp^* - cl(\mu) \quad (using (d.1))$$

□

4 Separation Axioms via *Fuzzy gp**-open Set

Definition 4.1. *A Fuzzy topological space* (Z, τ) *is* $Fgp^* - T_0$ *if for arbitrary Fuzzy singletons* x^1_λ *and* x^2_μ, *their exists a Fuzzy* gp^**-open set* Z *such that* $x^1_\lambda \leq Z \leq (1 - x^2_\mu)$ *or* $x^2_\mu \leq Z \leq (1 - x^1_\lambda)$.

Theorem 4.1. *A Fuzzy topological space* (Z, τ) *is* $Fgp^* - T_0$ *iff Fuzzy-gp** *closure of any two Fuzzy crisp singletons with different supports is distinct.*

Proof. Suppose (Z, τ) is $Fgp^* - T_0$ and x^1, x^2 are two *Fuzzy* crisp singletons with different supports. Now (Z, τ) being $Fgp^* - T_0$ implies that $\exists$ a *Fuzzy-gp** open set Z such that $x^1 \leq Z \leq (1 - x^2)$, implying $x^2 \leq Fgp^* - cl(x^2) \leq 1 - Z$. Since $x^1 \nleq 1 - Z$ so $x^1 \nleq Fgp^* - cl(x^2)$, but $x^1 \leq Fgp^* - cl(x^1)$ implies $Fgp^* - cl(x^1) \neq Fgp^* - cl(x^2)$.

Conversely, suppose x^1 and x^2 be two *Fuzzy* crisp singletons with different supports z_1 and z_2, respectively such that $x^1(z_1) = x^2(z_2) = 1$. Also let l_1 and l_2 be *Fuzzy* singletons with different supports z_1 and z_2, so by hypothesis $1_Z - Fgp^* - cl\{x^1\} \leq 1_Z - \{x^1\}$ and so $(1_Z - Fgp^* - cl(x^1)) \leq (1_Z - \{l_1\})$. Now $(1_Z - Fgp^* - cl(x^1))$ is a Fgp^*-open set such that $l_2 \leq (1_Z - Fgp^* - cl(x^1) \leq (1_Z - \{l_1\})$. Implying (Z, τ) is $Fgp^* - T_0$. □

Definition 4.2. *A Fuzzy topological space* (Z, τ) *is* $Fgp^* - T_1$ *if for arbitrary Fuzzy singletons* X^1_λ *and* x^2_μ, *their exists Fuzzy* gp^* *open sets* Z_1 *&* Z_2 *such that* $x^1_\lambda \leq Z_1 \leq (1 - x^2_\mu)$ *and* $x^2_\mu \leq Z_2 \leq (1 - x^1_\lambda)$.

Obviously every $Fgp^* - T_1$ *space is a* $Fgp^* - T_0$ *space.*

Theorem 4.2. *A Fuzzy topological space* (Z, τ) *is* $Fgp^* - T_1$ *iff every Fuzzy crisp singleton is Fuzzy-gp** *closed.*

Proof. Consider (Z, τ) is $Fgp^* - T_1$ and l_1 is a *Fuzzy* singleton with support z_1 such that $l_1(z_1) = 1$. So for any arbitrary *Fuzzy* singleton l_2 with support $z_2 \neq z_1$, their exists *Fuzzy*-gp* open sets α and β such that $l_1 \leq \alpha \leq 1_Z - l_2$ and $l_2 \leq \beta \leq 1_Z - l_1$. Now, as every *Fuzzy* set can be written as the union of *Fuzzy* singletons contained in it [2]. So $1_Z - l_1 = \vee_{l_2 \leq 1_Z - l_1} l_2$. From $1 - l_1(z_1) = 0$ it is clear that $1_Z - l_1 = \vee_{l_2 \leq 1_Z - l_1} \beta$, implying $1_Z - l_1$ is *Fuzzy*-gp* open. Conversely suppose that l_1 and m_1 are *Fuzzy* singletons with support z_1

such that $m_1(z_1) = 1$ and $l_1(z_1) \neq 1$ & l_2,m_2 are *Fuzzy* singletons with support z_2 such that $m_2(z_2) = 1$ and $l_2(z_2) \neq 1$. Now the *Fuzzy* sets $1_Z - m_1$ & $1_Z - m_2$ are *Fuzzy* gp * open sets satisfying $l_1 \leq 1_Z - m_2 \leq 1_Z - l_2$ & $l_2 \leq 1_Z - m_1 \leq 1_Z - l_1$ implying (Z, τ) is $Fgp^* - T_1$. □

Definition 4.3. *A Fuzzy topological space (Z, τ) is Fgp^*-Hausdorff or $Fgp^* - T_2$ if for arbitrary Fuzzy singletons X^1_λ and x^2_μ, their exists Fuzzy gp *-open sets Z_1 & Z_2 such that $x^1_\lambda \leq Z_1 \leq (1 - x^2_\mu)$, $x^2_\mu \leq Z_2 \leq (1 - x^1_\lambda)$ and $Z_1 \leq 1 - Z_2$.*
It is obvious that every $Fgp^* - T_2$ space is $Fgp^* - T_1$ space.

Definition 4.4. *A Fuzzy topological space (Z, τ) is Fgp *-Uryshon or $Fgp^* - T_{2\frac{1}{2}}$ if for arbitrary Fuzzy singletons x^1_λ and x^2_μ, their exists Fuzzy gp *-open sets Z_1 & Z_2 such that $x^1_\lambda \leq Z_1 \leq (1 - x^2_\mu)$, $x^2_\mu \leq Z_2 \leq (1 - x^1_\lambda)$ and $Fgp^* - cl(Z_1) \leq 1 - (Fgp^* - cl(Z_2))$.*

Remark 4.1. Every *Fuzzy* pre-open set in fts (Z, τ) is a *Fuzzy gp**-open set in (Z,τ).

Proof. Suppose α is a *Fuzzy* pre-open set in (Z,τ), so $1 - \alpha$ is *Fuzzy* pre-closed. Now by Theorem 3.1 every *Fuzzy* pre-closed set is *Fuzzy gp**-closed, implying $1 - \alpha$ is *Fuzzy gp**-closed & so α is a *Fuzzy gp**-open set in (Z, τ). □

Theorem 4.3. *Every FPT_0 space is $Fgp^* - T_0$ space.*

Proof. Suppose (Z, τ) is a FPT_0-space, so by [2] for *Fuzzy* singletons l_1 & l_2 with supports z_1,z_2 ($z_1 \neq z_2$) their exists a *Fuzzy* pre-open set ν such that $l_1 \leq \nu \leq 1_Z - l_2$ or $l_2 \leq \nu \leq 1_Z - l_1$. Now by Remark 4.1 ν is a *Fuzzy gp**-open set satisfying $l_1 \leq \nu \leq 1_Z - l_2$ or $l_2 \leq \nu \leq 1_Z - l_1$. Hence (Z,$\tau$) is a $Fgp^* - T_0$ space. □

Remark 4.2. The converse of the above theorem need not be true, for proof the following example is given.

Example 4.1. If $Z = \{z_1, z_2, z_3, z_4\}$ is a space with *Fuzzy* topology $\tau = \{0_Z, 1_Z, l, m, n, o\}$ where $l, m, n, o : Z \to [0, 1]$ are defined as

$$l(z) = \begin{cases} 1 & \text{if } z = z_1 \\ 0 & \text{otherwise,} \end{cases}$$

$$m(z) = \begin{cases} 1 & \text{if } z = z_2 \\ 0 & \text{otherwise,} \end{cases}$$

$$n(z) = \begin{cases} 1 & \text{if } z = z_1, z_2 \\ 0 & \text{otherwise,} \end{cases}$$

$$o(z) = \begin{cases} 1 & \text{if } z = z_1, z_2, z_3 \\ 0 & \text{otherwise.} \end{cases}$$

In this space Z with such kind of topology τ, the *Fuzzy* set p defined below is Fgp^*-open but not *Fuzzy* pre-open, implying that the space (Z, τ) is $Fgp^* - T_0$ but not FPT_0.

$$p(z) = \begin{cases} 1 & \text{if } z = z_1, z_3, z_4 \\ 0 & \text{otherwise/} \end{cases}$$

Theorem 4.4. *All FPT_1 spaces are $Fgp^* - T_1$ spaces.*

Proof. Suppose (Z,τ) is a FPT_1 space, so by the definition of FPT_1 for arbitrary singletons l_1 and l_2, $l_1 \leq \nu_1 \leq 1 - l_2$ & $l_2 \leq \nu_2 \leq 1 - l_1$ where ν_1 and ν_2 are *Fuzzy* pre-open sets. Now by Remark 4.1 ν_1 and ν_2 are *Fuzzy* gp *-open, concluding that (Z, τ) is a $Fgp^* - T_1$ spaces. □

Remark 4.3. The converse of the above theorem may not be true as shown in the following example.

Example 4.2. In the *Fuzzy* topological space defined in Example 4.1, the *Fuzzy* sets p & q defined below are Fgp *-open but not *Fuzzy* pre-open, implying that the space (Z, τ) is $Fgp^* - T_1$ but not FPT_1.

$$p(z) = \begin{cases} 1 & \text{if } z = z_1, z_3, z_4 \\ 0 & \text{otherwise,} \end{cases}$$

$$q(z) = \begin{cases} 1 & \text{if } z = z_1, z_4 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.5. *All FPT_2 spaces are $Fgp^* - T_2$ spaces.*

Proof. From the definition of FPT_2 spaces in [11] and from Remark 4.1, the proof is obvious. □

Remark 4.4. The converse of the above theorem need not be true as shown in the given example.

Example 4.3. In the *Fuzzy* topological space defined in Example 4.1, the *Fuzzy* sets r & s defined below are Fgp^*-open but not *Fuzzy* pre-open, implying that the space (Z, τ) is $Fgp^* - T_2$ but not FPT_2.

$$r(z) = \begin{cases} 1 & \text{if } z = z_3 \\ 0 & \text{otherwise,} \end{cases}$$

$$s(z) = \begin{cases} 1 & \text{if } z = z_4 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.6. *Every FT_0 space is $Fgp^* - T_0$ space.*

Proof. Suppose (Z, τ) is a FT_0 -space, so by [12] for *Fuzzy* singletons l_1 & l_2 with different supports, their exists a *Fuzzy* open set ν such that $l_1 \le \nu \le 1_Z - l_2$ or $l_2 \le \nu \le 1_Z - l_1$. Now from Remark 2.3 every *Fuzzy* open set is *Fuzzy* gp *-open, implying that ν is a *Fuzzy* gp^*-open set satisfying $l_1 \le \nu \le 1_Z - l_2$ or $l_2 \le \nu \le 1_Z - l_1$. Hence (Z, τ) is a $Fgp^* - T_0$ space. □

Remark 4.5. The converse of the above theorem need not be true as shown in the following example.

Example 4.4. If $Z = \{z_1, z_2, z_3, z_4, z_5\}$ is a space with *Fuzzy* topology $\tau = \{0_Z, 1_Z, \lambda_1, \lambda_2, \lambda_3\}$ where $\lambda_1, \lambda_2, \lambda_3 : Z \to [0, 1]$ are defined as

$$\lambda_1(z) = \begin{cases} 1 & \text{if } z = z_1, z_2 \\ 0 & \text{otherwise,} \end{cases}$$

$$\lambda_2(z) = \begin{cases} 1 & \text{if } z = z_3, z_4 \\ 0 & \text{otherwise,} \end{cases}$$

$$\lambda_3(z) = \begin{cases} 1 & \text{if } z = z_1, z_2, z_3, z_4 \\ 0 & \text{otherwise.} \end{cases}$$

In this *Fuzzy* topological space, the *Fuzzy* set λ_4 defined below is a *Fuzzy* gp *-open set but not *Fuzzy* open, implying that the space (Z, τ) is $Fgp^* - T_0$ but not FT_0.

$$\lambda_4(z) = \begin{cases} 1 & \text{if } z = z_1, z_2, z_4, z_5 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.7. *Every FT_1 space is $Fgp^* - T_1$ space.*

Proof. The proof is trivial from the definitions of FT_1 and $Fgp^* - T_1$ spaces and from the result that every *Fuzzy* open set is *Fuzzy* gp^*-open [5]. □

Remark 4.6. The converse that every $Fgp^* - T_1$ space is a FT_1 space is not true, for proof the following example is given

Example 4.5. In *Fuzzy* topological space (Z, τ) defined in Example 4.4, the *Fuzzy* sets λ_4 and λ_5 defined below are *Fuzzy* gp^*-open sets but not *Fuzzy* open sets, implying the *Fuzzy* space (Z, τ) is a $Fgp^* - T_1$ space but not a FT_1.

$$\lambda_4(z) = \begin{cases} 1 & \text{if } z = z_1, z_2, z_4, z_5 \\ 0 & \text{otherwise,} \end{cases}$$

$$\lambda_5(z) = \begin{cases} 1 & \text{if } z = z_2, z_3, z_4, z_5 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.8. *Every FT_2 space is $Fgp^* - T_2$ space.*

Proof. The proof is straightforward. □

Remark 4.7. The converse of the above theorem need not be true as shown in the following example.

Example 4.6. In *Fuzzy* topological space (Z, τ) defined in Example 4.4, the *Fuzzy* sets λ_4 and λ_5 defined below are *Fuzzy gp**-open sets but not *Fuzzy* open sets, implying the *Fuzzy* space (Z, τ) is a $Fgp^* - T_2$ space but not a FT_2.

$$\lambda_4(z) = \begin{cases} 1 & \text{if } z = z_1, z_2, z_4, z_5 \\ 0 & \text{otherwise.} \end{cases}$$

$$\lambda_5(z) = \begin{cases} 1 & \text{if } z = z_2, z_3, z_4, z_5 \\ 0 & \text{otherwise} \end{cases}$$

From the above discussion, we have the following diagram of implications

$$\begin{array}{ccccc} FT_0 & \Leftarrow & FT_1 & \Leftarrow & FT_2 \\ \Downarrow \not\Uparrow & & \Downarrow \not\Uparrow & & \Downarrow \not\Uparrow \\ Fgp^*-T_0 & \Leftarrow & Fgp^*-T_1 & \Leftarrow & Fgp^*-T_2 \\ \not\Downarrow \Uparrow & & \not\Downarrow \Uparrow & & \not\Downarrow \Uparrow \\ FPT_0 & \Leftarrow & FPT_1 & \Leftarrow & FPT_2 \end{array}$$

5 Conclusion

The main portion of this manuscript is dedicated to *Fuzzy* separation axioms via *Fuzzy gp**-open sets, We introduced these axioms and find out their relation with *Fuzzy* separation axioms and *Fuzzy* pre separation axioms introduced earlier. We can further investigate these spaces and relate the new results with the results already in trending in this area.

Acknowledgement. I am highly thankful to the Editors and anonymous Reviewers for their valuable suggestions to improve the paper in its present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 44-54**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

AN EOQ MODEL WITH TRADE CREDIT BASED DEMAND UNDER INFLATION

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(Received: October 23, 2022; In format: January 10, 2023; Revised: January 12, 2023; Accepted: January 23, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53105

Abstract

In this work, an EOQ model is presented with trade credit period and time dependent demand under inflation and delay in payments for deteriorating items. Shortages are permitted and partially backlogged that depends on the waiting time of next replenishment cycle. The holding cost and deterioration rate is considered constant. The aim of this study is to maximize the total profit. An algorithm is presented to get the optimal values of total profit, total inventory and stock-out period. To illustrate theoretical model numerical assessments, graphical representation and sensitivity analysis is also discussed.

2020 Mathematical Sciences Classification: 90B05, 90B10, 90B50.
Keywords and Phrases: Trade credit period and time dependent demand, inflation, delay in payments, deteriorating items, partially backlogged shortages.

1 Introduction

There are so many factors such as demand, deterioration, inflation, holding cost, shortages etc. that effects a business directly. Deterioration of products during storage time is a common problem that industries face. Deterioration is described as decay or spoilage of products that affects the value of products. Similarly, demand of a product is defined as how much consumers want a companys product in a given duration. Price of product itself and related complementary goods, income of consumers, fashion trends are some factors that decides the demand of a product. Storehouses costs such as rents, salaries of these storehouses employees are termed as carrying cost and costs of financing, damage, handling inventory are some aspects that determine holding cost of an inventory system. Increase in prices and fall in the purchasing value of money is termed as inflation and it plays a major role in todays business world. There are a number of researchers that developed inventory models with including these aspects. With constant demand and decay rate, [5] presented A two-storage inventory model for perishable products under trade credit policy and shortages. Similarly, [33] introduced ordering policies with constant demand, deterioration and carrying cost under shortages. [18] and [4] investigated the effect of inflation on an inventory model with constant demand rate. Considering constant type of demand, [7], [1] and [35] also established ordering policies under trade credit policy. Researchers such as [8] and [42] developed inventory models with price linked demand for Weibull deteriorate commodities whereas for such products [28] analysed inventory control model with linear demand and carrying cost. [17], [15] and [30] developed inventory models for perishable products with price sensitive demand, linearly time linked deterioration rate and holding cost under fully backlogged shortages. [10] studied the effects of inflation and time value of money on an inventory model with linearly time linked demand under shortages, later [13] redevelop [10]s model by modifying the hypothesis of uniform inventory in each replenishment cycle. Considering linear carrying cost and time sensitive demand rate under without shortages, [9] represented inventory model which is applicable for food grains, fashion clothes and electronic products and with same assumptions [22] derived inventory model for those industries that use preservation techniques to control the deterioration. [2] and [12] derived inventory models for perishable commodities with stock sensitive demand under without shortages with storage time linked and constant carrying cost respectively and similarly, [43] also presented inventory model with stock induced demand and carrying cost and discussed it with and without shortages. For retail business [14] presented inventory model for perishable products with shortages and stock dependent demand under inflation and time discounting. With inventory induced demand and linearly time sensitive carrying cost, [41] developed inventory model for deteriorating commodities under inflation. [6] and [36] presented two-storehouse inventory model for single perishable product with time, selling price and recurrence of advertisement linked demand rate under partially backlogged shortages and in addition [36] also take account of alternative trade credit policy. Considering uniform demand and production rate, [32] presented inventory model with two-level trade credit strategy where supplier provide retailer a full trade-credit policy whereas retailer give their customers a partial trade credit policy for exponentially deteriorating products. [39] discussed two cases with time sensitive and uniform demand and time dependent and uniform carrying cost for case 1 and case 2 respectively under without shortages. For the products that deteriorate with time, [20] developed inventory model with stock sensitive demand and [3] studied the effect of price induced demand and default risk on optimal customer credit duration and cycle length under shortages. [19] discussed a production inventory model which comprise an unfilled-order backlog for an inventory system for exponential deteriorating products and later [29] presented an optional method to get the optimal aspects of [19]s model. [16] and [34] discussed inventory models under inflation with ramp-type demand and advertisement sensitive demand respectively. Under partially backlogged shortages, [23] discussed inventory model with exponentially decreasing time sensitive demand and [37] presented inventory model where demand depends on inventory level and time during storage duration and shortage duration respectively. For company possessed storehouses where deteriorating products stored for extended time with extra caution, [40] developed an EOQ model with linearly time induced demand and discussed it with both exponentially and linearly time sensitive carrying cost. For perishable products, [31] presented inventory model with time linked increasing demand and fixed production rate under without shortages and on the other hand, considering shortages, [11] discussed a cost minimization framework with promotional work and selling price induced demand. For non-spontaneous perishable products under inflation and shortages, [38] studied the effect of linearly time linked carrying cost on life time inventory model with price and stock sensitive Demand whereas [24] presented inventory model with price and advertisement sensitive demand under trade credit policy. Sometimes, to promote market competitiveness supplier and retailer both accept trade credit policy and provides price discount, considering these facts, for perishable products under shortages, [25] derived inventory model with stock linked demand and linearly time induced carrying cost and [27] presented two-level inventory model with price and stock sensitive demand. [26] analysed the retailers replenishment policies for perishable products with delay in payments where the demand rate decreases with time without shortages. [21] developed inventory model with time and selling price dependent demand and shortages under inflation and trade credit policy for constantly deteriorate commodities.

Demand is not always constant this varies with time. Just like the demand for coolers and fans goes high in summer, the demand for heater-geyser is more in winter. Apart from the season, celebrations also affect the demand, for example, the demand for clothes and jewelry is high during weddings and festivals. Nowadays fashion has also become a factor in generating demand. The increase in demand for masks, sanitizers, and other medical items in covid-19 is another example of time-dependent demand. When the credit period is offered by the supplier to the wholesaler, then a demand can be increased indirectly like the wholesaler can generate sales (like Diwali sale). Apart from this he/she can reduce inventory costs by ordering more goods in quantity and he/she can generate demand by selling goods to the customer at a lower price.

Considering the above facts in the present study, an inventory model is developed for spontaneous perishable products with trade credit period and time induced demand, constant carrying cost and deterioration rate under inflation. Delay in payments and shortages are tolerated and shortages are partially backlogged. The optimal ordering policies are established by optimizing the total profit and stock-out duration. The theoretical model is discussed with examples and sensitivity analysis of various parameters.

This work is arranges in the following manner: in section 2, the postulates and symbols of this study are mentioned. Mathematical representation with solution and solution procedure of this model is presented in section 3 and 4. Numerical examples, sensitivity analysis and results with observations are discussed in section 5, 6 and 7 respectively. Conclusion and future work in this direction is discussed in section 8.

2 Assumptions and Notations

The following notations and assumptions are applied to develop our model.

2.1 Assumptions

We have considered the upcoming hypothesis to construct the mathematical model of present inventory model

- Lead time is minimal.
- Replenishment rate is infinite.
- The infinite planning horizon is considered.
- The demand rate D_p depends on time and credit period π that is, $D_p(\pi, t) = \alpha\pi^\lambda e^{\sigma t}$, where α is the selling parameter and $\lambda > 0, 0 < \sigma < 1$.
- In present work inventory model is derived for single spontaneous perishing products.
- Supplier didnt offered the replacement or return or repair policy.
- Shortages are permitted and the fraction of shortages backordered depends on the awaiting time for the upcoming replenishment and $S(t) = e^{-\delta_p(T-t)}$ where $0 \le \delta_p \le 1$.
- During the trade credit period, the retailer need not to clear the account with the supplier. This policy is provided by the supplier to the retailer under terms and conditions for a fixed duration.

2.2 Notations

Symbols	P_0	ordering cost per order
	H_p	holding cost per item per order
	C_p	purchasing cost per item
	S_p	unit selling cost ($S_p > C_p$)
	C_2	shortage cost per unit per order
	C_0	lost sales cost per order
	R_0	inflation rate
	π	trade credit period
	P_c	interest charged per \$ /year
	P_e	interest earned per \$ /year
	θ_p	deterioration rate, $0 \le \theta_p < 1$
	S	maximum inventory level
	P	maximum demand backlogged/cycle
	Q	total order quantity
	T	cycle length
Decision variables	t_P	stock out time period
Functions	$D_p(\pi, t)$	The demand rate D_p depends on time and credit period π that is, $D_p(\pi, t) = \alpha\pi^\lambda e^{\sigma t}$, where α is the selling parameter and $\lambda > 0, 0 < \sigma < 1$
	$S(t)$	$S(t) = e^{-\delta_p(T-t)}$ where $0 \le \delta_p \le 1$
	$I_p(t)$	inventory level, $0 \le t \le t_p$
	$I_s(t)$	inventory level, $t_p \le t \le T$
	$TP(t_p)$	total profit

3 Model Formulation

The inventory level for this paper is drafted in Figure 3.1. During the period $(0, t_p)$ the inventory level I_p depends on both demand and deterioration. It is governed by the equation:

$$\frac{dI_p(t)}{dt} = -D_p - \theta_p I_p(t) \qquad ; 0 \le t \le t_p. \tag{3.1}$$

The solution of the equation (3.1) with boundary condition $I_p(t_p) = 0$ is

$$I_p(t) = x_3 e^{-\theta_p t}(e^{x_2 t_p} - e^{x_2 t}), \tag{3.2}$$

where $x_1 = \alpha\pi^\lambda$, $x_2 = \sigma + \theta_p$ and $x_3 = \dfrac{x_1}{x_2}$.

The maximum inventory level is S, where

$$S = I_p(0) = x_3(e^{x_2 t_p} - 1). \tag{3.3}$$

During the period (t_p, T), the inventory level $I_s(t)$ is given by the differential equation

$$\frac{dI_s(t)}{dt} = -D_p e^{-\delta_p(T-t)} \qquad ; t_p \le t \le T. \tag{3.4}$$

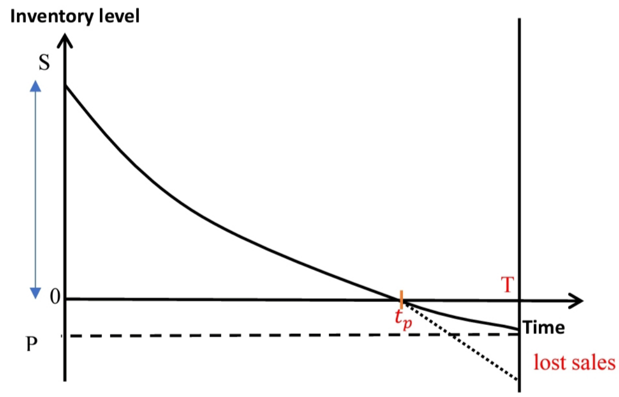

***Figure* 3.1:** Graphical representation of the inventory model

Using boundary condition $I_s(t_p) = 0$, the solution of the above Equation 3.4 is given by

$$I_s(t) = x_5 e^{-\delta_p T}(e^{x_4 t_p} - e^{x_4 t}), \tag{3.5}$$

where $x_4 = \sigma + \delta_p$ and $x_5 = \dfrac{x_1}{x_4}$.

The negative inventory is P, where

$$P = -I_s(T) = x_5 e^{-\delta_p T}(e^{x_4 T} - e^{x_4 t_p}). \tag{3.6}$$

Total order quantity $Q = S + P$.

$$Q = x_3(e^{x_2 t_p} - 1) + x_5 e^{-\delta_p T}(e^{x_4 T} - e^{x_4 t_p}). \tag{3.7}$$

The total cost per cycle depends on the following:

Ordering cost

$$f_1 = P_0,$$

Purchase cost

$$f_2 = C_p[S + P],$$

Holding cost

$$\begin{aligned} f_3 &= H_p \int_0^{t_p} I_p\,(t)\, e^{-R_0 t} dt \\ &= H_p x_3 \left[\frac{e^{x_2 t_p}(1 - e^{-x_6 t_p})}{x_6} + \frac{(1 - e^{-x_7 t_p})}{x_7} \right], \end{aligned}$$

where $x_6 = R_0 + \theta_p$ and $x_7 = R_0 - \sigma$.

Sales revenue

$$\begin{aligned} f_4 &= S_p \left[\int_0^{t_p} D_p e^{-R_0 t} dt + \int_{t_p}^{T} e^{-R_0 T} D_p e^{-\delta_p (T-t)} dt \right] \\ &= S_p x_1 \left[\frac{e^{-x_7 T} - e^{(x_8 t_p - x_9 T)}}{x_6} + \frac{(1 - e^{-x_7 t_p})}{x_7} \right], \end{aligned}$$

where $x_8 = \sigma + \delta_p$ and $x_9 = R_0 + \delta_p$.

Shortage cost

$$f_5 = \; -C_2 \int_{t_p}^{T} I_s\,(t)\, e^{-R_0 t} dt$$

$$= C_2 x_5 e^{-\delta_p T}\left[\frac{e^{-x_{10}t_p} - e^{x_{10}T}}{x_{10}} + \frac{e^{-t_p x_4}(e^{-R_0 T} - e^{-R_0 t_p})}{R_0}\right],$$

where $x_{10} = x_4 + R_0$.

Lost sales cost

$$\begin{aligned} f_6 &= C_0\left[\int_{t_p}^{T} e^{-R_0 t} D_p (1 - e^{-\delta_p (T-t)}) dt\right] \\ &= C_0 x_1\left[\frac{(e^{x_7 T} - e^{x_7 t_p})}{x_7} + \frac{e^{-\delta_p T}(e^{x_{11}T} - e^{-x_{11}t_p})}{x_{11}}\right], \end{aligned}$$

where $x_{11} = \sigma + \delta_p - R_0$.

Interest payable

***Case* 1.** $0 \le \pi \le t_p$

$$\begin{aligned} SP_1 &= C_p P_c \int_{\pi}^{t_p} I_p(t) e^{-R_0 t} dt \\ &= C_p P_c x_3\left[\frac{(e^{-x_{12}t_p} - e^{-x_{12}\pi})}{x_{12}} + \frac{e^{t_p x_2}(e^{-x_6 \pi} - e^{-x_6 t_p})}{x_6}\right], \end{aligned}$$

where $x_{12} = x_2 + \theta_p - R_0$.

***Case* 2.** $t_p \le \pi \le T$

$$SP_2 = 0.$$

Interest earned

***Case* 1.** $0 \le \pi \le t_p$

$$\begin{aligned} SE_1 &= S_p P_e \int_0^{\pi} D_p t e^{-R_0 t} dt \\ &= S_p P_e x_{13}[1 + e^{-x_7 \pi}(x_7 \pi - 1)], \end{aligned}$$

where

$$x_{13} = \frac{x_1}{x_7^2}$$

***Case* 2.** $t_p \le \pi \le T$

$$\begin{aligned} SE_2 &= S_p P_e\left[\int_0^{t_p} D_p t e^{-R_0 t} dt + (\pi - t_p)\int_0^{t_p} D_p e^{-R_0 t} dt\right] \\ &= S_p P_e x_{13}[(1 + e^{-x_7 t_p}(x_7 t_p - 1)) + x_7(\pi - t_p)(1 - e^{-x_7 t_p})]. \end{aligned}$$

The total profit per unit time, is described as

$$TP(t_p) = \begin{cases} TP_1(t_p); 0 \le \pi \le t_p \\ TP_2(t_p); t_p \le \pi \le T, \end{cases}$$

where

$$TP_1(t_p) = \frac{(f_4 + SE_1 - f_1 - f_2 - f_3 - f_5 - f_6 - SP_1)}{T}, \tag{3.8}$$

$$TP_1(t_p) = \frac{1}{T}\left\langle\left\{S_p x_1\left[\frac{e^{-x_7 T} - e^{(x_8 t_p - x_9 T)}}{x_6} + \frac{(1 - e^{-x_7 t_p})}{x_7}\right]\right\} + S_p P_e x_{13}\big[1 + e^{-x_7 \pi}(x_7 \pi - 1)\big]\right. \tag{3.9}$$

$$
\begin{aligned}
&- P_0 - C_p\big[x_3(e^{x_2t_p} - 1) + x_5e^{-\delta_pT}(e^{x_4T} - e^{x_4t_p})\big] - H_px_3\left[\frac{e^{x_2t_p}(1 - e^{-x_6t_p})}{x_6} + \frac{(1 - e^{-x_7t_p})}{x_7}\right]\\
&- C_2x_5e^{-\delta_pT}\left[\frac{(e^{-x_{10}t_p} - e^{x_{10}T})}{x_{10}} + \frac{e^{-t_px_4}(e^{-R_0T} - e^{-R_0t_p})}{R_0}\right]\\
&- C_0x_1\left[\frac{(e^{x_7T} - e^{x_7t_p})}{x_7} + \frac{e^{-\delta_pT}(e^{x_{11}T} - e^{-x_{11}t_p})}{x_{11}}\right]\\
&- C_pP_cx_3\left[\frac{(e^{-x_{12}t_p} - e^{-x_{12}\pi})}{x_{12}} + \frac{e^{t_px_2}(e^{-x_6\pi} - e^{-x_6t_p})}{x_6}\right]\Bigg\rangle,
\end{aligned}
$$

$$
TP_2(t_p) = \frac{(f_4 + SE_2 - f_1 - f_2 - f_3 - f_5 - f_6 - SP_2)}{T}, \tag{3.10}
$$

$$
\begin{aligned}
TP_2(t_p) = \frac{1}{T}\Bigg\langle\Bigg\{&S_px_1\left[\frac{e^{-x_7T} - e^{(x_8t_p - x_9T)}}{x_6} + \frac{(1 - e^{-x_7t_p})}{x_7}\right]\Bigg\} \qquad (3.11)\\
&+ S_pP_ex_{13}\big[(1 + e^{-x_7t_p}(x_7t_p - 1)) + x_7(\pi - t_p)(1 - e^{-x_7t_p})\big]\\
&- P_0 - C_p\big[x_3(e^{x_2t_p} - 1) + x_5e^{-\delta_pT}(e^{x_4T} - e^{x_4t_p})\big]\\
&- H_px_3\left[\frac{e^{x_2t_p}(1 - e^{-x_6t_p})}{x_6} + \frac{(1 - e^{-x_7t_p})}{x_7}\right]\\
&- C_2x_5e^{-\delta_pT}\left[\frac{(e^{-x_{10}t_p} - e^{x_{10}T})}{x_{10}} + \frac{e^{-t_px_4}(e^{-R_0T} - e^{-R_0t_p})}{R_0}\right]\\
&- C_0x_1\left[\frac{(e^{x_7T} - e^{x_7t_p})}{x_7} + \frac{e^{-\delta_pT}(e^{x_{11}T} - e^{-x_{11}t_p})}{x_{11}}\right] - 0\Bigg\rangle.
\end{aligned}
$$

4 Solution Procedure

***Step* 1.** In the beginning differentiate TP_i with respect to t_p ,i.e. $\dfrac{d(TP_i)}{dt_p}$ $i = 1, 2$ respectively.

***Step* 2.** Putting the above derivative equal to zero, i.e. $\dfrac{d(TP_i)}{dt_p} = 0$.

***Step* 3.** Find the value of t_p.

***Step* 4.** Find $\dfrac{d^2(TP_i)}{dt_p^2}$.

***Step* 5.** If $\dfrac{d^2(TP_i)}{dt_p^2} < 0$ at t_p then TP_i will be maximum.

With the help of MATLAB software, the optimal value of t_p which is denoted by t_p^* can be obtained. Then from Equations (3.7),(3.9) and (3.11), the values of TP^* and Q^* can be found. Here we assume suitable values for $P_0, H_p, C_P, S_p, C_0, C_2, R_0, \pi, P_c, P_e, \theta_p, T, \alpha, \lambda, \sigma$ and δ_p with appropriate units.

5 Numerical Examples

The trial and error method has been used for the numerical data of this paper. Tried increasing and decreasing all fixed values and finally the set of fixed values which gives maximum total profit is as follows

P_0	10000	R_0	0.7	H_p	5
θ_p	.5	C_p	5	S_p	40
σ	0.3	λ	4	α	5000
δ_p	0.3	C_2	3	C_0	3
P_c	2	P_e	4	T	1

.

***Example* 5.1.** When $0 \le \pi \le t_p$.

Using the above data with $\pi = 0.02$, we find the optimal values as $t_p^* = 0.6733, TP_1{}^* = 8607.6$ *and* $Q^* = 29.3899$. For the data taken in this, we get $\dfrac{d(TP_1)}{dt_p} = 0.00$. and $\dfrac{d^2(TP_1)}{dt_p^2} = -0.02$. i.e. $\dfrac{d^2(TP_1)}{dt_p^2} < 0$.

***Example* 5.2.** When $t_p \le \pi \le T$.

In this case we consider $\pi = 0.46$. Using the above data we obtain the optimal values $t_p^* = 0.4541, {TP_2}^* = 359.8152$ *and* $Q^* = 63.7898$. For the data taken in this, we get $\dfrac{d(TP_2)}{dt_p} = 0$ and $\dfrac{d^2(TP_2)}{dt_p^2} < 0$.

The examples are solved by MATLAB software.

6 Sensitivity Analysis

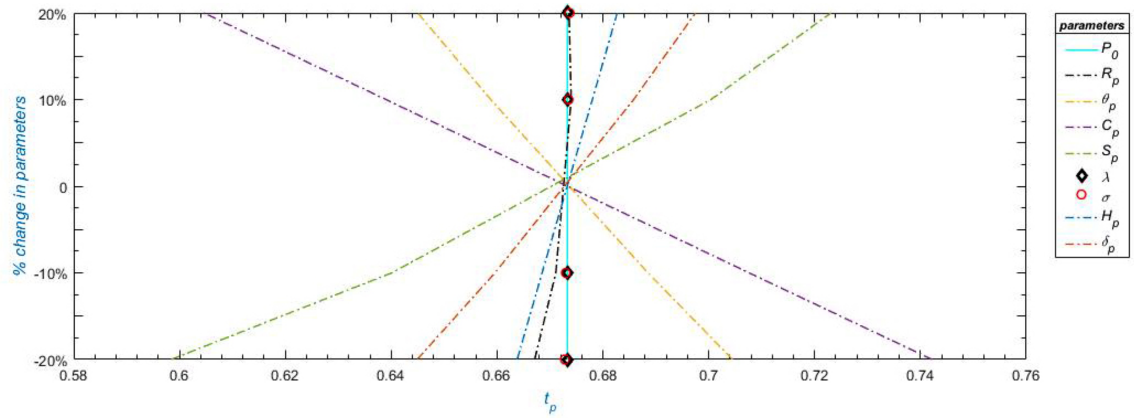

***Figure* 6.1:** Analysis of t_p with respect to the parameters

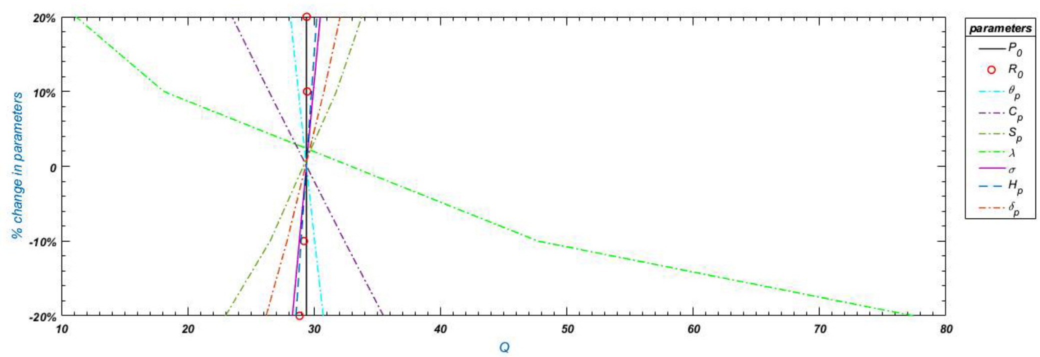

***Figure* 6.2:** Analysis of Q with respect to the parameters

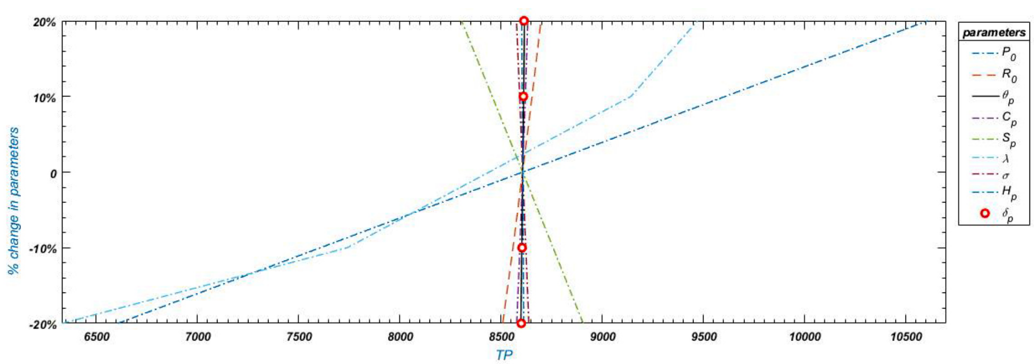

***Figure* 6.3:** Analysis of TP with respect to the parameters

Now, using ***Example* 5.1** a sensitivity analysis is examined to analyze the effects of changes in parameters on optimal values of $t_p^*,\ Q^* and\ TP_1{}^*$. The results are described in Table 6.1.

***Table* 6.1:** Sensitivity analysis of key parameters

parameters	% change	t_p	TP_1	Q
P_0	-20	0.6733	6607.6 0	29.3899
	-10	0.6733	7607.60	29.3899
	+10	0.6733	9607.60	29.3899
	+20	0.6733	10608.0	29.3899
R_0	-20	0.6671	8508.60	28.8508
	-10	0.6711	8559.30	29.1984
	+10	0.6740	8653.60	29.4509
	+20	0.6736	8697.50	29.4160
θ_p	-20	0.7045	8598.20	30.7251
	-10	0.6885	8603.00	30.0463
	+10	0.6588	8612.00	28.7514
	+20	0.6450	8616.20	28.1343
C_p	-20	0.7422	8577.70	35.4926
	-10	0.7076	8593.30	32.4018
	+10	0.6390	8620.50	26.4278
	+20	0.6046	8632.20	23.5051
S_p	-20	0.5984	8905.80	22.9833
	-10	0.6401	8757.40	26.5221
	+10	0.7004	8456.80	31.7653
	+20	0.7231	8305.20	33.7798
λ	-20	0.6733	6332.30	77.4150
	-10	0.6733	7740.20	47.6993
	+10	0.6733	9142.10	18.1086
	+20	0.6733	9471.40	11.1576
σ	-20	0.6728	8637.40	28.2880
	-10	0.6730	8622.70	28.8336
	+10	0.6735	8592.20	29.9397
	+20	0.6737	8576.50	30.4920
H_p	-20	0.6638	8612.50	28.5646
	-10	0.6685	8610.10	28.9724
	+10	0.6780	8605.10	29.7996
	+20	0.6827	8602.60	30.2103
δ_p	-20	0.6449	8600.40	26.2033
	-10	0.6596	8604.20	27.8604
	+10	0.6858	8610.80	30.7790
	+20	0.6975	8613.60	32.0666

7 Results and observation

Effect of % change in parameters on t_p, Q and TP is described as:

- As we raise the parameter P_0, total inventory Q and stock-out period t_p remains unchanged and total profit TP increases rapidly.
- Hike in the parameter R_0 remains t_p and Q almost unchanged and there is a slight increase in total profit TP.
- Total profit TP and total inventory Q behaves proportional to the parameter θ_p whereas t_p behaves inversely proportional.
- As we increase the parameter S_p, Q and t_p boosts and TP declines, on the other hand, total inventory Q, total profit TP and t_p behaves exactly opposite for the parameter C_p.
- When the parameter λ grows, t_p remains unchanged, TP and Q rapidly raises and drops respectively.

- Raising the parameter σ results t_p remains unchanged whereas TP and Q slightly hikes and declines respectively.
- As the parameters H_p and δ_p grows, t_p and Q rises whereas TP remains almost unchanged.

8. Conclusion and Future Research Direction

In this article, we presented an inventory model for spontaneous perishable products with trade credit period and time dependent demand rate under the effect of inflation. The carrying cost and deterioration of products are considered constant over the ordering cycle time. Partially backlogged shortages and delay in payments is allowed. The objective of this model is to maximizing the total profit by optimizing total inventory and stock-out period. Numerical example is discussed to demonstrate this model. The major findings of this study are

- If we increase the trade credit duration, it results more profit.
- To get maximum total profit, retailer should raise ordering cost.
- Hike in selling price of commodities, reduces total profit.

The further study in this direction can be done by considering variable holding cost and deterioration rate and selling price dependent demand rate. Also, this study can be performed for non-instantaneous deteriorating items.

Acknowledgement. The authors are very grateful to the anonymous Editors and reviewers for their remarkable proposals.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 55-61**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

STRUCTURE OF WEAKLY SEMI-I-OPEN SETS VIA SEMI LOCAL FUNCTIONS

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(Received: May 09, 2022; In format: May 15, 2022; Revised January 19, 2023; Accepted; February 05, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53106

Abstract

We introduce, study and investigate the concepts of weakly semi -I_s- open sets and some properties of the set. We introduce weakly semi -I_s- open functions and weakly semi -I_s- closed functions. Also, we introduced notion of weakly semi -I_s- open sets and weakly semi -I_s- closed sets. We discussed its properties and its relationship between other sets in topological spaces as said in below introduction. We also furnish decomposition of continuity in this paper.

2020 Mathematical Sciences Classification: 54A05, 54A10.

Keywords and Phrases: weakly semi-I_s-open, weakly semi-I_s-closed, weakly semi-I_s-continuity, weakly semi-I_s-sets.

1 Introduction

Topology as a well-defined mathematical discipline dates from the early twentieth century, though some isolated results can be traced back several centuries. An ideal topological space is a triplet (X,τ,I), where X is a nonempty set, τ is a topology on X and I is an ideal of subsets of X. Levine [13] introduced and investigated the concept of semi-open sets and semi-continuity in 1963. In 2006, in his paper *on weakly semi-I-open sets and another decomposition of continuity via ideals*, Hatir and Jafari [6] introduced the notions of weakly semi-I-open sets and weakly semi-I-continuous functions and obtained a decomposition of continuity. Khan and Noiri [11] introduced and investigated the concept of semilocal functions in his paper Semi-local functions in ideal topological spaces in 2010. Santhi and Rameshkumar [16] obtained several characterizations of semi-I_s-open sets and semi-I_s-continuous functions in 2013. Also, they introduce new semi-I_s-open and semi-I_s-closed functions as well. In 2014, Santhi and Rameshkumar [17] presented B_{Is}-sets, C_{Is}-sets, S_{Is}-sets, α-I_s-sets, semi-I_s-sets, and pre-I_s-sets to obtain a decomposition of continuity in ideal topological spaces using semi-local functions.

In this paper, we are introducing some properties of weakly semi-I_s-open sets and weakly semi-I_s-closed sets in ideal topological space via semilocal functions. We will study the relationship between weakly semi-I_s-open sets and weakly semi-I_s-closed sets, weakly semi-I_s-open sets and preopen set, weakly semi-I_s-open sets and α-I_s-open set, etc,.

2 Preliminaries

Let A be the subset of a topological space *(X,τ)* then *cl(A)* and *int(A)* denote closure and interior of A in *(X,τ)* respectively.

An Ideal I on a topological space *(X,τ)* is a non-empty collection of subsets of X which satisfies :

1. $A \in I$ and $B \subseteq A$ implies $B \in I$.
2. $A \in I$ and $B \in I$ implies $A \cup B \in I$.

The space *(X,τ,I)* is called an Ideal topological space or Ideal space.

Definition 2.1. *Let $P(X)$ be the power set of X. Then the operator $()^*$: $P(X) \to P(X)$ called a local function* [12] *of A with respect to τ and I is defined as follows : for $A \subseteq X$, $A^*(I,\tau) = \{x \in X \,|\, U \cap A \notin I$ for every open set U containing x $\}$. We simply write A^* instead of $A^*(I,\tau)$.*

Definition 2.2. *For $A \subseteq X$, $A_*(I,\tau) = \{ x \in X \,|\, U \cap A \notin I$ for every $U \in SO(X)\}$ is called semi-local function* [11] *of A with respect to I and τ, where $SO(X,x) = \{U \in SO(X) \,|x \in U\}$. We simply write A_* instead of $A_*(I,\tau)$.*

Definition 2.3. *It is given in* [4] *that $\tau^{*s}(I)$ is a topology on X, generated by a sub basis* $\{U - E{:}U \in SO(X)$ *and* $E \in I\}$ *or equivalently* $\tau^{*s}(I) = \{U \subseteq X{:}cl^{*s}(X - U)=X - U\}$.

Definition 2.4. *The closure operator* [4] cl^{*s} *for a topology* $\tau^{*s}(I)$ *is defined as follows: for* $A\subseteq X$, $cl^{*s}(A) = A \cup A_*$ *and* int^{*s} *denotes the interior of the set A in* (X,τ^{*s},I). *It is known that* $\tau\subseteq \tau^*(I)\subseteq\tau^{*s}(I)$.

Definition 2.5. *A subset A of (X,τ,I) is called semi-*-perfect* [10] *if* $A = A_*$. *A subset A of (X,τ,I) is called *-semi dense in-itself* [10] *if* $A \subset A_*$. *A subset A of (X,τ,I) is called semi-*-closed in-itself* [10] *if* $A_* \subseteq A$.

Definition 2.6. *A subset A of a space (X,τ) is said to be*

1. *regular closed* [15] *if cl(int(A)) =A.*
2. *semi-open* [13] *if* $A \subset cl(int(A))$. *The complement of semi open set is said to be semi-closed.*
3. *semi-closed* [13] *iff int(A) = int(cl(A)).*
4. *semi-closure* [13] *if intersections of all semi-closed sets containing A and it is denoted by scl(A).*

Definition 2.7 ([11])**.** *Let (X,τ,I) be an ideal topological space and A, B subsets of X. Then for the semi-local function the following properties hold:*

1. *If* $A \subseteq B$ *then* $A_* \subseteq B_*$.
2. *If* $U \in \tau$ *then* $U \cap A_* \subseteq (U \cap A)_*$.
3. $A_* = scl(A_*) \subseteq scl(A)$ *and* A_* *is semi-closed in X.*
4. $(A_*)_* \subseteq A_*$.
5. $(A \cup B)_* = A_* \cup B_*$.
6. *If* $I = \{\emptyset\}$, *then* $A_* = scl(A)$.

Definition 2.8. *A subset A of a topological space X is said to be*

1. *α-open* [14] *if* $A \subseteq int(cl(int(A)))$,
2. *pre-open* [3] *if* $A \subseteq int(cl(A))$,
3. *β-open* [5] *if* $A \subseteq cl(int(cl(A)))$.

Definition 2.9. *A subset A of an ideal topological space (X,τ,I) is said to be*

1. *α-I-open* [8] *if* $A \subseteq int(cl^*(int(A)))$,
2. *semi-I-open* [8] *if* $A \subseteq cl^*(int(A))$,
3. *pre-I-open* [1] *if* $A \subseteq int(cl^*(A))$,
4. *almost strong I-open* [7] *if* $A \subset cl^*(int(A^*))$,
5. *almost I-open* [2] *if* $A \subset cl(int(A^*))$,
6. *β-I-open* [8] *if* $A \subset cl(int(cl^*(A)))$,
7. *strong β-I-open* [7] *if* $A \subset cl^*(int(cl^*(A)))$,
8. *weakly semi-I-open* [15] *if* $A \subset cl^*(int(cl(A)))$.

Definition 2.10. *A subset A of an ideal topological space (X,τ,I) is said to be*

1. *α-I_s-open* [18] *if* $A \subseteq int(cl^{*s}(int(A)))$,
2. *s-I_s-set* [18] *if* $cl^{*s}(int(A)) = int(A)$,
3. *α^*-I_s-set* [18] *if* $int(cl^{*s}(int(A))) = int(A)$.

Corollary 2.1. *A subset A of an ideal topological space (X,τ,I) is said to be*

1. *Every almost strong I-open set is almost I-open but not converse* [7],
2. *Every almost strong I-open set is a strong β-I-open set but not converse* [7],
3. *Every strong β-I-open set is a β-I-open set but not converse* [7],
4. *Every β-I-open set is a β-open set but not converse* [7],
5. *Every almost I-open set is a β-I-open set but not converse* [7],
6. *Every weakly semi-I-open set is a β-open set but not converse* [6],
7. *Every strong β-I-open set is a weakly semi-I-open set but not converse* [6].

Definition 2.11. *Let (X,τ,I) be an ideal space and M be a *-semi dense in itself* [10] *subset of X. Then* $A_* = cl(A) = cl^{*s}(A)$.

Definition 2.12. *Let (X,τ,I) be an ideal space and* $A \subset X$.
Then $cl^{*s}(int(cl^{*s}(int(A)))) = cl^{*s}(int(A))$.

Definition 2.13. *A subset A of an ideal space (X,τ,I) is said to be semi-I_s-open* [18] *if $A \subseteq cl^{*s}(int(A)))$.*

Definition 2.14. *A subset A of an ideal space (X,τ,I) is said to be semi-I_s-open* [16] *iff there exists $U \in \tau$ such that $U \subseteq A \subseteq cl^{*s}(U)$. A subset H of an ideal space (X,τ,I) is said to be semi-I_s-closed* [16] if its complement is semi-I_s-open.

Definition 2.15. *A subset A of an ideal space (X,τ,I) is said to be pre-I_s-open* [18] *if $A \subseteq (int(cl^{*s}(A)))$.*

Definition 2.16. *A subset F of an ideal space (X,τ,I) is said to be pre-I_s-closed* [17] *if its complement is pre-I_s-open.*

Definition 2.17. *A subset A of an ideal space (X,τ,I) is called*

1. *An A_{IS}-set* [9] *if $A = U \cap V$, where U is open and $cl^{*s}(int(V)) = V$.*
2. *A B_{1IS}-set* [9] *if $A = U \cap V$, where U is α-I_s-open and $cl^{*s}(int(V)) = X$.*
3. *A B_{2IS}-set* [9] *if $A = U \cap V$, where U is α-I_s-open and $cl^{*s}(V) = X$.*
4. *An αA_{IS}-set* [9] *if $A = U \cap V$, where U is α-I_s-open and $cl^{*s}(int(V)) = V$.*
5. *An αC_{IS}-set* [9] *if $A = U \cap V$, where U is α-I_s-open and $int(cl^{*s}(int(V))) \subset V$.*
6. *A WLC_{IS}-set* [9] *if $A = U \cap V$, where U is open and $cl^{*s}(V) = V$.*
7. *A S_{IS}-set* [18] *if $A = U \cap V$, where $U \in \tau$ and V is S-I_s-set.*

3 Weakly semi-I_s-open sets

Definition 3.1. *A subset M of an ideal space (X,τ,I) is said to be weakly semi-I_s-open if $M \subseteq cl^{*s}(int(cl(M)))$.*

Example 3.1. Consider X = {m,n,o} in an ideal space *(X,τ,I)*, where τ = {$\emptyset$, {m}, {m,o}, X} and I ={$\emptyset$, {m}}. Let the semi open set of τ be B = {$\emptyset$, X} and M = {m,n}. Then $cl^{*s}(int(cl(M))) = cl^{*s}(int(cl(\{m,n\}))) = cl^{*s}(int(X)) = cl^{*s}(X) = X \supset M$ and so M is weakly semi-I_s-open.

Example 3.2. Consider X = $\{1,2,3,4\}$ in an ideal space *(X,τ,I)*, where $\tau = \{\emptyset, \{1\}, \{3\}, \{1,3\}, X\}$ and $I = \{\emptyset, \{1\}\}$. Let the semi open set of τ be B = $\{\emptyset, \{1,2,4\}, \{1,3,4\}, X\}$, M = $\{1,3\}$, $M_* = \{3\}$. Then $cl^{*s}(int(cl(M))) = cl^{*s}(int(cl(\{1,3\}))) = cl^{*s}(int(X)) = cl^{*s}(X) = X \supset M$ and so M is weakly semi-I_s-open.

Lemma 3.1. *Every semi-I_s-open set is weakly semi-I_s-open set, but converse doesn't hold.*

Example 3.3. Consider X = {m,n,o} in an ideal space *(X,τ,I)*, where τ = {$\emptyset$, {m,n}, X} and I ={$\emptyset$, {o}}. Then $M = \{m\}$, $cl^{*s}(int(cl(M))) = cl^{*s}(int(cl(\{m\}))) = cl^{*s}(int(X)) = cl^{*s}(X) = X \supset M$ and so M is weakly semi-I_s-open, but $cl^{*s}(int(M)) = cl^{*s}(int\{m\}) = cl^{*s}(\emptyset) = \emptyset \not\supset M$ and so M is not semi-I_s-open.

Theorem 3.1. *Let (X,τ,I) be an ideal topological space. If M is weakly semi-I_s-open set then M is β-open, but not conversely.*

Proof. If M is weakly semi-I_s-open, then $M \subset cl^{*s}(int(cl(M))) = (int(cl(M)))_* \cup (int(cl(M))) \subset cl(int(cl(M))) \cup int(cl(M)) = cl(int(cl(M)))$. Therefore M is β-open and converse doesn't hold. ☐

Example 3.4. Consider X = {m,n,o} in an ideal space *(X,τ,I)*, where τ = {$\emptyset$, {m}, {n}, {m,n}, X} and I ={$\emptyset$, {m}}. Then M = {m,o} is β-open, but not weakly semi-I_s-open.

Example 3.5. Consider X = {m,n,o,p} in an ideal space *(X,τ,I)*, where τ = {$\emptyset$, {m}, {o}, {m,o}, X} and I ={$\emptyset$, {m}}. Let the semi open set of τ be B = {$\emptyset$, {m,n,p}, {m,o,p}, X}, M = {m,o}, M_* = {c}. Then cl^{*s}(int(cl(M))) = cl^{*s}(int(cl({m,o}))) = cl^{*s}(int(X)) = cl^{*s}(X) = X $\supset M$ and so M is weakly semi-I_s-open. Also $M \neq M_*$, hence M is not semi-*-perfect. $M \not\subset M_*$, hence M is not *-semidense. $M \subseteq M_*$, hence M is semi-*-closed.

Corollary 3.1. *Let (X,τ,I) be an ideal space and M is *-semi dense in itself, then the following are equivalent :*

(a) M is β-open,

(b) M is weakly semi-I_s-open.

Theorem 3.2. *Let the ideal topological space be (X,τ,I) and M, N be the subsets of X. If M is weakly semi-I_s-open set and $N \in \tau$, then $M \cap N$ is weakly semi-I_s-open.*

Proof. Let M is weakly semi-I_s-open and $N \in \tau$. If $M \subset cl^{*s}(int(cl(M)))$, then $M \cap N \subset cl^{*s}(int(cl(M))) \cap N = ((int(cl(M)))_* \cup int(cl(M))) \cap N = (int(cl(M)))_* \cap N \cup int(cl(M)) \cap N \subset (int(cl(M)) \cap N_*) \cup int(cl(M \cap N)) = (int(cl(M \cap N)))_* \cup int(cl(M \cap N)) = cl^{*s}(int(cl(M \cap N)))$. This shows that $M \cap N$ is weakly semi-I_s-open. □

Remark 3.1. In general, the finite intersection of weakly semi-I_s-open sets need not be weakly semi-I_s-open.

Lemma 3.2. *Let the ideal topological space be (X,τ,I), where $M \subset X$ and $U \in$ semiopen set of τ. Then $cl^{*s}(M) \cap U = cl^{*s}(M \cap U)$.*

Proof. $cl^{*s}(M) \cap (U) = (M_* \cup M) \cap U = (M_* \cap U) \cup (M \cap U) \subset (M \cap U)_* \cup (M \cap U) = cl^{*s}(M \cap U)$. □

Example 3.6. Consider X = $\{1,2,3,4\}$ in an ideal space (X,τ,I), where $\tau = \{\emptyset, \{1\}, \{3\}, \{1,3\}, X\}$ and $I = \{\emptyset, \{1\}\}$. Let $M = \{1,3\}$ and $M_* = \{3\}$. From example 3.2, M is weakly semi-I_s-open. $M \not\subset M_*$, hence M is not *-semidense.

Example 3.7. Consider X = {m,n,o} in an ideal space (X,τ,I), where τ = {$\emptyset$, {m}, {n}, {m,n}, X} and I ={$\emptyset$, {m}}. Let the semi open set of τ be B = {$\emptyset$, {m,o}, {n,o}, X} and M = {m,o}, where $cl^{*s}(int(cl(M))) = cl^{*s}(int(cl(\{m,o\}))) = cl^{*s}(int(\{m,o\})) = cl^{*s}(\{m\}) = \emptyset \not\supset M$ and so M is not weakly semi-I_s-open. since $cl(int(cl^{*s}(M))) = cl(int(cl^{*s}(\{m,o\}))) = cl(int(X)) = cl(X) = X \supset M$ and so M is $\beta - I_s$-open.

The above example shows that weakly semi-I_s-openness and β-I_s-openness are independent concepts.

Theorem 3.3. *Let an ideal space be (X,τ,I). If M is pre-open, then M is weakly semi-I_s-open.*

Proof. If M is pre-open, then $M \subset int(cl(M))$ and so $M \subset cl^{*s}(int(cl(M)))$ which implies that M is weakly semi-I_s-open. □

Example 3.8. Consider X = {m,n,o} in an ideal space (X,τ,I), where τ = {$\emptyset$, {m}, {m,o}, X} and I ={$\emptyset$, {m}}. Let the semi open set of τ be B = {$\emptyset$, X} and M = {m,n}. Then $cl^{*s}(int(cl(M))) = cl^{*s}(int(cl(\{m,n\}))) = cl^{*s}(int(X)) = cl^{*s}(X) = X \supset M$ and so M is weakly semi-I_s-open. Also, $int(cl(M)) = int(cl(\{m,n\})) = int(X) = X \supset M$ and therefore, M is pre-open.

Theorem 3.4. *Let an ideal space be (X,τ,I). If $M \subset N \subset cl^{*s}(M)$ and M is weakly semi-I_s-open, then N is weakly semi-I_s-open. In particular, if M is weakly semi-I_s-open, then $cl^{*s}(M)$ is weakly semi-I_s-open.*

Proof. If M is weakly semi-I_s-open, then $M \subset cl^{*s}(int(cl(M)))$. Since $N \subset cl^{*s}(M) \subset cl^{*s}(cl^{*s}(int(cl(M)))) = cl^{*s}(int(cl(M))) \subset cl^{*s}(int(cl(N)))$. Hence N is weakly semi-I_s-open. □

Theorem 3.5. *Let the ideal space be (X,τ,I). If M is α-I_s-open and N is weakly semi-I_s-open, then $M \cap N$ is weakly semi-I_s-open.*

Proof. Since M is α-I_s-open, $M \subset int(cl^{*s}(int(M)))$ and N is weakly semi-I_s-open, $N \subset cl^{*s}(int(cl(N)))$. Now $M \cap N \subset int(cl^{*s}(int(M))) \cap cl^{*s}(int(cl(N))) \subset cl^{*s}(int(cl^{*s}(int(M))) \cap int(cl(N))) = cl^{*s}(int(cl^{*s}(int(M) \cap int(cl(N))))) \subset cl^{*s}(int(cl^{*s}(int(M) \cap int(cl(N))))) = cl^{*s}(int(cl^{*s}(int(int(M) \cap cl(N))))) \subset cl^{*s}(int(cl^{*s}(int(cl(int(M) \cap N))))) \subset cl^{*s}(int(cl^{*s}(int(cl (M \cap N))))) = cl^{*s}(int(cl(M \cap N)))$ by Definition 2.12, which implies that $M \cap N$ is weakly semi-I_s-open. □

Theorem 3.6. *Let the ideal space be (X,τ,I) and $M \subset X$ be weakly semi-I_s-open. If M is either semiclosed or I_s-locally closed, then M is semi-I_s-open.*

Proof. Suppose M is I_s-locally closed. M is I_s-locally closed implies that $M = U \cap M_*$ for some semi open set U. M is weakly semi-I_s-open implies that $M \subset cl^{*s}(int(cl(M)))$. Now $M = U \cap M_* \subset U \cap (cl^{*s}(int(cl(M))))_* \subset U \cap cl^{*s}(cl^{*s}(int(cl(U \cap M_*)))) = U \cap cl^{*s}(int(cl(U \cap M_*))) \subset cl^{*s}(U \cap int(cl(U \cap M_*))) = cl^{*s}(int(U \cap cl(U \cap M_*))) \subset cl^{*s}(int(U \cap cl(U) \cap cl(M_*))) = cl^{*s}(int(U \cap M_*)) = cl^{*s}(int(M))$. Hence M is semi-I_s-open. Suppose M is semiclosed. Then $int(cl(M)) = int(M)$. Since M is weakly semi-I_s-open implies that $M \subset cl^{*s}(int(cl(M))) = cl^{*s}(int(M))$. Hence M is semi-I_s-open. □

Example 3.9. Consider X = {m,n,o} in an ideal space *(X,τ,I)*, where $\tau = \{\emptyset, \{m,n\}, X\}$ and $I = \{\emptyset, \{o\}\}$. Let the semi open set of τ be B = {$\emptyset$, X} and $M = \{m\}$, then $M_* =$ X and $cl^{*s}(int(cl(M))) = cl^{*s}(int(X)) = X \subset M$ and so M is weakly semi-I_s-open. Also, $cl^{*s}(int(M)) = cl^{*s}(\emptyset) = \emptyset$. Hence M is not semi-I_s-open. Moreover, M is neither I_s-locally closed nor semiclosed.

Definition 3.2. *A subset M of an ideal space (X,τ,I) is said to be weakly semi-I_s-closed if $M \subseteq int^{*s}(cl(int(M)))$.*

Theorem 3.7. *A subset M of a space (X,τ,I) is weakly semi-I_s-closed iff $int^{*s}(cl(int(M))) \subset M$. Also, if M is weakly semi-I_s-closed subset of X, then M is an α^*-I_s-set.*

Proof. Let M be weakly semi-I_s-closed set of *(X,τ,I)*. Then X - M is weakly semi-I_s-open and hence X - $M \subset cl^{*s}(int(cl(X-M))) = X - int^{*s}(cl(int(M)))$. Therefore, we have $int^{*s}(cl(int(M))) \subset M$. Conversely, let $int^{*s}(cl(int(M))) \subset M$. Then X - $M \subset cl^{*s}(int(cl(X-M)))$ and hence X - M is weakly semi-I_s-open. Therefore, M is weakly semi-I_s-closed. Also $int^{*s}(cl(int(M))) \subset M$ and so $int^{*s}(cl(int(M))) \subset int(M)$. Hence it follows that $int^{*s}(cl(int(M))) = int(M)$ which implies that M is α^*-I_s-set. □

Definition 3.3. *A subset M of an ideal space (X,τ,I) is said to be weakly S_{IS}-set (resp. C_{IS}-set* [18]*) if $M = G \cap V$ where G is open and V is weakly semi-I_s-closed (resp. α^*-I_s-set).*

Remark 3.2. Every open set is a weakly S_{IS}-set and every weakly S_{IS}-set is a C_{IS}-set.

Theorem 3.8. *Let (X,τ,I) be an ideal space. Then the following are equivalent :*
(a) M is open,
(b) M is α-I_s-open and a weakly S_{IS}-set,
(c) M is α-I_s-open and a C_{IS}-set.

Proof. If M is open, (a) implies (b) and (b) implies (c) are clear. Then (c) implies (a) follows from the preposition 4.16 of [18]. □

Definition 3.4. *A subset M of a space (X,τ,I) is called Strong $s - I_s$-set if $cl^{*s}(int(cl(M))) = int(M)$.*

Definition 3.5. *A subset M of a space (X,τ,I) is called Strong S_{IS}-set if $M = U \cap V$, where $U \in \tau$ and V is Strong $S - I_s$-set.*

Remark 3.3. a) Every strong $s - I_s$-set is $S - I_s$-set.
b) Every strong S_{IS}-set is S_{IS}-set.
c) Every open set is strong S_{IS}-set.

Proposition 3.1. *For a subset M of a topological space (X,τ,I), the following holds equivalently :*
a) M is open,
b) M is weakly semi-I_s-open and strong S_{IS}-set,
c) M is semi-I_s-open and strong S_{IS}-set.

Proof. By the above remarks we prove this as follows:

If M is a semi-I_s-open set and also a strong S_{IS}-set, then $M \subseteq cl^{*s}(int(cl(M))) = cl^{*s}(int(cl(U \cap V)))$, where $U \in \tau$ and V is strong S_{IS}-set. Hence $M \subset U \cap M \subset U \cap (cl^{*s}(int(cl(U))) \cap cl^{*s}(int(cl(V)))) = U \cap int(V) = int(M)$, shows that M is open □

4 Weakly semi-I_s-open and Weakly semi-I_s-closed functions

Definition 4.1. *Let $f : (M, \tau, I) \to (N, \sigma, J)$ be a function of weakly semi-I_s-open if the image of every open set in (M, τ, I) is weakly semi-I_s-open in (N, σ, J).*

Theorem 4.1. *A function $f : (M, \tau, I) \to (N, \sigma, J)$ is weakly semi-I_s-open iff for each point m of X and each neighbourhood U of m, there exists a weakly semi-I_s-open set V in N containing $f(m)$ such that $V \subset f(U)$.*

Theorem 4.2. *A function $f : (M, \tau, I) \to (N, \sigma, J)$ is weakly semi-I_s-open function such that $F \subset N$ and $G \subset M$ is a closed set containing $f^{-1}(F)$, then there exists a weakly semi-I_s-open set $W \subset N$ containing F such that $f^{-1}(W) \subset G$.*

Definition 4.2. *Let $f : (M, \tau, I) \to (N, \sigma, J)$ be a function of weakly semi-I_s-closed if the image of every closed set in (M, τ, I) is weakly semi-I_s-closed in (N, σ, J).*

Theorem 4.3. *A function $f : (M, \tau, I) \to (N, \sigma, J)$ is weakly semi-I_s-closed function such that $F \subset N$ and $G \subset M$ is a open set containing $f^{-1}(F)$, then there exists a weakly semi-I_s-closed set $W \subset N$ containing F such that $f^{-1}(W) \subset G$.*

Definition 4.3. *A function $f : (M, \tau, I) \to (N, \sigma, J)$ is said to be weakly semi-I_s-continuous if for every $V \in \sigma$, $f^{-1}(V)$ is an ws-I_s-set of (M, τ, I).*

Proposition 4.1. *$f : (M, \tau, I) \to (N, \sigma, J)$ be bijective function then the following condition holds:*
(1) *f^{-1} is weakly semi-I_s-continuous,*
(2) *f is weakly semi-I_s-open,*
(3) *f is weakly semi-I_s-closed.*

Theorem 4.4. *Consider the functions $f : (M, \tau.I) \to (N, \sigma, J)$ and $g : (N, \sigma, J) \to (O, \nu, K)$, whre I, J and K are ideals on M, N and O, respectively. The following statement holds:*
(1) *If f is open and g is weakly semi-I_s-open then $g \circ f$ is weakly semi-I_s-open,*
(2) *If $g \circ f$ is open and g is weakly semi-I_s-continuous injection then f is weakly semi-I_s-open.*

5 Conclusion

In this paper, we obtained several characterization of weakly semi-I_s-open sets. we introduced weakly semi-I_s-open sets and weakly semi-I_s-closed sets using semi local functions. Also we introduced weakly semi-I_s-open functions and weakly semi-I_s-closed functions. We discussed their relationship with various sets.

Acknowledgment
Authors are thankful to the editor and referee for their valuable consideration, guidance and suggestions.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 62-67**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

FUZZY SEMI-SEPARATION AXIOMS AND FUZZY SEMI-CONNECTEDNESS IN FUZYY BICLOSURE SPACES

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(Received: June 22, 2022; In format: September 28, 2022; Revised: February 02, 2023; Accepted: February 09, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53107

Abstract

The purpose of this paper is to introduce the notion of fuzzy semi-separation axioms and fuzzy semi-connectedness in fuzzy biclosure spaces. Further we investigate their characterizations and find relations with other already existing definitions. We generalize the results of semi-connectedness in fuzzy setting. Here we follow the definition of closure operator given by Birkhoff [5].

2020 Mathematical Sciences Classification: 54A40, 03E72

Keywords and Phrases: fuzzy semi-connectedness, fuzzy q-separated sets, fuzzy semi-separated sets, fuzzy semi-separation axioms.

1 Introduction

The notion of fuzzy semi-open sets as well as fuzzy semi-closed sets plays a very significant role in fuzzy topology. The concept of semi-open sets were introduced by Levine [8]. Azad [1] introduced the concept of fuzzy semi-open sets and fuzzy semi-separation axioms in 1981.

We introduce and study the concept of fuzzy semi-separation axioms in different way. Here we introduce fuzzy semi-connectedness in fuzzy biclosure spaces and also study their basic properties.

2 Preliminaries

The concept of fuzzy set was introduced by Zadeh (1965) in his classical paper [18]. A fuzzy set 'A' in a non-empty set X is a mapping from X to $[0,1]$.

A fuzzy point x_r is a fuzzy set in X taking value $r \in (0,1)$ at x and zero otherwise. A fuzzy point x_r is said to belong to a fuzzy set A i.e. $x_r \in A$ iff $r \leq A(x)$ [13]. A fuzzy singleton x_r is a fuzzy set in X taking value $r \in (0,1]$ at x and 0 elsewhere. A non-empty set X together with two fuzzy topologies $\tau_1,\ \tau_2$ is called fuzzy bitopological space. It is denoted by (X, τ_1, τ_2).

A fuzzy point x_r is said to be quasi-coincident with A denoted by $x_r qA$ iff $r + A(x) > 1$. A fuzzy set A is said to be quasi-coincident with another fuzzy set B denoted by AqB iff $\exists\ x \in X$ such that $A(x) + B(x) > 1$ similarly we say that $A\bar{q}B$ iff $A \subseteq coB$ i.e. $A(x) + B(x) \leq 1$. Obviously, if A and B are quasi-coincident at x both $A(x)$ and $B(x)$ are not zero and here A and B intersect at x. Here we follow the Lowen's definition [9] of fuzzy topology as a family τ of fuzzy sets of a non-empty X is said to form a fuzzy topology on X if it is preserved under Arbitrary union, finite intersection and contains all constant fuzzy sets. The members of τ are called fuzzy open sets and their complements are called fuzzy closed sets. All the definitions, results and terminology used here is taken from Ming and Ming [13]. In this paper, we introduce the concept of semi-separation axioms and semi-connectedness in fuzzy biclosure spaces. We use abbreviation fbcs for fuzzy biclosure space.

The concept of closure operator was given by Čech [7] and Birkhoff [5] separately. A lot of works has been done on Čech closure operator. We are using here Birkhoff closure operator. Now we mention the definitions of closure operator as:

Definition 2.1 ([7]). *(Čech closure operator)*
An operator $C : 2^X \to 2^X$ is called closure operator if it satisfies the following axioms:
1. $C(\phi) = \phi$,
2. $A \subseteq C(A),\ \forall A \in 2^X$,
3. $C(A \cup B) = C(A) \cup C(B),\ \forall A, B \in 2^X$.

Here C is called a closure operator and (X, C) is known as Čech closure space.

Definition 2.2 ([5])**.** *(Birkhoff closure operator)*
An operator $C : 2^X \to 2^X$ is called closure operator on a non-empty set X if it satisfies the following axioms:
1. $C(\phi) = \phi$,
2. $A \subseteq C(A),\ \forall A \in 2^X$,
3. $A \subseteq B \Rightarrow C(A) \subseteq C(B), \forall A, B \in 2^X$,
4. $C(C(A)) = C(A),\ \forall A \in 2^X$,

Here C is called Birkhoff closure operator and the pair (X, C) is called a closure space.

Definition 2.3 ([10])**.** *A fuzzy closure operator on a set X is a function $c : I^X \to I^X$ satisfying the following three axioms:*
1. $c(\phi) = \phi$,
2. $c(A) \subset A,\ \forall A \in I^X$,
3. $c(A \cup B) = c(A) \cup c(B) <\ \forall A, B \in I^X$.

The pair (X, c) is a fuzzy closure space (in short fcs).

Definition 2.4 ([16])**.** *A fuzzy closure operator c on a set X is a function $c : I^X \to I^X$ satisfying following the axioms:*
1. $c(\underline{\alpha}) = \alpha; \alpha \in [0, 1]$,
2. $A \subseteq c(A),\ \forall A \in I^X$,
3. $A \subseteq B \Rightarrow c(A) \subseteq c(B),\ \forall A, B \in I^X$,
4. $c(c(A)) = c(A),\ \forall A \in I^X$,

Here (X, c) is known as fuzzy closure space.

Definition 2.5 ([17])**.** *A function $c_i : I^X \to I^X (i = 1, 2)$ is called a fuzzy biclosure operator on X if the following postulates are satisfied:*
1. $c_i(\underline{\alpha}) = \underline{\alpha}, \alpha \in [0, 1]$,
2. $A \subseteq c_i(A), \forall A \in I^X$,
3. $A \subseteq B \Rightarrow c_i(A) \subseteq c_i(B), \forall A, B \in I^X$,
4. $c_i(c_i(A)) = c_i(A),\ \forall A \in I^X$.

Then (X, c_1, c_2) is called a fuzzy biclosure spaces.

Definition 2.6. *Let (X, c_1, c_2) be a fuzzy biclosure space. If the closure operator satisfies the condition $c_i(A \cup B) = c_i(A) \cup c_i(B)$ then it is called additive property.*

The concept of semi-open sets was introduced by Levine [8] as "A set R in a topological space X will be termed semi-open (simply written as s.o.) iff there exist an open set P such that $P \subset R \subset c(P)$ where c denote the closure operator in X [8]." Also we know that a fuzzy set R is said to be fuzzy semi-open iff $R \subseteq cl$ (int R). The complement of fuzzy semi-open set is called fuzzy semi-closed set. So, a fuzzy set A is said to be fuzzy semi-closed iff int$(cl(R) \subseteq R$. Let $\{R_\alpha\}\ \alpha \in \Lambda$ be a collection of semi-open set in X then $\cup_{\alpha \in \Lambda} R_\alpha$ is semiopen.

Definition 2.7. *Let (X, c_1, c_2) be a fuzzy biclosure space and the semi closure of a fuzzy set A in X is defined as:*

c_i *- scl*$(A) = \cap\{B : B$ *is fuzzy closed set and* $B \supset A\}$

Similarly the semi interior of a fuzzy set A in X is defined as:

c_i *- sint*$(A) = \cap\{B : B$ *is fuzzy open set and* $B \subset A\}$.

3 Fuzzy semi-separation axioms in fuzzy biclosure space

In this section, we define the concepts of fuzzy semi-separation axioms in fuzzy biclosure spaces:

Definition 3.1. *A fuzzy biclosure space (X, c_1, c_2) is said to be*
1. Fuzzy Pairwise Semi T_0 if $\forall x, y \in X, x \neq y\ \exists$ a fuzzy semi-open set U such that $U(x) \neq U(y)$,
2. Fuzzy Pairwise Semi T_1 if $\forall x, y \in X, x \neq y\ \exists$ fuzzy semi-open sets U, V in X such that $U(x) = 1, U(y) = 0$ and $V(x) = 0, V(y) = 1$,

3. Fuzzy Pairwise Weakly semi T_1 if $\exists$ a c_1-fuzzy semi-open set or a c_2-fuzzy semi-open set U such that $U(x) = 1, U(y) = 0$,
4. Fuzzy Pairwise Semi T_2 if for every pair of distinct fuzzy points x_r, y_s in X their exists fuzzy semi-open sets U and V such that $x_r \in U, y_s \in V$ and $U \cap V = \emptyset$,
5. Fuzzy Pairwise Semi-Regular if for each fuzzy point x_r and each fuzzy closed set F such that $x_r \bar{q} F$ $\exists$ fuzzy semi-open sets U and V such that $x_r \subseteq U, F \subseteq V$ and $U \bar{q} V$,
6. Fuzzy Pairwise Semi-Normal if for every pair of fuzzy closed set F_1 and F_2 such that $F_1 \bar{q} F_2$, $\exists$ fuzzy semi-open sets U, V such that $F_1 \subseteq U$ and $F_2 \subseteq V$ and $U \bar{q} V$.

Clearly fuzzy semi $T_2 \Rightarrow$ fuzzy semi $T_1 \Rightarrow$ fuzzy semi T_0 but not conversely.

Theorem 3.1. *A fbcs (X, c_1, c_2) is fuzzy pairwise semi T_0 if either (X, c_1) or (X, c_2) is semi T_0.*

Proof. It is given that (X, c_1) or (X, c_2) is fuzzy semi T_0. If (X, c_1) is semi T_0 then we have $x, y \in X, x \neq y$ $\exists$ *a* c_1-fuzzy semi-open set U such that $U(x) \neq U(y)$. Now if (X, c_2) is fuzzy semi T_0 then $\exists$ *a* c_2-fuzzy semi-open set V such that $V(x) \neq V(y)$. Thus for $x, y \in X, x \neq y$ $\exists$ a fuzzy semi-open set U in c_1 or c_2 such that $U(x) \neq U(y)$. Hence (X, c_1, c_2) is fuzzy pairwise semi T_0. □

Theorem 3.2. *A fbcs (X, c_1, c_2) is fuzzy pairwise semi T. iff (X, c_1) and (X, c_2) are fuzzy semi T_1.*

Proof. First let the fbcs (X, c_1, c_2) is fuzzy pairwise semi T_1. then for $x, y \in X, x \neq y$ $\exists$ $U_1 \in c_1$ and $V_1 \in c_1$ such that $U_1(x) = 1, V_1(y) = 0$ and $U_1(x) = 0, V_1(y) = 1$. If we take $x, y \in X$ then $U_2, V_2 \in X$ such that $U_2(x) = 0, U_2(y) = 1$ and $V_2(x) = 1, V_2(y) = 0$ .Therefore $x, y \in X, x \neq y$ we have $U_1, U_2 \in c_2$ such that $U_1(x) = 1, U_1(y) = 0$ and $U_2(x) = 0, U_2(y) = 1$ implies that (X, c_1) is fuzzy semi T_1. Similarly (X, c_2) is fuzzy semi T_1.

Conversely suppose that (X, c_1) and (X, c_2) are fuzzy semi T_1 then (X, c_1) is fuzzy semi T_1 for $x, y \in X, x \neq y$ $\exists$ a c_1-fuzzy semi-open set U such that $U(x) = 1, U(y) = 0$ and since (X, c_2) is fuzzy semi T_1 for $x, y \in X, x \neq y$ $\exists$ a c_2-fuzzy semi-open set V such that $V(x) = 0, V(y) = 1$. Then for $x, y \in X, x \neq y$ $\exists$ a c_1-fuzzy semi-open set U and a c_2-fuzzy semi-open set V such that $U(x) = 1, V(y) = 0$ and $V(x) = 0, U(y) = 1$ implies that (X, c_1, c_2) is fuzzy pairwise semi T_1. □

Theorem 3.3. *A fbcs (X, c_1, c_2) is fuzzy weakly pairwise semi T_1 iff c_1-$scl\{x\} \cap c_2$-$scl\{x\} = \{x\}$ for every $x \in X$.*

Proof. Let (X, c_1, c_2) be a fuzzy pairwise semi T_1. Let $x \in X$ and choose any $y \neq x$, then $\exists$ a c_1-fuzzy semi-open set or a c_2-fuzzy semi-open set U such that $U(y) = 1, U(x) = 0$.

First let us consider U as a c_1-fuzzy semi-open set then clearly coU is c_1-fuzzy semi-closed set such that $coU(x) = 1$, $coU(y) = 0$. Thus c_1 - $scl\{x\} \supseteq \{x\}$ for $i = 1, 2$ where $\{x\}$ is semi-closed.

Hence $\{c_1\text{-}scl\{x\}\}(x) = 1$ for $i = 1, 2$ which implies that $(c_1\text{-}scl\{x\} \cap c_2\text{-}scl\{x\})(x) = 1$. let $c_1\text{-}scl\{x\} = \cap \left\{F \in I^X : F \supseteq \{x\}\right\}$ and F is c_1-fuzzy semi-closed set. Let $\left\{F \in I^X : F \supseteq \{x\}\right\}$ be defined by $\mathcal{F}$. Then $coU \in \mathcal{F}$ and since $coU(y) = 0$, we have $(c_1\text{-}scl\{x\})(y) = 0$ which implies that $(c_1\text{-}scl\{x\} \cap c_2\text{-}scl\{x\})(y) = 0$. Let (X, c_1, c_2) be a fuzzy pairwise semi T_1. Hence (X, c_1, c_2) is fuzzy weakly pairwise semi T_1.

Conversely let $x, y \in X$ $x \neq y$ then $(c_1\text{-}scl\{x\} \cap c_2\text{-}scl\{x\})(x) = 1$ and $(c_1\text{-}scl\{x\} \cap c_2\text{-}scl\{x\})(y) = 0$.

Taking complement of both the sides we get

$[(X - c_1\text{-}scl\{x\}) \cup (X - c_2\text{-}scl\{x\})](x) = 0, [(X - c_1\text{-}scl\{x\}) \cup (X - c_2\text{-}scl\{x\})](y) = 1$ also $[(X - c_1 - scl\{x\})(y) = 1$ or $(X - c_2\text{-}scl\{x\})(y) = 1$.

Let us suppose that $(X - c_1\text{-}scl\{x\})(y) = 1$, hence $X - c_1\text{-}scl\{x\}$ and $X - c_2\text{-}scl\{x\}$ are fuzzy open sets such that $(X - c_1\text{-}scl\{x\})(x) = 0$ and $(X - c_2\text{-}scl\{x\})(x) = 0$ and

$(X - c_1\text{-}scl\{x\})(y) = 1$ or $(X - c_2\text{-}scl\{x\})(y) = 1$. Thus we have fuzzy semi-open set in c_1 viz. $X - c_1\text{-}scl\{x\}$ such that $(X - c_1\text{-}scl\{x\})(x) = 0$ and $(X - c_1\text{-}scl\{x\})(y) = 1$ which implies that (X, c_1, c_2) is fuzzy weakly pairwise semi T_1. □

Theorem 3.4. *A fuzzy biclosure space (X, c_1, c_2) is fuzzy pairwise semi T_2 iff the diagonal set Δ_X is fuzzy semi-closed in $(X \times X, c_1 \times c_2)$.*

Proof. Let (X, c_1, c_2) be fuzzy pairwise semi T_2. We have to show that Δ_X is fuzzy semiclosed in $X \times X$. In other side, we have to show that $X \times X - \Delta_X$ is fuzzy open. Let x_r, y_r be any two distinct fuzzy points in X because (X, c_1, c_2) is fuzzy pairwise semi T_2, $\exists$ *a* c_1-fuzzy semi-open set U and a c_2-fuzzy semi-open set V

such that $x_r \in U, y_r \in V$ and $U \cap V = \emptyset$. Consider the basic fuzzy semi-open set $U \times V$ in $(X \times X, c_1 \times c_2)$ then $(x_r, y_r) \in U \times V \subseteq X \times X - \Delta_X$ which implies that $X \times X - \Delta_X$ is fuzzy semi-open in $c_1 \times c_2$ i.e. Δ_X is fuzzy semi-closed.

Conversely, let Δ_X be fuzzy semi-closed in $(X \times X, c_1 \times c_2)$ i.e. $X \times X - \Delta_X$ be fuzzy open in $X \times X$. We show that (X, c_1, c_2) is fuzzy pairwise semi T_2. Let $x_r, y_r \in X, x \neq y$.

Let $r \leq s$ and consider $(x, y)_S$ then $(x, y)_S$ is a fuzzy point in. $X \times X - \Delta_X$ therefore $\exists$ a basic fuzzy semi-open set $U \times V$ in $c_1 \times c_2$ such that $(x, y)_S \in U \times V \subseteq X \times X - \Delta_X$.

Here $U \in c_1$ and $V \in c_2$ where $U = \bigcup_{i \in A_1} U_i$ and $V = \bigcup_{j \in A_2} V_j$ where $U_i \in c_1$ and $V_j \in c_2$ thus $(x, y)_s \in \cup_i U_i \times \cup_j V_j \subseteq X \times X - \Delta_X$

This implies that $\exists$ i, j such that $(x, y)_s \epsilon U_i \times V_j \subseteq X \times X - \Delta_X$. Thus we may say that $x_r \in U_i$ and $y_s \in V_j$ and $U_i \cap V_j = \emptyset$ which implies that (X, c_1, c_2) is fuzzy pairwise semi T_2. □

Theorem 3.5. *The fbcs (X, c_1, c_2) is fuzzy pairwise semi-regular iff for each c_i-fuzzy open set F $\exists$ a c_i-fuzzy semi-open set U such that $x_r \subseteq U \subseteq c_1$-$sclU$ $\bar{q}F$.*

Proof. Let (X, c_1, c_2) be fuzzy pairwise semi-regular. Then for every c_i-fuzzy open set F and each fuzzy point x_r such that $x_r \subseteq F$, $\exists$ *a* c_1-fuzzy semi-open set U and c_2-fuzzy semi-open set V such that $x_r \subseteq U, coF \subseteq V$ and $U \subseteq coV$ then $x_r \subseteq U \subseteq coV \subseteq F$. Since coV is a c_i-fuzzy semi-closed set such that $U\bar{q}V$ then $x_r \subseteq U$ and $sclU\bar{q}F$. This can also be written as $x_r \subseteq U \subseteq c_1$-$sclU\bar{q}F$. Conversely, let x_r be a fuzzy point and F be a c_1-fuzzy semi-closed set such that $x_r\bar{q}F$. Thus $\exists$ a c_i-fuzzy semi-open set U such that $x_r \subseteq U \subseteq c_i$-$sclU\bar{q}F$. Consider fuzzy sets U_1 and V_1 where $U_1 = U$ and $V_1 = 1 - c_i$-scl U clearly U_1 is c_1-fuzzy semi-open set and V_1 is c_j-fuzzy semi-open set such that $x_r \subseteq U_1, F \subseteq V_1$ and $U_1\bar{q}V_1$ since for any $z \in X, U_1(z) + V_1(z) = U(z) + 1 - c_i - sclU(z)$ which is obviously ≤ 1. □

Theorem 3.6. *A fbcs (X, c_1, c_2) is fuzzy pairwise semi-normal iff for any c_i-fuzzy semi-closed set A and a c_j-fuzzy semi-open set B such that $A \subseteq B$ $\exists$ a c_i-fuzzy semi-open set U such that $A \subseteq U$ and c_i-$sclU \subseteq B$*

Proof. First let the fbcs (X, c_1, c_2) be fuzzy pairwise semi-normal then for any c_i-fuzzy semi-closed set A and a c_j-fuzzy semi-open set B such that $A \subseteq B$ $\exists$ *a* c_j-fuzzy semi-open sets U and a c_i-fuzzy semi-open set V such that $A \subseteq U$ and co$B \subseteq V$ and $U\bar{q}V$ thus $A \subseteq U$, $U\bar{q}V$ and $coV \subseteq B$ or $V \subseteq$ coB. We can also write $A \subseteq U \subseteq c_1$-$sclU \subseteq B$.

Conversely, let A be any c_i-fuzzy semi-closed set and B be any c_j-fuzzy semi-closed set such that $A\bar{q}B$ then $\exists$ a c_j-fuzzy semi-open set U such that $A \subseteq U \subseteq c_1$-$sclU \subseteq B$.

Consider the fuzzy sets U_1 and V_1 such that $U_1 = U$ and $V_1 = 1 - c_1$-$sclU$. Obviously, U_1 is a c_j-fuzzy semi-open set and V_1 is a c_j-fuzzy semi-open set such that $A \subseteq U_1, B \subseteq V_1$ and $U_1 \subseteq coV_1$ i.e. $U\bar{q}V_1$ as for any $x \in X, U_1(x) + V_1(x) = U(x) + 1 - c_1$-$scl(x) \leq 1$. □

4 Fuzzy pairwise semi-continuous maps in fuzzy biclosure space

The concepts of fuzzy biclosed maps, pairwise fuzzy bicontinuous maps and generalized fuzzy continuous maps in fbcs were introduced by Navalakhe [14, 15]. Pairwise continuity between fuzzy closure spaces was introduced by Azad [1]. Further using fuzzy semi-open sets Azad [1] introduced and studied fuzzy pairwise s-continuous mapping between fuzzy closure spaces. We introduce and study two more definitions using fuzzy semi-open sets and compare all the definitions with each other. Let (X, c_1, c_2) and (Y, c_1, c_2) be any two fuzzy biclosure spaces then

Definition 4.1 ([1])**.** *A map $f : X \to Y$ is said to be fuzzy pairwise continuous if $f^{-1}(V)$ is fuzzy open in X whenever V is fuzzy open set in Y.*

Definition 4.2 ([1])**.** *A map $f : X \to Y$ is said to be fuzzy pairwise s-continuous if $f^{-1}(V)$ is semi-open in X whenever V is semi-open set in Y.*

Definition 4.3. *A map $f : X \to Y$ is said to be fuzzy pairwise semi-continuous if $f^{-1}(V)$ is open in X whenever V is semi-open set in Y.*

Definition 4.4. *A map $f : X \to Y$ is said to be fuzzy pairwise s^*-continuous if $f^{-1}(V)$ is semi-open in X whenever V is open set in Y.*

Comparing them with each other we have

$$\text{Fuzzy pairwise-continuous}$$
$$\swarrow \quad \Downarrow \quad \searrow$$
$$\text{Fuzzy semi-continuous} \Rightarrow \text{Fuzzy s-continuous} \Leftarrow \text{Fuzzy s}^*\text{-continuous.}$$

5 Fuzzy semi-connectedness in fuzzy biclosure spaces

In this section, we introduce and study the concept of q-separated sets. This concept is earlier introduced by Ming-Ming in 1980.

Definition 5.1 ([13]). *The fuzzy sets A_1 and A_2 in fts (X,T) are said to be separated iff $\exists\, U_i \in T(i=1,2):$ $U_i \supset A_i$ and $U_1 \cap A_2 = \emptyset = U_2 \cap A_1$.*

Definition 5.2 ([13]). *Two fuzzy sets A_1 and A_2 in fts (X,T) are said to be Q-separated iff $\exists$ T closed sets $H_i : H_i \supset A_i$ and $H_1 \cap A_2 = \emptyset = H_2 \cap A_1$.*

It is obvious that A_1 and A_2 are Q-Separated iff $\bar{A}_1 \cap A_2 = \emptyset = \bar{A}_2 \cap A_1$

Definition 5.3 ([13]). *A fuzzy set Y in (X,c) is called disconnected iff there exist two non-void sets A and B in the subspace Y_0 (i.e supp Y) such that A and B are Q-separated and $Y = A \cup B$. A fuzzy set is called connected iff it is not disconnected.*

Definition 5.4. *The fuzzy sets A and B are said to be q-separated iff $cl(A)\bar{q}B$ and $A\bar{q}cl(B)$*

Definition 5.5. *The fuzzy sets A and B in a fbcs (X, c_1, c_2) are said to be c_i - q-separated (or simply q-separated) iff $c_i(A)\bar{q}B$ and $c_i(B)\bar{q}A$.*

We introduce here the definition of semi-connectedness using $c_i - q$-separated sets in a fuzzy biclosure space:

Definition 5.6. *A fuzzy set in a fuzzy closure space is said to be fuzzy semi-connected iff there doesn't exist two non-empty semi-open sets A and B in the subspace Y_0 (supp Y) such that $Y = A \cup B$ and A, B are q-separated.*

Theorem 5.1. *The s-continuous onto image of a fuzzy semi-connected biclosure space which has additive property also is fuzzy semi-connected biclosure space with additive property.*

Proof. Let X and Y be two fbcs and X be fuzzy semi-connected. Suppose Y is not fuzzy semi-connected then Y is fuzzy semi disconnected. Then $\exists$ two non-empty fuzzy sets A and B in the subspace Y_0 (supp Y) such that A and B are q-separated and $Y_0 = A \cup B$ (supp Y) or $c_i(A)\bar{q}B$ and $A\bar{q}c_i(B)$. Now f is semi continuous $f^{-1}(A)$ and $f^{-1}(B)$ are subsets of X therefore $X = f^{-1}(A) \cup f^{-1}(B)$ where $c_i\left(f^{-1}(A)\right)\bar{q}f^{-1}(B)$ and $f^{-1}(A)\bar{q}c_i\left(f^{-1}(B)\right)$.

We know from the additive property that $c_i(A \cup B) = c_i(A) \cup c_i(B)$. Then $f^{-1}(Y_0) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ which is a contradiction since $f^{-1}(A), f^{-1}(B)$ are q-separated. It means our assumption is wrong. Thus Y is fuzzy semi-conneced biclosure space with additive property. □

Theorem 5.2. *Let (X, c_1, c_2) be semi disconnected let $c_i \subset c_i^*(i=1,2)$ then (X, c_1^*, c_2^*) is semi disconnected.*

Proof. Let (X, c_1, c_2) be semi disconnected fbcs then $\exists$ two non-empty fuzzy semi-open sets A and B in the subspace Y_0 (supp Y) such that $Y = A \bigcup B$ where A and B are c_i-q-separated. Since $c_i \subset c_i^*$ then A and B are c_i^*-q-separated also. Thus we have two non void sets A and B in the space Y_0(supp$Y$) such that $Y = A \cup B$ where A and B are c_i^*-q-separated also. Hence (X, c_1^*, c_2^*) is semi-connected. □

Theorem 5.3. *Let (X, c_1, c_2) be a fbcs. Let $c_i^* \subset c_i$ then (X, c_1^*, c_2^*) is also fuzzy semi-connected.*

Proof. Let (X, c_1, c_2) be a semi-connected fbcs. Then it cannot be written as union of two non-empty q-separated sets. Let $c_i^* \subset c_i$ and suppose that (X, c_1^*, c_2^*) is disconnected. Since (X, c_1^*, c_2^*) is disconnected $\exists$ two non-empty semi-open set A and B in the subspace Y_0 (supp Y) such that $Y = A \cup B$ where A and B are c_i-q-separated also. Then (X, c_1, c_2) is also disconnected. Since $\exists$ two non-empty semi-open sets A and B such that $Y_0 = A \cup B$ (supp Y) where A and B are c_i-q-separated sets which is a contradiction. Hence (X, c_1^*, c_2^*) is semi-connected. □

6 Conclusion

In this paper, we introduce and study fuzzy semi separation axioms in fuzzy biclosure space. Though this is weaker than separation axioms already introduced and studied in fuzzy setting by various researchers but it doesn't affect its importance. It is used in Boolean algebra, convex set etc in pure mathematics. Here we introduce various types of continuities using semi-open sets and compare them with each other. Here we introduce semi-separated sets and semi connectedness in fuzzy biclosure space. Further scope of this research is in the field of medical sciences using fuzzy semi-open sets.

Acknowledgement. Authors wish to express their thanks to the Editors and Reviewers for their help to bring the paper in its present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 68-76**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

META-GAME THEORETIC ANALYSIS OF SOME STANDARD GAME THEORETIC PROBLEMS

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(Received: September 30, 2022; In Format: September 07, 2022; Revised: October 07, 2022; Accepted: February 12, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53108

Abstract

Meta game theory is a non-quantitative reconstruction of mathematical game theory. This paper attempts to adapt meta-game theory for conflict analysis. A conflict is a situation where parties with opposing goals affect one another. A simple approach for performing meta-game analysis is adapted in this paper and illustrated on various games standard in game theory literature. The approach presented yields the desired results, although the computation required is much lesser than the standard Game theory analysis. Even a person without detailed knowledge about meta-game analysis or game theory can implement this method.

2020 Mathematical Sciences Classification: 91A40.
Keywords and Phrases: Stability analysis, Unilateral improvement (*UI*), Preference vectors.

1 Introduction

Game playing can be used to couple direct competition with the intellectual activity. The chances of a person's winning more games improve if a person has better thinking abilities and learning skills. The opportunity to test and refine his/her intellectual skills are provided by playing the game [19]. A scientific method that reconstructs classical Game- Theory on a non-quantitative basis is Meta Game theory. Its application to actual conflicts is called Meta-Game analysis or the analysis of options [7]. The Meta- Game theory has been applied to various problems, including the fall of France, an international water allocation conflict, and the Garrison Diversion Unit *(GDU)* irrigation project in North Dakota, U.S.A. [7], the Vietnam war and arms control, and the Arab-Israeli conflict using the method of Meta- Game analysis. The technique has also been applied to environmental management [12], and analysing political conflicts, particularly water resources problems [11]. The conflict analysis uses Meta-Game theory to make non-quantitative predictions instead of long mathematical calculations like Game-Theory. Conflict analysis can be employed to perform decision-making on problems that are hard to deal with quantitatively. Further, conflict analysis avoids the assumptions taken in Game Theory studies [25].

Various standard Game-Theoretical problems find applications in day-to-day life. The GameTheoretic analysis of many such problems is described in the following text.

For the Game of Chicken, Cooper et al. [3] establish that the Pareto optimal outcome is only sometimes selected in the games. If the opponent plays a dominated strategy, the equilibrium selection disturbs [3]. Similarly, other methods are discussed by Fox et al. [8], Carbon et al. [4], and Mehta et al. [16].

For the Prisoners' dilemma, Holler et al. [13] consider the real-world cases of a state choosing a dominant strategy. Using the method of general Meta-Games, Howard [14] showed that cooperation is an equilibrium of the full Meta-Game. Zhang et al. [26] study the effect of memory on the evolution of Prisoners' dilemma. The authors construct different kinds of two-layer networks. Miettinen et al. [18] experimentally investigate behaviour and beliefs in a sequential prisoners' dilemma. Proto et al. [20] used a repeated Prisoners' dilemma. The role of attention and memory is used to show that social interactions are likely to be mediated by cognitive skills in heterogeneous groups.

The Stag-Hunt was invented by philosopher Jean-Jacques Rousseau [32] in his discourses on inequality. Boudreau et al. [1] study a three-party game of conflict. They study the potential alliance formation introducing the concept of Stag-Hunt alliance. They further use this concept as a novel solution to the alliance formation puzzle in contests. Girtz et al. [9] establish that risk-averse individuals tend to make riskier choices less often than risk-loving individuals do in a series of economic games. Guarin et al. [10] study whether the co-worker's gender affects coordination on the mutually beneficial outcome in a socially risky environment. Marsh et al. [1] conduct the first ever study to influence human cooperation by means of positive imagery. Riedl [21] perform an experiment with human subjects. They study how free neighborhood choice affects coordination. Luo et al. [15] study divide-and-conquer strategy with the Stag-Hunt Game to model a terrorism and counter terrorism environment. The authors establish the universal characteristics of cooperative dynamics in different scenarios for the N-person Stag-Hunt Game. Dong et al. [6] propose a memory bases Stag-Hunt Game. The work is concerned with the study of evolutionary games with memory effect. Capraro et al. [5] study two sets of experimental data ($N = 523$) to investigate Stag-Hunt cooperation. The authors find that the cooperation is dependent on efficiency rather than moral Preferences. Belloc et al. [2] establish that intuition and deliberation play a significant role in strategic situations that entail social coordination.

A simple approach for performing Meta-Game analysis is adapted in this paper and illustrated on various games popular in Game-Theory literature. The approach presented yields the desired results, although the computation required is much lesser than the standard Game-Theory analysis. This method can be implemented even by a person without detailed knowledge about Meta-Game analysis or Game-Theory.

2 Methodology

The information in the problem and the payoff matrices are transformed into binary to construct Preference vectors. The Preference vectors contain the possible outcomes of the problem in descending order of the players' Preferences. The equally preferred outcomes are denoted by placing a bridge on the top. The Preference vectors can be cross-checked logically. The Preference vectors are transformed into decimal form, called decimalized Preference vectors, by multiplying the entry in the upper row by 2^0 and in the lower row by 2^1. It is checked whether a player can improve his/her position while keeping the other player's binary values fixed. This is termed as *UI*. If an outcome has no *UI*, r is written on top of the column and u otherwise. Stability analysis is conducted for individual players and amongst the players to obtain an outcome from which no player wants to deviate. The analysis is done for various problems which were earlier solved using Game-Theory analysis only. The results obtained using the (extension of) Meta-game theoretic analysis is consistent with those obtained using GameTheoretic analysis. The significant advantage of using (extension of) Meta-game theoretic analysis over Game-Theoretic analysis is less rigorous calculations.

***Table* 2.1:** The algorithm to perform stability analysis [7]

1. Model the conflict
   (a) for a particular point in time,
   (b) as a game with players and options
   (c) create a meaningful ordering of options.
2. Construct the tableau for the conflict
   (a) Order outcomes by Preferences for each player
   (b) and list UI under each outcome.
3. Perform the stability analysis
   (a) and mark as rational (r) all outcomes with no UI
   (b) for each successive outcome, determine if it is "reasonable" for a player to improve. If it is reasonable, mark the outcome as unstable (u), or if not reasonable to improve, mark it as stable (s),
   (c) if an outcome is unstable for two or more players, check for "stability by simultaneity,"
   (d) If an outcome is stable for all players, it is an equilibrium. All other outcomes are not equilibria.
4. Return to step 1) if necessary.

3 Stability Analysis

A stable outcome is one from which no player wants to deviate. The stability can be for a player or all the players. Write 'r' (rational) above every outcome that does not have a UI, as the outcomes with no UI are stable. (a) Suppose player A has a UI and player B does not have a UI from A 's UI. In that case, that outcome is unstable for A. (b) If player A has a UI and player B has a UI from A 's UI then (i) if A prefers that outcome to B's outcome, then it is unstable. (ii) It is stable if A does not prefer that outcome to B's.

4 Game of Chicken

In the Game of Chicken, two drivers must pass a single road. If they keep driving straight, they will collide, so at least one must swerve to avoid a head-on collision. If a person swerves and the other does not (i.e., goes straight), the one who swerves is called a chicken (loser). The relative payoffs in the different scenarios that can emerge in this interaction are depicted in Table 4.1.

***Table* 4.1:** The payoff matrix

		Player B	
		Swerve	Straight
Player A	Swerve	0,0	$-1,+1$
	Straight	$+1,-1$	$-1000,-1000$

Preference vector for Player A and Player B

Let Swerve correspond to 1 and straight correspond to 0 .

For player A, in Table 4.2 (a) the first Preference vector is $(1,0)$, which means player A goes straight, and player B swerves, (b) the second Preference vector is $(0,0)$, which means both player A and player B swerve (c) the third Preference vector is $(0,1)$ which means player A swerves and player B goes straight (d) the fourth Preference vector is $(1,1)$ which means both Player A goes straight and Player B goes straight.

***Table* 4.2:** The Preference vector for player A

Outcomes					
Player A	1	0	0	1	$2^0=1$
Player B	0	0	1	1	$2^1=2$
	1	0	2	3	.

In Table 4.3, for player B (a) the first Preference vector is $(0,1)$, which means player A swerves and player B goes straight, (b) the second Preference vector is $(0,0)$, which means both players A and player B swerve (c) the third Preference vector is $(1,0)$ which means player A goes straight and player B swerves (d) the fourth Preference vector is $(1,1)$ which means both players A and B go straight.

***Table* 4.3:** The Preference vector for player B

Outcomes					
Player A	0	0	1	1	$2^0=1$
Player B	1	0	0	1	$2^1=2$
	2	0	1	3	

Decimalized Preference vector for Player A and Player B

In Tables 4.5 and 4.6, decimalized Preference vectors are obtained by multiplying the first row of the Preference vectors of A and B by 2^0 and the second row by 2^1.

***Table* 4.4:** Decimalized Preference vector for player A

Player A	1	0	2	3

***Table* 4.5:** Decimalized Preference vector for player B

Player B	2	0	1	3

Stability Analysis for Player A and Player B
The outcomes of player B are kept fixed in Table 4.3 and then checked for UI in the outcomes for player A. Player A has UI from column two to column one and column four to three (Table 4.6).

***Table* 4.6:** UI for player A

Player A	1	0	2	3
		1		2

The outcomes of player A are kept fixed in Table 4.3. The authors check for UI in the outcomes for player B. Player B has UI from column two to column one and four to three (Table 4.7).

***Table* 4.7:** UI for player B

Player B	2	0	1	3
		2		1

Player A, (Table 4.8) has UI from column two to column one, and player B (Table 4.9) has no UI from column three, which means player A can improve from column two to column one. Thus, this is an unstable outcome. Similarly, player A, (Table 4.8) has UI from column four to column three, and player B (Table 4.9) has no UI from column one, which means player A can improve from column four to column three. Thus, this too is an unstable outcome.

***Table* 4.8:** Stability analysis for player

	E		E	
	r	u	r	u
Player A	1	0	2	3
		1		2

For player B, from column two to column one (Table 4.9) and player A (Table 4.8) has no UI from column three, which means player B can improve from column two to column one. Thus, this is an unstable outcome. Player B, (Table 4.9) has UI from column four to column three, and player A (Table 4.8) has no UI from column one, which means player B can improve from column four to column three. Thus, this too is an unstable outcome.

***Table* 4.9:** Stability analysis vector for player B

	E		E	
	r	u	r	u
Player B	2	0	1	3
		2		1

Player A has a UI from column two to column one, and player B has no UI from column three. Hence, column two is unstable for player A. Player B has UI from column two to column one, and player A has no UI from column three. Hence, column two is unstable for B (Table 4.10).

Solutions
There are two stable equilibrium solutions Straight-Straight and Swerve-Swerve. When analyzed using Game-Theoretic techniques, the Nash equilibria obtained for the game are identical [23].

5 Prisoners' dilemma

There are two accused of a crime, and they are not allowed to communicate. The options left to the two accused i.e., A and B are (i) If both defect, a two-year prison sentence is awarded to both (ii) If A defects and B cooperates, A is released, and B gets a three-year prison sentence (iii) If A cooperates and B defects,

A will get three years in prison and B will be released (iv) If A and B both cooperate, they both get one-year prison sentence (Table 5.1).

***Table* 5.1:** The payoff matrix

		Player B	
		Cooperate	Defect
Player A	Cooperate	$-1, -1$	$-3, 0$
	Defect	$0, -3$	$-2, -2$

Preference vector for Player A and Player B

Let Cooperate correspond to 1 and Defect correspond to 0 .

In Table 5.2, for player A (a) the first Preference vector is $(0, 1)$, which means player A defects and player B cooperates, (b) the second Preference vector is $(1, 1)$, which means player A cooperates and player B cooperates (c) the third Preference vector is $(0, 0)$ which means player A defects and player B defects (d) the fourth Preference vector is $(1, 0)$ which means player A cooperates and player B defects.

***Table* 5.2:** The Preference vector for player A

Outcomes					
Player A	0	1	0	1	$2^0 = 1$
Player B	1	1	0	0	$2^1 = 2$
	2	3	0	1	

In Table 5.3, for player A (a) the first Preference vector is $(1, 0)$, which means player A cooperates and player B defects, (b) the second Preference vector is $(1, 1)$, which means player A cooperates and player B cooperates (c) the third Preference vector is $(0, 0)$ which means player A defects and player B defects (d) the fourth Preference vector is $(0, 1)$ which means player A defects and player B cooperates.

***Table* 5.3:** The Preference vector for player B

Outcomes					
Player A	1	1	0	0	$2^0 = 1$
Player B	0	1	0	1	$2^1 = 2$
	1	3	0	2	

Decimalized Preference vector for Player *A* and Player B

In Tables 5.4 and 5.5, the authors obtain decimalized Preference vectors by multiplying the first row by 2^0 and the second row by 2^1.

***Table* 5.4:** Decimalized Preference vector for player A

Player A	2	3	0	1

***Table* 5.5:** Decimalized Preference vector for player B

Player B	1	3	0	2

Stability Analysis for Player A and Player B

The outcomes of player B are kept fixed in Table 5.2, then the authors check for UI in the outcomes for player A. Player A has UI from two to one and four to three (Table 5.6).

***Table* 5.6:** UI for player A

Player A	2	3	0	1
		2		0

The outcomes of player A are kept fixed in Table 5.3. The authors check for UI in the outcomes for player B. Player B has UI from column two to column one and column four to column three (Table 5.7).

***Table* 5.7:** UI for player B

Player B	1	3	0	2
		1		0

In Table 5.8, Player A has UI from column two to one, and player B has UI from column four to three as column two is more preferred over column three. Therefore, it is a stable outcome. Player A has UI from column four to three, and player B has no UI from column three, so this is an unstable outcome.

***Table* 5.8:** Stability analysis for player A

	E		E	
	r	s	r	u
Player A	2	3	0	1
		2		0

Player B has UI from column two to column one, player A has UI from column four to column three, and player B prefers column two to column three. Therefore, it is a stable outcome. Player B has UI from column four to three, and player A has no UI from column three. Thus, it is an unstable outcome (Table 5.9).

***Table* 5.9:** Stability analysis for player B

	E		E	
	r	s	r	u
Player B	1	3	0	2
		1		0

Solution

The solution to the Prisoners' dilemma is (i) column one and one, which means both players defect, (ii) column three and three, which also means both players defect, (iii) column two and two, which means both the players cooperate.

When analyzed using Game-Theoretic techniques, the same Nash equilibria are obtained [24].

6 Stag-Hunt Game

Imagine two hunters, they can hunt a stag or a hare. They can independently hunt a hare. For hunting a stag, they need each other's help. The (Stag, Stag) is the pareto optimal outcome. But in experimental games, people choose (Hare, Hare).

***Table* 6.1:** The payoff matrix

Player A	Player B		
		Stag	Hare
	Stag	10,10	1,8
	Hare	8,1	5,5

Preference Vector for Player A and Player B

Let Stag correspond to 1 and Hare correspond to 0.

In Table 6.2, for player A (a) the first Preference vector is $(1,1)$ which means both player A and player B want to hunt a stag, (b) the second Preference vector is $(0,1)$ which means player A wants to hunt a Hare and player B wants to hunt a Stag (c) the third Preference vector is $(0,0)$ which means both player A and player B want to hunt a hare (d) the fourth Preference vector is $(1,0)$ which means player A wants to hunt a stag and player B wants to hunt a hare.

***Table* 6.2:** The Preference vector for player A

Outcomes					
Player A	1	0	0	1	$2^0 = 1$
Player B	1	1	0	0	$2^1 = 2$
	3	2	0	1	

In Table 6.3, for player A (a) the first Preference vector is $(1,1)$ which means player A and player B wants to hunt a stag, (b) the second Preference vector is $(1,0)$ which means player A wants to hunt a stag and player B wants to hunt a hare (c) the third Preference vector is $(0,0)$ which means player A and player B want to hunt a hare (d) the fourth Preference vector is $(0,1)$ which means player A wants to hunt a hare and player B wants to hunt a stag.

***Table* 6.3:** The Preference vector for player B

Outcomes					
Player A	1	1	0	0	$2^0 = 1$
Player B	1	0	0	1	$2^1 = 2$
	3	1	0	2	

Decimalized Preference vector for Player *A* and Player B

In Table 6.4 and 6.5, the authors obtain decimalized Preference vectors by multiplying first row by 2^0 and second row by 2^1.

***Table* 6.4:** Decimalized Preference vector for player A

Player A	3	2	0	1

***Table* 6.5:** Decimalized Preference vector for player B

Player B	3	1	0	2

Stability Analysis for Player A and Player B

The outcomes of player B are kept fixed in Table 6.2, then the authors check for UI in the outcomes for player A. Player A has UI from column two to column one and from column four to column three.

***Table* 6.6:** UI for player A

Player A	3	2	0	1
		3		0

The outcomes of player A are kept fixed in Table 6.3, then the authors check for UI in the outcomes for player B. Player B has UI from column two to column one and from four to three.

***Table* 6.7:** UI for player B

Player B	3	1	0	2
		3		0

The player A has UI from column two to column one and player B has no UI from column one, so it is an unstable outcome. The player A has UI from column four to column three and player B has no UI from column three, so it is an unstable outcome.

***Table* 6.8:** Stability Analysis for player A

	E		E	
		u		u
Player A	3	2	0	1
		3		0

The player B has UI from column two to column one to 3 and player A has no UI from column one, so it is an unstable outcome. The player B has UI from column four to column three and player A has no UI from column three, so it is an unstable outcome.

***Table* 6.9:** Stability Analysis for player B

	E		E	
		u		u
Player B	3	1	0	2
		3		0

Solutions

There are two stable equilibrium values, Stag- Stag and Hare- Hare.

When analyzed using Game-Theoretic techniques, the game yields exactly same values as the Nash equilibrium values [22].

7 Conclusion

The authors attempted to do a novel analysis of standard Game-Theoretic problems using the MetaGame analysis techniques. The Meta-Game analysis has many advantages. It includes all the information about the conflict, is easy to do by hand, and can be used for hyper games and very complex conflicts. The results obtained show the stability analysis for various standard Game Theoretic problems. These solutions obtained in the three cases viz., Game of Chicken, Prisoners' dilemma, and Stag-Hunt Game are consistent with the Nash equilibrium values obtained from the Game-Theoretic analysis [22-24]. Thus, Meta-game theory can replace Game-Theory in these situations as it includes no tedious mathematical calculations.

Acknowledgement. The authors are grateful to Professor D. S. Mishra and Professor S. K. Gaur, DEI for their valuable inputs. We are also very much thankful to the Editors and Reviewers for valuable suggestions to bring the paper in its present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 77-86**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

DETECTION OF RELATIVELY PRIME INTEGER SOLUTIONS FOR TWO DISPARATE FORMS OF MORDELL CURVES

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(Received: December 02, 2021*; In format: March* 08, 2022*; Revised : March* 26, 2022*; Accepted April* 04, 2023*)*

DOI: https://doi.org/10.58250/jnanabha.2023.53109

Abstract

In this paper, two unrelated natures of Mordell curves $y^3 = x^2 + k$ where k is a multinomial of degree four and six are scrutinized for many sets of relatively prime integer solutions. The geometrical description of the curves for each set of solutions are also exhibited with an assistance of *MATLAB* tools.

2020 Mathematical Sciences Classification: Primary 11; Secondary D25.
Keywords and Phrases: Mordell equation, relatively prime, integer solutions.

1 Introduction

A Diophantine problem is one in which the solutions are mandatory to be integers. If a Diophantine equation has a supplementary variable or variables occurring as exponents, it is known as an exponential Diophantine equation [3, 4, 5, 6]. In [7], the author considered the Diophantine equation $Y^3 = X^2 + C$ and found out the numerical solutions for $C = 9, 36, -16$. In [1, 2, 8] authors discovered consecutive integer solutions to the Diophantine equation $y^3 = x^2 + k$.

In this communication, two dissimilar forms of Mordell type curves $y^3 = x^2 + k$ where k is a polynomial of degree four and six are examined for various sets of relatively prime integer solutions. The geometrical representation of the curves for each set of solutions are also displayed with the help of *MATLAB* tools.

2 Evaluation of relatively prime integer solutions to Mordell type equations

It is well recognized that the Mordell equation is

$$a^2 = b^3 + r, \tag{2.1}$$

where r is a constant. If $S'(r)$ is the number of relatively prime integral solutions $(a, b) \in Z(u)$ of (2.1) where Z is the ring of integer, then $\limsup_{r\to\infty} S'(r) \geq 1$. Here the solutions to (2.1) are denoted by $M_j = (a_j, b_j)$, $j = 1, 2, 3$ etc.

In this paper two different kinds of Mordell type equations are considered in (2.1) and (2.2) for sleuthing relatively prime integer solutions.

2.1 Equation of the form $a^2 = b^3 + r$ where $r = 64u^4 + 64u^3 + 16u^2 + 1$

Consider the equation of type (2.1) as

$$a_j^2 = b_j^3 + 64u^4 + 64u^3 + 16u^2 + 1,\ u \in N. \tag{2.2}$$

The probable three sets of relatively prime integer solutions to (2.2) which are identified by

$$M_1 : a_1 = 8u^2 + 1,\ b_1 = -4u,$$
$$M_2 : a_2 = 8u^2 + 4u,\ b_2 = -1,$$
$$M_3 : a_3 = 8u^2 + 8u + 3,\ b_3 = 4u + 2.$$

Since (2.2) may have more than three solutions, $\limsup_{r\to\infty} S'(r) \geq 3$.

The following *MATLAB* program supports to treasure the numerical values for all variables.

```
clear all;close all;clc;
disp('the Equation is a^2 = b^3+(64*u^4)+(64*u^3)+(16*u^2)+1)');
u = -50:50;
k = (64*u.^4)+(64*u.^3)+(16*u.^2)+1;
a1 = 8*u.^2+1;b1 = -4*u;
a2 =(8*u.^2)+4*u;
[r,c]=size(a2);
b2 = -1*ones(r,c);a3 = (8*u.^2)+(8*u)+3;
b3 = (4*u)+2;
fprintf('The first solution \n');fprintf('a1 = %d\n',a1);
fprintf('b1 = %d\n',b1);fprintf('The second  solution \n');
fprintf('a2 = %d\n',a2);fprintf('b2 = %d\n',b2);
fprintf('The third solution\n');fprintf('a3 = %d\n',a3);
fprintf('b3 = %d\n',b3);
z1=[b1;a1;k];
surf(z1)
colormap(cool)
title('8*u^2+1,-4*u,(64*u^4)+(64*u^3)+(16*u^2)+1)')
figure
z2=[b2;a2;k];
surf(z2)
colormap(cool)
title('(8*u^2)+4*u,-1,(64*u^4)+(64*u^3)+(16*u^2)+1)')
figure
z3=[b3;a3;k];
surf(z3)
colormap(cool)
title('(8*u^2)+(8*u)+3,(4*u)+2,(64*u^4)+(64*u^3)+(16*u^2)+1)')
```

For easy verification the arithmetic values of $(a_j b_j)$ for few natural numbers u are tabulated in Table 2.1.

***Table* 2.1**

u	r	$M_1 = (a_1, b_1)$	$M_2 = (a_2, b_2)$	$M_3 = a_3, b_3)$
1	145	$(9, -4)$	$(12, -1)$	$(19, 6)$
2	1601	$(33. - 8)$	$(40, -1)$	$(51, 10)$
3	7057	$(73, -12)$	$(84, -1)$	$(99, 14)$
4	20737	$(129, -16)$	$(144, -1)$	$(163, 18)$
5	48401	$(201, -20)$	$(220, -1)$	$(243, 22)$

The subsequent table (Table 2.2) displays the left-hand and right-hand side values of (2.2) for all the above three sets of solutions.

***Table* 2.2**

u	M_1		M_2		M_3	
	a_1^2	$b_1^3 + r$	a_2^2	$b_2^3 + r$	a_3^2	$b_3^3 + r$
1	81	81	144	144	361	361
2	1089	1089	1600	1600	2601	2601
3	5329	5329	7056	7056	9801	9801
4	16641	16641	20736	20736	26569	26569
5	40401	40401	48400	48400	59049	59049

For all other values of u the values of (a_j, b_j) can be calculated by using the above *MATLAB* algorithm.

The geometrical representation of (2.2) for the above three sets of solutions are visualized in Figures 2.1, 2.2 and 2.3.

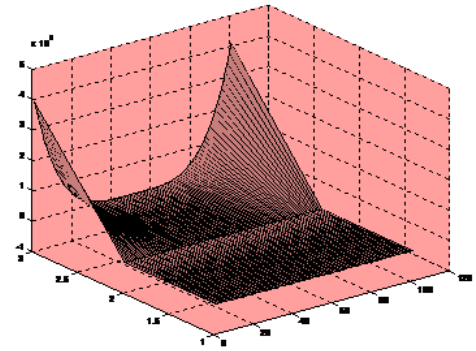

***Figure* 2.1:** Visualization of (2.2) for the solution M_1

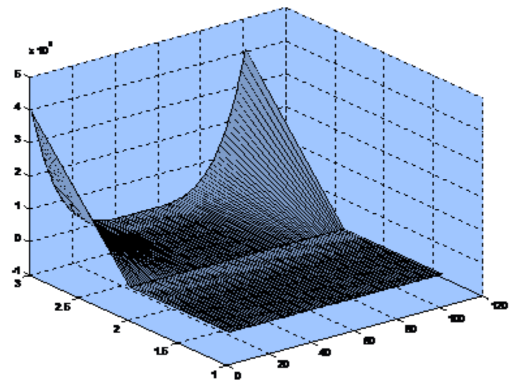

***Figure* 2.2:** Visualization of (2.2) for the solution M_2

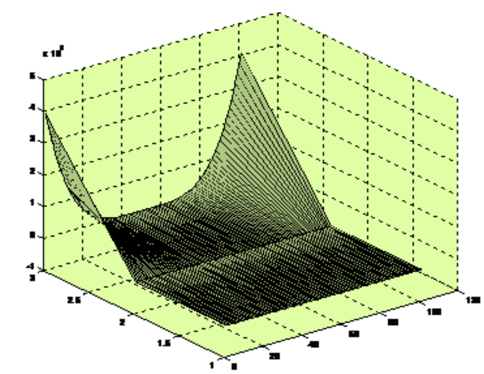

***Figure* 2.3:** Visualization of (2.2) for the solution M_3

2.2 Equation of the form $a^2 = b^3 + r$ where $r = \left(16(n+1)^6u^6 + 1\right)$

Consider an additional Mordell kind equation as

$$a_j^2 = b_j^3 + \left(16(n+1)^6u^6 + 1\right)u, \; n \in N. \tag{2.3}$$

It is experiential that (2.3) is fulfilled by six pair of values of (a_j, b_j). They are denoted by $\pm M_j, \; j = 1, 2, 3$ where $M_j = (a_j, b_j)$ and $-M_j = (a_j - b_j)$.

Among six set of solutions, the first three set of values of $(a_j, b_j), j = 1, 2, 3$ are pointed out by the following equations:

$$M_1 : a_1 = 4(n+1)^3u^3 + 1, \; b_1 = 2(n+1)u.$$

$$M_2 : a_2 = 4(n+1)^3u^3, \; b_2 = -1.$$

$$M_3 : a_3 = 4(n+1)^3 u^3 - 1,\ b_3 = -2(n+1)u.$$

Let us find the remaining three pairs (a_j, b_j), $j = 4, 5, 6$. If (a_1, b_1) and (a_2, b_2) are two distinct points on E, their sum (a', b') is given by

$$a' = \left(\frac{b_2 - b_1}{a_2 - a_1}\right)^2 - a_1 - a_2,$$
$$b' = \left(\frac{b_2 - b_1}{a_2 - a_1}\right)(a' - a_1) + b_1.$$

Since $-M_j = (a_j, -b_j)$ the co-ordinates of a_i, b_j for $M_j \pm M_k, 1 \le j < k \le 3$ such that only three out of these six points sustaining (2.3) are scrutinized that

$$\begin{aligned}
&M_4 = M_1 - M_2 : \\
&a_4 = -(64(n+1)^6u^6 - 96(n+1)^5u^5 + 96(n+1)^4u^4 \\
&\qquad - 68(n+1)^3u^3 + 36(n+1)^2u^2 - 12(n+1)u + 3), \\
&b_4 = 16(n+1)^4u^4 - 16(n+1)^3u^3 + 12(n+1)^2u^2 - 18(n+1)u + 2, \\
&M_5 = M_2 - M_3 : \\
&a_5 = -(64(n+1)^6u^6 + 96\,(n+1)^5\,u^5 + 96(n+1)^4u^4 + 68(n+1)^3u^3 \\
&\qquad + 36(n+1)^2u^2 + 12(n+1)u + 3), \\
&b_5 = 16(n+1)^4u^4 + 16(n+1)^3u^3 + 12(n+1)^2u^2 + 18(n+1)u + 2, \\
&M_6 = M_1 - M_3 : \\
&a_6 = -\left(8(n+1)^6u^6 + 1\right),\ b_6 = 4(n+1)^6u^6.
\end{aligned}$$

To check all the above coordinates (a_j, b_j) are coprime for all integer u, Euclid's algorithm may be applied. Example for $M_1 - M_2$,

$$\begin{aligned}
&(16(n+1)^4u^4 - 16(n+1)^3u^3 + 12(n+1)^2u^2 - 18(n+1)u + 2, \\
&\quad \Big(- 64(n+1)^6u^6 + 96(n+1)^5u^5 - 96(n+1)^4u^4 \\
&\quad + 68(n+1)^3u^3 - 36(n+1)^2u^2 + 12(n+1)u - 3\Big) \\
&= (8(n+1)^3u^3 + 8(n+1)^2u^2 + 2(n+1)u + 1, \\
&\qquad 6(n+1)^4u^4 + 16(n+1)^3u^3 + 12(n+1)^2u^2 + 18(n+1)u + 2) \\
&= (8(n+1)^2u^2 + 4(n+1) + 2, 4(n+1) + 1) \\
&= (4(n+1), 1) \\
&= 1.
\end{aligned}$$

Thus for $M_1 - M_2$, $\gcd(a_4, b_4) = 1$ and subsequently the pair $(a_4 b_4)$ is relatively prime to each other.

Similarly, it is evidenced that all other pairs (a_j, b_j) for the remaining five sets of solutions are relatively prime by utilizing Euclid's algorithm. Since, it can be able to find more than six groups of solutions in co-prime integers $\limsup_{r\to\infty} S'(r) \ge 6$.

3 *MATLAB* program for finding (a_j, b_j)

MATLAB program for finding (a_j, b_j) for distinct values of u are illustrated below:

```
clear all;close all;clc;
disp('the Equation is a^2 = b^3+(16(n+1)^6*u^6+1)');
n = 1;u = -50:50;k =(16*(n+1)^6*u.^6+1)
disp('the following 6 solution');
a1 = 4*(n+1)^3*u.^3+1;b1 = 2*(n+1)*u;
a2 = 4*(n+1)^3*u.^3;[r,c]=size(a2);
b2 = -1*ones(r,c);a3 = 4*(n+1)^3*u.^3-1;b3 = -2*(n+1)*u;
a4 = (-64*(n+1)^6*u.^6 )-(96*(n+1)^5*u.^5 )+(96*(n+1)^4*u.^4 )
-(68*(n+1)^3*u.^3)+(36*(n+1)^2*u.^2)-(12*(n+1)*u)+3;
```

```
b4 = (16*(n+1)^4*u.^4)-(16*(n+1)^3*u.^3)+(12*(n+1)^2*u.^2)
-(6*(n+1)*u)+2;
a5 = -((64*(n+1)^6*u.^6)+(96*(n+1)^5*u.^5)+(96*(n+1)^4*u.^4)
+(68*(n+1)^3*u.^3)+(36*(n+1)^2*u.^2)+(12*(n+1)*u)+3);
b5 = (16*(n+1)^4*u.^4)+(16*(n+1)^3*u.^3)+(12*(n+1)^2*u.^2)
+(6*(n+1)*u)+2;
a6 = -(8*(n+1)^6*u.^6+1);b6 = 4*(n+1)^4*u.^4;
fprintf('the first solution \n');fprintf('a1 = %d\n^',a1);
fprintf('b1 = %d\n',b1);fprintf('the second  solution \n');
fprintf('a2 = %d\n^',a2);fprintf('b2 = %d\n',b2);
fprintf('the third solution\n');fprintf('a3 = %d\n^',a3);
fprintf('b3 = %d\n',b3);fprintf('the fourth solution \n');
fprintf('a4= %d\n',a4);fprintf('b4 = %d\n',b4);
fprintf('the fifth solution \n');fprintf('a5 = %d\n^',a5);
fprintf('b5 = %d\n',b5);fprintf('the Sixth solution \n');
fprintf('a6 = %d\n^',a6);fprintf('b6 = %d\n',b6);
z1=[b1;a1;k];surf(z1)
colormap(cool)
title('4*(n+1)^3*u^3+1,2*(n+1)*u,(16(n+1)^6*u^6+1)')
figure
z2=[b2;a2;k];surf(z2)
colormap(cool)
title('4*(n+1)^3*u^3,-1,(16(n+1)^6*u^6+1)')
figure
z3=[b3;a3;k];
surf(z3)
colormap(cool)
title('4*(n+1)^3*u^3+1,-2*(n+1)*u,(16(n+1)^6*u^6+1)')
figure
z4=[b4;a4;k];surf(z4)
colormap(cool)
title('(-64*(n+1)^6*u^6 )-(96*(n+1)^5*u^5 )+(96*(n+1)^4*u^4 )
-(68*(n+1)^3*u^3)+(36*(n+1)^2*u^2)-(12*(n+1)*u)+3,
(16*(n+1)^4*u.^4)-(16*(n+1)^3*u.^3)+(12*(n+1)^2*u^2)
-(6*(n+1)*u)+2,(16(n+1)^6*u^6+1)')
figure
z5=[b5;a5;k];
surf(z5)
colormap(cool)
title('-(64*(n+1)^6*u^6)+(96*(n+1)^5*u^5)+(96*(n+1)^4*u^4)
+(68*(n+1)^3*u^3)+(36*(n+1)^2*u^2)+(12*(n+1)*u)+3),
(16*(n+1)^4*u^4)+(16*(n+1)^3*u^3)+(12*(n+1)^2*u^2)
+(6*(n+1)*u)+2,(16(n+1)^6*u^6+1)')
figure
z6=[b6;a6;k];
surf(z6)
colormap(cool)
title('-(8*(n+1)^6*u^6+1),4*(n+1)^4*u^4,(16(n+1)^6*u^6+1)')
```

Illustration: A

If $n = 1$, then the corresponding representation of (2.3) is

$$a_j^2 = b_j^3 + \left(1024u^6 + 1\right). \tag{3.1}$$

The six pair of solutions to (2.3) are viewed by

$$M_1 : a_1 = 32u^3 + 1,\ b_1 = 4u.$$

$M_2 : a_2 = 32u^3,\ b_2 = -1.$

$M_3 : a_3 = 32u^3 - 1,\ b_3 = -4u.$

$M_4 = M_1 - M_2 : a_4 = -\left(4096u^6 - 3072u^5 + 1536u^4 - 544u^3 + 144u^2 - 24u + 3\right).$

$b_4 = 256u^4 - 128u^3 + 48u^2 - 12u + 2.$

$M_5 = M_2 - M_3 : a_5 = -\left(4096u^6 + 3072u^5 + 1536u^4 + 544u^3 + 144u^2 + 24u + 3\right).$

$b_5 = 256u^4 + 128u^3 + 48u^2 + 12u + 2.$

$M_6 = M_1 - M_3 : a_6 = -\left(512u^6 + 1\right),\ b_6 = 64u^4.$

Table 3.1 shows the six values of (a_j, b_j) equivalent to $n = 1$.

***Table* 3.1**

u	r	M_1	M_2	M_3
1	1025	(33,4)	$(32, -1)$	$(31, -4)$
2	65537	(257,8)	$(256, -1)$	$(255, -8)$
3	746497	(865,12)	$(864, -1)$	$(863, -12)$
4	4194305	(2049,16)	$(2048, -1)$	$(2047, -16)$
5	16000001	(4001,20)	$(4000, -1)$	$(3999, -20)$
		M_4	M_5	M_6
		$(-2139, 166)$	$(-9419, 446)$	$(-513, 64)$
		$(-184595, 3242)$	$(-390003, 5338)$	$(-32769, 1024)$
		$(-2350443, 17678)$	$(-3872955, 24662)$	$(-373249, 5184)$
		$(-13992099, 58066)$	$(-20353379, 74546)$	$(-2097153, 16384)$
		$(-55295483, 145142)$	$(-74631723, 177262)$	$(-8000001, 40000)$

Table 3.2 displays the left-hand and right-hand side values of (2.3) for all the above six sets of solutions equivalent to $n = 1$.

***Table* 3.2**

u	M_1		M_2		M_3	
	a_1^2	$b_1^3 + r$	a_2^2	$b_2^3 + r$	a_3^2	$b_3^3 + r$
1	1089	1089	1024	1024	961	961
2	66049	66049	65536	65536	65025	65025
3	748225	748225	746496	746496	744769	744769
4	4198401	4198401	4190209	4190209	4190209	4190209
	M_4		M_5		M_6	
	a_4^2	$b_4^3 + r$	a_5^2	$b_5^3 + r$	a_6^2	$b_6^3 + r$
	4575321	4575321	88717561	88717561	263169	263169
	3.407×10^{10}	3.407×10^{10}	1.521×10^{11}	1.521×10^{11}	107380736	107380736
	5.524×10^{12}	5.524×10^{12}	1.499×10^{13}	1.499×10^{13}	1.393×10^{11}	1.393×10^{11}
	1.957×10^{14}	1.957×10^{14}	4.142×10^{14}	4.142×10^{14}	4.398×10^{12}	4.398×10^{12}

Illustration: B

If $n = 2$, then the required equation is

$$a_j^2 = b_j^3 + \left(11664u^6 + 1\right). \tag{3.2}$$

The equivalent solutions to (3.2) are given by

$M_1 : a_1 = 108u^3 + 1,\ b_1 = 6u.$

$M_2 : a_2 = 108u^3,\ b_2 = -1.$

$M_3 : a_3 = 108u^3 - 1,\ b_3 = -6u.$

$M_4 = M_1 - M_2 :$

$$a_4 = -\left(46656u^6 - 23328u^5 + 7776u^4 - 1836u^3 + 324u^2 - 36u + 3\right).$$
$$b_4 = 1296u^4 - 432u^3 + 108u^2 - 18u + 2.$$
$$M_5 = M_2 - M_3 :$$
$$a_5 = -\left(46656u^6 + 23328u^5 + 7776u^4 + 1836u^3 + 324u^2 + 36u + 3\right).$$
$$b_5 = 1296u^4 + 432u^3 + 108u^2 + 18u + 2.$$
$$M_6 = M_1 - M_3 : a_6 = -\left(5832u^6 + 1\right).$$
$$b_6 = 324u^4.$$

Table 3.3 shows the six values of (a_j, b_j) equivalent $n = 2$.

***Table* 3.3**

u	r	M_1	M_2	M_3
1	11665	(109,6)	$(108, -1)$	$(107, -6)$
2	746497	(865,12)	$(864, -1)$	$(863, -12)$
3	8503057	(2917,18)	$(2916, -1)$	$(2915, -18)$
4	47775745	(6913,24)	$(6912, -1)$	$(6911, -24)$
		M_4	M_5	M_6
		$(-29559, 956)$	$(-79959, 1856)$	$(-5833, 324)$
		$(-2350443, 17678)$	$(-3872955, 24662)$	$(-373249, 5184)$
		$(-28926615, 94232)$	$(-40363383, 117668)$	$(-4251529, 26244)$
		$(-169093299, 365786)$	$(-217104339, 361226)$	$(-23887873, 82944)$

Table 3.4 displays the left-hand and right-hand side values of (2.3) for all the above six sets of solutions equivalent to $n = 2$.

***Table* 3.4**

u	M_1		M_2		M_3	
	a_1^2	$b_1^3 + r$	a_2^2	$b_2^3 + r$	a_3^2	$b_3^3 + r$
1	11881	11881	11664	11664	11449	11449
2	748225	748225	746496	746496	744769	744769
3	8508889	8508889	8503056	8503056	8497225	8497225
4	4.8×10^4	4.8×10^4	4.8×10^4	4.8×10^4	1.82×10^8	1.82×10^8
	M_4		M_5		M_6	
	a_4^2	$b_4^3 + r$	a_5^2	$b_5^3 + r$	a_6^2	$b_6^3 + r$
	873734481	873734481	6393441681	6393441681	34023889	34023889
	5.524×10^{12}	5.524×10^{12}	1.499×10^{13}	1.499×10^{13}	1.393×10^{11}	1.393×10^{11}
	8.367×10^{14}	8.367×10^{14}	1.629×10^{15}	1.629×10^{15}	1.807×10^{13}	1.807×10^{13}
	2.859×10^{16}	2.859×10^{16}	4.713×10^{16}	4.713×10^{16}	5.706×10^{14}	5.706×10^{14}

The three-dimensional shape of the original equation (2.3) for each of the above six set of solutions are envisaged below.

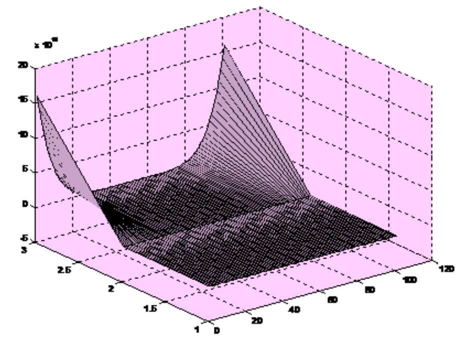

***Figure* 3.1:** Visualization of (2.3) for the solution M_1

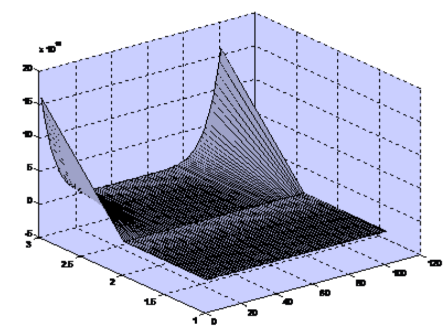

***Figure* 3.2:** Visualization of (2.3) for the solution M_2

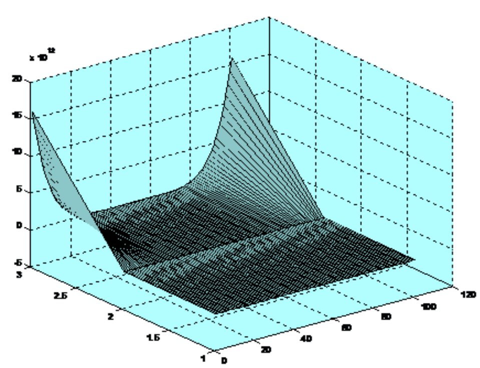

***Figure* 3.3:** Visualization of (2.3) for the solution M_3

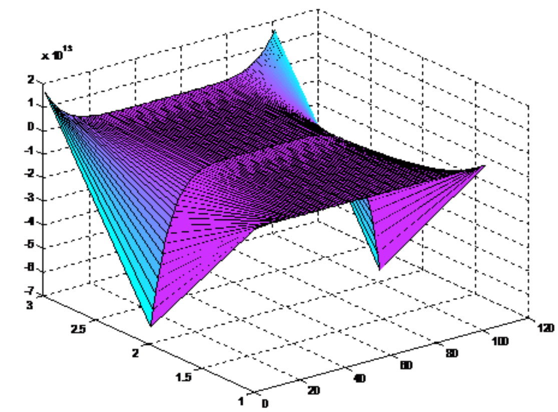

Figure **3.4:** Visualization of (2.3) for the solution M_4

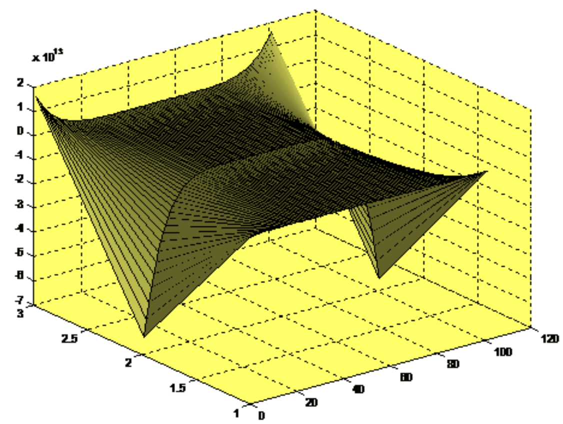

Figure **3.5:** Visualization of (2.3) for the solution M_5

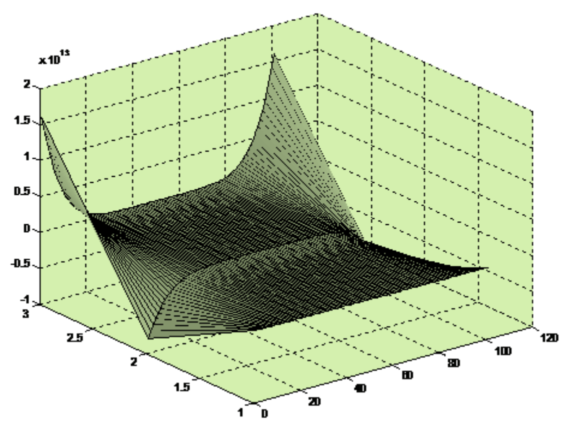

Figure **3.6:** Visualization of (2.3) for the solution M_6

4 Conclusion

Two discrete Mordell kinds equations $y^3 = x^2 + k$ where k is a polynomial of degree four and six are scrutinized for finite sets of relatively prime integer solutions. The pictures of the curves for each pair of solutions are also unveiled with the support of *MATLAB* program. In this manner, one can search varieties of such types of equations for consecutive odd and even integers.

Acknowledgment

We are very much thankful to the Editor and Referee for their valuable suggestions.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 87-91**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

HYPERGEOMETRIC FORM OF $(1+x^2)^{\frac{ib}{2}}\exp(b\tan^{-1}x)$ AND ITS APPLICATIONS

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(Received: March 05, 2022; Informat: May 10, 2022; Revised: January 30, 2023; Accepted January 31, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53110

Abstract

In this article, we obtain hypergeometric forms (not available in the literature) of some composite functions like:

$$(1-y^2)^{\frac{d}{2}}\exp(d\tanh^{-1}y),\quad (1+x^2)^{\frac{g}{2}}\cos(g\tan^{-1}x),\quad (1+x^2)^{\frac{g}{2}}\sin(g\tan^{-1}x),$$

$$(1+x^2)^{\frac{ik}{2}}\cosh(k\tan^{-1}x),\quad (1+x^2)^{\frac{ik}{2}}\sinh(k\tan^{-1}x),\quad (1-y^2)^{\frac{g}{2}}\cosh(g\tanh^{-1}y),$$

$$(1-y^2)^{\frac{g}{2}}\sinh(g\tanh^{-1}y),\quad (1-y^2)^{\frac{ik}{2}}\cos(k\tanh^{-1}y),\quad (1-y^2)^{\frac{ik}{2}}\sin(k\tanh^{-1}y),$$

by using Leibniz theorem for successive differentiation and Maclaurin's series expansion. Some applications are also discussed.
2020 Mathematical Sciences Classification: 33C05, 34A35, 41A58, 33B10.
Keywords and Phrases: Hypergeometric function; Maclaurin series; Leibniz theorem.

1 Introduction and Preliminaries

In this paper, we shall use the following standard notations:

$\mathbb{N} := \{1,2,3,\cdots\}$; $\mathbb{N}_0 := \mathbb{N}\bigcup\{0\}$; and $\mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0,-1,-2,-3,\cdots\}$.

The symbols $\mathbb{C},\mathbb{R},\mathbb{N},\mathbb{Z},\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

The Pochhammer symbol $(\alpha)_p(\alpha,p\in\mathbb{C})$ is defined by ([15, p.22 Eq.(1), p.32, Q.N.(8) and Q.N.(9)],see also [17, p.23, Eq.(22) and Eq.(23)]).

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters [17, p.42, Eq.(1)].

Relations between hyperbolic and trigonometric functions are:

$$\cos(i\theta)=\cosh(\theta),\quad \sin(i\theta)=i\sinh(\theta), \tag{1.1}$$

$$\tan^{-1}(ix)=i\tanh^{-1}(x). \tag{1.2}$$

The Maclaurin's series is a particular case of Taylor's series expansion of a function about the origin, the Maclaurin series is given as:

$$\begin{aligned} y(x) &= (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + \cdots \\ &= \sum_{n=0}^{\infty}\frac{x^n}{n!}(y_n)_0 \qquad (1.3)\\ &= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}(y_{2n})_0 + \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}(y_{2n+1})_0, \qquad (1.4)\end{aligned}$$

where, $(y_m)_0 = \left(\frac{d^my}{dx^m}\right)_{x=0}$.

The general Leibniz rule, named after Gottfried Wilhelm Leibniz, generalizes the product rule (which is also known as "Leibniz's rule"), which states that if $U(x)$ and $T(x)$ are n-times differentiable functions, then the product $U(x).T(x)$ is also n-times differentiable and its nth derivative is given by:

$$D^n[U(x)\ T(x)] = ({}^nC_0)(D^n\ U)(D^0\ T) + ({}^nC_1)(D^{n-1}\ U)(D^1\ T) + ({}^nC_2)(D^{n-2}\ U)(D^2\ T) + \cdots +$$
$$+({}^nC_{n-1})(D^1\ U)(D^{n-1}\ T) + ({}^nC_n)(D^0\ U)(D^n\ T), \tag{1.5}$$
$$= \sum_{r=0}^{n} {}^nC_r(D^rT)(D^{n-r}U), \tag{1.6}$$
$$= \sum_{r=0}^{n} {}^nC_r(D^{n-r}T)(D^rU), \tag{1.7}$$

where $D = \frac{d}{dx}$.

Euler's formula is

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta). \tag{1.8}$$

The present article is organized as follows. In section 3 we have given the proof of presented composite function. In section 4 we have discussed some applications using the relations between inverse trigonometric and inverse hyperbolic functions. The proof of the presented function is not available in the literature[1, 2, 3, 4, 6, 7, 9, 10, 5, 8] see also [11, 13, 12, 14, 16]. So we are interested to give the proof of hypergeometric form using Maclaurin series.

2 Hypergeometric Form of Composite Function

When the values of numerator, denominator parameters and arguments leading to the results which do not make sense are tacitly excluded, then the following hypergeometric form holds true:

$$(1+x^2)^{\frac{ib}{2}} \exp(b\tan^{-1}x) = {}_2F_1\left[\begin{array}{ccc} -\frac{ib}{2}, & \frac{-ib+1}{2}; & \\ & & -x^2 \\ & \frac{1}{2}; & \end{array}\right] + bx\,{}_2F_1\left[\begin{array}{ccc} \frac{1-ib}{2}, & \frac{2-ib}{2}; & \\ & & -x^2 \\ & \frac{3}{2}; & \end{array}\right]. \tag{2.1}$$

Note:In above hypergeometric function x and b can be purely real or purely imaginary or complex numbers.

3 Independent Proof of Hypergeometric Form

$Proof of$ (2.1).

Let

$$y = (1+x^2)^{\frac{ib}{2}} \exp(b\tan^{-1}x). \tag{3.1}$$

Put $x = 0$ in equation (3.1), we get

$$(y)_0 = 1. \tag{3.2}$$

Differentiate equation (3.1) w.r.t. x and put $x = 0$, we get

$$(1+x^2)y_1 - (xi+1)yb = 0. \tag{3.3}$$
$$(y_1)_0 = b = i(-ib). \tag{3.4}$$

Differentiate equation (3.3) n-times w.r.t. x, and applying Leibniz theorem, we get

$$D^n\left\{(1+x^2)y_1\right\} - bD^n\left\{(xi+1)y\right\} = 0; \qquad n \geq 1.$$
$$(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} - b(xi+1)y_n - bin(1-b)y_{n-1} = 0; \quad n \geq 1. \tag{3.5}$$

Put $x = 0$ in equation(3.5) we get

$$(y_{n+1})_0 = -\left[n(n-1) - bin\right](y_{n-1})_0 + b(y_n)_0; \quad n \geq 1. \tag{3.6}$$

Put $n = 1, 2, 3, 4, 5, 6, 7, 8....$ in equation (3.6), we get

$$(y_2)_0 = b(b+i) = ib(1-ib), \tag{3.7}$$
$$(y_3)_0 = b(b+i)(b+2i) = i^2b(1-ib)(2-ib), \tag{3.8}$$
$$(y_4)_0 = b(b+i)(b+2i)(b+3i) = i^3b(1-ib)(2-ib)(3-ib), \tag{3.9}$$

$$(y_5)_0 = b(b+i)(b+2i)(b+3i)(b+4i) = i^4 b(1-ib)(2-ib)(3-ib)(4-ib), \tag{3.10}$$

$$(y_6)_0 = b(b+i)(b+2i)(b+3i)(b+4i)(b+5i) = i^5 b(1-ib)(2-ib)(3-ib)(4-ib)(5-ib) \tag{3.11}$$

$$(y_n)_0 = \prod_{k=1}^{n} \{b+(k-1)i\}, \tag{3.12}$$

$$(y_n)_0 = (i)^{n-1} b(1-ib)(2-ib)(3-ib)....(n-1-ib), \tag{3.13}$$

$$(y_n)_0 = i^n (-ib)_n. \tag{3.14}$$

We know by Maclaurin series expansion

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + ... \tag{3.15}$$

$$y = \sum_{n=0}^{\infty} \frac{(y_n)_0 \; x^n}{n!}, \tag{3.16}$$

$$y = \sum_{n=0}^{\infty} \frac{(-ib)_n (xi)^n}{n!}, \tag{3.17}$$

$$y = \sum_{n=0}^{\infty} \frac{(-ib)_{2n}(xi)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-ib)_{2n+1}(xi)^{2n+1}}{(2n+1)!}, \tag{3.18}$$

$$y = \sum_{n=0}^{\infty} \frac{\left(-\frac{ib}{2}\right)_n \left(-\frac{ib+1}{2}\right)_n (-x^2)^n}{\left(\frac{1}{2}\right)_n n!} + bx \sum_{n=0}^{\infty} \frac{\left(-\frac{ib+1}{2}\right)_n \left(-\frac{ib+2}{2}\right)_n (-x^2)^n}{\left(\frac{3}{2}\right)_n n!}. \tag{3.19}$$

Using definition of generalized hypergeometric function of one variable, we get the required result (2.1).

4 Applications

Suppose $x \in \mathbb{R}$ and b is purely imaginary in equation (2.1), then putting $x = iy$ and $b = -id$ in equation (2.1), where y is purely imaginary and d is purely real, we get

$$(1-y^2)^{\frac{d}{2}} \exp(d \tanh^{-1} y) = {}_2F_1 \left[\begin{matrix} -\frac{d}{2}, & \frac{1-d}{2}; & \\ & & y^2 \\ & \frac{1}{2}; & \end{matrix} \right] + dy \; {}_2F_1 \left[\begin{matrix} \frac{1-d}{2}, & \frac{2-d}{2}; & \\ & & y^2 \\ & \frac{3}{2}; & \end{matrix} \right]. \tag{4.1}$$

Putting $b = -ig$ in the equation (2.1), where g is purely real, we get

$$(1+x^2)^{\frac{g}{2}} \exp(-ig \tan^{-1} x) = {}_2F_1 \left[\begin{matrix} -\frac{g}{2}, & \frac{-g+1}{2}; & \\ & & -x^2 \\ & \frac{1}{2}; & \end{matrix} \right] - igx {}_2F_1 \left[\begin{matrix} \frac{1-g}{2}, & \frac{2-g}{2}; & \\ & & -x^2 \\ & \frac{3}{2}; & \end{matrix} \right]. \tag{4.2}$$

Applying Euler's formula on left hand side of equation (4.2), then on equating real and imaginary parts, we get

$$(1+x^2)^{\frac{g}{2}} \cos(g \tan^{-1} x) = {}_2F_1 \left[\begin{matrix} -\frac{g}{2}, & \frac{-g+1}{2}; & \\ & & -x^2 \\ & \frac{1}{2}; & \end{matrix} \right], \tag{4.3}$$

$$(1+x^2)^{\frac{g}{2}} \sin(g \tan^{-1} x) = gx \; {}_2F_1 \left[\begin{matrix} \frac{1-g}{2}, & \frac{2-g}{2}; & \\ & & -x^2 \\ & \frac{3}{2}; & \end{matrix} \right]. \tag{4.4}$$

Put $g = ik$ in equation (4.3) and (4.4), where k is purely imaginary, we get

$$(1+x^2)^{\frac{ik}{2}} \cosh(k \tan^{-1} x) = {}_2F_1 \left[\begin{matrix} -\frac{ik}{2}, & \frac{-ik+1}{2}; & \\ & & -x^2 \\ & \frac{1}{2}; & \end{matrix} \right], \tag{4.5}$$

$$(1+x^2)^{\frac{ik}{2}} \sinh(k \tan^{-1} x) = kx \; {}_2F_1 \left[\begin{matrix} \frac{1-ik}{2}, & \frac{2-ik}{2}; & \\ & & -x^2 \\ & \frac{3}{2}; & \end{matrix} \right]. \tag{4.6}$$

Putting $x = iy$ in equation (4.3) and (4.4) , where y is purely imaginary, we get

$$(1-y^2)^{\frac{g}{2}} \cosh(g \tanh^{-1} y) = {}_2F_1 \left[\begin{array}{c} -\frac{g}{2}, \quad \frac{-g+1}{2}; \\ \qquad\qquad\qquad y^2 \\ \frac{1}{2}; \end{array} \right], \tag{4.7}$$

$$(1-y^2)^{\frac{g}{2}} \sinh(g \tanh^{-1} y) = gy \; {}_2F_1 \left[\begin{array}{c} \frac{1-g}{2}, \quad \frac{2-g}{2}; \\ \qquad\qquad\qquad y^2 \\ \frac{3}{2}; \end{array} \right]. \tag{4.8}$$

Putting $x = iy$ and $g = ik$ in equation (4.3) and (4.4), where y and k are purely imaginary, we get

$$(1-y^2)^{\frac{ik}{2}} \cos(k \tanh^{-1} y) = {}_2F_1 \left[\begin{array}{c} -\frac{ik}{2}, \quad \frac{-ik+1}{2}; \\ \qquad\qquad\qquad y^2 \\ \frac{1}{2}; \end{array} \right], \tag{4.9}$$

$$(1-y^2)^{\frac{ik}{2}} \sin(k \tanh^{-1} y) = ky {}_2F_1 \left[\begin{array}{c} \frac{1-ik}{2}, \quad \frac{2-ik}{2}; \\ \qquad\qquad\qquad y^2 \\ \frac{3}{2}; \end{array} \right]. \tag{4.10}$$

5 Conclusion

In our present investigation, we have obtained hypergeometric forms of some composite functions using Maclaurin's series expansion and Leibniz theorem. We conclude our present investigation by observing that hypergeometric form of some other functions can be derived in an analogous manner. More over the results derived are significant. These are expected to find some potential applications in the fields of Applied Mathematics and Engineering Sciences.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 92-100**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

A GEOMETRIC PROGRAMMING APPROACH TO CONVEX MULTI-OBJECTIVE PROGRAMMING PROBLEMS

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(Received: January 11, 2023; In format January 19, 2023; Revised April 18, 2023; Accepted: April 21, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53111

Abstract

Over the past few years, convex optimization has played a vital role in the study of complex engineering problems in different fields. Geometric programming is one of the available techniques particularly used for solving nonconvex programming problems. But in this article, a suitable attempt has been made to solve a real-life model on convex multi-objective using geometric programming technique with help of the ϵ-constraint method and result is compared with the solutions obtained by fuzzy technique. Finally, a conclusion is presented by analyzing the solutions to a numerical problem.

2020 Mathematical Sciences Classification: 90C25,90C29,90C30,90C70.

Keywords and Phrases: Convex optimization; Multi-objective optimization; Geometric programming; ϵ-constraint method; Fuzzy.

1 Introduction

There is hardly ever a situation that arises where one can expect only one goal at the same time. For example, while purchasing something we are expecting a high-quality product at a low price. In the same manner, most of the technical problems involve more than one goal to maximize quality versus minimizing the cost. This ambiguity proceeds to the field of multi-objective optimization. There exists an infinite number of optimal solutions to multi-objective problems because of conflicting objectives. The group of these agreement solutions is called the pareto set[8] and solutions are called pareto solutions. But the question arises how to combine different objectives to yield optimal solutions for our modeled problems. In this article geometric programming technique has been discussed for solving different engineering applications, which was developed by Duffin, Peterson and Zener [9]. Nowadays, it can be used in various fields like circuit design [4, 5], production, and constructing models for market planning [2, 11]. Many important problems in engineering need to solve non-convex multi-objective optimization problems to achieve the proper results. But in this article, we have tried to discuss convex multi-objective optimization problems. The optimization problem in which objective functions, as well as, constraints are convexly is known as the convex problems. Recently convex optimization methods are widely used in the design and analysis of communication systems and signal processing algorithms because in convex problems local optimum is considered as global optimum. Luo et al.[12], in their recent paper have shown how convex optimization is useful for communications and signal processing. Different applications in the field of automatic control systems, electronic circuit design, data analysis, statistics, and finance has been discovered since its development. The basic advantages of the convex optimization problem for solving a problem very reliably and efficiently using interior-point methods or other special methods have been shown by Boyd et al.[6]. The connectedness properties of quasi-convex problems using cone-efficient set of the solution have been shown by Zhou[16]. An and Liu[1] have proven different necessary and sufficient conditions for getting weakly Pareto solutions and weakly efficient solutions of convex multi-objective programming problems. For deriving the solutions of multi-objective convex problems using both equality and inequality constraints, Shang et al.[15] have discussed the homotopy method which does not require any starting point to be the feasible point.

The paper is structured as follows: beginning with the introduction, the basic concept of convex optimization has been discussed in sec 2. The modeling of multi-objective convex geometric problems discussed in sec 3 and corresponding solution procedure by ϵ-constraint method discussed in sec 4. The rule of convergence of solutions by ϵ-constraint method and a suitable example based on our discussion given in sec 5 and sec 6 respectively. Finally, conclusion drawn is presented in Sec 7.

2 Basic concepts of convex set, convex function and convex programming

The study of fundamental concepts of convexity and its use in the construction of mathematical models[7, 14] related to various physical problems are key for everyone.

Convex sets:

A set $S \subset R^n$ is said to be convex if for any two points $x, y \in S$, their convex linear combination also lies in S. Mathematically, it is represented as.

λ x + (1-λ)y$\in$ S for all $\lambda \in$ [0, 1] and $x, y \in$ S

Since the line segment joining any two distinct points is no longer on the unit sphere, the unit sphere is not convex whereas the unit ball is a convex set. Generally, a convex set is a solid object having no holes and always curved outward. An important property regarding convex sets is that the intersection of more than one number of convex sets is again convex.

Convex functions

A function $f(x) : R^n \to R$ is called convex if for any two points $x, y \in R^n$ the following condition must be satisfied

f(λ x + (1-λ)y)$\leq$ λ $f(x)$ + (1-λ)$f(y)$ for all $\lambda \in$ [0, 1].

Geometrically, a function is called convex if the line joining x and y lies above the graph of f that is called an epigraph.

Theorem 2.1. Let f is a function which is defined and differentiable on $dom f$. Then f is called convex if and only if $f(y) \geq f(x) + \nabla f(x)^T(y-x)$ for all $x, y \in dom f$.

It will be strictly convex if and only if $dom f$ is convex for every $x, y \in dom f$and $x \neq y$. Then we have $f(y) > f(x) + \nabla f(x)^T(y-x)$.

Theorem 2.2. *Let f be a twice differentiable function on its convex domain $dom f$. Then f is called convex if and only if the hessian of the function should be positive semi definite: $\nabla^2 f(x) \geq 0$ for all$x \in dom f$. However, if $\nabla^2 f(x) > 0$ for all $x \in dom f$, then f will be strictly convex.*

Basic Properties of convex functions

· f is called strictly convex if the strict inequality holds and $x \neq y$.

· if f is concave then $-f$ will be convex.

· if f is convex then its epigraph epi f is also a convex set.

· If f is a convex function over a convex set S, then the local minimum will be the global minimum.

Convex Optimization

The optimization problem of the form

$$min f(x) \tag{2.1}$$

subject to

$$f_i(x) \leq 0, \quad i = 1, 2, ..., m, \tag{2.2}$$

$$g_j(x) = 0, \quad j = 1, 2, 3, ..., n, \qquad x \in S, \tag{2.3}$$

is called convex optimization problem if inequality constraints are convex and equality constraints are affine where S is a convex set. The optimization problem will be non-convex if one of the conditions is violated. Convex optimization problems have three important properties that make them fundamentally more powerful than any generic non-convex optimization problems.

· Local optimum is necessarily a global optimum;

· Detection of exact infeasibility ;

· Very large problems can be handled by efficient numerical solution methods.

Theorem 2.3. *(Local optima implies global optima)Let Q be a convex optimization problem and let $x^* \in S$ be a point such that $f(x^*) \leq f(y)$ for all feasible y with $||x^* - y|| \leq \rho$. Then $f(x^*) \leq f(y)$ for all feasible y.*

proof: The proof is by contradiction. Assume that there is some feasible t such that $f(t) < f(x^*)$. Then take $y = \alpha x^* + (1-\alpha)t$ for $\alpha \in (0, 1)$ close to 1. We claim this point is feasible. The affine constraints are satisfied due to linearity, since both $x^*$and t are feasible. As for the inequality constraints, by convexity we get

$g_i(\alpha x^* + (1-\alpha)t) \leq \alpha g_i(x^*) + (1-\alpha)g_i(t) \leq 0$.

Hence y is feasible. However, the objective value is strictly smaller than $f(x^*)$, since

$f(\alpha x^* + (1-\alpha)t) \leq \alpha f(x^*) + (1-\alpha)f(t) < f(x^*)$.

For α close to one, we will get $||x^* - y|| \leq \rho$, which is a contradiction.

3 Multi-objective convex geometric programming problem *(MOCGPP)*

Optimization of multiple objective functions subject to given constraints is known as multi-objective optimization. In this process a solution which is optimal with respect to one objective function may not be same for remaining objectives. As a result, we can not find only one global optimal solution. Therefore optimizing a problem means, find a solution such that it should acceptable to all the decision makers. A multi-objective geometric problem will be convex if all objective functions as well as constraints are convex. It can be defined mathematically as follows:

To determine $x = (x_1, x_2, ..., x_n)^T$ in order to

$$\min : f_k(x) = \sum_{i=1}^{\alpha^k} \beta_{0i}^k \prod_{j=1}^{n} x_j^{c_{0ij}^k}, \quad k = 1, 2, ..., p. \tag{3.1}$$

Subject to

$$g_t(x) = \sum_{i=1}^{\alpha_t} \beta_{it} \prod_{j=1}^{n} x_j^{c_{itj}} \leq 1, \quad t = 1, 2, ..., m, \tag{3.2}$$

$$x_j > 0, \quad j = 1, 2, ..., n, \tag{3.3}$$

where all objective functions and constraints are convex and
$\beta_{0i}^k \geq 0 \; \forall \; k$ and i,
$\beta_{it} \geq 0 \; \forall \; i$ and t,
c_{0tj}^k and c_{itj} are real numbers $\forall \; i, j, k, t$,
$\alpha^k =$ no. of terms in the k^{th} objective function $f_k(x)$,
$\alpha_t =$ no. of terms in the t^{th} constraint.

4 The ϵ-constraintmethod

This method was developed first by Haimes et al.[10] for generating pareto optimal solutions for the multi-objective optimization problems. In this method at a time, only one of the objective functions solved expressing other objective functions as constraints. This method can be stated as:

$$\min : \; f_0^l(x), \quad where \quad l \in \{1, 2, ..., k\}, \tag{4.1}$$

subject to

$$f_0^p(x) \leq \epsilon_p, \quad p = 1, 2, ..., k, \quad p \neq l, \tag{4.2}$$

$$g_i(x) \leq 1, \quad i = 1, 2, ..., m. \tag{4.3}$$

We define

$$L_p \leq \epsilon_p \leq U_p, \quad p = 1, 2, ..., k, \quad p \neq l.$$

where

$$L_p = \min_{\forall x \in X} \; f_0^p(x), \quad p = 1, 2, ..., k.$$

and

$$U_p = \max_{\forall x \in X} \; f_0^p(x), \quad p = 1, 2, ..., k,$$

$$x \in X, \quad X \quad being \;\; the \;\; feasible \;\; region \; .$$

Compromise solutions of the problems can be generated considering the values of ϵ_p in the interval $[L_p, U_p]$ for $p = 1, 2, ..., k$.

5 Test of convergence of solutions by ϵ-constraint method

Sometimes, it is necessary to check the convergence of optimal solutions in multi-objective problems. Regarding this, Ojha and Biswal[13] in their recent paper have shown how the pareto solutions converges using ϵ-constraint method. Below given steps are some of the notes representing converges of the solutions.

Step 1. Determine the bounds of the objectives $(f_0^{(l)}(x), l = 1, 2, ..., k)$ using ideal points $X^{(1)}, X^{(2)}, ..., X^{(k)}$ as obtained by geometric programming technique. Let L_l and U_l are the least and best values of $f_0^{(l)}$ i.e $L_l \leq f_0^{(l)}(x) \leq U_l, \quad l = 1, 2, ..., k$.

Step 2. Consider ϵ_l be a point in the interval $[L_l, U_l]$ such that $L_l \leq \epsilon_l \leq U_l, \quad l = 1, 2, ..., k$.

Step 3. If we assign different values to ϵ_l in the interval $[L_l, U_l]$, it will initiate a set of pareto optimal

solution.
Step 4. Weigh the differences between the pareto solutions with the solution obtained by fuzzy method.
Step 5. If the solution obtained in step 3 is same as that obtained in step 4, then stop and accept the solution of the problem.

6 Numerical example

Let's consider and solve the following example on the basis of our above discussion.
Example 6.1.
Find x_1, x_2 and x_3 to

$$\min f_1(x) = 2x_1^{-1}x_2^{-1} + 20x_2 + 12x_3^{-1}, \tag{6.1}$$

$$\min f_2(x) = 4x_1^{-1}x_2^{-1}x_3^{-1} + 20x_2^2 + 10x_3^{-1}, \tag{6.2}$$

subject to

$$2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1, \tag{6.3}$$

$$where \;\; x_1, x_2, x_3 \geq 0. \tag{6.4}$$

Verification of convexity of objective functions

It can be shown that a function $f(x_1, x_2, ...x_n)$ is a convex function if and only if the matrix of second order derivatives or Hessian matrix is positive semi-definite and principal minor determinants of this matrix are all non negative.

for example, if $H(x) = [\frac{\partial^2 f}{\partial_{x_i}.\partial_{x_j}}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, where i=1, 2, 3 and j=1, 2, 3

Let us verify the convexity of 1st objective function $f_1(x) = 2x_1^{-1}x_2^{-1} + 20x_2 + 12x_3^{-1}$
As per the definition, Hessian matrix for the given problem will be

$$H(x) = \begin{pmatrix} 4x_1^{-3}x_2^{-1} & 2x_1^{-2}x_2^{-2} & 0 \\ 2x_1^{-2}x_2^{-2} & 4x_1^{-1}x_2^{-3} & 0 \\ 0 & 0 & 24x_3^{-3} \end{pmatrix}.$$

Corresponding to the above hessian matrix, the determinant of the principal minors $D_1 = 4x_1^{-3}x_2^{-1}, D_2 = 12x_1^{-4}x_2^{-4}$ and $D_3 = 288x_1^{-4}x_2^{-4}x_3^{-3}$ are positive for the design variables x_1, x_2 and x_3. So it is a convex function.

Similarly for the 2nd objective function $f_2(x) = 4x_1^{-1}x_2^{-1}x_3^{-1} + 20x_2^2 + 10x_3^{-1}$, hessian matrix will be

$$H(x) = \begin{pmatrix} 8x_1^{-3}x_2^{-1}x_3^{-1} & 4x_1^{-2}x_2^{-2}x_3^{-1} & 4x_1^{-2}x_2^{-1}x_3^{-2} \\ 4x_1^{-2}x_2^{-2}x_3^{-1} & 84x_1^{-1}x_2^{-3}x_3^{-1} + 40 & 4x_1^{-1}x_2^{-2}x_3^{-2} \\ 4x_1^{-2}x_2^{-1}x_3^{-2} & 4x_1^{-1}x_2^{-2}x_3^{-2} & 8x_1^{-1}x_2^{-1}x_3^{-3} + 20x_3 \end{pmatrix}.$$

Corresponding to the above hessian matrix, the determinant of the principal minors $D_1 = 8x_1^{-3}x_2^{-1}x_3^{-1}, D_2 = 64x_1^{-4}x_2^{-4}x_3^{-2} + 320x_1^{-3}x_2^{-1}x_3^{-1} - 16x_1^{-4}x_2^{-4}x_3^{-2}$ and $D_3 = 2564x_1^{-5}x_2^{-5}x_3^{-5} + 19208x_1^{-4}x_2^{-2}x_3^{-4} + 6400x_1^{-3}x_2^{-1}x_3^{-4} - 192x_1^{-4}x_2^{-4}x_3^{-5}$ are positive for the design variables x_1, x_2 and x_3. So it is a convex function.

For the given constraint $2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1$, the hessian matrix will be

$$H(x) = \begin{pmatrix} 4x_2^{-1/2} & -2x_1x_2^{-3/2} & 0 \\ -2x_1x_2^{-3/2} & 3/2x_1^2x_2^{-5/2} + 4x_2^{-3}x_3^2 & -4x_2^{-2}x_3 \\ 0 & -4x_2^{-2}x_3 & 4x_2^{-1} \end{pmatrix}.$$

Now this hessian matrix can be checked for convexity of constraints. We found it is also convex function. Therefore the given optimization problem is a convex optimization problem.

Here we have divided this problem into two sub problems as primal(i) and primal(ii)in order to find its optimal solutions.

Primal (i) Corresponding dual and their solutions
Primal(i).
Find x_1, x_2 and x_3 to

$$\min f_1(x) = 2x_1^{-1}x_2^{-1} + 20x_2 + 12x_3^{-1}, \tag{6.5}$$

subject to

$$2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1, \tag{6.6}$$

$$where \;\; x_1,\; x_2,\; x_3 \geq 0. \tag{6.7}$$

Dual:
The Dual of the above primal will be as follows:

$$\max_t : V(t) = \left(\frac{2}{t_{01}}\right)^{t_{01}} \left(\frac{20}{t_{02}}\right)^{t_{02}} \left(\frac{12}{t_{03}}\right)^{t_{03}} \left(\frac{2}{t_{11}}\right)^{t_{11}} \left(\frac{2}{t_{12}}\right)^{t_{12}} (t_{11}+t_{12})^{(t_{11}+t_{12})}. \tag{6.8}$$

Subject to

$$\begin{aligned} t_{01}+t_{02}+t_{03} &= 1,\\ -t_{01}+2t_{11} &= 0,\\ -t_{01}+t_{02}-\frac{1}{2}t_{11}-t_{12} &= 0,\\ -t_{03}+2t_{12} &= 0,\\ t_{01},\; t_{02},\; t_{03},\; t_{11},\; t_{12} &\geq 0, \end{aligned}$$

Solution of primal(i) is $f_1(x) = 45.10214$ for $x_1 = 0.3390946$, $x_2 = 0.9131737$ and $x_3 = 0.5888183$ where as its corresponding dual is $f_1^* = 45.10214$ for $t_{01} = 0.1432052,\; t_{02} = 0.4049359,\; t_{03} = 0.4518589,$ '$t_{11} = 0.0716025,\; t_{12} = 0.2259295$.

Primal(ii) Corresponding dual and their solutions

Primal(ii).

Find x_1, x_2 and x_3 to minimize,

$$\min f_2(x) = 4x_1^{-1}x_2^{-1}x_3^{-1} + 20x_2^2 + 10x_3^{-1}, \tag{6.9}$$

subject to

$$2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1, \tag{6.10}$$

$$where \;\; x_1,\; x_2,\; x_3 \geq 0. \tag{6.11}$$

Dual.

$$\max_t : V(t) = \left(\frac{4}{t_{01}}\right)^{t_{01}} \left(\frac{20}{t_{02}}\right)^{t_{02}} \left(\frac{10}{t_{03}}\right)^{t_{03}} \left(\frac{2}{t_{11}}\right)^{t_{11}} \left(\frac{2}{t_{12}}\right)^{t_{12}} (t_{11}+t_{12})^{(t_{11}+t_{12})}. \tag{6.12}$$

Subject to

$$\begin{aligned} t_{01}+t_{02}+t_{03} &= 1,\\ -t_{01}+2t_{11} &= 0,\\ -t_{01}+2t_{02}-\frac{1}{2}t_{11}-t_{12} &= 0,\\ -t_{01}-t_{03}+2t_{12} &= 0,\\ t_{01}, t_{02}, t_{03}, t_{11}, t_{12} &\geq 0. \end{aligned}$$

Solution of primal $f_2(x) = 54.28115$ for $x_1 = 0.40671$, $x_2 = 0.9880575$ and $x_3 = 0.5741136$ where as its dual will be $f_2^* = 54.28115$ for $t_{01} = 0.3194082,\; t_{02} = 0.3597041,\; t_{03} = 0.3208877,\; t_{11} = 0.1597041,\; t_{12} = 0.3201480$.

Replacing the value of f_1 in f_2 and f_2 in f_1, we find both lower and upper bound of the functions:

$L_1 = 45.10214 \leq f_1 \leq 45.63988 = U_1$,

and $L_2 = 54.28115 \leq f_2 \leq 55.64130 = U_2$.

Defining ϵ_1 and ϵ_2 based on the values of f_1 and f_2, we have

$45.10214 \leq \epsilon_1 \leq 45.63988$ and $54.28115 \leq \epsilon_2 \leq 55.64130$.

We can observe, as the value of ϵ_1 and ϵ_2 changes within their range, the value of objective functions f_1 and f_2 also changes and are converging towards their suitable compromise values.

Primal(i) and its solution by ϵ-constraint method

Find $x_1,\; x_2$ and x_3 to

$$\min f_1(x) = 2x_1^{-1}x_2^{-1} + 20x_2 + 12x_3^{-1}, \tag{6.13}$$

subject to

$$4x_1^{-1}x_2^{-1}x_3^{-1} + 20x_2^2 + 10x_3^{-1} \leq \epsilon_2, \tag{6.14}$$

$$2x_1^2 x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1, \tag{6.15}$$

$$where \;\; x_1, \; x_2, \; x_3 > 0. \tag{6.16}$$

Different values of the Primal(i) will be obtained by changing ϵ_2 between 54.28115 to 55.63988 given in Table 6.1.

***Table* 6.1:** (Optimal solution of Primal(i))

ϵ_2	x_1	x_2	x_3	primalf_1
54.3	0.3968714	0.9834149	0.5792341	45.50971
54.5	0.3744542	0.9677492	0.5881633	45.27651
54.7	0.3634936	0.9560997	0.5906395	45.19375
54.9	0.3557495	0.9455062	0.5913476	45.14871
55.1	0.3497639	0.9355950	0.5911580	45.12281
55.3	0.3449175	0.9262459	0.5904456	45.10875
55.5	0.3408725	0.9173958	0.5894118	45.10279
55.6	0.3390946	0.9131737	0.5888183	45.10214

Dual

The Dual of the above primal will be:

$$\max_t : V(t) = \left(\frac{1}{t_{01}}\right)^{t_{01}} \left(\frac{1}{4t_{02}}\right)^{t_{02}} \left(\frac{3}{4t_{11}}\right)^{t_{11}} \left(\frac{3}{8t_{12}}\right)^{t_{12}} (t_{11}+t_{12})^{(t_{11}+t_{12})}$$
$$\left(\frac{2}{\epsilon_2 t_{21}}\right)^{t_{21}} \left(\frac{2}{\epsilon_2 t_{22}}\right)^{t_{22}} (t_{21}+t_{22})^{(t_{21}+t_{22})}, \tag{6.17}$$

subject to

$$\begin{aligned} t_{01} + t_{02} &= 1, \\ -2t_{01} + 2t_{11} - t_{21} + t_{22} &= 0, \\ 2t_{02} - 2t_{11} + t_{12} - t_{21} + t_{22} &= 0, \\ -t_{02} + 2t_{12} - t_{21} &= 0, \\ t_{01}, \; t_{02}, \; t_{11}, \; t_{12}, \; t_{21}, \; t_{22} &\geq 0. \end{aligned} \tag{6.18}$$

As the value of ϵ_2 will change between 54.28115 to 55.64130, the changes occur in the dual value is given in the Table 6.2.

***Table* 6.2:** (Dual Solution)

ϵ_2	t_{01}	t_{02}	t_{03}	t_{11}	t_{12}	t_{13}	t_{21}	t_{22}	$Dual f_1$
54.3	0.1126002	0.432175	0.455221	1.23822	1.35357	1.208159	0.675410	1.45080	45.50971
54.5	0.1218983	0.427481	0.450619	0.240704	0.240232	0.218063	0.181301	0.454694	45.27651
54.7	0.1273361	0.423111	0.449552	0.127157	0.119299	0.110479	0.127246	0.343594	45.19375
54.9	0.1316970	0.418840	0.449462	0.075176	0.668398	0.063216	0.103437	0.293928	45.14871
55.1	0.1354475	0.414688	0.449864	0.043940	0.037172	0.035917	0.089675	0.264842	45.12281
55.3	0.1387802	0.410672	0.450547	0.022454	0.018169	0.0017934	0.080617	0.245468	45.10875
55.5	0.1418002	0.406802	0.451397	0.006534	0.005608	0.0005108	0.074167	0.231528	45.10279
55.6	0.1432051	0.404935	0.451859	0.22940×10^{-7}	0.17625×10^{-7}	0.18190×10^{-7}	0.071600	0.225929	45.10214

From table 6.2, we found the optimal value of dual, that is 45.10214 for $t_{01} = 0.143051$, $t_{02} = 0.4049359$, $t_{03} = 0.4518591$, $t_{11} = 0.22940 \times 10^{-7}$, $t_{12} = 0.17625 \times 10^{-7}$, $t_{13} = 0.18190 \times 10^{-7}$, $t_{21} = 0.071600$, $t_{22} = 0.225929$

Primal(ii) and its solution by ϵ-constraint method
Find x_1, x_2 and x_3 to

$$\min f_2(x) = 4x_1^{-1}x_2^{-1}x_3^{-1} + 20x_2^2 + 10x_3^{-1}, \tag{6.19}$$

subject to

$$2x_1^{-1}x_2^{-1} + 20x_2 + 12x_3^{-1} \leq, \epsilon_1 \tag{6.20}$$

$$2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1, \tag{6.21}$$

$$where \;\; x_1,\; x_2,\; x_3 \geq 0. \tag{6.22}$$

Solutions of the Primal(ii) obtained by changing the values of ϵ_1 between 45.10214 to 45.63988 given in Table 6.3.

***Table* 6.3:** (Optimal solution of Primal(ii))

ϵ_1	x_1	x_2	x_3	primalf_2
45.11	0.345469	0.927377	0.590554	55.2752
45.3	0.377145	0.970123	0.587336	54.46271
45.5	0.396079	0.982994	0.579618	54.30320
45.6	0.403834	0.986803	0.575673	54.28274
45.61	0.404565	0.987130	0.575281	54.28203
45.63	0.406007	0.987758	0.574499	54.28125
45.639	0.406647	0.988031	0.574148	54.28115

Dual
The Dual of the above Primal will be

$$\max_t : V(t) = \left(\frac{1}{t_{01}}\right)^{t_{01}} \left(\frac{1}{4t_{02}}\right)^{t_{02}} \left(\frac{3}{4t_{11}}\right)^{t_{11}} \left(\frac{3}{8t_{12}}\right)^{t_{12}} (t_{11}+t_{12})^{(t_{11}+t_{12})}$$
$$\left(\frac{2}{\epsilon_2 t_{21}}\right)^{t_{21}} \left(\frac{2}{\epsilon_2 t_{22}}\right)^{t_{22}} (t_{21}+t_{22})^{(t_{21}+t_{22})}. \tag{6.23}$$

Subject to

$$\begin{aligned} t_{01} + t_{02} &= 1 \\ -2t_{01} + 2t_{11} - t_{21} + t_{22} &= 0, \\ 2t_{02} - 2t_{11} + t_{12} - t_{21} + t_{22} &= 0, \\ -t_{02} + 2t_{12} - t_{21} &= 0, \\ t_{01},\; t_{02},\; t_{11},\; t_{12},\; t_{21},\; t_{22} &\geq 0. \end{aligned} \tag{6.24}$$

As the value of ϵ_1 will change between 45.10214 to 45.63988,the changes occur in dual value is given in Table 6.4.

***Table* 6.4:** (Dual Solution)

ϵ_1	t_{01}	t_{02}	t_{03}	t_{11}	t_{12}	t_{13}	t_{21}	t_{22}	$Dual f_2$
45.11	0.327074	0.350879	0.322046	0.0075578	0.028352	0.031227	0.167315	0.340174	54.29273
45.3	0.322068	0.356626	0.321305	0.0025116	0.0097014	0.010488	0.162289	0.326931	54.28255
45.5	0.319408	0.359704	0.320887	0.116862×10^{-7}	0.453369×10^{-7}	0.487989×10^{-7}	0.159704	0.320148	54.28115
45.6	0.319408	0.359704	0.320887	0.116030×10^{-7}	0.453361×10^{-7}	0.491941×10^{-7}	0.159704	0.320148	54.28115

From table 6.4, it can be found that the optimal value of the dual is 54.28115 for $t_{01} = 0.319408,\; t_{02} = 0.359704,\; t_{03} = 0.320887,\; t_{11} = 0.116030 \times 10^{-7},\; t_{12} = 0.453361 \times 10^{-7},\; t_{13} = 0.491941 \times 10^{-7},\; t_{21} = 0.159704,\; t_{22} = 0.320148$

Solution by fuzzy method

Case-1.Solution of f_1
The crisp model of f_1 using fuzzy method can be stated as:
$max : \theta$
subject to

$$2x_1^{-1}x_2^{-1} + 20x_2 + 12x_3^{-1} + (45.63988 - 45.10214)\theta \leq 45.63988,$$
$$2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1,$$
$$\theta > 0,\ x_i > 0\ for\ \ i = 1, 2, 3. \tag{6.25}$$

The optimal value of $f_1 = 45.10214$ for $\theta = 1.00008$, $x_1 = 0.3390946$, $x_2 = 0.9131737$, $x_3 = 0.5888183$.

Case-2.Solution of f_2
The crisp model of f_2 using fuzzy method is defined as follows:
$max : \theta$
s.to

$$4x_1^{-1}x_2^{-1}x_3^{-1} + 20x_2^2 + 10x_3^{-1} + (55.64130 - 54.28115)\theta \leq 55.64130,$$
$$2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1,$$
$$\theta > 0,\ x_i > 0\ for\ \ i = 1, 2, 3. \tag{6.26}$$

The optimal value of $f_2 = 54.28115$ for $\theta = 1.0000$, $x_1 = 0.4067100$, $x_2 = 0.9880575$ and $x_3 = 0.5741136$.

Result Analysis
Usually, the geometric programming problems are non-convex in nature. In this paper, the problem taken for our research purposes is a convex problem. The main aim of taking convex problem is that global minima of a problem will be global optima if the considered test problem is convex.
The above work out shows how the solutions converging to $f_1 = 45.10214$ for $x_1 = 0.3390946$, $x_2 = 0.9131737$, $x_3 = 0.588183$ and $f_2 = 54.281159$ for $x_1 = 0.4067100$, $x_2 = 0.9880575$, $x_3 = 0.5741136$ by obtained by ϵ-constraint method which is exactly same as obtained by fuzzy method. However, the decision makers have multiple choices in ϵ-constraint method But there is only one choice in fuzzy method.

7 Conclusion

It is very interesting to search a suitable solution for the multi-objective problems. But only one difficulty arises because of conflicting of objectives. Due to non-convexity nature,sometimes it is difficult to find a best compromise solutions for multi-objective problems. Here we are not interested to explain whether a generic multi-objective optimization problem is efficiently solvable or not. However, we are interested,how to solve the problem efficiently. As far as the solutions of the problem is concerned, there exists optimization problems in which both objective and the constraints are convex. Under the given conditions a convex optimization problem can be solve up to to a given accuracy. In contrast, a non-convex problems is difficult to solve. The computational effort required to solve such problems by the best known numerical methods grows fast with the dimensions of the problems and therefore it is difficult to study an intrinsic nature of non-convex problems. Because of this, we have considered a multi-objective convex geometric problem to study its behaviour in order to find best compromise solutions.
Acknowledgement. We are very much grateful to the Editor and Referee for their valuable suggestions to bring the paper in its present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 101-105**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

CREATION OF SEQUENCES OF SINGULAR 3-TUPLES THROUGH ABEL AND CYCLOTOMIC POLYNOMIAL WITH COMMENSURABLE PROPERTY

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(Received: June 06, 2022, In format: July 28, 2022; Revised: August 24, 2022; Accepted: April 08, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53112

Abstract

In this paper, the sequence of 3-tuples named as singular 3-tuples $\{f(x), g(x), h(x)\}$, $\{g(x), h(x), i(x)\}$ etc concerning Abel's polynomial and Cyclotomic polynomial such that the arithmetic mean of any two polynomials increased by a monomial with integer coefficients provides square of a particular polynomial is enumerated. Furthermore, Python program for conformation of each of an evaluated singular 3-tuples is also exemplified.

2020 Mathematical Sciences Classification: 11B83
Keywords and Phrases: Singular 3-tuples, Abel's polynomial, Cyclotomic polynomial.

1 Introduction

"A set of s positive integers $\{r_1, r_2, \ldots, r_s\}$ is called a Diophantine s-tuples with the property $D(l), l \in Z-\{0\}$ if $r_m r_n + l$ is a perfect square for all $1 \leq m < n \leq s$." In [1, 2, 4], the writers originate limited number of Diophantine triples with precise properties. For an inclusive analysis of numerous problems on Diophantine 3-tuples with appropriate properties, see [5-11, 15-18]. In this communication, the process of finding sequence of singular 3-tuples $\{f(x), g(x), h(x)\}$, $\{g(x), h(x), i(x)\}$ etc consisting Abel's polynomial and Cyclotomic polynomial such that the arithmetic mean of any two polynomials in each 3-tuples enlarged by a monomial with integer coefficients remains square of a polynomial is demonstrated. Moreover, Python program for checking each of such singular 3-tuples with numerical values is illustrated.

2 Basic Definitions

2.1 Abel's Polynomial

The n^{th} term of Abel's Polynomial is defined by

$$p_{a,n}(x) = x(x - an)^{n-1}.$$

Therefore

$$P_{1,2}(x) = x^2 - 2x, \; P_{2,2}(x) = x^2 - 4x.$$

2.2 Cyclotomic Polynomial

The n^{th} term of Cyclotomic Polynomial is defined by

$$\varphi_n(x) = \begin{cases} \sum\limits_{k=0}^{n-1} x^k, & \text{if } n \textit{ is a prime number} \\ \sum\limits_{k=0}^{p-1} (-x)^k, & \text{if } n = 2p \textit{ where } p \textit{ is an odd prime number.} \end{cases}$$

Then, $\varphi_3(x) = x^2 + x + 1$, $\varphi_6(x) = x^2 - x + 1$.

2.3 Singular 3-tuples

A set of 3-tuples $\{f(x), g(x), h(x)\}$ is called singular 3-tuples with property $D[r(x)]$ if the average of any two polynomials in the set added with $r(x)$ is a square of some other polynomial where $f(x), g(x), h(x), r(x)$ are polynomials with integer coefficients.

3 Process of Receiving Singular 3-tuples

The method of discovering two different types of singular 3-tuples in which the elements are Abel's polynomial and Cyclotomic polynomial such that average of any two polynomials added with certain monomial gives a square of a polynomial is expressed in Section 3.1 and Section 3.2.

3.1 Singular 3-tuples with Ables Polynomial

Choose $f(x) = P_{1,2}(16x) = 256x^2 - 32x$ and $g(x) = P_{2,2}(16x) = 256x^2 - 64x$ be such that average of $f(x)$ and $g(x)$ added with $80x + 1$ is a square of a polynomial.

Mathematically, the above hypothesis is expressed by

$$\left[\frac{f(x) + g(x)}{2}\right] + 80x + 1 = (16x + 1)^2 = [\alpha(x)]^2 \text{ (say)}. \tag{3.1}$$

Let $h(x)$ be another polynomial collected with the resulting two conditions that

$$\left[\frac{g(x) + h(x)}{2}\right] + 80x + 1 = [\beta(x)]^2, \tag{3.2}$$

$$\left[\frac{f(x) + h(x)}{2}\right] + 80x + 1 = [\gamma(x)]^2. \tag{3.3}$$

Subtraction of (3.3) from (3.2) affords the succeeding equation

$$\left[\frac{g(x) - f(x)}{2}\right] = [\beta(x)]^2 - [\gamma(x)]^2. \tag{3.4}$$

For finding the third element $h(x)$ in an essential triple, let us select the proper conversions as specified below.

$$\beta(x) = A + 2 \text{ and } \gamma(x) = A. \tag{3.5}$$

Replacing the chosen values of $f(x)$, $g(x)$ and the alterations (3.5) in (3.4), the possibility of A and hence $\gamma(x)$ is attained by

$$\gamma(x) = A = 4x - 1.$$

Retaining $f(x)$ and the above derived value of $\gamma(x)$ in (3.3), it is scrutinized that

$$h(x) = -224x^2 - 112x.$$

Note that, $\{f(x), g(x), h(x)\} = \left\{256x^2 - 32x, 256x^2 - 64x, -224x^2 - 112x\right\}$ is a gorgeous singular 3-tuples with property $D(80x + 1)$.

Similarly opening with patterns of singular 2-tuples $\{g(x), h(x)\}$, $\{h(x), i(x)\}$ etc, it is possible to extend patterns of singular 3-tuples $\{g(x), h(x), i(x)\}$ $\{h(x), i(x), j(x)\}$ etc with an equivalent condition $D(80x+1)$. Here

$$i(x) = 7200x^4 + 1440x^3 + 56x^2 - 72x,$$
$$j(x) = 1620000x^8+648000x^7+190800x^6+43200x^5+2450x^4-20x^3+134x^2-68x.$$

Table 3.1 demonstrates singular 3-tuples for few values of x for easy understanding.

***Table* 3.1**

x	$D(80x+1)$	$\{f(x), g(x), h(x)\}$	$\{g(x), h(x)i(x)\}$	$\{h(x)i(x), j(x)\}$
1	$D(81)$	$\{224, 192, -336\}$	$\{192, -336, 8624\}$	$\{-336, 8624, 2504496\}$
2	$D(161)$	$\{960, 896, -1120\}$	$\{896, -1120, 126800\}$	$\{-1120, 126800, 511297040\}$
3	$D(241)$	$\{2208, 2112, -2352\}$	$\{2112, -2352, 622368\}$	$\{-2352, 622368, 12195785712\}$
4	D(321)	$\{3968, 3840, -4032\}$	$\{3840, -4032, 1935968\}$	$\{-4032, 1935968, 117611533392\}$
5	$D(401)$	$\{6240, 6080, -6160\}$	$\{6080, -6160, 4681040\}$	$\{-6160, 4681040, 686555281760\}$

3.2 Singular 3-tuples with Cyclotomic Polynomial

Let $\{k(x), l(x)\} = \{\varphi_3(16x), \varphi_6(16x)\} = \left\{256x^2 + 16x + 1,\ 256x^2 - 16x + 1\right\}$ be a pair comprising Cyclotomic Polynomials such that the average of these two polynomials augmented by the monomial $32x$ is a square of some other polynomial. Following the procedure as explained in Section 3.1, this pair is extended into singular triple $\{k(x), l(x), m(x)\}$ with property $D(32x)$. Here $m(x) = -224x^2 - 64x + 1$.

Similarly, if $\{l(x), m(x)\}, \{m(x), n(x)\}$ etc are pairs in which the elements are certain polynomials, then as in Section 3.1 each pair can be protracted into singular triples $\{l(x), m(x), n(x)\}, \{m(x), n(x), p(x)\}$ etc with the similar property $D(32x)$. Here

$$n(x) = 7200x^4 + 1440x^3 + 56x^2 - 24x + 1,$$

$$p(x) = 1620000x^8 + 648000x^7 + 190800x^6 + 43200x^5 + 2450x^4 - 20x^3 + 134x^2 - 20x + 1.$$

The ensuing Table 3.2 establishes the prescribed singular 3-tuples for limited values of x.

***Table* 3.2**

x	$D(32x)$	$\{k(x), l(x), m(x)\}$	$\{l(x), m(x), n(x)\}$	$\{m(x), n(x), p(x)\}$
1	$D(32)$	$\{273, 241, -287\}$	$\{241, -287, 8673\}$	$\{-287, 8673, 2504545\}$
2	$D(64)$	$\{1057, 993, -1023\}$	$\{993, -1023, 126897\}$	$\{-1023, 126897, 511297137\}$
3	$D(96)$	$\{2353, 2257, -2207\}$	$\{2257, -2207, 622513\}$	$\{-2207, 622513, 12195785857\}$
4	$D(128)$	$\{4161, 4033, -3839\}$	$\{4033, -3839, 1936161\}$	$\{-3839, 1936161, 117611533585\}$
5	$D(160)$	$\{6481, 6321, -5919\}$	$\{6321, -5919, 4681281\}$	$\{-5919, 4681281, 686555282001\}$

4 Python program

Python program is displayed below for endorsement of each of the singular 3-tuples with arithmetic values.

```
import math
Section=int(input('ENTER THE VALUE OF SECTION'))
if Section == 1:
  x=int(input('ENTER THE VALUE OF x = '))
  f=256 * x * x-32 * x
  g=256 * x * x-64 * x
  h=-224 * x * x-112 * x
  i=7200 * x * x * x * x+1440 * x * x * x+56 * x * x-72 * x
  j=1620000 * (x * x * x * x * x * x * x * x) + 648000 * (x * x * x * x * x * x * x)
    + 190800 * (x * x * x * x * x * x)+43200 * (x * x * x * x * x)
    + 2450 * (x * x * x * x)-20 * (x * x * x)+134 * (x * x)-68 * x
print('f(x)=', f,'g(x)=', g,'h(x)=', h,'i(x)=', i,'j(x)=', j)
  R=((f+g)/2)+80 * x+1
  Y=((g+h)/2)+80 * x+1
  Z=((h+f)/2)+80 * x+1
  M=((g+i)/2)+80 * x+1
  N=((h+i)/2)+80 * x+1
  P=((h+j)/2)+80 * x+1
  Q=((i+j)/2)+80 * x+1
root1=mathsqrt(R)
root2=mathsqrt(Y)
root3=mathsqrt(Z)
root4=mathsqrt(M)
root5=mathsqrt(N)
root6=mathsqrt(P)
root7=mathsqrt(Q)
if (int(root1+0.5) ** 2==R) and (int(root2+0.5) ** 2==Y) and (int(root3+0.5) ** 2==Z):
print('(f(x), g(x), h(x))=',(f, g, h), "is a Singular triple with D(80x+1)")
else:
print('(f(x), g(x), h(x))=',(f, g, h), "is not a Singular triple with D(80x+1)")
if (int(root2+0.5) ** 2==Y) and (int(root4+0.5) ** 2==M) and (int(root5+0.5) ** 2==N):
print('(g(x), h(x), i(x))=',(g, h, i), "is a Singular triple with D(80x+1)")
else:
print('(g(x), h(x), i(x))=',(g, h, i), "is not a Singular triple with D(80x+1)")
if (int(root5+0.5) ** 2==N) and (int(root6+0.5) * * 2==P) and (int(root7+0.5) ** 2==Q):
print('(h(x), i(x), j(x))=',(h, i, j), "is a Singular triple with D(80x+1)")
else:
print('(h(x), i(x), j(x))=',(h, i, j), "is not a Singular triple with D(80x+1)")
elif Section == 2:
X=int(input('ENTER THE VALUE OF X = '))
k=256 * X * X+16 * X+1
l=256 * X * X-16 * X+1
```

```
m=-224 * X * X-64 * X+1
n=7200 * X * X * X * X+1440 * X * X * X+56 * X * X-24 * X+1
p=1620000 * (X * X * X * X * X * X * X * X) + 648000 * (X * X * X * X * X * X * X)
  + 190800 * (X * X * X * X * X * X)+43200 * (X * X * X * X * X)
  + 2450 * (X * X * X * X)-20 * (X * X * X)+134 * (X * X)-20 * X+1
print('k(x)=', k,'l(x)=', l,'m(x)=', m,'n(x)=', n,'p(x)=', p)
A=((k+l)/2)+32 * X
B=((l+m)/2)+32 * X
C=((m+k)/2)+32 * X
D=((l+n)/2)+32 * X
E=((m+n)/2)+32 * X
F=((m+p)/2)+32 * X
G=((n+p)/2)+32 * X
root1=mathsqrt(A)
root2=mathsqrt(B)
root3=mathsqrt(C)
root4=mathsqrt(D)
root5=mathsqrt(E)
root6=mathsqrt(F)
root7=mathsqrt(G)
if (int(root1+0.5) ** 2==A) and (int(root2+0.5) ** 2==B) and (int(root3+0.5) ** 2==C):
print('(k(x), l(x), m(x))=',(k,l,m), "is a Singular triple with D(32x)")
else:
print('(k(x), l(x)m(x))=',(k,l,m), "is not a Singular triple with D(32x)")
if (int(root2+0.5) ** 2==B) and (int(root4+0.5) ** 2==D) and (int(root5+0.5) ** 2==E):
print('(l(x), m(x), n(x))=',(l,m,n), "is a Singular triple with D(32x)")
else:
print('(l(x), m(x), n(x))=',(l,m,n), "is not a Singular triple with D(32x)")
if (int(root5+0.5) ** 2==E) and (int(root6+0.5) ** 2==F)
  and (int(root7+0.5) ** 2==G):
print('(m(x), n(x), p(x))=',(m, n, p), "is a Singular triple with D(32x)")
else:
print('(m(x), n(x), p(x))=',(m, n, p), "is not a Singular triple with D(32x)")
```

Output of Some Examples

```
ENTER THE VALUE OF SECTION 1
ENTER THE VALUE OF x = 1
f(x)= 224 g(x)= 192 h(x)=-336 i(x)= 8624 j(x)= 2504496
(f(x), g(x), h(x))= (224, 192,-336) is a Singular triple with D(80x+1)
(g(x), h(x), i(x))= (192, -336, 8624) is a Singular triple with D(80x+1)
(h(x), i(x), j(x))= (-336, 8624, 2504496) is a Singular triple with D(80x+1)

ENTER THE VALUE OF SECTION 1
ENTER THE VALUE OF x = 2
f(x)= 960 g(x)= 896 h(x)=-1120 i(x)= 126800 j(x)= 511297040
(f(x), g(x), h(x))= (960, 896,-1120) is a Singular triple with D(80x+1)
(g(x), h(x), i(x))= (896, -1120, 126800) is a Singular triple with D(80x+1)
(h(x), i(x), j(x))= (-1120,126800,511297040) is a Singular triple with D(80x+1)

ENTER THE VALUE OF SECTION 2
ENTER THE VALUE OF X = 1
k(x)= 273 l(x)= 241 m(x)=-287 n(x)= 8673 p(x)= 2504545
(k(x), l(x), m(x))= (273, 241,-287) is a Singular triple with D(32x)
(l(x), m(x), n(x))= (241, -287, 8673) is a Singulartriple with D(32x)
(m(x), n(x), p(x))= (-287, 8673, 2504545) is a Singular triple with D(32x)

ENTER THE VALUE OF SECTION 2
ENTER THE VALUE OF X = 2
k(x)= 1057 l(x)= 993 m(x)=-1023 n(x)= 126897 p(x)= 511297137
(k(x), l(x), m(x))= (1057, 993, -1023) is a Singular triple with D(32x)
(l(x), m(x), n(x))= (993,-1023, 126897) is a Singular triple with D(32x)
(m(x), n(x), p(x))= (-1023, 126897, 511297137) is a Singular{triple with D(32x)
```

5 Conclusion

In this paper, the development of finding order of singular 3-tuples $\{f(x), g(x), h(x)\}$, $\{g(x), h(x), i(x)\}$ etc entailing Abel's polynomial and Cyclotomic polynomial in which the arithmetic mean of any two polynomials in each 3-tuples added by a monomial with integer coefficients leftovers square of a polynomial is recognized. Additionally, Python program for inspection of each of such singular 3-tuples with numerical values is presented. To conclude this, one can pursuit varieties of 3-tuples nourishing innumerable exciting features.

Acknowledgement. Authors are grateful to the Editor and Reviewer for their fruitful suggestions to improve the paper.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 106-109**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

CRYPTANALYSIS USING LAPLACE TRANSFORM OF ERROR FUNCTION

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(Received: August 05, 2022*; In format: August* 13, 2022*; Revised: January* 30, 2023*; Accepted: February* 04, 2023*)*

DOI: https://doi.org/10.58250/jnanabha.2023.53113

Abstract

This paper presents a new cryptographic scheme for encryption and decryption by introducing the Laplace transform of error function. Here we have used the concept of congruence relation and modular arithmetic to find symmetric key, cipher text and Decryption process. The implementation has been done using Matlab.
2020 Mathematical Sciences Classification: 11A07, 44A10, 94A60, 11T71
Keywords and Phrases: Laplace transform, network security, error function, cipher text, symmetric key, congruence relation.

1 Introduction

Cryptography is a scientific technique related to aspects of information security such as data integrity, entity authentication and data origin authentication. Cryptography is a set of techniques to provide information security. It helps to store sensitive information, transmit it across insecure networks like internet so that it can't be read by anyone except the intended receiver.

Analysis of cryptographic security leads to using theoretical computer science especially complexity theory. The actual implementation of crypto systems and the hard work of carrying out security analysis for specific cryptosystems fall into engineering and practical computer science and computing. The persons or systems performing cryptanalysis in order to break a crypto system are called attackers. The process of such type of attacking is called hacking. Some cryptographic algorithms are very trivial to understand, replicate and therefore easily cracked. To secure the data from hackers it needs to be encrypted with high level of security.

Encryption and Decryption are carried forward using mathematical algorithms in cryptography. Initially Stanoyevitch [14] introduced cryptography with mathematical foundations and computer implementations. Overbey, Traves and Wojdylo [9] used technique of Hill cipher keyspace consist of all matrices that are invertible. Use of matrices for encryption and decryption were found by Dhanorkar and Hiwarekar[1]. After that [3,4, 5,8] used Laplace transform techniques for the cryptographic purpose by combining infinite series of various functions . Dhingra, Savalgi and Jain [2] presented new scheme for the cryptography by combining infinite series and Laplace transform using ASCII code. Genoglu [16] used a new method of cryptography using Lapalce transform of Hyperbolic function. Some interesting results are found in the literature $[7, 10-13]$ regarding modular arithmetic and cryptography.

In this paper, the process of encryption is expanded using series of error function and taking its Laplace transform. On the basis of literature survey we found while using the cryptography on the basis of Laplace transform only functions with positive terms were considered so far, but in error function we have an alternating series, so we used the concept of congruence relation to change the sign of transformed series terms coefficients , and use modular arithmetic to find symmetric key , cipher text and Decryption process.

2 Some Basic Terminologies

2.1 Plain text

It signifies a message that can be understood by the sender, the recipient and also by anyone else who gets access to that message.

2.2 Cipher text

When a plain text message is codified using any suitable scheme, the resulting message is called as cipher text.

2.3 Encryption and Decryption

Encryption transforms a plain text message into cipher text, whereas decryption transforms a cipher text message back into plain text.

2.4 Symmetric and Asymmetric Key

Cryptography algorithms classified mainly into two major types: Symmetric-key cryptography and public key (Asymmetric) cryptography [15]. In Symmetric-key cryptography, each sender and receiver shared the same key used to encrypt and decrypt data with disadvantage of key management required to keep the key secure. The Data Encryption Standard *(DES)* and the Advanced Encryption Standard *(AES)* are examples of Symmetrickey cryptography methods. In public-key cryptography, each sender and receiver use two different keys to encrypt and decrypt data - public key and private key-, the public key can be freely distributed, while its paired private key must remain secret. In public-key cryptography, we overcome the key management distribution issue of Symmetric-key cryptography, but at the expense of performance speed

2.5 Laplace transform

Laplace transform is useful out of many transformations that are used for security purposed and as per the requirement which is a useful factor for changing key where algorithm plays an important role. That's why it will be difficult for a hacker to trace the key by any mode. For any function $f(t), t \geq 0$ Laplace transform $L\{f(t)\}$ is defined as

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = f(s),$$

where t is known as time domain parameter and S (may be real or complex) is known as frequency domain parameter.

2.6 Error Function

$$erf(\sqrt{t})\} = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} du = \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} (-1)^i \frac{t^{(2i+1)/2}}{i!(2i+1)}.$$

3 Encryption Algorithm

The proposed algorithm uses the Laplace transform of error function to generate the cipher text and a sender key as symmetric encryption. In the beginning this secret key between sender and receiver is determined and shared on the basis of quotient reminder theorem and congruence relation.

Step 1: Sender and receiver agree on secret key.

Step 2: Select the message to be sent and convert each plain text alphabet into as a number in an increasing sequence as $A = 0, B = 1, C = 2, \ldots, Z = 25$.

Let the plain text is "*SUBJECT*" and it is equivalent to 1820194219.

Let $C_0 = 18, \mathrm{C}_1 = 20, \mathrm{C}_2 = 1, \mathrm{C}_3 = 9, \mathrm{C}_4 = 4, \mathrm{C}_5 = 2, \mathrm{C}_6 = 19$.

Step 3: Now writing these numbers as coefficients of $\mathrm{erf}(\sqrt{t})$, neglecting higher coefficients ($i \geq 7$) and considering $f(t) = C\,\mathrm{erf}(\sqrt{t})$,

we get

$$\begin{aligned} f(t) &= \frac{2}{\sqrt{\pi}} \left[C_0 t^{1/2} - \frac{C_1 t^{3/2}}{3} + \frac{C_2 t^{5/2}}{2!5} - \frac{C_3 t^{7/2}}{3!7} + \frac{C_4 t^{9/2}}{4!9} - \frac{C_5 t^{11/2}}{5!11} + \frac{C_6 t^{13/2}}{6!13} \right] \\ &= \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} (-1)^i \frac{t^{(2i+1)/2}}{i!(2i+1)} C_i. \end{aligned} \tag{3.1}$$

Step 4: Now taking Laplace transform of (3.1) , we get

$$L\{f(t)\} = \frac{2}{\sqrt{\pi}} L \left\{ C_0 t^{1/2} - \frac{C_1 t^{3/2}}{3} + \frac{C_2 t^{5/2}}{2!5} - \frac{C_3 t^{7/2}}{3!7} + \frac{C_4 t^{9/2}}{4!9} - \frac{C_5 t^{11/2}}{5!11} + \frac{C_6 t^{13/2}}{6!13} \right\}.$$

Using $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}, (n+1) > 0$,

$$L\{f(t)\} = \frac{C_0}{s^{3/2}} - \frac{C_1}{2s^{5/2}} + \frac{3C_2}{2^3 s^{7/2}} - \frac{5C_3}{2^4 s^{9/2}} + \frac{105C_4}{2^6 s^{11/2}} - \frac{63C_5}{2^8 s^{13/2}} + \frac{231C_6}{2^{10} s^{15/2}}.$$

Alternating terms of the series are converted using congruence relation to integer modulo 26 .

$$L\{f(t)\} = \frac{C_0}{s^{3/2}} + \frac{25C_1}{2s^{5/2}} + \frac{3C_2}{2^3 s^{7/2}} + \frac{21C_3}{2^4 s^{9/2}} + \frac{105C_4}{2^6 s^{11/2}} + \frac{15C_5}{2^8 s^{13/2}} + \frac{231C_6}{2^{10} s^{15/2}}. \tag{3.2}$$

Substituting the values of $C_i i = 0, 1, 2, \ldots, 6$ and simplifying, we get

$$2^{10} L\{f(t)\} = \frac{18432}{s^{3/2}} + \frac{256000}{s^{5/2}} + \frac{384}{s^{7/2}} + \frac{12096}{s^{9/2}} + \frac{6720}{s^{11/2}} + \frac{120}{s^{13/2}} + \frac{4389}{s^{15/2}}.$$

Step 5: Now we calculate r_i using $r_i = M_i \bmod (26)$ and q_i as quotient

i	M_i	r_i	q_i
0	18432	24	708
1	256000	4	9846
2	384	20	14
3	12096	6	465
4	6720	12	258
5	120	16	4
6	4389	21	168

Hence the message "*SUBJECT*" is encrypted to "*YEUGMQV*" as cipher text and the symmetric secret key as 708 9846 14 465 258 4 168, delivered to receiver.

4 Decryption Algorithm

Steps involved in Decryption are as follows:

Step 1. Consider the cipher text and key received from the sender. In the above example cipher text is "*YEUGMQV*" and the secret key 708 9846 14 465 258 4 168.

Step 2 . Convert the given cipher text to a corresponding finite sequence of numbers, 24 4 20 6 12 16 21, comparing with (3.2) using modular arithmetic equivalent in mod 26, we get

$$C_0 \cdot 2^{10} = 26.K_0 + 24 \Rightarrow C_0 = \frac{26.K_0 + 24}{2^{10}} \Rightarrow \begin{cases} K_0 = 196 & C_{0,1} = 5 \\ K_0 = 708 & C_{0,2} = 18 \end{cases}$$

$$C_1 \cdot 2^9 \cdot 25 = 26.K_1 + 4 \Rightarrow C_1 = \frac{26.K_1 + 4}{2^9.25} \Rightarrow \begin{cases} K_1 = 3446 & C_{1,1} = 7 \\ K_1 = 9846 & C_{1,2} = 20 \end{cases}$$

$$C_2 \cdot 2^7 \cdot 3 = 26.K_2 + 20 \Rightarrow C_2 = \frac{26.K_2 + 20}{2^7 \cdot 3} \Rightarrow \begin{cases} K_2 = 14 & C_{2,1} = 1 \\ K_2 = 206 & C_{2,2} = 14 \end{cases}$$

$$C_3.2^6.21 = 26.K_3 + 6 \Rightarrow C_3 = \frac{26.K_3 + 6}{2^6.21} \Rightarrow \begin{cases} K_3 = 465 & C_{3,1} = 9 \\ K_3 = 1137 & C_{3,2} = 22 \end{cases}$$

$$C_4 \cdot 2^4.105 = 26.K_4 + 12 \Rightarrow C_4 = \frac{26.K_4 + 12}{2^4.105} \Rightarrow \begin{cases} K_4 = 258 & C_{4,1} = 4 \\ K_4 = 1098 & C_{4,2} = 17 \end{cases}$$

$$C_5 \cdot 2^2 \cdot 15 = 26.K_5 + 16 \Rightarrow C_5 = \frac{26.K_5 + 16}{2^2.15} \Rightarrow \begin{cases} K_5 = 4 & C_{5,1} = 2 \\ K_5 = 34 & C_{5,2} = 15 \end{cases}$$

$$=$$

Step 3. Now using the secret key 708 9846 14 465 258 4 168 we get required C_i's as 18,20 , $1, 9, 4, 2, 19$, now convert the numbers of above finite sequence to alphabets the original plain text is obtained as "*SUBJECT*".

5 Conclusion

The proposed algorithm give us an encrypted cipher text and after decryption we are getting a original plain text, thus the proposed method is valid and helpful in the case when an alternative series such as error function is used in cryptography. On the basis of modular arithmetic we get proper results instead of using sequential numbers as coefficients we can use ASCII code, with the same function for getting more secured chipper text and key.

In the proposed work we expand an innovative cryptographic scheme using Laplace transforms of error function and modular arithmetic functions.

Acknowledgement. We are thankful to Editors and Reviewers for their valuable suggestions to improve the article.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 110-117**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

GROWTH PROPERTIES OF AN ENTIRE FUNCTION OF SEVERAL COMPLEX VARIABLES ON THE BASIS OF RELATIVE ORDER

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(Received: January 06, 2023; In format: January 08, 2023; Revised : February 04, 2023; Accepted: February 12, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53114

Abstract

In this paper, we study some comparative growth properties of composite entire function of several complex variables, on the basis of relative order and relative lower order of an entire function with respect to an entire function. Here we are defining some definitions related to relative order and relative lower order in terms of central index. A Rastogi, Biswas, Pramanik, Somsundaram many authors work on central index.

2020 Mathematical Sciences Classification: 32A05, 32C15.
Keywords and Phrases: Entire function, maximum term, several complex variables, maximum modulus, composition, generalized relative order, generalized relative L order, generalized relative L^* order, central index.

1 Introduction

Let $f(z) = \sum_{m_j=0, j=1,2,\cdots,n}^{\infty} a_{m_j} z_j^{m_j}$ where $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, be an entire function of n-complex variables and $M_f(r_1, r_2, \cdots, r_n) := \max_{\{|z_j|=r_j; j=1,2,\cdots,n\}} |f(z_j)|$ be the maximum modulus of f and let $\mu_f(r_1, r_2, \cdots, r_n) := \max_{m_j \geq 0, j=1,2,\cdots,n} |a_{m_j}| r_j^{m_j}$ be the maximum term of f. The central index $\nu_f(r_1, r_2, \cdots, r_n) := \{m_k \mid \mu_f(r_1, r_2, \cdots, r_n) = |a_{m_k}| r^{m_k}\}$ or $|a_{\nu_f(r_1,r_2,\cdots,r_n)}| r^{\nu_f(r_1,r_2,\cdots,r_n)} = \mu_f(r_1, r_2, \cdots, r_n)$. Clearly $\mu_f(r_1, r_2, \cdots, r_n)$ is non decreasing function and $\mu_f(r_1, r_2, \cdots, r_n) \leq M_f(r_1, r_2, \cdots, r_n)$.

Let g be an entire function. Then the ratio $\frac{M_f(r_1,r_2,\cdots,r_n)}{M_g(r_1,r_2,\cdots,r_n)}$, where $r_k \to \infty$, $k = 1, 2, \cdots, n$ is called the growth of f with respect to g in term of maximum moduli.
In fact $\mu_f(r_1, r_2, \cdots, r_n)$ is much weaker than $M_f(r_1, r_2, \cdots, r_n)$ in some sense. So from another angle of view $\frac{\mu_f(r_1,r_2,\cdots,r_n)}{\mu_g(r_1,r_2,\cdots,r_n)}$, where $r_k \to \infty$, $k = 1, 2, \cdots, n$ is called growth of f with respect to g, in term of maximum terms, now in similar way we get growth of f with respect to g in terms of central index, $\frac{\nu_f(r_1,r_2,\cdots,r_n)}{\nu_g(r_1,r_2,\cdots,r_n)}$ where $r_k \to \infty$, $k = 1, 2, \cdots, n$. Rastogi [7], Biswas [1] and Pramanik [10] worked on central index. The details of the notations of maximum modulus, entire functions, growth, maximum term and central index for one variable appear in [1,2,3,4,6,7,8,9,11,13].

To start our paper we just recall the following definitions:

2 Definitions

Definition 2.1 ([12]). *Let f and g be entire functions. The relative order of f with respect to g is defined by*

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \nu_g^{-1} \nu_f(r)}{\log[r]},$$

and relative lower order is defined by

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log \nu_g^{-1} \nu_f(r)}{\log[r]}.$$

Definition 2.2 ([12]). *L order and L lower order for an entire function, where $L \equiv L(r)$ is a positive continuous function such that $L(ar) \sim L(r)$ as $r \to \infty$, for every positive constant a, on the basis of maximum modulus $M(r, f)$. The relative L order of an entire function f with respect to g, in terms of central index is defined by*

$$\rho_g^L(f) = \limsup_{r\to\infty} \frac{\log \nu_g^{-1}\nu_f(r)}{\log[rL(r)]},$$

and relative L lower order is defined by

$$\lambda_g^L(f) = \liminf_{r\to\infty} \frac{\log \nu_g^{-1}\nu_f(r)}{\log[rL(r)]}.$$

In the light of Definition 2.2 and from the concept of several complex variables [5], we would like to introduce the following definitions for several complex variables:

Definition 2.3. *Let f and g be entire functions of n complex variables. The relative order and relative lower order of f with respect to g are defined by*

$$\rho_g(f) = \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_g^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log(r_1 r_2 \cdots, r_n)},$$

and

$$\lambda_g(f) = \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_g^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log(r_1 r_2 \cdots, r_n)},$$

respectively.

Definition 2.4. *Let f and g be entire functions of n complex variables. The relative L order and relative L lower order of f with respect to g, are defined by*

$$\rho_g^L(f) = \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_g^{-1}\nu_f(r_1, r_2 \cdots, r_n)}{\log[(r_1 r_2 \cdots, r_n)L(r_1 r_2 \cdots, r_n)]},$$

and

$$\lambda_g^L(f) = \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_g^{-1}\nu_f(r_1, r_2 \cdots, r_n)}{\log[(r_1 r_2 \cdots, r_n)L(r_1 r_2 \cdots, r_n)]},$$

respectively. Here idea of L order (respectively, L lower order) of entire function is defined in [10,12].

Definition 2.5. *Let f and g be entire functions of n complex variables. The relative L^* order and relative L^* lower order of f with respect to g is defined by*

$$\rho_g^{L^*}(f) = \limsup_{r_1,r_2,\cdots,r_n\to\infty} \frac{\log \nu_g^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log[(r_1 r_2 \cdots, r_n)\exp^{L(r_1 r_2 \cdots, r_n)}]}.$$

Here idea of L^* order (respectively, L^* lower order) of an entire function where L^* is nothing but a weaker assumption of L [13]*. The relative L^* lower order is defined by*

$$\lambda_g^{L^*}(f) = \liminf_{(r_1,r_2,\cdots,r_n\to\infty)} \frac{\log \nu_g^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log[(r_1 r_2 \cdots, r_n)\exp^{L(r_1 r_2 \cdots, r_n)}]}.$$

3 Results

In this section, we establish some interesting results.

Theorem 3.1. *Let f, g and h be entire functions such that $0 < \lambda_h(f \circ g) \leq \rho_h(f \circ g) < \infty$ and $0 < \lambda_h(f) \leq \rho_h(f) < \infty$.*
Then

$$\frac{\lambda_h(f\circ g)}{\rho_h f} \leq \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq min\left\{\frac{\lambda_h(f\circ g)}{\lambda_h f}, \frac{\rho_h(f\circ g)}{\rho_h{}^{p,q} f}\right\}$$
$$\leq max\left\{\frac{\lambda_h(f\circ g)}{\lambda_h f}, \frac{\rho_h(f\circ g)}{\rho_h f}\right\} \leq \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h{}^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h(f\circ g)}{\lambda_h f}.$$

Proof. From the definition of relative order defined in (2.3), for arbitrary $\epsilon > 0$, we get the following

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \leq (\rho_h(f\circ g) + \epsilon)\log(r_1 r_2 \cdots, r_n), \tag{3.1}$$

and

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \geq (\lambda_h(f\circ g) - \epsilon)\log(r_1 r_2 \cdots, r_n), \tag{3.2}$$

when $r_k \to \infty$, where $k = 1, 2, \cdots, n$

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \leq (\lambda_h(f\circ g) + \epsilon)\log(r_1 r_2 \cdots, r_n), \tag{3.3}$$

and

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \geq (\rho_h(f\circ g) - \epsilon)\log(r_1 r_2 \cdots, r_n). \tag{3.4}$$

Similarly when we replace $f\circ g$ by f in the above equation, we get the following

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \leq (\rho_h(f) + \epsilon)\log(r_1 r_2 \cdots r_n), \tag{3.5}$$

and

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \geq (\lambda_h(f) - \epsilon)\log(r_1 r_2 \cdots r_n). \tag{3.6}$$

When $r_k \to \infty$, where $k = 1, 2, \cdots, n$

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \leq (\lambda_h(f) + \epsilon)\log(r_1 r_2 \cdots r_n), \tag{3.7}$$

and

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \geq (\rho_h(f) - \epsilon)\log(r_1 r_2 \cdots r_n). \tag{3.8}$$

From (3.2) and (3.5) it follows for sufficiently large value of $(r_1, r_2, \cdots, r_n)$ that

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\lambda_h(f\circ g) - \epsilon}{\rho_h(f) + \epsilon}.$$

Since ϵ is arbitrary

$$\liminf_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\lambda_h(f\circ g)}{\rho_h(f)}. \tag{3.9}$$

From (3.3) and (3.6), we obtain

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\lambda_h(f\circ g) + \epsilon}{\lambda_h(f) - \epsilon}.$$

Since ϵ is arbitrary

$$\liminf_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\lambda_h(f\circ g)}{\lambda_h(f)}, \tag{3.10}$$

from (3.1) and (3.8)

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h(f\circ g) + \epsilon}{\rho_h(f) - \epsilon}.$$

Since ϵ is arbitrary

$$\liminf_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h(f\circ g)}{\rho_h(f)}. \tag{3.11}$$

From (3.9), (3.10) and (3.11), we obtain

$$\frac{\lambda_h(f\circ g)}{\rho_h(f)} \leq \liminf_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq min\left\{\frac{\lambda_h(f\circ g)}{\lambda_h(f)}, \frac{\rho_h(f\circ g)}{\rho_h(f)}\right\}. \tag{3.12}$$

From (3.2) and (3.7) for $r_k \to \infty$, where $k = 1, 2, \cdots, n$, we get

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\lambda_h(f\circ g) - \epsilon}{\lambda_h(f) + \epsilon}.$$

As $\epsilon > 0$ is arbitrary, we obtain

$$\limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \geq \frac{\lambda_h(f\circ g))}{\lambda_h(f))}. \tag{3.13}$$

From (3.1) and (3.6), we obtain the following

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq \frac{\rho_h(f\circ g)+\epsilon}{\lambda_h(f)-\epsilon}.$$

As $\epsilon > 0$, we obtain

$$\limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq \frac{\rho_h(f\circ g)}{\lambda_h(f)}. \tag{3.14}$$

Similarly from (3.4) and (3.5)

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \geq \frac{\rho_h(f\circ g)-\epsilon}{\rho_h(f)+\epsilon}.$$

As ϵ is arbitrary

$$\limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \geq \frac{\rho_h(f\circ g)}{\rho_h(f)}. \tag{3.15}$$

Combining (3.13), (3.14) and (3.15), we obtain

$$max\left\{\frac{\lambda_h{}^{p,q}(f\circ g)}{\lambda_h f}, \frac{\rho_h(f\circ g)}{\rho_h f}\right\} \leq \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu f(r_1,r_2,\cdots,r_n)} \leq \frac{\rho_h(f\circ g)}{\lambda_h(f)}. \tag{3.16}$$

Hence, from (3.12) and (3.16), we obtain

$$\frac{\lambda_h(f\circ g)}{\rho_h(f)} \leq \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq min\left\{\frac{\lambda_h(f\circ g)}{\lambda_h f}, \frac{\rho_h(f\circ g)}{\rho_h f}\right\}$$

$$\leq max\left\{\frac{\lambda_h(f\circ g)}{\lambda_h f}, \frac{\rho_h(f\circ g)}{\rho_h f}\right\} \leq \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h{}^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq \frac{\rho_h(f\circ g)}{\lambda_h(f)}.$$

□

Theorem 3.2. *Let f, g and h be entire functions such that $0 < \lambda_h(f\circ g) \leq \rho_h(f\circ g) < \infty$ and $0 < \lambda_h(f) \leq \rho_h(f) < \infty$.*
Then

$$\frac{\lambda_h(f\circ g)}{\rho_h(f)} \leq \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq \frac{\lambda_h(f\circ g)}{\lambda_h f}$$

$$\leq \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h{}^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq \frac{\rho_h(f\circ g)}{\lambda_h(f)}.$$

Proof. The above theorem follows from (3.9), (3.10), (3.13) and (3.16). □

Theorem 3.3. *Let f, g and h be entire functions such that $0 < \lambda_h(f\circ g) \leq \rho_h(f\circ g) < \infty$ and $0 < \lambda_h(f) \leq \rho_h(f) < \infty$.*
Then

$$\liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq \frac{\rho_h(f\circ g)}{\rho_h f}$$

$$\leq \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h{}^{-1}\nu_f(r_1,r_2,\cdots,r_n)}.$$

Proof. From the definition of relative order of f for $\epsilon > 0$ and $r_k \to \infty$, where $k = 1,2,\cdots,n$

$$\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n) \geq (\rho_h(f)-\epsilon)\log(r_1r_2\cdots r_n), \tag{3.17}$$

and

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n) \leq (\rho_h(f\circ g)+\epsilon)\log(r_1r_2\cdots r_n). \tag{3.18}$$

From (3.17) and (3.18) we obtain

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h(f\circ g)+\epsilon}{\rho_h f - \epsilon}.$$

As $\epsilon > 0$

$$\liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h(f\circ g)}{\rho_h f}. \tag{3.19}$$

Since $r_k \to \infty$, where $k = 1, 2, \cdots, n$

$$\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \geq (\rho_h^{p,q}(f\circ g) - \epsilon) \log^{[q]}(r_1 r_2 . \cdots, r_n). \tag{3.20}$$

Now combining form of (3.5) and (3.20) is the following:

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\rho_h(f\circ g)+\epsilon}{\rho_h f - \epsilon}.$$

As $\epsilon > 0$ is arbitrary

$$\limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\rho_h(f\circ g)}{\rho_h(f)}. \tag{3.21}$$

Hence, from (3.21) and (3.19), we obtain

$$\begin{aligned}\liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} &\leq \frac{\rho_h(f\circ g)}{\rho_h f} \\ &\leq \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_{f\circ g}^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)}.\end{aligned}$$

□

Theorem 3.4. *Let f, g and h be entire functions such that $0 < {\lambda_h}^L(f\circ g) \leq {\rho_h}^L(f\circ g) < \infty$ and $0 < {\lambda_h}^L(f) \leq {\rho_h}^L(f) < \infty$.*

Then

$$\begin{aligned}\frac{{\lambda_h}^L(f\circ g)}{{\rho_h}^L(f)} \leq \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq min\left\{\frac{{\lambda_h}^L(f\circ g)}{{\lambda_h}^L f}, \frac{{\rho_h}^L(f\circ g)}{{\rho_h}^L f}\right\} \\ \leq max\left\{\frac{{\lambda_h}^L(f\circ g)}{{\lambda_h}^L f}, \frac{{\rho_h}^L(f\circ g)}{{\rho_h}^L f}\right\} \leq \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log {\nu_h}^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{{\rho_h}^L(f\circ g)}{{\lambda_h}^L(f)}.\end{aligned}$$

Proof. From the definition (2.4) and arbitrary $\epsilon > 0$

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \leq (\rho_h^L(f\circ g) + \epsilon) \log[(r_1 r_2 \cdots r_n) L(r_1 r_2 \cdots r_n)], \tag{3.22}$$

and

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \geq (\lambda_h^L(f\circ g) - \epsilon) \log[(r_1 r_2 \cdots r_n) L(r_1 r_2 \cdots r_n)]. \tag{3.23}$$

Since $r_k \to \infty$, where $k = 1, 2, \cdots, n$

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \leq (\lambda_h^L(f\circ g) + \epsilon) \log[(r_1 r_2 \cdots r_n) L(r_1 r_2 \cdots r_n)], \tag{3.24}$$

and

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \geq (\rho_h^L(f\circ g) - \epsilon) \log[(r_1 r_2 \cdots r_n) L(r_1 r_2 \cdots r_n)]. \tag{3.25}$$

Similarly for the function f, we obtain

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \leq (\rho_h^L(f) + \epsilon) \log[(r_1 r_2 \cdots r_n) L(r_1 r_2 \cdots r_n)], \tag{3.26}$$

and

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \ldots, r_n) \geq (\lambda_h^L(f) - \epsilon) \log[(r_1 r_2 \cdots r_n) L(r_1 r_2 \cdots r_n)]. \tag{3.27}$$

Since $r_k \to \infty$, where $k = 1, 2, \cdots, n$

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \leq (\lambda_h^L(f) + \epsilon) \log[(r_1 r_2 \cdots r_n) L(r_1 r_2 \cdots r_n)], \tag{3.28}$$

and

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \geq (\rho_h^L(f) - \epsilon) \log[(r_1 r_2 \cdots r_n) L(r_1 r_2 \cdots r_n)]. \tag{3.29}$$

From (3.23) and (3.26) it follows for sufficiently large value of $(r_1, r_2, \cdots, r_n)$

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\lambda_h^L(f\circ g) - \epsilon}{\rho_h^L(f) + \epsilon}.$$

Since $\epsilon > 0$ is arbitrary

$$\liminf_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\lambda_h^L(f\circ g)}{\rho_h^L(f)}. \tag{3.30}$$

From (3.24) and (3.27), we obtain

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \ldots r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\lambda_h^L(f\circ g) + \epsilon}{\lambda_h^L(f) - \epsilon}.$$

Since $\epsilon > 0$ is arbitrary

$$\liminf_{r_1, r_2, \cdots, r_n\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\lambda_h^L(f\circ g)}{\lambda_h^L(f)}. \tag{3.31}$$

From (3.22) and (3.29), we obtain

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h^L(f\circ g) + \epsilon}{\rho_h^L(f) - \epsilon}.$$

As $\epsilon > 0$

$$\liminf_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h^L(f\circ g)}{\rho_h^L(f)}. \tag{3.32}$$

From (3.30), (3.31) and (3.32), we get the following

$$\frac{\lambda_h^L(f\circ g)}{\rho_h^L(f)} \leq \liminf_{r_1, r_2, \cdots, r_n\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq min\left\{\frac{\lambda_h^L(f\circ g)}{\lambda_h^L(f)}, \frac{\rho_h^L(f\circ g)}{\rho_h^L(f)}\right\}. \tag{3.33}$$

From (3.23) and (3.28) for $r_k \to \infty$, where $k = 1, 2, \cdots, n$, then

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\lambda_h^L(f\circ g) - \epsilon}{\lambda_h^L(f) + \epsilon}.$$

As $\epsilon > 0$ is arbitrary

$$\limsup_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \ldots, r_n)} \geq \frac{\lambda_h^L(f\circ g)}{\lambda_h^L(f)}. \tag{3.34}$$

From (3.22) and (3.27)

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h^L(f\circ g) + \epsilon}{\lambda_h^L(f) - \epsilon}.$$

As $\epsilon > 0$ is arbitrary

$$\limsup_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h^L(f\circ g)}{\lambda_h^L(f)}. \tag{3.35}$$

Similarly from (3.25) and (3.26), we get the following:

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\rho_h^L(f\circ g) - \epsilon}{\rho_h^L(f) + \epsilon}.$$

As $\epsilon > 0$ is arbitrary

$$\limsup_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\rho_h^L(f\circ g)}{\rho_h^L(f)}. \tag{3.36}$$

From (3.34), (3.35) and (3.36) we obtain

$$max\left\{\frac{\lambda_h{}^L(f\circ g)}{\lambda_h{}^L f}, \frac{\rho_h{}^L(f\circ g)}{\rho_h{}^L f}\right\} \leq \limsup_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h{}^L(f\circ g)}{\lambda_h{}^L(f)}. \tag{3.37}$$

From (3.33) and (3.37)

$$\frac{\lambda_h{}^L(f\circ g)}{\rho_h{}^L(f)} \leq \liminf_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq min\left\{\frac{\lambda_h{}^L(f\circ g)}{\lambda_h{}^L f}, \frac{\rho_h{}^L(f\circ g)}{\rho_h{}^L f}\right\}$$

$$\leq max\left\{\frac{\lambda_h{}^L(f\circ g)}{\lambda_h{}^L f}, \frac{\rho_h{}^L(f\circ g)}{\rho_h{}^L f}\right\} \leq \limsup_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \ldots, r_n)}{\log \nu_h{}^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h{}^L(f\circ g)}{\lambda_h{}^L(f)}.$$

□

From the above theorems we can obtain the following corollaries:

Corollary 3.1. *Let f, g and h be entire functions such that $0 < {\lambda_h}^L(f \circ g) \le {\rho_h}^L(f \circ g) < \infty$ and $0 < {\lambda_h}^L(f) \le {\rho_h}^L(f) < \infty$.}*

$$\frac{{\lambda_h}^L(f \circ g)}{{\rho_h}^L(f)} \le \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \le \frac{{\lambda_h}^L(f \circ g)}{{\lambda_h}^L f}$$
$$\le \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log {\nu_h}^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \le \frac{{\rho_h}^L(f \circ g)}{{\lambda_h}^L(f)}.$$

Proof. When we take L-lower order and L-order in Theorem 3.2 then we get the corollary. ☐

Corollary 3.2. *Let f, g and h be entire functions such that $0 < {\lambda_h}^L(f \circ g) \le {\rho_h}^L(f \circ g) < \infty$ and $0 < {\lambda_h}^L(f) \le {\rho_h}^L(f) < \infty$.*

Then

$$\liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \le \frac{{\rho_h}^L(f \circ g)}{{\rho_h}^L f}$$
$$\le \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log {\nu_h}^{-1}\nu_f(r_1,r_2,\cdots,r_n)}.$$

Proof. When we take L-order and L- lower order in Theorem 3.3 we can get the result. ☐

Corollary 3.3. *Let f, g and h be entire functions such that $0 < {\lambda_h}^{L^*}(f \circ g) \le {\rho_h}^{L^*}(f \circ g) < \infty$ and $0 < {\lambda_h}^{L^*}(f) \le {\rho_h}^{L^*}(f) < \infty$.*

Then

$$\frac{{\lambda_h}^{L^*}(f \circ g)}{{\rho_h}^{L^*}(f)} \le \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \le \frac{{\lambda_h}^{L^*}(f \circ g)}{{\lambda_h}^{L^*} f}$$
$$\le \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log {\nu_h}^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \le \frac{{\rho_h}^{L^*}(f \circ g)}{{\lambda_h}^{L^*}(f)}.$$

Proof. When we take L^* order and L^* lower order in Theorem 3.2 we can get the result. ☐

Corollary 3.4. *Let f, g and h be entire functions such that $0 < {\lambda_h}^{L^*}(f \circ g) \le {\rho_h}^{L^*}(f \circ g) < \infty$ and $0 < {\lambda_h}^{L^*}(f) \le {\rho_h}^{L^*}(f) < \infty$.*

Then

$$\liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \le \frac{{\rho_h}^{L^*}(f \circ g)}{{\rho_h}^{L^*} f}$$
$$\le \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log {\nu_h}^{-1}\nu_f(r_1,r_2,\ldots,r_n)}.$$

Proof. When we take L^* order and L^* lower order in Theorem 3.3 we can get the result. ☐

4 Conclusion

In this paper we have established some inequalities between relative order and relative lower order of entire functions of several complex variables in terms of central index. Further we have obtained some corollaries of the above theorems.

Acknowledgement. We are very much grateful to the Editor and Referee for their fruitful suggestions to bring the paper in its present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 118-124**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

SOME FIXED POINT RESULTS FOR CYCLIC $(\psi, \phi, Z)-$ CONTRACTION IN PARTIAL METRIC SPACES

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(Received: January 19, 2023; In format: February 03, 2023; Revised: February 07, 2023; Accepted: February 08, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53115

Abstract

In this paper, we present a new type of cyclic $(\psi, \phi, Z)-$ contraction which is a combination of cyclic $(\psi, \phi, A, B)-$ contraction and $Z-$contraction in the framework of complete partial metric space with the help of simulation function. We investigate the existence of fixed point result using cyclic $(\psi, \phi, Z)-$ contraction in the setting of complete partial metric space. Also we give an example to clarify the main result.

2020 Mathematical Sciences Classification: 47H09, 47H10, 54H25.
Keywords and Phrases: Partial metric spaces, Simulation function, Cyclic mapping, Cyclic (ψ, ϕ, Z)-contraction.

1 Introduction

The idea of partial metric space was introduced by Mathews ([19]) and it is defined as the same point in partial metric does not necessarily need to be zero. In 2003, Kirk ([17]) introduced the notion of cyclic contraction. Karapinar ([14]) explored cyclic contraction in partial metric space in 2012 while Agarwal ([2]) defined a very useful cyclic generalized contractions on the complete partial metric space in the same year. Khojasteh ([16]) introduced new approach in fixed point theory by using a simulation function. This paper inspired us to find a different type of cyclic contraction in complete partial metric space. Many authors have already demonstrated different types of contractions in partial metric spaces (see [4, 5, 6, 7, 8, 11]).

In this paper, we establish a cyclic $(\psi, \phi, Z)-$ contraction in complete partial metric space to determine a unique fixed point.

On the other hand, the concept of simulation function was established in [16] to unify the existing fixed point results.

2 Preliminaries

Definition 2.1 ([16]). *A function $\xi : [0, \infty) \to [0, \infty)$ satisfying the following conditions*

(ξ_1) $\xi(0, 0) = 0$*;*

(ξ_2) $\xi(t, s) < t - s$ *for all* $t, s > 0$*;*

(ξ_3) $\{t_n\}, \{s_n\}$ *are sequences in* $(0, \infty)$ *such that* $lim_{n\to\infty} t_n = lim_{n\to\infty} s_n > 0$*, then* $\limsup_{n\to\infty} \xi(t_n, s_n) < 0$, *is called a simulation function.*

Due to the axiom (ξ_2), we have $\xi(t, t) < 0$ for all $t > 0$.

Example 2.1 ([3, 16, 20]). Let $\phi_1 : [0, \infty) \to [0, \infty)$ be a continuous functions with $\phi_i(t) = 0$ if and only if $t = 0$. For $i = 1, 2, 3, 4, 5, 6$, we define the mappings $\xi_i : [0, \infty) \times [0, \infty) \to \mathbb{R}$ as follows

(i) $\xi_1(t, s) = \phi_1(s) - \phi_1(t)$ for all $t, s \in [0, \infty)$, where $\phi_1(t) < t \leq \phi_2(t)$ for all $t > 0$;

(ii) $\xi_1(t, s) = s - \frac{f(t,s)}{g(t,s)} t$ for all $t, s \in [0, \infty)$, where $f, g : [0, \infty)^2 \to [0, \infty)$ are two continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$;

(iii) $\xi_3(t, s) = s - \phi_3(s) - t$ for all $t, s \in [0, \infty)$;

(iv) If $\psi : [0, \infty) \to [0, 1)$ is a function such that $\limsup_{t\to r^+} \psi(t) < 1$ for all $r > 0$ and define

$$\xi_4(t, s) = s\psi(s) - t \text{ for all } s, t \in [0, \infty);$$

(v) If $\eta : [0, \infty) \to [0, \infty)$ is an upper semi-continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$ and define

$$\xi_5(t, s) = \eta(s) - t \text{ for all } s, t \in [0, \infty);$$

(vi) If $\phi : [0, \infty) \to [0, \infty)$ is a function such that $\int_0^\epsilon \phi(u)du$ exists and $\int_0^\epsilon \phi(u)du > \epsilon$ for each $\epsilon > 0$ and define

$$\xi_6(t, s) = s - \int_0^t \phi(u)du \text{ for all } s, t \in [0, \infty).$$

It is clear that each function $\xi_i (i = 1, 2, 3, 4, 5, 6)$ forms a simulation function.

Definition 2.2 ([19])**.** *A partial metric on a non empty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y \in X$*

(p_1) $x = y \iff p(x, x) = p(y, y) = p(x, y)$;
(p_2) $p(x, x) \leq p(x, y)$;
(p_3) $p(x, y) = p(y, x)$;
(p_4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A pair (X, p) is called a partial metric space. Each partial metric on X generates T_0 topology τ_p on X which is the family of $p-$open balls $\{B_p(x, \delta) : x \in X, \delta > 0\}$, where $B_p(x, \delta) = \{y \in X : p(x, y) < p(x, x) + \delta\}$ for all $x \in X$ and $\delta > 0$. If p is partial metric on X, then the function $d_p : X \times X \to R^+$ given by $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X.

Definition 2.3. *Let (X, p) be a partial metric space. Then*

(1) *A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ if and only if*
$$p(x, x) = \lim_{n\to\infty} p(x, x_n);$$
(2) *A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if and only if*
$$\lim_{n,m\to\infty} p(x_n, x_m)$$
exists (finite);
(3) *A partial metric space $(X, p)$is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that*
$$p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m);$$
(4) *A subset A of a partial metric space (X, p) is closed if whenever $\{x_n\}$is a sequence in A such that $\{x_n\}$ converges to some $x \in X$, then $x \in A$.*

Definition 2.4. *Let A and B be non-empty subset of a metric space (X, d) and $T : A \cup B \to A \cup B$. Then T is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$.*

Theorem 2.1 ([17])**.** *Let A and B be non empty closed subsets of a complete metric space (X, d). Suppose that $T : A \cup B \to A \cup B$ is a cyclic map such that*
$$d(Tx, Ty) \leq kd(x, y).$$
If $k \in [0, 1)$, then T has a unique fixed point in $A \cap B$.

To see [12], Karapinar and Erhan showed different types of cyclic contractions in usual metric space.

Definition 2.5 ([15])**.** *The function $\phi : [0, \infty) \to [0, \infty)$ is called an altering distance functions if the following conditions are satisfied:*

(1) *ϕ is continuous and non decreasing;*
(2) *$\phi(t) = 0$ if and only if $t = 0$.*

3 Main Results

Definition 3.1. *Let (X, p) be a partial metric space and A, B be a non empty closed subsets of (X, p). A mapping $T : A \cup B \to A \cup B$ is called cyclic (ψ, ϕ, Z)-contraction if*

(i) *$A \cup B$ has a cyclic representation with respect to T, i.e) $T(A) \subseteq B$ and $T(B) \subseteq A$;*
(ii) *If ψ and ϕ are altering distance functions,*
$$\xi(\psi(p(Tx, Ty)), \phi(\max(p(x, Tx), p(y, Ty)))) \geq 0 \quad \forall x \in A \text{ and } y \in B. \tag{3.1}$$

Theorem 3.1. *Let A, B be non empty closed subsets of a complete partial metric space (X, p). if $T : A\cup B \to A \cup B$ is a cyclic (ψ, ϕ, Z)-contraction. Then T has a unique fixed point $v \in A \cap B$.*

Proof. Fix any $x_0 \in A$. We choose $x_1 \in B$, since $T(A) \subseteq B$ such that $Tx_0 = x_1$. Again we choose $x_2 \in A$ such that $Tx_1 = x_2$, since $T(B) \subseteq A$. Continuing on this way, we construct a sequence $\{x_n\}$ in X such that $x_{2n} \in A, x_{2n+1} \in B$, i.e) $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$. if $x_{2n_0+1} = Tx_{2n_0+1}$. Thus x_{2n_0+1} is a fixed point of T in $A \cap B$.

In this above manner we assume that $x_{2n+1} \neq x_{2n+2}$ for all $n \in \mathbb{N}$. If n is even, then $n = 2j$ for some $j \in \mathbb{N}$. Let $x_{2j+1} \neq x_{2j+2}$ and from equation (3.1), we have

$$\xi(\psi(p(Tx_{2j}, Tx_{2j+1})), \phi(\max(p(x_{2j}, Tx_{2j}), p(x_{2j+1}, Tx_{2j+1})))) \geq 0.$$

Using (ξ_2), we have

$$\begin{aligned}\xi(\psi(p(x_{2j+1}, x_{2j+2})), \phi(\max(p(x_{2j}, x_{2j+1}), p(x_{2j+1}, x_{2j+2}))))\\ < \phi(\max(p(x_{2j}, x_{2j+1}), p(x_{2j+1}, x_{2j+2}))) - \psi(p(x_{2j+1}, x_{2j+2})).\end{aligned} \tag{3.2}$$

From the above, we have

$$\psi(p(x_{2j+1}, x_{2j+2})) < \phi(\max(p(x_{2j}, x_{2j+1}), p(x_{2j+1}, x_{2j+2}))), \tag{3.3}$$

if $\max(p(x_{2j}, x_{2j+1}), p(x_{2j+1}, x_{2j+2})) = p(x_{2j+1}, x_{2j+2})$,
$p(x_{2j}, x_{2j+1}) < p(x_{2j+1}, x_{2j+2})$,
$\psi(p(x_{2j}, x_{2j+1})) < \phi(p(x_{2j+1}, x_{2j+2}))$.
Since ϕ is non-decreasing function
$\phi(p(x_{2j+1}, x_{2j+2})) = 0$, hence $p(x_{2j+1}, x_{2j+2}) = 0$.
By (p_1) and (p_2), $x_{2j+1} = x_{2j+2}$,
which is a contradiction to our assumption

$$\max(p(x_{2j}, x_{2j+1}), p(x_{2j+1}, x_{2j+2})) = p(x_{2j}, x_{2j+1})$$

From (3.3), we get

$$\psi(p(x_{2j+1}, x_{2j+2})) < \phi(p(x_{2j}, x_{2j+1})). \tag{3.4}$$

If n is odd, then $n = 2j + 1$ for some $j \in \mathbb{N}$. By equation (3.1), we get

$$\xi(\psi(p(Tx_{2j+1}, Tx_{2j+2})), \phi(\max(p(x_{2j+1}, Tx_{2j+1}), p(x_{2j+2}, Tx_{2j+2})))) \geq 0.$$

Using (ξ_2), we get

$$\psi(p(x_{2j+2}, x_{2j+3})) < \phi(\max(p(x_{2j+1}, x_{2j+2}), p(x_{2j+2}, x_{2j+3}))),$$

if

$$\max(p(x_{2j+1}, x_{2j+2}), p(x_{2j+2}, x_{2j+3})) = p(x_{2j+2}, x_{2j+3})$$

i.e)

$$p(x_{2j+2}, x_{2j+3}) < p(x_{2j+2}, x_{2j+3})$$

$$\psi(p(x_{2j+2}, x_{2j+3})) < \phi(p(x_{2j+2}, x_{2j+3})).$$

Since ϕ is non-decreasing function.
$\phi(p(x_{2j+2}, x_{2j+3})) = 0$ and hence $p(x_{2j+2}, x_{2j+3}) = 0$,by (p_1) and (p_2).
It implies that, $x_{2j+2} = x_{2j+3}$
which contradicts to our assumption
Therefore,

$$\max(p(x_{2j+1}, x_{2j+2}), p(x_{2j+2}, x_{2j+3})) = p(x_{2j+1}, x_{2j+2}),$$

$$\psi(p(x_{2j+2}, x_{2j+3})) < \phi(p(x_{2j+1}, x_{2j+2})). \tag{3.5}$$

From equation (3.4) and (3.5), we get

$$\psi(p(x_{n+1}, x_{n+2})) < \phi(p(x_n, x_{n+1})). \tag{3.6}$$

In the above $\{p(x_n, x_{n+1})/n \in \mathbb{N}\}$ is a non-increasing sequence and hence there exist $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r. \tag{3.7}$$

Let $n \to \infty$ in equation (3.6) and also using the fact ψ and ϕ are continuous, we get $\psi(r) < \phi(r)$. It gives

$$\xi(\psi(r), \phi(r)) \geq 0, \xi(\psi(r), \phi(r)) < \phi(r) - \psi(r).$$

From (ξ_1), $\xi(\psi(r), \phi(r)) = 0$ and hence $\psi(r) = \phi(r) = 0$, by altering distance function, $\psi(r) = \phi(r) = 0$ iff $r = 0$.
By equation (3.7), we get

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0, \tag{3.8}$$

by (p_2), we get

$$\lim_{n\to\infty} p(x_n, x_n) = 0, \tag{3.9}$$

since $d_p(x, y) = 2p(x, y)$ for all $x, y \in X$.

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0. \tag{3.10}$$

Next we show that $\{x_n\}$ is a Cauchy sequence in metric space $(A \cup B, d_P)$. It is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence in $(A \cup B, d_P)$. Suppose to the contrary $\{x_{2n}\}$ is not a Cauchy sequence in $(A \cup B, d_P)$, there exist $\epsilon > 0$ and two subsequences $\{x_{2n(k)}\}$ and $\{x_{2m(k)}\}$ of $\{x_{2n}\}$ with $m(k) > n(k) > k$. $m(k)$ is the smallest index in $\mathbb{N}$ such that

$$d_p(x_{2m(k)}, x_{2n(k)}) \geq \epsilon, \tag{3.11}$$

this means that

$$d_p(x_{2m(k)}, x_{2n(k)-2}) < \epsilon, \tag{3.12}$$

from equation (3.10), (3.11) and triangle inequality, we get

$$\begin{aligned} \epsilon &\leq d_p(x_{2m(k)}, x_{2n(k)}) \\ &\leq d_p(x_{2m(k)}, x_{2n(k)-2}) + d_p(x_{2n(k)-2}, x_{2n(k)}) \\ &< \epsilon + d_p(x_{2n(k)-2}, x_{2n(k)-1}) + d_p(x_{2n(k)-1}, x_{2n(k)}). \end{aligned}$$

As $k \to \infty$ and using (3.8) we have

$$\lim_{k\to\infty} d_p(x_{2m(k)}, x_{2n(k)}) = \epsilon. \tag{3.13}$$

Again from (3.10) and we use triangle inequality we get

$$\begin{aligned} \epsilon &\leq d_P(x_{2m(k)}, x_{2n(k)}) \\ &\leq d_p(x_{2n(k)}, x_{2n(k)-1}) + d_p(x_{2n(k)-1}, x_{2m(k)}) \\ &\leq d_p(x_{2n(k)}, x_{2n(k)-1}) + d_p(x_{2n(k)}, x_{2m(k)+1}) + d_p(x_{2m(k)+1}, x_{2m(k)}) \\ &\leq d_p(x_{2n(k)}, x_{2n(k)-1}) + d_p(x_{2n(k)-1}, x_{2m(k)}) + 2d_p(x_{2m(k)+1}, x_{2m(k)}) \\ &\leq 2d_p(x_{2n(k)}, x_{2n(k)-1}) + d_p(x_{2m(k)}, x_{2n(k)}) + 2d_p(x_{2m(k)+1}, x_{2m(k)}). \end{aligned}$$

Using limit $n \to \infty$ in the above inequality and using equation (3.8), (3.10), we get

$$\begin{aligned} \lim_{k\to\infty} d_p(x_{2m(k)}, x_{2n(k)}) &= \lim_{k\to\infty} d_p(x_{2m(k)+1}, x_{2n(k)-1}) \\ &= \lim_{k\to\infty} d_p(x_{2m(k)+1}, x_{2n(k)}) \\ &= \lim_{k\to\infty} d_p(x_{2m(k)}, x_{2n(k)-1}). \end{aligned}$$

Since $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$, therefore

$$\begin{aligned} \lim_{k\to\infty} d_p(x_{2m(k)}, x_{2n(k)}) &= \lim_{k\to\infty} d_p(x_{2m(k)+1}, x_{2n(k)-1}) \\ &= \lim_{k\to\infty} d_p(x_{2m(k)+1}, x_{2n(k)}) \\ &= \lim_{k\to\infty} d_p(x_{2m(k)}, x_{2n(k)-1}) \\ &= \frac{\epsilon}{2}. \end{aligned}$$

By equation (3.1) , we have

$$\begin{aligned} &\xi(\psi(p(x_{2m(k)+1}, x_{2n(k)-1})), \phi(\max(p(x_{2m(k)}, Tx_{2m(k)}), p(x_{n(k)-2}, Tx_{2n(k)-2})))) \\ &\qquad < \phi(\max(p(x_{2m(k)}, Tx_{2m(k)}), p(x_{n(k)-2}, Tx_{2n(k)-2}))) - \psi(p(x_{2m(k)+1}, x_{2n(k)-1}) \end{aligned}$$

$$\xi(\psi(p(x_{2m(k)+1},x_{2n(k)-1})) < \phi(\max(p(x_{2m(k)},Tx_{2m(k)}),p(x_{n(k)-2},Tx_{2n(k)-2}))).$$

Therefore

$$\xi(\psi(p(x_{2m(k)+1},x_{2n(k)-1})) = 0.$$

Also $\xi(\psi(p(x_{2m(k)+1},x_{2n(k)-1})) = 0$ if and only if $x_{2m(k)+1} = x_{2n(k)-1}$, hence $\psi(\frac{\epsilon}{2}) = 0$ iff $\frac{\epsilon}{2} = 0$ and $\epsilon = 0$. It is a contradiction to our assumption, thus $\{x_{2n}\}$ is a Cauchy sequence in $(A\cup B, d_p)$. Since (X, d) is complete and $A \cup B$ is a closed subspace of (X, p), then $(A \cup B, p)$ is complete.
Therefore $\{x_n\}$ converges in the metric space $(A \cup B, d_p)$,

$$\lim_{n\to\infty} d_p(x_n, v) = 0.$$

Hence

$$p(v,v) = \lim_{n\to\infty} p(x_n,v) = \lim_{n,m\to\infty} p(x_n,x_m). \tag{3.14}$$

Since $\{x_n\}$ is Cauchy in $(A\cup B, d_p)$ and $(A\cup B, p)$ if and only if it is Cauchy in $(A\cup B, d_p)$ and $(A\cup B, p)$ is complete iff $(A \cup B, d_p)$ is complete.

$$\lim_{n,m\to\infty} d_p(x_n,x_m) = 0,$$

$$d_p(x_m,x_n) = 2p(x_m,x_n) - p(x_m,x_m) - p(x_n,x_n). \tag{3.15}$$

As $m,n\to\infty$ and using equation (3.9) and equation (3.15) in the above we get

$$\lim_{n,m\to\infty} d_p(x_m,x_n) = 2p(x_m,x_n) = 0.$$

By equation (3.14), we have

$$\lim_{n\to\infty} p(x_n,v) = p(v,v) = 0.$$

Since $p(x_{2n}, v) \to 0$, x_{2n} is belongs to A and A is closed in (X,p),$v \in A$, ie) $v \in A \cap B$.
From definition of p, we have

$$\begin{aligned} p(x_n,Tv) &\leq p(x_n,v) + p(v,Tv) - p(v,v) \\ &\leq p(x_n,v) + p(v,x_n) + p(x_n,Tv) - p(v,v) - p(x_n,x_n). \end{aligned}$$

Taking limit $n\to\infty$ in the above inequality, we get

$$\lim_{n\to\infty} p(x_n,Tv) = p(v,Tv).$$

Now, we claim that $Tv = v$.
Since $x_{2n} \in A$ and $v \in B$ by equation (3.1), we have

$$\begin{aligned} \xi(\psi(p(x_{2n+1},Tv),\phi(\max(p(x_{2n},Tx_{2n}),p(v,Tv))))) &< \phi(\max(p(x_{2n},Tx_{2n}),p(v,Tv)) \\ &- \psi(p(x_{2n+1},Tv))), \\ \psi(p(x_{2n+1},Tv)) &\leq \phi(\max(p(x_{2n},Tx_{2n}),p(v,Tv))) \\ &= \phi(p(v,Tv)). \end{aligned}$$

Since ϕ is an altering distance function, $\phi(v,Tv) = 0 \iff p(v,Tv) = 0$,
ie) $Tv = v$.
Hence v is a fixed point of T.
To prove uniqueness:
Let w be any other fixed point of T in $A\cap B$.It is easy to prove $p(v,w) = 0$.

$$\begin{aligned} \xi(\psi(p(Tv,Tw),\phi(\max(p(v,Tv),p(w,Tw))))) &< \phi(\max(p(v,Tv),p(w,Tw)) \\ &- \psi(p(Tv,Tw) \\ \psi(p(Tv,Tw) &\leq \phi(\max(p(v,Tv),p(w,Tw))). \end{aligned}$$

Thus $\psi(p(Tv,Tw)) = 0$ and hence $p(Tv,Tw) = 0$, $p(v,w) = 0$. Hence $v = w$. $\square$

4 Conclusion

In this paper, the main result determines a fixed point using cyclic $(\psi, \phi, Z)-$ contraction in partial metric spaces. Suppose, if we use this contraction in quasi-partial metric space, it satisfies the conditions (QPM1), (QPM2), (QPM3), (QPM4) in [13]. As a result, this contraction has a unique fixed point in quasi-partial metric space as well.

Acknowledgement. Authors are very much thankful to the Editor and Reviewer for their valuable suggestions to bring the paper in its present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 125-145**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

SOLUTION TO EQUAL SUM OF FIFTH POWER DIOPHANTINE EQUATIONS – A NEW APPROACH

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(Received: November 23, 2022; In format: November 17, 2022; Revised: February 09, 2023; Accepted; February 17, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53116

Abstract

Purpose of writing this paper is to introduce simple parametric solutions to quintic Diophantine equations 5.n.m where integer $n > 2$ and integer $m > 3$. Methodology applied is writing numbers in algebraic form as $a_i x + b_i$ with variable x, then writing fifth power Diophantine equation, in algebraic form with one variable and then transforming it to a linear equation by vanishing its four terms. For achieving this purpose, values to a_i and b_i of algebraic numbers are assigned so as to vanish constant term and coefficient of fifth power of x. Then equating with zero the coefficient of second power and coefficient of x, vanishes other two terms. These operations yield two relations between various a_i and b_i and also a linear equation in x. On putting the value of x obtained from this linear equation in given Diophantine equation, provides solution. Paper provides a single direct parametric solution to all quintic Diophantine equations $5.n.n$ where $\infty > n > 5$ and is simple, easily comprehensible and didactic.

2020 Mathematical Sciences Classification: 11D4.

Keywords and Phrases: Integers, Rational quantity, Linear equation, Diophantine equation of fifth power.

1 Introduction

In this paper, integer solutions to the Diophantine equations 5.4.4, 5.3.4, 5.3.5, 5.5.5, 5.4.5, 5.6.6, 5.5.7, 5.5.6, $5.n.n$ (where n is integer such that $\infty > n > 5$) as detailed below

$$A^5 + B^5 + C^5 + D^5 = E^5 + F^5 + G^5 + H^5, \tag{1.1}$$

$$A^5 + B^5 + C^5 + D^5 = F^5 + G^5 + H^5, \tag{1.2}$$

$$A^5 + B^5 + C^5 + D^5 + E^5 = F^5 + G^5 + H^5, \tag{1.3}$$

$$A^5 + B^5 + C^5 + D^5 + E^5 = F^5 + G^5 + H^5 + I^5 + J^5, \tag{1.4}$$

$$A^5 + B^5 + C^5 + D^5 + E^5 = F^5 + G^5 + H^5 + I^5, \tag{1.5}$$

$$A^5 + B^5 + C^5 + D^5 + E^5 + F^5 = G^5 + H^5 + I^5 + J^5 + K^5 + L^5, \tag{1.6}$$

$$A^5 + B^5 + C^5 + D^5 + E^5 + F^5 + G^5 = H^5 + I^5 + J^5 + K^5 + L^5, \tag{1.7}$$

$$A^5 + B^5 + C^5 + D^5 + E^5 + F^5 = G^5 + H^5 + I^5 + J^5 + K^5, \tag{1.8}$$

and

$$A_1^5 + A_2^5 + A_3^5 + \ldots + A_{n-2}^5 + A_{n-1}^5 + A_n^5 = B_1^5 + B_2^5 + B_3^5 + \ldots + B_{n-2}^5 + B_{n-1}^5 + B_n^5, \tag{1.9}$$

have been determined using numbers in algebraic form. Alphabets $A, B, C, \ldots, L$ used in Equations (1.1) to (1.8) and alphabets $A_1, A_2, A_3, \ldots, A_n$, $B_1, B_2, B_3, \ldots, B_n$ used in equation (1.9) denote integers. W. Eric Weisstein, [8] gave history of solutions to some of these Diophantine equation of fifth power. Swinnerton-Dyer [7] also solved Diophantine equations 5.3.3 using a method of transformation of the equation but the method being presented in the paper is easy, simple, comprehensible and didactic also. In addition to individual parametric solution specific to the equation, we have provided in the paper, there is a single direct parametric solution to all quintic Diophantine equations $5.n.n$, where $\infty > n > 5$. Xeroudakes and Moessner [9], and Gloden [5] determined parametric solutions to solve equation 5.3.4. using two parameters. Rao [6] gave the smallest solution to this equation 5.3.4. Again Xeroudakes and Moessner [9] found several parametric solutions to the equation 5.4.4. Xeroudakes and Moessner [9] and Gloden [5] found the solution to the equation 5.5.6. Chen Shuwen [4] has found the solution to equation 5.6.6. Chaudhry [2, 3] while presenting methods of solution to fifth power Diophantine equation, gave method for representation of every rational number by an algebraic sum of fifth powers of rational numbers. Notwithstanding above works, a geometric approach to solve Diophantine equations of fifth power was also adopted by Bremner [1].

Nomenclature $5.n.m$ of Diophantine equation indicates, it is a fifth degree equation with larger number of terms m and smaller number of terms n. It is already stated in the Abstract, an algebraic equation of power five with variable x has been obtained by assigning algebraic form $(a_i \cdot x + b_i)$ to the integers of Diophantine equation. This equation is then, transformed to a linear equation. Although method of transformation of algebraic equation of fifth power, has been used earlier, the method being presented in the paper is easy, simple, comprehensible and didactic also. We have also provided parametric solutions, followed by exhaustive examples to illustrate and corroborate the results derived. To start with, we express a number, say n as $a \cdot x + b$ where a and b are real rational quantities as assigned by us and x is a real rational variable quantity. For example, a number, say 7, can be written as $3 \cdot x + 1$ assigning values 3 for a and 1 for b where $x = 2$. Similarly, 7 can be written $(3/4)x + 11/2$ where $a = 3/4, b = 11/2$ and $x = 2$. From above, it can be stated, $7 = 3 \cdot 2 + 1 = (3/4) \cdot 2 + 1$. In general,

$$n = a \cdot x + b. \tag{1.10}$$

This proves Lemma 1.1.

Lemma 1.1. *A number n is always expressible in algebraic form as $a \cdot x + b$ where a and b are fixed rational quantities neither zero nor infinity and x is a variable.*

Using Lemma 1.1*, equation* (1.1) *can be written as*

$$\begin{aligned}(ax + p)^5 &+ (bx + q)^5 + (cx + r)^5 + (dx + s)^5 \\ &= (ex + t)^5 + (fx + u)^5 + (gx + v)^5 + (hx + w)^5,\end{aligned} \tag{1.11}$$

where $a, b, c, \ldots, h,\ p, q, r, \ldots, v$ are arbitrary rational numbers and x is a real variable. On expanding and rearranging equation (1.11)*,*

$$\begin{aligned}x^5(a^5 &+ b^5 + c^5 + d^5 - e^5 - f^5 - g^5 - h^5) \\ &+ 5x^4(a^4p + b^4q + c^4r + d^4s - e^4t - f^4u - g^4v - h^4w) \\ &+ 10x^3(a^3p^2 + b^3q^2 + c^3r^2 + d^3s^2 - e^3t^2 - f^3u^2 - g^3v^2 - h^3w^2) \\ &+ 10x^2(a^2p^3 + b^2q^3 + c^2r^3 + d^2s^3 - e^2t^3 - f^2u^3 - g^2v^3 - h^2w^3) \\ &+ 5x(ap^4 + bq^4 + cr^4 + ds^4 - et^4 - fu^4 - gv^4 - hw^4) \\ &+ (p^4 + q^4 + r^4 + s^4 - t^4 - u^4 - v^4 - w^4) = 0,\end{aligned} \tag{1.12}$$

it is found, resultant equation (1.12) *is too tedious and difficult to solve for x. This equation is, therefore, transformed into a linear equation so as to solve it easily. Obviously, constant term and term containing x^4 can be ridden of, if*

$$\big(a^5 + b^5 + c^5 + d^5 - e^5 - f^5 - g^5 - h^5\big) = 0 \tag{1.13}$$

and

$$\big(p^4 + q^4 + r^4 + s^4 - t^4 - u^4 - v^4 - w^4\big) = 0. \tag{1.14}$$

To achieve this motive, we replace e, f, g, h with a, b, c, d and $t, u, v, w$with p, q, r, s respectively and accordingly, equation (1.1) is written in algebraic form,

$$\begin{aligned}(ax + p)^5 &+ (bx + q)^5 + (cx + r)^5 + (dx + s)^5 \\ &= (ax + q)^5 + (bx + r)^5 + (cx + s)^5 + (dx + p)^5.\end{aligned} \tag{1.15}$$

With this introduction, further steps will be taken to vanish other two terms to transform this equation into a linear equation.

2 Transformation into linear equation

Expansion of equation (1.13) on expansion, yields

$$\begin{aligned}5x^4&\{p(a^4-d^4)+q(b^4-a^4)+r(c^4-b^4)+s(d^4-c^4)\}\\&+10x^3\{p^2(a^3-d^3)+q^2(b^3-a^3)+r^2(c^3-b^3)+s^2(d^3-c^3)\}\\&+10x^2\{p^3(a^2-d^2)+q^3(b^2-a^2)+r^3(c^2-b^2)+s^3(d^2-c^2)\}\\&+5x\{p^4(a-d)+q^4(b-a)+r^4(c-b)+s^4(d-c)\}=0.\end{aligned}\tag{2.1}$$

It has solution at $x=0$ but that yields $p^5+q^5+r^5+s^5=p^5+q^5+r^5+s^5$, which is a trivial solution and is ignored. Equation (2.1) can, then be written as

$$\begin{aligned}x^3&\{p(a^4-d^4)+q(b^4-a^4)+r(c^4-b^4)+s(d^4-c^4)\}\\&+2x^2\{p^2(a^3-d^3)+q^2(b^3-a^3)+r^2(c^3-b^3)+s^2(d^3-c^3)\}\\&+2x\{p^3(a^2-d^2)+q^3(b^2-a^2)+r^3(c^2-b^2)+s^3(d^2-c^2)\}\\&+\{p^4(a-d)+q^4(b-a)+r^4(c-b)+s^4(d-c)\}=0.\end{aligned}\tag{2.2}$$

It is a cubic equation and to transform it into a linear equation, its constant term and coefficient of x are equated to zero. That is

$$\left\{p^4\left(a-d\right)+q^4\left(b-a\right)+r^4\left(c-b\right)+s^4\left(d-c\right)\right\}=0$$

and

$$2\left\{p^3\left(a^2-d^2\right)+q^3\left(b^2-a^2\right)+r^3\left(c^2-b^2\right)+s^3\left(d^2-c^2\right)\right\}=0.$$

On simplification,

$$a=-\frac{b\left(q^4-r^4\right)+c\left(r^4-s^4\right)+d\left(s^4-p^4\right)}{p^4-q^4},\tag{2.3}$$

and

$$a^2=-\frac{b^2\left(q^3-r^3\right)+c^2\left(r^3-s^3\right)+d^2\left(s^3-p^3\right)}{p^3-q^3},\tag{2.4}$$

where $p\neq q$ and also $p\neq -q$.

2.1 Determination of a and b from Equations (2.3) and (2.4)

Equations (2.3) and (2.4) impose certain conditions on values of a and b that make these dependent upon c,d,p,q,r and s. From equations (2.3) and (2.4),

$$\left\{-\frac{b\left(q^4-r^4\right)+c\left(r^4-s^4\right)+d\left(s^4-p^4\right)}{p^4-q^4}\right\}^2=-\frac{b^2\left(q^3-r^3\right)+c^2\left(r^3-s^3\right)+d^2\left(s^3-p^3\right)}{p^3-q^3}.\tag{2.5}$$

For its easy solvability for b, term with coefficient of b^2 in equation (2.5) is eliminated by equating its coefficients to zero and that yields

$$\left(\frac{q^4-r^4}{p^4-q^4}\right)^2=-\left(\frac{q^3-r^3}{p^3-q^3}\right).$$

Considering $q=0$, yields $r=p$ and on simplifying equation (2.5) by substituting r with p and $q=0$, we obtain

$$b=\frac{1}{2}(c-d)\left(1-t^4\right)+\frac{1}{2}(c+d)\frac{\left(1+t+t^2\right)}{\left(1+t^2\right)\left(1+t\right)},\tag{2.6}$$

where $s/p=t$ and $t\neq -1$. On simplifying equation (2.3) by substituting r with p, $q=0$ and $s/p=t$,

$$a=b-(c-d)\left(1-t^4\right).\tag{2.7}$$

On putting value of b from equation (2.6) in equation (2.7),

$$a=-\frac{1}{2}(c-d)\left(1-t^4\right)+\frac{1}{2}(c+d)\frac{\left(1+t+t^2\right)}{\left(1+t^2\right)\left(1+t\right)}.\tag{2.8}$$

2.2 Determination of value of x and solution to Equation (1.1)

When equation (2.3) and (2.4) are satisfied, then equation (2.2) transforms into

$$x^3\left\{p\left(a^4-d^4\right)+q\left(b^4-a^4\right)+r\left(c^4-b^4\right)+s\left(d^4-c^4\right)\right\}$$
$$+2x^2\left\{p^2\left(a^3-d^3\right)+q^2\left(b^3-a^3\right)+r^2\left(c^3-b^3\right)+s^2\left(d^3-c^3\right)\right\}=0. \quad (2.9)$$

On simplifying after putting $r=p, s/p=t$ and $q=0$, it transforms into linear equation

$$x=-2p(P/Q), \quad (2.10)$$

where

$$P=a^3-b^3+\left(c^3-d^3\right)\left(1-t^2\right) \quad (2.11)$$

and

$$Q=a^4-b^4+\left(c^4-d^4\right)(1-t). \quad (2.12)$$

By putting values of a and b from equations (2.8) and (2.6) in equations (2.10) and (2.11), x is determined by equation (2.9). Substituting this value x in equation (1.13), solution to equation (1.10) is obtained after normalisation. On putting $r=p$ and $q=0$, equation (1.11) takes the form

$(ax+p)^5+(\text{ bx})^5+(cx+p)^5+(dx+s)^5=(\text{ax})^5+(bx+p)^5+(cx+s)^5+(dx+p)^5$. (2.12)

Normalisation, wherever, the word normalisation appears in this paper, it will mean converting-fractions to integers by multiplying these with lowest, common multiplier abbreviated LCM. Based on the method discussed in foregoing paragraphs, some solutions are given in Table 2.1.

***Table* 2.1:** Solution to Diophantine equation $A^5+B^5+C^5+D^5=E^5+F^5+G^5+H^5$

S. N.	Values of c,d,p,s	Calculated a,b,t and x	Normalized and rearranged $A^5+B^5+C^5+D^5=E^5+F^5+G^5+H^5$
1	$c=1,\ d=2,$ $p=1, s=-2$	$a=-\frac{42}{5}, b=\frac{33}{5},$ $t=-2, x=\frac{7160}{12651}$	$(47256)^5+(19811)^5+(60144)^5+(18142)^5$ $=(47493)^5+(59907)^5+(10982)^5$ $+(26971)^5$
2	$c=1,\ d=-2,$ $p=-1, s=2$	$a=\frac{114}{5}, b=-\frac{111}{5},$ $t=-2, x=2040/1223$	$(45289)^5+(817)^5+(5303)^5+(46511)^5$ $=(46512)^5+(4486)^5+(1634)^5+(45288)^5$
3	$c=3,\ d=1,$ $p=1, s=-2$	$a=69/5, b=-81/5,$ $t=-2, x=\frac{7085}{16858}$	$(114777)^5+(26631)^5+(97773)^5+(23943)^5$ $=(114631)^5+(38113)^5+(97919)^5$ $+(12461)^5$
4	$c=3,\ d=-5,$ $p=1, s=-2$	$a=303/5, b=-297/5,$ $t=-2, x=-\frac{112415}{134801}$	$(6677548)^5+(202444)^5+(6812252)^5$ $+(696876)^5$ $=(6677451)^5+(292473)^5+(6812349)^5$ $+(606847)^5$
5	$c=3,\ d=2,$ $p=1, s=-2$	$a=6, b=-9,$ $t=-2, x=\frac{296}{845}$	$(2664)^5+(1098)^5+(1776)^5+(1437)^5$ $=(2621)^5+(1733)^5+(1819)^5+(802)^5$
6	$c=4,\ d=1,$ $p=1, s=-2$	$a=21, b=-24,$ $t=-2, x=\frac{2544}{7585}$	$(61056)^5+(12626)^5+(53424)^5+(10129)^5$ $=(61009)^5+(17761)^5+(53471)^5+(4994)5$
7	$c=0,\ d=1,$ $p=1, s=-2$	$a=-39/5, b=36/5,$ $t=-2, x=640/383$	$(4609)^5+(126)^5+(4991)^5+(1023)^5$ $=(4608)^5+(383)^5+(4992)^5+(766)5$

This proves Lemma 2.1 and Lemma 2.2.

Lemma 2.1. *A Diophantine equation* 5.4.4, $(ax+p)^5+(\ bx)^5+(cx+p)^5+(dx+s)^5=(ax)^5+(bx+p)^5+(cx+s)^5+(dx+p)^5$ *is always transformable into a linear equation,* $x\left\{a^3-b^3+\left(c^3-d^3\right)\left(1-t^2\right)\right\}+2p\{a^3-b^3+\left(c^3-d^3\right)\left(1-t^2\right)\}=0$, *where* $a,b,c,d,\ p,q,s$ *are rational quantities,* a,b,t *are given by equations* (2.6), (2.8) *and* $t=s/p$.

Lemma 2.2. *After normalisation, a Diophantine equation* 5.4.4, $(ax+p)^5+(\ bx)^5+(cx+p)^5+(dx+s)^5=(ax)^5+(bx+p)^5+(cx+s)^5+(dx+p)^5$ *is always true and, in fact, is an identity, when* $a=-\frac{1}{2}(c-d)\left(1-t^4\right)+\frac{1}{2}(c+d)\frac{(1+t+t^2)}{(1+t^2)(1+t)}$, $b=\frac{1}{2}(c-d)\left(1-t^4\right)+\frac{1}{2}(c+d)\frac{(1+t+t^2)}{(1+t^2)(1+t)}$, $x=-2p\frac{\left\{a^3-b^3+\left(c^3-d^3\right)\left(1-t^2\right)\right\}}{\{a^4-b^4+(c^4-d^4)(1-t)\}}$, $t=s/p$, c,d,p *are real rational quantities and* $t\neq -1$.

However, it was observed, solution to Diophantine equations 5.4.4 *also yields solutions to Diophantine equation* 5.5.3. *These solutions are given in Table* 2.2.

Table 2.2: Solution to Diophantine equation $A^5+B^5+C^5+D^5+E^5=F^5+G^5+H^5$

S. N.	Values of c,d,p,s	Calculated a,b,t and x	Normalized and rearranged $A^5+B^5+C^5+D^5+E^5=F^5+G^5+H^5$
1	$c=3,\ d=-1,\ p=1, s=-2$	$a=147/5,\ b=-153/5,\ t=-2, x=\frac{28115}{33701}$	$(860319)^5+(95517)^5+(826581)^5+(16943)^5+5586^5=(860282)^5+(118046)^5+(826618)^5$
2	$c=4,\ d=-1,\ p=1, s=-2$	$a=183/5,\ b=-192/5,\ t=-2, x=\frac{175760}{315951}$	$(6749184)^5+(807662)^5+(6432816)^5+(71138)^5+(140191)^5=(6748767)^5+(1018991)^5+(6433233)^5$
3	$c=6,\ d=-1,\ p=1, s=-2$	$a=51, b=-54,\ t=-2, x=\frac{13784}{41285}$	$(744336)^5+(96354)^5+(702984)^5+(134)^5+(27501)^5=(744269)^5+(123989)^5+(703051)^5$
4	$c=5,\ d=-1,\ p=1, s=-2$	$a=219/5,\ b=-231/5,\ t=-2, x=\frac{63285}{151658}$	$(2923767)^5+(366601)^5+(2771883)^5+(13109)^5+(88373)^5=(2923541)^5+(468083)^5+(2772109)^5$

2.3 Solution to Diophantine Equation 5.4.3, $A^5+B^5+C^5+D^5=F^5+G^5+H^5$

On putting $a=0$, in equation (2.12), it transforms into Diophantine equation 5.4.3

$$(p)^5+(\ \mathrm{bx})^5+(cx+p)^5+(dx+s)^5=(bx+p)^5+(cx+s)^5+(dx+p)^5 \tag{2.13}$$

and equation (2.8), (2.6) transform into

$$\frac{c}{d}=\frac{\left(1-t^4\right)^2+\left(1-t^3\right)}{\left(1-t^4\right)^2-\left(1-t^3\right)}, \tag{2.14}$$

$$b=2d\cdot\frac{\left(1-t^4\right)\left(1-t^3\right)}{\left(1-t^4\right)^2-\left(1-t^3\right)}. \tag{2.15}$$

where b,c,d,p,s are real rational quantities, t is neither equal to one nor equal to zero and also $d\neq 0$. Value of x is, then given by relation

$$x=-2p(P/Q), \tag{2.16}$$

where

$$P = -d^3 \left\{ 2 \cdot \frac{\left(1-t^3\right)\left(1-t^4\right)}{\left(1-t^4\right)^2 - \left(1-t^3\right)} \right\}^3 + d^3 \left(1-t^2\right) \left[\left\{ \frac{\left(1-t^4\right)^2 + \left(1-t^3\right)}{\left(1-t^4\right)^2 - \left(1-t^3\right)} \right\}^3 - 1 \right], \tag{2.17}$$

$$Q = -d^4 \left\{ 2 \cdot \frac{\left(1-t^3\right)\left(1-t^4\right)}{\left(1-t^4\right)^2 - \left(1-t^3\right)} \right\}^4 + d^4 \left(1-t\right) \left[\left\{ \frac{\left(1-t^4\right)^2 + \left(1-t^3\right)}{\left(1-t^4\right)^2 - \left(1-t^3\right)} \right\}^4 - 1 \right]. \tag{2.18}$$

Based on the method discussed in foregoing paragraphs, some solutions are given in Table 2.3.

***Table* 2.3:** Solution to Diophantine Equation $A^5 + B^5 + C^5 + D^5 = F^5 + G^5 + H^5$

S.N.	Values of t, p, d	Calculated $c, b,$ and x	Normalized And Rearranged $A^5 + B^5 + C^5 + D^5 = F^5 + G^5 + H^5$
1	$t = -2,\ p = 1,$ $d = 1$	$c = 13/12,$ $b = -5/4,$ $x = 7872/4525$	$(4525)^5 + (13053)^5 + (5315)^5 + (522)^5$ $= (9840)^5 + (1178)^5 + (12397)^5$
2	$t = -1/2,$ $p = 1,$ $d = 1$	$c = -57/7,$ $b = -60/7,$ x $= -11207/29900$	$(26157)^5 + (125960)^5 + (76307)^5 + (18693)^5$ $= (29900)^5 + (96060)^5 + (121157)^5$

This proves Lemmas 2.3 and 2.4.

Lemma 2.3. *A Diophantine equation* 5.4.3, $(p)^5 + (\ bx)^5 + (cx+p)^5 + (dx+s)^5 = (bx+p)^5 + (cx+s)^5 + (dx+p)^5$ *is always transformable into a linear equation* $x = -2p(P/Q)$ *where* P *and* Q *are given by Equations* (2.17) *and* (2.18). *Real rational quantities* b, c, *are given by Equations* (2.15), (2.14) *respectively,* $d, s,$ p *are real rational quantities,* $t = s/p$ *which is neither equal to one nor equal to zero and also* $d \neq 0$.

Lemma 2.4. *After normalisation, a Diophantine equation* 5.4.3, $(p)^5 + (\ bx)^5 + (cx+p)^5 + (dx+s)^5 = (bx+p)^5 + (cx+s)^5 + (dx+p)^5$ *is always true and, in fact, is an identity when* $x, P, Q, b,$ c *are given by Equations* (2.16), (2.17), (2.18), (2.15), (2.14) *respectively,* $d, s,$ p *are real rational quantities,* $t = s/p$ *which is neither equal to one nor equal to zero and also* $d \neq 0$.

Next Diophantine equation 5.5.5 *is taken up. Procedure applied to equation* 5.4.4 *will also be used here.*

3 Transformation of Diophantine Equation 5.5.5 $A^5+B^5+C^5+D^5+E^5=F^5+G^5+H^5+I^5+J^5$ into linear equation and its solution

Diophantine equation $A^5 + B^5 + C^5 + D^5 + E^5 = F^5 + G^5 + H^5 + I^5 + J^5$ can be written in algebraic form

$$(ax+p)^5+(bx+q)^5+(cx+r)^5+(dx+s)^5+(ex+u)^5 = (ax+q)^5+(bx+r)^5+(cx+s)^5+(dx+u)^5+(ex+p)^5, \tag{3.1}$$

where $a, b, c, d, e, p, q, r, s, u$ are real rational quantities ,$p \neq q$ and also $p \neq -q$. On expansion, it takes the form

$$\begin{aligned} &5x^4 \left\{p\left(a^4-e^4\right)+q\left(b^4-a^4\right)+r\left(c^4-b^4\right)+s\left(d^4-c^4\right)+u\left(e^4-d^4\right)\right\} \\ &\quad + 10x^3 \left\{p^2\left(a^3-e^3\right)+q^2\left(b^3-a^3\right)+r^2\left(c^3-b^3\right)+s^2\left(d^3-c^3\right)+u^2\left(e^3-d^3\right)\right\} \\ &\quad + 10x^2 \left\{p^3\left(a^2-e^2\right)+q^3\left(c^2-b^2\right)+r^3\left(c^2-b^2\right)+s^3\left(d^2-c^2\right)+u^3\left(e^2-d^2\right)\right\} \\ &\quad + 5x\left\{p^4\left(a-e\right)+q^4\left(b-a\right)+r^4\left(c-b\right)+s^4\left(d-c\right)+u^4(e-d)\right\} = 0. \end{aligned} \tag{3.2}$$

On equating coefficient of x and x^2 with zero,

$$\left\{p^4\left(a-e\right)+q^4\left(b-a\right)+r^4\left(c-b\right)+s^4\left(d-c\right)+u^4(e-d)\right\} = 0 \tag{3.3}$$

and

$$\left\{p^3\left(a^2-e^2\right)+q^3\left(b^2-a^2\right)+r^3\left(c^2-b^2\right)+s^3\left(d^2-c^2\right)+u^3\left(e^2-d^2\right)\right\} = 0. \tag{3.4}$$

From Equations (3.3) and (3.4),

$$a=-\frac{b\left(q^4-r^4\right)+c\left(r^4-s^4\right)+d\left(s^4-u^4\right)+e\left(u^4-p^4\right)}{p^4-q^4}, \tag{3.5}$$

$$a^2=-\frac{b^2\left(q^3-r^3\right)+c^2\left(r^3-s^3\right)+d^2\left(s^3-u^3\right)+e^2\left(u^3-p^3\right)}{p^3-q^3}. \tag{3.6}$$

That results in

$$\left\{-\frac{b\left(q^4-r^4\right)+c\left(r^4-s^4\right)+d\left(s^4-u^4\right)+e\left(u^4-p^4\right)}{p^4-q^4}\right\}^2=-\frac{b^2\left(q^3-r^3\right)+c^2\left(r^3-s^3\right)+d^2\left(s^3-u^3\right)+e^2\left(u^3-p^3\right)}{p^3-q^3}. \tag{3.7}$$

For easy solvability, quadratic equation (3.7) in b is transformed into linear equation by equating coefficient of b^2 to zero. That yields

$$\left(\frac{q^4-r^4}{p^4-q^4}\right)^2=-\left(\frac{q^3-r^3}{p^3-q^3}\right).$$

Let $q=0, s=0$, then $r=p$ and equation (3.1) transforms into

$$\begin{aligned}(ax+p)^5+(\text{ bx})^5+(cx+p)^5+(\text{dx})^3+(ex+u)^5\\ =(\text{ax})^5+(bx+p)^5+(\text{cx})^5+(dx+u)^5+(ex+p)^5.\end{aligned} \tag{3.8}$$

Also equation (3.2) transforms into

$$\begin{aligned}&5x^4\left\{p\left(a^4+c^4-b^4-e^4\right)+u\left(e^4-d^4\right)\right\}+10x^3\left\{p^2\left(a^3+c^3-b^3-e^3\right)+u^2\left(e^3-d^3\right)\right\}\\ &+10x^2\left\{p^3\left(a^2+c^2-b^2-e^2\right)+u^3\left(e^2-d^2\right)\right\}\\ &+5x\left\{p^4\left(a+c-b-e\right)+u^4\left(e-d\right)\right\}=0,\end{aligned} \tag{3.9}$$

and equation (3.3) and (3.4) transform into

$$\left\{p^4\left(a-b+c-e\right)+u^4(e-d)\right\}=0, \tag{3.10}$$

$$\left\{p^3\left(a^2-b^2+c^2-e^2\right)+u^3\left(e^2-d^2\right)\right\}=0. \tag{3.11}$$

From equation (3.10) and (3.11),

$$a=(b+e-c)-t^4\left(e-d\right), \tag{3.12}$$

$$a^2=\left(b^2+e^2-c^2\right)-t^3\left(e^2-d^2\right), \tag{3.13}$$

where $u/p=t$. Therefore,

$$\left\{(b+e-c)-t^4\left(e-d\right)\right\}^2=\left(b^2+e^2-c^2\right)-t^3\left(e^2-d^2\right).$$

On simplification,

$$b=\frac{c^2+\left\{c+t^4(e-d)\right\}^2+t^3\left(e^2-d^2\right)-2e\left\{c+t^4\left(e-d\right)\right\}}{2\left\{c-e+t^4\left(e-d\right)\right\}}, \tag{3.14}$$

where $c-e+t^4(e-d)\neq 0$ and also $c\neq d$. For easy solvability, we assign $t=-1$ and simplify equations (3.14) and (3.15) and obtain

$$b=c-d+\frac{cd-e^2}{c-d} \tag{3.15}$$

and

$$a=\frac{cd-e^2}{c-d}. \tag{3.16}$$

When a and b have values as given by equations (3.15) and (3.16), then equation (3.9) transforms into linear equation

$$x=-2p\left(\frac{a^3-b^3+c^3-d^3}{a^4-b^4+c^4+d^4-2e^4}\right), \tag{3.17}$$

wherec, d, e are arbitrary real rational quantities such that $c\neq d$ and $a^4+c^4+d^4\neq b^4+2e^4$. On putting the value of x given by equation (3.16) in equation (3.8), gives solution to Diophantine Equation 5.5.5.

That also proves Lemmas 3.1 and 3.2. In the Table 3.1, p is neither considered nor assigned any value except one owing to the fact that it does not appear in final equation when value of x is put in equation (3.8).

***Table* 3.1:** Solution to Diophantine equation $A^5+B^5+C^5+D^5+E^5=F^5+G^5+H^5+I^5+J^5$

S .N.	Values of c, d, e	Calculated a, b and x	$(ax+p)^5+(bx)^5+(cx+p)^5+(dx)^3+(ex-p)^5$ $=(ax)^5+(bx+p)^5+(cx)^5+(dx-p)^5+(ex+p)^5$.
1	$2,3,5$	$19,18,-1/12$	$7^5+18^5+3^5+17^5+7^5=10^5+19^5+6^5+2^5+15^5$
2	$-2,3,5$	$31/5,6/5,-5/4$	$27^5+6^5+15^5+29^5+10^5=14^5+31^5+2^5+19^5+21^5$
3	$3,-4,5$	$-37/7,12/5,$ $-7/8$	$12^5+13^5+43^5+37^5+20^5$ $=45^5+28^5+4^5+21^5+27^5$
4	$3,-4,1$	$-13/7,36/7,$ $-7/24$	$36^5+31^5+13^5+4^5+17^5=37^5+3^5+28^5+12^5+21^5$
5	$3,-4,$ 0	$-12/7,37/7,$ $-21/74$	$111^5+74^5+36^5+10^5+74^5$ $=110^5+11^5+84^5+37^5+63^5$
6	$3,4,0$	$-12,1\,3,-3/26$	$10^5+39^5+26^5+9^5+26^5$ $=35^5+12^5+36^5+13^5+14^5$
7	$3,4,6$	$24,2\,3\,,-3/46$	$26^5+69^5+12^5+64^5+28^5$ $=37^5+72^5+23^5+9^5+58^5$
8	$-4,5,6$	$-56/9,$ $-2\,5/9,27/50$	$75^5+58^5+168^5+85^5+212^5$ $=218^5+135^5+112^5+25^5+108^5$
9	$-4,5,0$	$20/9,$ $-61/9,27/122$	$183^5+122^5+60^5+13^5+122^5$ $=182^5+14^5+135^5+61^5+108^5$
10	$-4,5,1$	$7/3,-20/3,9/40$	$60^5+31^5+21^5+5^5+49^5=61^5+4^5+45^5+20^5+36^5$

Lemma 3.1. *A Diophantine equation* 5.5.5, $(ax+p)^5+(bx)^5+(cx+p)^5+(dx)^5+(ex-p)^5=(ax)^5+(bx+p)^5+(cx)^5+(dx-p)^5+(ex+p)^5$ *is always transformable into a linear equation,* $x=-2p\left(\frac{a^3-b^3+c^3-d^3}{a^4-b^4+c^4+d^4-2e^4}\right)$, *where* a, b *are given by Equations* (3.16), (3.15) *respectively,* c, d, e *andp are arbitrary real rational quantities such that* $c \neq d$ *and* $a^4+c^4+d^4 \neq b^4+2e^4$.

Lemma 3.2. *After normalisation, a Diophantine equation* 5.5.5, $(ax+p)^5+(bx)^5+(cx+p)^5+(dx)^5+(ex-p)^5=(ax)^5+(bx+p)^5+(cx)^5+(dx-p)^5+(ex+p)^5$ *is always true and, in fact, is an identity, when* $x=-2p\left(\frac{a^3-b^3+c^3-d^3}{a^4-b^4+c^4+d^4-2e^4}\right)$, $a=\frac{cd-e^2}{c-d}$, $b=c-d+\frac{cd-e^2}{c-d}$, c, d, e *andp are arbitrary real rational quantities such that* $c \neq d$ *and* $a^4+c^4+d^4 \neq b^4+2e^4$.

4 Solution to Diophantine Equation 5.5.4, $A^5+B^5+C^5+D^5+E^5=G^5+H^5+I^5+J^5$

Procedure of Diophantine equations $A^5+B^5+C^5+D^5+E^5=F^5+G^5+H^5+I^5+J^5$ as determined in foregoing paragraphs will be utilised here by putting $a=0$. In that case, the equation

$$(ax+p)^5+(\mathrm{bx})^5+(cx+p)^5+(\mathrm{dx})^5+(ex-p)^5=(\mathrm{ax})^5+(bx+p)^5+(\mathrm{cx})^5+(dx-p)^5+(ex+p)^5$$

transforms into

$$(p)^5+(\mathrm{bx})^5+(cx+p)^5+(\mathrm{dx})^5+(ex-p)^5=(bx+p)^5+(\mathrm{cx})^5+(dx-p)^5+(ex+p)^5, \tag{4.1}$$

and equations (3.15), (3.14) and (3.16) transform into

$$d=e^2/c, \tag{4.2}$$

$$b=c-e^2/c, \tag{4.3}$$

$$x=-2p\left(\frac{-b^3+c^3-d^3}{-b^4+c^4+d^4-2e^4}\right), \tag{4.4}$$

respectively where $c \neq 0$. On putting the value of b and d in equation (4.4) and simplifying

$$x=-\frac{3pc}{2\left(c^2-e^2\right)}. \tag{4.5}$$

where p, c and e are arbitrarily assigned real rational quantities such that $c \neq 0$ and also $c^2 = e^2$ Putting the value of x given by equation (4.5) in equation (4.1) gives solutions to Diophantine equation 5.5.4. Based on these equations, some solutions are given in Table 4.1. Here also p is not considered on the basis of explanation already given.

***Table* 4.1:** Solution to Diophantine equation $A^5 + B^5 + C^5 + D^5 + E^5 = G^5 + H^5 + I^5 + J^5$

S.N.	Values of c, e	Calculated value of b, d and x	$A^5 + B^5 + C^5 + D^5 + E^5 = G^5 + H^5 + I + J$
1	3,4	$-7/3, 16/3, 9/14$	$14^5 + 41^5 + 48^5 + 22^5 + 7^5 = 21^5 + 27^5 + 34^5 + 50^5$
2	5, 4	$9/5, 16/5, -5/6$	$6^5 + 3^5 + 25^5 + 22^5 + 14^5 = 9^5 + 19^5 + 16^5 + 26^5$
3	7, 4	$33/7, 16/7, -7/22$	$22^5 + 11^5 + 49^5 + 38^5 + 6^5 = 33^5 + 27^5 + 16^5 + 50^5$
4	3, 5	$-16/3, 25/3, 9/32$	$32^5 + 59^5 + 75^5 + 13^5 + 16^5 = 48^5 + 27^5 + 43^5 + 77^5$
5	5, 8	$-39/5, 64/5, 5/26$	$26^5 + 51^5 + 64^5 + 14^5 + 13^5 = 39^5 + 25^5 + 38^5 + 66^5$

Lemma 4.1. *A Diophantine equation* 5.5.4, $(p)^5 + (\ bx)^5 + (cx + p) + (dx)^5 + (ex - p)^5 = (bx + p)^5 + (cx)^5 + (dx - p)^5 + (ex + p)^5$ *is always transformable into linear equation,* $x = -2p\left(\frac{-b^3+c^3-d^3}{-b^4+c^4+d^4-2e^4}\right)$*, where* b *and* d *are given by equations* (4.3), (4.2), *and* c, e *and* p *are real rational quantities.*

Lemma 4.2. *After normalisation, a Diophantine equation* 5.5.4, $(p)^5 + (bx)^5 + (cx + p)^5 + (dx)^5 + (ex - p)^5 = (bx + p)^5 + (cx)^5 + (dx - p)^5 + (ex + p)^5$ *is always true and, in fact, is an identity, when* $x = -2p\left(\frac{-b^3+c^3-d^3}{-b^4+c^4+d^4-2e^4}\right)$, $d = \frac{e^2}{c}$, $b = c - \frac{e^2}{c}$, *and* c, e *and* p *are arbitrary real rational quantities such that* $c \neq 0$ *and* $c^4 + d^4 \neq b^4 + 2e^4$.

5 Solution to Diophantine Equation 5.6.6 $A^5{+}B^5{+}C^5{+}D^5{+}E^5{+}F^5{=}G^5{+}H^5{+}I^5{+}J^5{+}K^5{+}L^5$

Equation $A^5 + B^5 + C^5 + D^5 + E^5 + F^5 = G^5 + H^5 + I^5 + J^5 + K^5 + L^5$ can be written in algebraic form as

$$(a_1x + p)^5 + (a_2x)^5 + (a_3x + p)^5 + (a_4x)^5 + (a_5x + p)^5 + (a_6x)^5$$
$$= (a_1x)^5 + (a_2x + p)^5 + (a_3x)^5 + (a_4x + p)^5 + (a_5x)^5 + (a_6x + p)^5 . \quad (5.1)$$

On expansion,

$$(5x^4p\left(a_1^4 + a_3^4 + a_5^4 - a_2^4 - a_4^4 - a_6^4\right) + 10x^3p^2\left(a_1^3 + a_3^3 + a_5^3 - a_2^3 - a_4^3 - a_6^3\right)$$
$$+ 10x^2p^3\left(a_1^2 + a_3^2 + a_5^2 - a_2^2 - a_4^2 - a_6^2\right)$$
$$+ 5xp^4\left(a_1 + a_3 + a_5 - a_2 - a_4 - a_6\right) = 0. \quad (5.2)$$

where $a_1, a_2, a_3, \ldots, a_7$ and p are real rational quantities. Equating coefficients of x and x^2 equal to zero,

$$(a_1 + a_3 + a_5 - a_2 - a_4 - a_6) = 0$$

and

$$\left(a_1^2 + a_3^2 + a_5^2 - a_2^2 - a_4^2 - a_6^2\right) = 0.$$

From aforementioned equations,

$$a_1 = -a_3 - a_5 + a_2 + a_4 + a_6 \quad (5.3)$$

and

$$a_1^2 = -a_3^2 - a_5^2 + a_2^2 + a_4^2 + a_6^2. \quad (5.4)$$

From Equations (5.3) and (5.4)

$$(a_2 - a_3 + a_4 - a_5 + a_6)^2 = -a_3^2 - a_5^2 + a_2^2 + a_4^2 + a_6^2 .$$

On simplification, it yields

$$a_2 = a_3 + a_5 - \frac{a_3a_5 - a_4a_6}{a_3 - a_4 + a_5 - a_6}. \quad (5.5)$$

where $a_3 + a_5 \neq a_4 + a_6$. On putting the value of a_2 in equation (5.3),

$$a_1 = a_4 + a_6 - \frac{a_3a_5 - a_4a_6}{a_3 - a_4 + a_5 - a_6}. \quad (5.6)$$

When equations (5.5) and (5.6) are satisfied, equation (5.2) transforms into

$$x = -2p\frac{(a_1^3 + a_3^3 + a_5^3 - a_2^3 - a_4^3 - a_6^3)}{(a_1^4 + a_3^4 + a_5^4 - a_2^4 - a_4^4 - a_6^4)} = -2p\frac{\left(\sum_{n=1}^{3} a_{2n-1}^3\right) - \left(\sum_{n=1}^{3} a_{2n}^3\right)}{\left(\sum_{n=1}^{3} a_{2n-1}^4\right) - \left(\sum_{n=1}^{3} a_{2n}^4\right)}, \tag{5.7}$$

where $\sum_{n=1}^{3} a_{2n-1}^4 \neq \sum_{n=1}^{3} a_{2n}^4$. Sign$\sum_{n=1}^{3} a_{2n-1}^3$ used in this paper, means summation of terms a_{2n-1}^3 and n varies from 1 to 3. On putting the values of a_1 and a_2 in equation (5.7) and, then putting the value of x in equation (5.1), gives solutions to Diophantine Equation 5.6.6. Based on this method, some solutions are given in the Table 5.1.

***Table* 5.1:** Solution to Diophantine equation $A^5 + B^5 + C^5 + D^5 + E^5 + F^5 = G^5 + H^5 + I^5 + J^5 + K^5 + L^5$

S. N.	Ass. Values of a_3, a_4, a_5 and a_6	Calculated values of a_1, a_2, x	$(a_1x+p)^5 + (a_2x)^5 + (a_3x+p)^5 + (a_4x)^5 + (a_5x+p)^5 + (a_6x)^5 = (a_1x)^5 + (a_2x+p)^5 + (a_3x)^5 + (a_4x+p)^5 + (a_5x)^5 + (a_6x+p)^5$
1	$3, 5, 6, 7$	$19/3, 10/3, -9/92$	$35^5 + 65^5 + 38^5 + 57^5 + 27^5 + 54^5 = 30^5 + 45^5 + 63^5 + 62^5 + 47^5 + 29^5$
2	$2, 5, 6, 7$	$25/4, 9/4, -2/19$	$13^5 + 30^5 + 14^5 + 25^5 + 8^5 + 24^5 = 9^5 + 20^5 + 28^5 + 29^5 + 18^5 + 10^5$
3	$-2, -5, -6, 7$	$67/10, -33/10, 15/13$	$99^5 + 34^5 + 150^5 + 154^5 + 201^5 + 236^5 = 227^5 + 210^5 + 73^5 + 60^5 + 124^5 + 180^5$
4	$-4, -5, -6, 7$	$83/12, -61/12, 18/37$	$183^5 + 70^5 + 180^5 + 142^5 + 249^5 + 326^5 = 323^5 + 252^5 + 109^5 + 144^5 + 106^5 + 216^5$
5	$-3, 5, 6, 7$	$55/9, -26/9, -27/164$	$78^5 + 245^5 + 2^5 + 165^5 + 162^5 + 25^5 = 1^5 + 135^5 + 189^5 + 242^5 + 81^5 + 29^5$
6	$-3, -5, -6, 7$	$75/11, -46/11, 11/16$	$46^5 + 17^5 + 55^5 + 50^5 + 75^5 + 93^5 = 91^5 + 77^5 + 30^5 + 33^5 + 39^5 + 66^5$
7	$-3, -5, 6, 1$	$-15/7, 34/7, -7/4$	$19^5 + 25^5 + 35^5 + 30^5 + 42^5 + 3^5 = 34^5 + 38^5 + 7^5 + 15^5 + 21^5 + 39^5$
8	$-3, 5, 6, 1$	$-5/3, -14/3, -9/8$	$23^5 + 42^5 + 35^5 + 37^5 + 54^5 + 1^5 = 45^5 + 46^5 + 9^5 + 15^5 + 50^5 + 27^5$
9	$3, -5, -6, -1/2$	$27/10, 26/5, 5$	$29^5 + 52^5 + 32^5 + 48^5 + 60^5 + 3^5 = 50^5 + 58^5 + 5^5 + 37^5 + 54^5 + 30^5$

That also proves Lemmas 5.1 and 5.2.

Lemma 5.1. *A Diophantine equation* 5.6.6, $(a_1x+p)^5 + (a_2x)^5 + (a_3x+p)^5 + (a_4x)^5 + (a_5x+p)^5 + (a_6x)^5 = (a_1x)^5 + (a_2x+p)^5 + (a_3x)^5 + (a_4x+p)^5 + (a_5x)^5 + (a_6x+p)^5$ *is always transformable into linear equation,*

$x = -2p\frac{\left(a_1^3+a_3^3+a_5^3-a_2^3-a_4^3-a_6^3\right)}{\left(a_1^4+a_3^4+a_5^4-a_2^4-a_4^4-a_6^4\right)} = -2p\frac{\left(\sum_{n=1}^{3} a_{2n-1}^3\right)-\left(\sum_{n=1}^{3} a_{2n}^3\right)}{\left(\sum_{n=1}^{3} a_{2n-1}^4\right)-\left(\sum_{n=1}^{3} a_{2n}^4\right)}$, *where* a_1 *and* a_2 *are given by equations* (5.5), (5.6) *and* $a_3, a_4,\ a_5, a_6$ *and* p *are arbitrary real rational quantities such that* $\sum_{n=1}^{3} a_{2n-1}^4 \neq \sum_{n=1}^{3} a_{2n}^4$.

Lemma 5.2. *After normalisation, a Diophantine equation* 5.6.6, $(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+(a_4x)^5+(a_5x+p)^5+(a_6x)^5 = (a_1x)^5+(a_2x+p)^5+(a_3x)^5+(a_4x+p)^5+(a_5x)^5+(a_6x+p)^5$ *is always true and, in fact, is an identity, when*

$$x = -2p\frac{\left(a_1^3+a_3^3+a_5^3-a_2^3-a_4^3-a_6^3\right)}{\left(a_1^4+a_3^4+a_5^4-a_2^4-a_4^4-a_6^4\right)} = -2p\frac{\left(\sum_{n=1}^{3} a_{2n-1}^3\right)-\left(\sum_{n=1}^{3} a_{2n}^3\right)}{\left(\sum_{n=1}^{3} a_{2n-1}^4\right)-\left(\sum_{n=1}^{3} a_{2n}^4\right)},$$

$$a_2 = a_3+a_5-\frac{a_3a_5-a_4a_6}{a_3-a_4+a_5-a_6}, a_1 = a_4+a_6-\frac{a_3a_5-a_4a_6}{a_3-a_4+a_5-a_6}$$

and $a_3, a_4,\ a_5, a_6$ *andp are arbitrary real rational quantities such that* $\sum_{n=1}^{3} a_{2n-1}^4 \neq \sum_{n=1}^{3} a_{2n}^4$.

6 Solution to Diophantine Equation 5.7.5 $A^5+B^5+C^5+D^5+E^5+F^5+G^5=H^5+I^5+J^5+K^5+L^5$

While solving Diophantine Equations 5.6.6, solutions to Diophantine Equations 5.7.5 are also obtained and are entered in Table 6.1.

***Table* 6.1:** Solution to Diophantine equation $A^5+B^5+\ C^5+D^5+E^5+F^5=G^5+H^5+I^5+J^5+K^5+L^5$

S.N.	Assigned values of a_3, a_4, a_5 and a_6	Calculated values of a_1, a_2, x	$(a_1x+p)^5+(a_2x)^5+\ (a_3x+p)^5+(a_4x)^5+\ (a_5x+p)^5+(a_6x)^5+(-a_1x)^5 = (a_2x+p)^5+(a_3x)^5+(a_4x+p)^5+(a_5x)^5+(a_6x+p)^5$
1	$-2, 5, 6, 7$	$49/8, -15/8, -4/27$	$5^5+15^5+70^5+6^5+49^5+48^5+2^5 = 40^5+56^5+69^5+16^5+14^5$
2	$-2, -5, 6, 7$	$-19/2, -15/2, 3/11$	$10^5+58^5+42^5+57^5+23^5+12^5+8^5 = 35^5+45^5+30^5+36^5+64^5$
3	$3, -5, 6, 7$	$-39/7, 10/7, -7/16$	$10^5+5^5+6^5+49^5+39^5+26^5+51^5 = 55^5+35^5+21^5+42^5+33^5$
4	$3, -5, -6, 7$	$27/5, 2/5, -5/8$	$19^5+2^5+7^5+35^5+6^5+33^5+30^5 = 25^5+38^5+27^5+15^5+27^5$
5	$-3, -5, -6, 1$	$3/5, -22/5, 5/28$	$22^5+25^5+2^5+3^5+6^5+3^5+33^5 = 31^5+13^5+5^5+15^5+30^5$
6	$3, -5, -6, 1/2$	$35/6, 22/3, -9/17$	$71^5+132^5+20^5+9^5+124^5+108^5+25^5 = 90^5+142^5+105^5+98^5+54^5$

7 Solution of Diophantine Equation 5.6.5 $A^5+B^5+C^5+D^5+E^5+F^5=H^5+I^5+J^5+K^5+L^5$

For solutions of Diophantine equation 5.6.5, equations (5.1), (5.5), (5.6) and (5.7) are referred to. When $a_1 = 0$, these equations take the forms after simplification as given below.

$$(p)^5+(a_2x)^5+(a_3x+p)^5+(a_4x)^5+(a_5x+p)^5+(a_6x)^5 = (a_2x+p)^5+(a_3x)^5+(a_4x+p)^5+(a_5x)^5+(a_6x+p)^5, \tag{7.1}$$

$$a_3 = a_4+a_6-\frac{a_4a_6}{(a_4-a_5+a_6)}, \tag{7.2}$$

$$a_2 = a_5-\frac{a_4a_6}{(a_4-a_5+a_6)}, \tag{7.3}$$

$$x = -2p\frac{\left(a_3^3+a_5^3-a_2^3-a_4^3-a_6^3\right)}{\left(a_3^4+a_5^4-a_2^4-a_4^4-a_6^4\right)} = -2p\frac{\left(\sum_{n=2}^{3} a_{2n-1}^3\right)-\left(\sum_{n=1}^{3} a_{2n}^3\right)}{\left(\sum_{n=2}^{3} a_{2n-1}^4\right)-\left(\sum_{n=1}^{3} a_{2n}^4\right)}, \tag{7.4}$$

where a_4, a_5, a_6 and p are arbitrary real quantities such that $\sum_{n=2}^{3} a_{2n-1}^4 \neq \sum_{n=1}^{3} a_{2n}^4$.

The values of a_2, a_3 obtained from equations (7.3), (7.2) and values of a_4, a_5 and a_6 as assigned by us, are then put in equation (7.4), that yields value of x. Putting the value of x so obtained in equation (7.1), gives its solution. Based on this method, some solutions to this equation are given in the Table 7.1.

***Table* 7.1:** Solution to Diophantine equation $A^5 + B^5 + C^5 + D^5 + E^5 + F^5 = H^5 + I^5 + J^5 + K^5 + L^5$

S.N.	Assigned a_4, a_5 and a_6	Calculated7 a_2, a_3, x	$(p)^5 + (a_2x)^5 + (a_3x+p)^5 + (a_4x)^5 + (a_5x+p)^5 + (a_6x)^5 = (a_2x+p)^5 + (a_3x)^5 + (a_4x+p)^5 + (a_5x)^5 + (a_6x+p)^5$
1	$4, 5, 6$	$1/5, 26/5, -5/34$	$1^5 + 20^5 + 30^5 + 33^5 + 14^5 + 4^5 = 34^5 + 8^5 + 9^5 + 26^5 + 25^5$
2	$4, -5, 6$	$-33/5, 42/5, -15/34$	$34^5 + 99^5 + 109^5 + 126^5 + 26^5 + 56^5 = 92^5 + 60^5 + 90^5 + 133^5 + 75^5$
3	$-4, -5, 6$	$-13/7, 38/7, -7/2$	$2^5 + 11^5 + 28^5 + 37^5 + 38^5 + 40^5 = 36^5 + 42^5 + 13^5 + 30^5 + 35^5$
4	$-4, -5, 7$	$-3/2, 13/2, -1$	$2^5 + 3^5 + 8^5 + 12^5 + 13^5 + 12^5 = 11^5 + 14^5 + 5^5 + 10^5 + 10^5$
5	$4, 5, 7$	$1/3, 19/3, -9/68$	$3^5 + 36^5 + 63^5 + 65^5 + 32^5 + 5^5 = 68^5 + 11^5 + 23^5 + 57^5 + 45^5$
6	$-4, 5, -7$	$27/4, -37/4, 6/17$	$34^5 + 81^5 + 94^5 + 111^5 + 14^5 + 50^5 = 77^5 + 48^5 + 84^5 + 115^5 + 60^5$
7	$-4, 5, -3$	$6, -6, 3/2$	$2^5 + 18^5 + 17^5 + 18^5 + 10^5 + 7^5 = 16^5 + 12^5 + 9^5 + 20^5 + 15^5$
8	$-4, 5, 3$	$3, -3, -3/4$	$4^5 + 13^5 + 12^5 + 5^5 + 15^5 + 5^5 = 9^5 + 11^5 + 9^5 + 9^5 + 16^5$

That also proves Lemma 7.1 and 7.2.

Lemma 7.1. *An Diophantine equation* 5.6.5, $(p)^5 + (a_2x)^5 + (a_3x+p)^5 + (a_4x)^5 + (a_5x+p)^5 + (a_6x)^5 = (a_2x+p)^5 + (a_3x)^5 + (a_4x+p)^5 + (a_5x)^5 + (a_6x+p)^5$ *is always transformable into linear equation,* $x = -2p\frac{(a_3^3+a_5^3-a_2^3-a_4^3-a_6^3)}{(a_3^4+a_5^4-a_2^4-a_4^4-a_6^4)} - 2p\frac{(\sum_{n=2}^{3} a_{2n-1}^3)-(\sum_{n=1}^{3} a_{2n}^3)}{(\sum_{n=2}^{3} a_{2n-1}^4)-(\sum_{n=1}^{3} a_{2n}^4)}$, *where* a_2 *and* a_3 *are given by equations* (7.3), (7.2) *and* $a_4,\ a_5, a_6$ *are arbitrarily assigned real rational quantities such that such that* $\sum_{n=2}^{3} a_{2n-1}^4 \neq \sum_{n=1}^{3} a_{2n}^4$.

Lemma 7.2. *After normalisation, a Diophantine equation* 5.6.5, $(p)^5 + (a_2x)^5 + (a_3x+p)^5 + (a_4x)^5 + (a_5x+p)^5 + (a_6x)^5 = (a_2x+p)^5 + (a_3x)^5 + (a_4x+p)^5 + (a_5x)^5 + (a_6x+p)^5$ *is always true and, in fact, is an identity, when* $x = -2p\frac{(a_3^3+a_5^3-a_2^3-a_4^3-a_6^3)}{(a_3^4+a_5^4-a_2^4-a_4^4-a_6^4)} - 2p\frac{(\sum_{n=2}^{3} a_{2n-1}^3)-(\sum_{n=1}^{3} a_{2n}^3)}{(\sum_{n=2}^{3} a_{2n-1}^4)-(\sum_{n=1}^{3} a_{2n}^4)}$, $a_2 = a_5 - \frac{a_4a_6}{(a_4-a_5+a_6)}$, $a_3 = a_4 + a_6 - \frac{a_4a_6}{(a_4-a_5+a_6)}$, *and* $a_4,\ a_5, a_6$

are arbitrarily assigned real rational quantities such that such that $\sum_{n=2}^{3} a_{2n-1}^4 \neq \sum_{n=1}^{3} a_{2n}^4$.

8 Solution to generalised form 5.*n*.*n* of Diophantine Equation 5.*n*.*n* $Y_1^5+Y_2^5+Y_3^5+\ldots+Y_{n-2}^5+Y_{n-1}^5+Y_n^5=Z_1^5+Z_2^5+Z_3^5+\ldots+Z_{n-2}^5+Z_{n-1}^5+Z_n^5$, where n is an integer ≥ 6.

For generalised form of Diophantine equation 5.*n*.*n*, two cases will be taken up, first case will be of the category, when $n = 2k$ (even integers) and second when $n = 2k - 1$ (odd integers) where $k \geq 3$.

a. When $n = 2k$,

Diophantine equation 5.8.8 as mentioned below, is taken up.

$$(a_1x+p)^5 + (a_2x)^5 + (a_3x+p)^5 + (a_4x)^5 + (a_5x+p)^5 + (a_6x)^5 + (a_7x+p)^5 + (a_8x)^5$$

$$= (a_1x)^5 + (a_2x+p)^5 + (a_3x)^5 + (a_4x+p)^5 + (a_5x)^5 + (a_6x+p)^5 + (a_7x)^5 + (a_8x+p)^5, \tag{8.1}$$

where $a_1, a_2, a_3, \ldots, a_8$ are arbitrarily assigned real rational quantities. On equating coefficients of x and x^2 equal to zero in equation (8.1) , following equations are obtained

$$a_1 = -(a_3 + a_5 + a_7 - a_2 - a_4 - a_6 - a_8), \tag{8.2}$$

and

$$a_1^2 = -\left(a_3^2 + a_5^2 + a_7^2 - a_2^2 - a_4^2 - a_6^2 - a_8^2\right), \tag{8.3}$$

$$5x^4p\left(a_1^4 + a_3^4 + a_5^4 + a_7^4 - a_2^4 - a_4^4 - a_6^4 - a_8^4\right) + 10x^3p^2\left(a_1^3 + a_3^3 + a_5^3 + a_7^4 - a_2^3 - a_4^3 - a_6^3 - a_8^3\right) = 0. \tag{8.4}$$

Elimination of a_1 from the equations (8.2) and (8.3) yields

$$a_2 = A_8 - \frac{P_8 - Q_8}{A_8 - B_8}, \tag{8.5}$$

and putting this value of a_2 in equation (8.2) results in

$$a_1 = B_8 - \frac{P_8 - Q_8}{A_8 - B_8}. \tag{8.6}$$

where $a_3, a_4, a_5, \ldots, a_8$ are arbitrarily assigned real rational quantities,

$$A_8 = a_3 + a_5 + a_7 = \sum_{i=2}^{4} a_{(2i-1)}, \tag{8.7}$$

$$B_8 = a_4 + a_6 + a_8 = \sum_{i=2}^{4} a_{2i}, \tag{8.8}$$

$$P_8 = a_3\left(a_5 + a_7\right) + a_5\left(a_7\right) = a_3 \cdot \sum_{i=3}^{4} a_{(2i-1)} + a_5 \cdot a_7, \tag{8.9}$$

$$Q_8 = a_4\left(a_6 + a_8\right) + a_6\left(a_8\right) = a_4 \cdot \sum_{i=3}^{4} a_{2i} + a_6 \cdot a_8, \tag{8.10}$$

$A_8 \neq B_8$ and also $\sum_{n=2}^{4} a_{2n-1}^4 \neq \sum_{n=1}^{4} a_{2n}^4$. Value of x obtained from equation (8.4) is then given by

$$x = -2p\frac{\left(a_1^3 + a_3^3 + a_5^3 + a_7^3 - a_2^3 - a_4^3 - a_6^3 - a_8^3\right)}{\left(a_1^4 + a_3^4 + a_5^4 + a_7^4 - a_2^4 - a_4^4 - a_6^4 - a_8^4\right)} = -2p\frac{\left(\sum_{n=1}^{4} a_{(2n-1)}^3\right) - \left(\sum_{n=1}^{4} a_{2n}^3\right)}{\left(\sum_{n=1}^{4} a_{(2n-1)}^4\right) - \left(\sum_{n=1}^{4} a_{2n}^4\right)}. \tag{8.11}$$

Substituting this value of x in equation (8.1), will give solutions to Diophantine equation 5.8.8.

Generalising it for$n = 2k$, where $n \geq 3$ for the Diophantine equation

$$(a_1x+p)^5 + (a_2x)^5 + (a_3x+p)^5 + \ldots + \left\{a_{(2k-2)} \cdot x\right\}^5 + \left\{a_{(2k-1)} \cdot x + p\right\}^5 + (a_{2k}x)^5$$
$$= (a_1x)^5 + (a_2x+p)^5 + (a_3x)^5 + \ldots + \left\{a_{(2k-2)} \cdot x + p\right\}^5 + \left\{a_{(2k-1)} \cdot x\right\}^5 + (a_{2k}x+p)^5, \tag{8.12}$$

and following the same procedure as that of Diophantine equation 5.8.8, equations for a_2 and a_1 can be derived as given below

$$a_2 = A_{2k} - \frac{P_{2k} - Q_{2k}}{A_{2k} - B_{2k}}, \tag{8.13}$$

$$a_1 = B_{2k} - \frac{P_{2k} - Q_{2k}}{A_{2k} - B_{2k}}. \tag{8.14}$$

where $a_3, a_4, a_5, \ldots, a_{2k}$ are arbitrarily assigned real rational quantities,

$$A_{2k} = a_3 + a_5 + a_7 + \ldots + a_{2k-5} + a_{2k-3} + a_{2k-1} = \sum_{i=2}^{k} a_{2i-1}, \tag{8.15}$$

$$B_{2k} = a_4 + a_6 + a_8 + \ldots + a_{2k-4} + a_{2k-2} + a_{2k} = \sum_{i=2}^{k} a_{2i}, \tag{8.16}$$

$$P_{2k} = a_3 (a_5 + a_7 + a_9 \ldots + a_{2k-1}) + a_5 (a_7 + a_9 + a_{11} + \ldots + a_{2k-1}) + a_7 (a_9 + a_{11} + a_{13} + \ldots + a_{2k-1}) + \ldots + a_{2k-3} \cdot a_{2k-1} \tag{8.17}$$

and

$$Q_{2k} = a_4 (a_6 + a_8 + a_{10} \ldots + a_{2k}) + a_6 (a_8 + a_{10} + a_{12} + \ldots + a_{2k}) + a_8 (a_{10} + a_{12} + a_{14} + \ldots + a_{2k-}) + \ldots + a_{2k-2}a_{2k}, \tag{8.18}$$

$A_{2k} \neq B_{2k}$ and also $\sum_{n=2}^{4} a_{2n-1}^4 \neq \sum_{n=1}^{4} a_{2n}^4$. In mathematical notations,

$$P_{2k} = a_3 \cdot \sum_{i=3}^{k} a_{2i-1} + a_5 \cdot \sum_{i=4}^{k} a_{2i-1} + a_7 \cdot \sum_{i=5}^{k} a_{2i-1} + \ldots + a_{2k-7} \cdot \sum_{i=k-2}^{k} a_{2i-1} + a_{2k-5} \cdot \sum_{i=k-1}^{k} a_{2i-1} + a_{2k-3}a_{2k-1},$$

$$Q_{2k} = a_4 \cdot \sum_{i=3}^{k} a_{2i} + a_6 \cdot \sum_{i=4}^{k} a_{2i} + a_8 \cdot \sum_{i=5}^{k} a_{2i} + \ldots + a_{2k-6} \cdot \sum_{i=k-2}^{k} a_{2i} + a_{2k-4} \cdot \sum_{i=k-1}^{k} a_{2i} + a_{2k-2}a_{2k}.$$

To avoid repetition , the procedure is not reiterated here for finding the relation of x which is given by

$$x = -2p\frac{(a_1^3 + a_3^3 + a_5^3 + \ldots + a_{2k-1}^3) - (a_2^3 + a_4^3 + a_6^3 + \ldots + a_{2k}^3)}{(a_1^4 + a_3^4 + a_5^4 + \ldots + a_{2k-1}^4) - (a_2^4 + a_4^4 + a_6^4 + \ldots + a_{2k}^4)} = -2p\frac{\left(\sum_{i=1}^{k} a_{2i-1}^3\right) - \left(\sum_{i=1}^{k} a_{2i}^3\right)}{\left(\sum_{i=1}^{k} a_{2i-1}^4\right) - \left(\sum_{i=1}^{k} a_{2i}^4\right)}. \tag{8.19}$$

where $\sum_{n=1}^{k} a_{2i-1}^4 \neq \sum_{n=1}^{4} a_{2i}^4$. Substitution of this value of x in equation (8.12), will give solution to Diophantine equation $5.n.n$where $n = 2k$ and $k \geq 3$.

b. When $n = 2k - 1$ and $k \geq 3$.

Such $5.n.n$ equations can be written as

$$(a_1x + p)^5 + (a_2x)^5 + (a_3x + p)^5 + \ldots + (a_{2k-3}x + p)^5 + (a_{2k-2}x)^5 + (a_{2k-1}x - p)^5 = (a_1x)^5 + (a_2x + p)^5 + (a_3x)^5 + \ldots + (a_{2k-3}x)^5 + (a_{2k-2}x - p)^5 + (a_{2k-1}x + p)^5 . \tag{8.20}$$

Kindly note in case of Diophantine equations where $n = 2k$, signs of p were always positive since number of terms containing p appearing in Left Hand Side was equal to those appearing in Right Hand Side, thus constant term containing p^5 vanished. But when n is odd, number of terms containing p in LHS is more by one than the correspondent terms in RHS, therefore, last term of LHS is taken as $-p$ and last but one term of RHS is also taken as $-p$. With this arrangement, constant term of equation (8.20) expansion vanishes. Assuming $n = 7$,Diophantine equation 5.7.7 can be written as

$$(a_1x + p)^5 + (a_2x)^5 + (a_3x + p)^5 + (a_4x)^5 + (a_5x + p)^5 + (a_6x)^5 + (a_7x - p)^5 = (a_1x)^5 + (a_2x + p)^5 + (a_3x)^5 + (a_4x + p)^5 + (a_5x)^5 + (a_6x - p)^5 + (a_7x + p)^5 . \tag{8.21}$$

Equating coefficients of x and x^2 with zero yields,

$$a_1 = -a_3 - a_5 + a_2 + a_4 + a_6, \tag{8.22}$$

$$a_1^2 = -a_3^2 - a_5^2 + 2a_7^2 + a_2^2 + a_4^2 - a_6^2, \tag{8.23}$$

and

$$x = -2p\frac{(a_1^3 + a_3^3 + a_5^3 - a_2^3 - a_4^3 - a_6^3)}{(a_1^4 + a_3^4 + a_5^4 - 2a_7^4 - a_2^4 - a_4^4 + a_6^4)} = -2p\frac{\left(\sum_{i=1}^{3} a_{2i-1}^3\right) - \left(\sum_{i=1}^{3} a_{2i}^3\right)}{\left(\sum_{i=1}^{3} a_{2i-1}^4\right) - 2\left(a_7^4 - a_6^4\right) - \left(\sum_{i=1}^{3} a_{2i}^4\right)}, \tag{8.24}$$

where $\sum_{n=1}^{3} a_{2i-1}^4 - \sum_{n=1}^{4} a_{2i}^4 \neq 2\left(a_7^4 - a_6^4\right)$. From Equations (8.23) and (8.24),

$$a_1 = B_7 - \frac{P_7 - Q_7 + a_7^2 - a_6^2}{A_7 - B_7}, \tag{8.25}$$

$$a_2 = A_7 - \frac{P_7 - Q_7 + a_7^2 - a_6^2}{A_7 - B_7}, \tag{8.26}$$

where

$$A_7 = a_3 + a_5, \tag{8.27}$$

$$B_7 = a_4 + a_6, \tag{8.28}$$

$$P_7 = a_3\left(a_5\right), \tag{8.29}$$

$$Q_7 = a_4\left(a_6\right). \tag{8.30}$$

Generalising it for Diophantine equation (8.21) where $n = 2k-1$, the equations, then can be written

$$a_2 = A_{2k-1} - \frac{P_{2k-1} - Q_{2k-1} + a_{2k-1}^2 - a_{2k-2}^2}{A_{2k-1} - B_{2k-1}}, \tag{8.31}$$

$$a_1 = B_{2k-1} - \frac{P_{2k-1} - Q_{2k-1} + a_{2k-1}^2 - a_{2k-2}^2}{A_{2k-1} - B_{2k-1}}, \tag{8.32}$$

where

$$A_{2k-1} = a_3 + a_5 + a_7 + \ldots a_{2k-3} = \sum_{i=2}^{k-1} a_{2i-1}, \tag{8.33}$$

$$B_{2k-1} = a_4 + a_6 + a_8 + \ldots a_{2k-2} = \sum_{i=2}^{k-1} a_{2i}, \tag{8.34}$$

$$\begin{aligned} P_{2k-1} &= a_3\left(a_5 + a_7 + a_9 + \ldots + a_{2k-3}\right) + a_5\left(a_7 + a_9 + a_{11} + \ldots + a_{2k-3}\right) \\ &\quad + a_7\left(a_9 + a_{11} + a_{13} + \ldots + a_{2k-3}\right) + \ldots + a_{2k-5}\left(a_{2k-3}\right), \end{aligned} \tag{8.35}$$

$$\begin{aligned} Q_{2k-1} &= a_4\left(a_6 + a_8 + a_{10} + \ldots + a_{2k-2}\right) + a_6\left(a_8 + a_{10} + a_{12} + \ldots + a_{2k-2}\right) \\ &\quad + a_8\left(a_{10} + a_{12} + a_{14} + \ldots + a_{2k-2}\right) + \ldots + a_{2k-4}\left(a_{2k-2}\right), \end{aligned} \tag{8.36}$$

and

$$x = -2p\frac{\left(a_1^3 + a_3^3 + a_5^3 + \ldots a_{2k-3}^3\right) - \left(a_2^3 + a_4^3 + a_6^3 + \ldots + a_{2k-2}^3\right)}{\left(a_1^4 + a_3^4 + a_5^4 + \ldots + a_{2k-3}^4 - 2a_{2k-1}^4\right) - \left(a_2^4 + a_4^4 + a_6^4 + \ldots + a_{2k-4}^4 - a_{2k-2}^4\right)}, \tag{8.37}$$

where $\sum_{n=1}^{3} a_{2i-1}^{k-1} - \sum_{n=1}^{k-1} a_{2i}^4 \neq 2\left(a_{2k-1}^4 - a_{2k-2}^4\right)$. In mathematical notations,

$$P_{2k-1} = a_3 \cdot \sum_{i=3}^{k-1} a_{2i-1} + a_5 \cdot \sum_{i=4}^{k-1} a_{2i-1} + a_7 \cdot \sum_{i=5}^{k-1} a_{2i-1} + \ldots + a_{2k-7} \cdot \sum_{i=k-2}^{k-1} a_{2i-1} + a_{2k-5} \cdot a_{2k-3},$$

$$Q_{2k-1} = a_4 \cdot \sum_{i=3}^{k-1} a_{2i} + a_6 \cdot \sum_{i=4}^{k-1} a_{2i} + a_8 \cdot \sum_{i=5}^{k-1} a_{2i} + \ldots + a_{2k-6} \cdot \sum_{i=k-2}^{k-1} a_{2i} + a_{2k-4} \cdot a_{2k-2},$$

and

$$x = -2p\frac{\left(\sum_{i=1}^{k-1} a_{2i-1}^3\right) - \left(\sum_{i=1}^{k-1} a_{2i}^3\right)}{\left(\sum_{i=1}^{k-1} a_{2i-1}^4\right) - 2\left(a_{2k-1}^4 - a_{2k-2}^4\right) - \left(\sum_{i=1}^{k-1} a_{2i}^4\right)}.$$

where $\sum_{n=1}^{3} a_{2i-1}^{k-1} - \sum_{n=1}^{k-1} a_{2i}^4 \neq 2\left(a_{2k-1}^4 - a_{2k-2}^4\right)$ and $a_1, a_2, a_3, \ldots, a_{2k-1}$ are arbitrarily assigned real rational quantities. Based on these equations, some Diophantine equations have been solved. Kindly refer to Table 8.1..

This proves Lemma 8.1, Lemma 8.2, Lemma 8.3 and Lemma 8.4. From figures mentioned in Table 8,1, it is observed that while solving Diophantine equations $5.n.n$, solutions to Diophantine equation $5.n-1.n$. were also obtained.

Lemma 8.1. *A Diophantine equation* $5.n.n$ *where* $n = 2k$ *and* $k \geq 3$, *then* $(a_1x + p)^5 + (a_2x)^5 + (a_3x + p)^5 + \ldots + (a_{2k-2}x)^5 + (a_{2k-1}x + p)^5 + (a_{2k}x)^5 = (a_1x)^5 + (a_2x + p)^5 + (a_3x)^5 + \ldots + (a_{2k-2}x + p)^5 + (a_{2k-1}x)^5 + (a_{2k}x + p)^5$ *is always transformable into linear equation* $x = -2p\frac{\left(a_1^3 + a_3^3 + a_5^3 + \ldots + a_{2k-1}^3\right) - \left(a_2^3 + a_4^3 + a_6^3 + \ldots + a_{2k}^3\right)}{\left(a_1^4 + a_3^4 + a_5^4 + \ldots + a_{2k-1}^4\right) - \left(a_2^4 + a_4^4 + a_6^4 + \ldots + a_{2k}^4\right)}$, *where* $a_{1,} a_2, A_{2k}, B_{2k}, P_{2k}$ *and* Q_{2k} *are given by equations* (8.14), (8.13), (8.15). (8.16), (8.17) *and* (8.18) *respectively and* $a_{1,} a_2, a_{3,} \ldots, a_{2k}$ *are all real rational quantities such that* $\sum_{n=1}^{k} a_{2i-1}^4 \neq \sum_{n=1}^{4} a_{2i}^4$.

***Table* 8.1:** Solution to Diophantine equation $Y_1^5+Y_2^5+Y_3^5+\ldots+Y_{n-2}^5+Y_{n-1}^5+Y_n^5=Z_1^5+Z_2^5+Z_3^5+\ldots+Z_{n-2}^5+Z_{n-1}^5+Z_n^5$

D.E.	Assigned $a_3, a_4.$ $a_5 \ldots a_n$	Calculated $A_n, B_n,$ P_n, Q_n	Calculated a_1, a_2 $, x$	$(a_1x+p)^5+(a_2x)^5+\ (a_3x+p)^5+\cdots+\ (a_nx)^5$ $=(a_1x)^5+\ (a_2x+p)^5+\ (a_3x)^5+\cdots+(a_nx+p)^5$
5.11.11	$1,-1,2,-2,$ $3,-3,4,-4,$ $6,$	$10,-10$ $35,35$	$-11,9$ $31/50$	$291^5+31^5+62^5+93^5+124^5+329^5+31^5+19^5$ $+62^5+93^5+124^5+236^5$ $=279^5+81^5+112^5+143^5+174^5+136^5+341^5$ $+12^5+43^5+174^5$
5.12.12	$1,-1,2,-2,$ $3,-3,$ $4,-4,$ $5,-6$	$15,-16,$ $85,95$	$-\frac{486}{31},$ $\frac{527}{176}$	$8086^5+527^5+1054^5+1581^5+2108^5+3162^5$ $+8251^5+527^5+1054^5+1581^5$ $+2108^5+2635^5$ $=8075^5+703^5+1230^5+1757^5+2284^5+2811^5$ $+8262^5+351^5$ $+878^5+1405^5+1932^5+2986^5$
5.15.15	$1,-1,$ $2,-2,$ $3,-3,$ $4,-4,$ $5,-5,$ $6,-6,$ 8	$21,-21,$ $175,175$	$-\frac{65}{3},\frac{6}{3}$ $\frac{2844}{6523}$	$58097^5+2844^5+5688^5+8532^5+11376^5$ $+14220^5+17064^5+61351^5$ $+2844^5+679^5+5688^5+8532^5+11376^5+14220^5$ $+17064^5+26275^5$ $\underline{=57828^5+6367^5+9211^5\ +12055^5+14899^5}$ $\underline{+17743^5+20587^5}$ $+19229^5+61620^5+2165^5+5009^5+7853^5$ $+10697^5+20587^5$

Lemma 8.2. *After normalisation, a Diophantine equation* 5.*n*.*n*, $(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+\ldots+(a_{2k-2}x)^5+(a_{2k-1}x+p)^5+(a_{2k}x)^5=(a_1x)^5+(a_2x+p)^5+(a_3x)^5+\ldots+(a_{2k-2}x+p)^5+(a_{2k-1}x)^5+(a_{2k}x+p)^5$ *is always true and, in fact, is an identity where where* $n=2k\ k\geq 3$, $x=-2p\frac{\left(a_1^3+a_3^3+a_5^3+\ldots+a_{2k-1}^3\right)-\left(a_2^3+a_4^3+a_6^3+\ldots+a_{2k}^3\right)}{\left(a_1^4+a_3^4+a_5^4+\ldots+a_{2k-1}^4\right)-\left(a_2^4+a_4^4+a_6^4+\ldots+a_{2k}^4\right)}$, $a_{1,}a_2, A_{2k}, B_{2k}, P_{2k}$ *and* Q_{2k} *are given by equations* (8.14), (8.13), (8.15). (8.16), (8.17) *and* (8.18) *respectively and* $a_{1,}a_2,a_{3,}\ldots,a_{2k}$ *are all real rational quantities such that* $\sum_{n=1}^{k}a_{2i-1}^4\neq\sum_{n=1}^{4}a_{2i}^4$.

Lemma 8.3. *An equation* 5.*n*.*n where* $n=2k-1$ *and* $k>3$, *then* $(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+\ldots+(a_{2k-2}x)^5+(a_{2k-1}x-p)^5=(a_1x)^5+(a_2x+p)^5+(a_3x)^5+\ldots+(a_{2k-2}x-p)^5+(a_{2k-1}x+p)^5$ *is always transformable into linear equation* $x=-2p\frac{\left(a_1^3+a_3^3+a_5^3+\ldots a_{2k-3}^3\right)-\left(a_2^3+a_4^3+a_6^3+\ldots+a_{2k-2}^3\right)}{\left(a_1^4+a_3^4+a_5^4+\ldots+a_{2k-3}^4\right)-2\left(a_{2k-1}^4-a_{2k-2}^4\right)-\left(a_2^4+a_4^4+a_6^4+\ldots+a_{2k-2}^4\right)}$, *where* $a_{1,}a_2, A_{2k-1}, B_{2k-1}, P_{2k-1}$ *and* Q_{2k-1} *are given by equations* (8.26), (8.25), (8.27), (8.28), (8.29) *and* (8.30) *respectively and* $a_{1,}a_2,a_{3,}\ldots,a_{2k-1}$ *are all real rational quantities such that* $\sum_{n=1}^{3}a_{2i-1}^{k-1}-\sum_{n=1}^{k-1}a_{2i}^4\neq 2\left(a_{2k-1}^4-a_{2k-2}^4\right)$.

Lemma 8.4. *After normalisation, a Diophantine equation* 5.*n*.*n*, $(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+\ldots+(a_{2k-2}x)^5+(a_{2k-1}x-p)^5=(a_1x)^5+(a_2x+p)^5+(a_3x)^5+\ldots+(a_{2k-2}x-p)^5+(a_{2k-1}x+p)^5$ *is always true and in fact is an identity where* $n=2k$ *and* $k\geq 3, x=-2p\frac{\left(a_1^3+a_3^3+a_5^3+\ldots a_{2k-3}^3\right)-\left(a_2^3+a_4^3+a_6^3+\ldots+a_{2k-2}^3\right)}{\left(a_1^4+a_3^4+a_5^4+\ldots+a_{2k-3}^4\right)-2\left(a_{2k-1}^4-a_{2k-2}^4\right)-\left(a_2^4+a_4^4+a_6^4+\ldots+a_{2k-2}^4\right)}$, $a_{1,}a_2, A_{2k-1}, B_{2k-1}, P_{2k-1}$ *and* Q_{2k-1} *are given by equations* (8.26), (8.25), (8.27). (8.28), (8.29) *and* (8.30) *respectively and* $a_{1,}a_2,a_{3,}\ldots,a_{2k-1}$ *are* *all real rational quantities such that* $\sum_{n=1}^{3} a_{2i-1}^{k-1} - \sum_{n=1}^{k-1} a_{2i}^{4} \neq 2\left(a_{2k-1}^{4} - a_{2k-2}^{4}\right).$

9 Parametrisation

9.1 Parametric solution to Diophantine Equation 5.4.4

For parametric solution to Diophantine equation 5.4.4, referring to equation (2.9), equation (2.12) can be written as

$$\begin{aligned}(2a \cdot P - Q)^5 + (2b \cdot P)^5 + (2c \cdot P - Q)^5 + (2d \cdot P - t \cdot Q)^3 \\ = (2a \cdot P)^5 + (2b \cdot P - Q)^5 + (2c \cdot P - t \cdot Q)^5 + (2d \cdot P - Q)^5,\end{aligned} \tag{9.1}$$

where a, b, P and Q are given by Equations (2.8), (2.6), (2.10) and (2.11) respectively and $t = s/p$. For the sake of brevity, these are not reiterated here. Evidently a and b are dependent upon c, d and t meaning thereby that by changing the value of any one out of three, will give new values of a and b, since P and Q are dependent upon upon a, b and t, therefore, varying c, d or t will give a new set of solution. Kindly peruse Table 2.1, where t is kept equal to -2 and values of c and d have been changed. Alternatively, all $c, d$and t can be varied, therefore, equation (9.1) can be transformed into two variables or one variable by keeping two or one parameter constant. That proves there can be infinite parametric solutions to Diophantine Equation 5.4.4.

9.2 Parametric solution to Diophantine Equation 5.4.3

For parametric solution to Diophantine equation 5.4.3, a is equated with zero then equation (9.1) transforms into

$$(-Q)^5 + (2b \cdot P)^5 + (2c \cdot P - Q)^5 + (2d \cdot P - tQ)^3 = (2b \cdot P - Q)^5 + (2c \cdot P - t \cdot Q)^5 + (2d \cdot P - Q)^5. \tag{9.2}$$

a in Diophantine equation 5.4.4 was dependent upon $c, d$and t and now a has been equated with zero, therefore, c now depends upon d and t by equation (2.14) meaning thereby that now there are two variable d and t. Value of b is dependent upon t by equation (2.15). Values of P and Q now are given by equation (2.17) and (2.18). By fixing one variable, say t, then parametric solution will be in one variable. This also has infinite parametric solutions by fixing t or d at different values. Based on this parametrisation, some solutions are given in Table 2.3.

9.3 Parametric solution to Diophantine Equation 5.5.5

For parametric solution of Diophantine equation 5.5.5, equations (3.16) and (3.15) can be written as $a = \frac{cd-e^2}{c-d} = c\left(\frac{z-y^2}{1-z}\right)$ and $b = c\left\{1 - z + \left(\frac{z-y^2}{1-z}\right)\right\}$ where $\frac{e}{c} = y$, $\frac{d}{c} = z$. Considering $z = -1$, these equations take the form, $a = -\frac{c}{2}\left(1 + y^2\right)$ and $b = c\left\{2 - \frac{1}{2}\left(1 + y^2\right)\right\}$. Putting these values of a and b in equation (3.17) and simplifying

$$x = \frac{3p}{c}\left(\frac{1}{y^2 - 3}\right), \tag{9.3}$$

where $c \neq 0$. Also Diophantine equation (3.8) takes the form

$$\begin{aligned}(ax + p)^5 + (\text{ bx})^5 + (cx + p)^5 + (-cx)^3 + (ex - p)^5 \\ = (\text{ax})^5 + (bx + p)^5 + (\text{cx})^5 + (-cx - p)^5 + (ex + p)^5.\end{aligned} \tag{9.4}$$

On putting the value of $x$given by equation (9.3) in equation (9.4) and simplifying,

$$\begin{aligned}\left\{-(y^2 + 9)\right\}^5 + \left\{3\left(3 - y^2\right)\right\}^5 + \left\{2y^2\right\}^5 + \left\{-6\right\}^5 + \left\{-2y^2 + 6y + 6\right\}^5 \\ = \left\{-3\left(y^2 + 1\right)\right\}^5 + \left\{-y^2 + 3\right\}^5 + \left\{6\right\}^5 + \left\{-2y^2\right\}^5 + \left\{2y^2 + 6y - 6\right\}^5.\end{aligned} \tag{9.5}$$

This is a parametric solution with one variable, however, by changing the value of z and t, an infinite parametric solutions can be had. On the basis of parametric solution given by equation (9.5), some solutions to Diophantine Equation 5.5.5 are given in Table 9.1. Since solutions to Diophantine equation 5.5.5 also give solution to equation 5.6.4 both are given in the Table 9.1. These solutions prove veracity of parametric solutions given by equation (9.5).

9.4 Parametric solution to Diophantine Equation 5.5.4

For parametric solution to Diophantine equation 5.5.4, d is equated with zero and equation (9.4) takes the form

$$\begin{aligned}(ax + p)^5 + (\text{ bx})^5 + (cx + p)^5 + (ex - p)^5 + p^5 \\ = (\text{ax})^5 + (bx + p)^5 + (\text{cx})^5 + (ex + p)^5\end{aligned} \tag{9.6}$$

***Table* 9.1:** Solution to Diophantine equation $A^5+B^5+C^5+D^5+E^5=F^5+G^5+H^5+I^5+J^5$ and $A^5+B^5+C^5+D^5+E^5+F^5=G^5+H^5+I^5+J^5$

S.N.	y	$A^5+B^5+C^5+D^5+E^5=F^5+G^5+H^5+I^5+J^5$ and $A^5+B^5+C^5+D^5+E^5+F^5=G^5+H^5+I^5+J^5$
1	2	$13^5+3^5+6^5+14^5+6^5=8^5+10^5+15^5+1^5+8^5$
2	4	$25^5+39^5+6^5+2^5+6^5+50^5=32^5+51^5+13^5+32^5$
3	5	$34^5+66^5+6^5+14^5+6^5+74^5=50^5+78^5+22^5+50^5$
4	6	$45^5+99^5+6^5+30^5+6^5+102^5=72^5+111^5+33^5+72^5$
5	7	$58^5+138^5+6^5+50^5+6^5+134^5=98^5+150^5+46^5+98^5$
6	8	$73^5+183^5+6^5+74^5+6^5+170^5=128^5+195^5+61^5+128^5$
7	9	$90^5+234^5+6^5+102^5+6^5+210^5=162^5+246^5+78^5+162^5$
8	10	$109^5+291^5+6^5+134^5+6^5+254^5=200^5+303^5+254^5+200^5$

and Equation (3.15) transforms into $b=c-e^2/c$ and equation (3.16) transforms to $a=-e^2/c$. On putting these values of $a, b$and d in Equation (3.17) and then after simplification,

$$x=\frac{3p}{2c}\left\{\frac{1}{(y^2-1)}\right\}, \tag{9.7}$$

where $c\neq 0$. and $y\neq 1$.Putting this value x in in equation (9.6) and after simplification,

$$\left[-(y^2+2)\right]^5+\left[3\left(1-y^2\right)\right]^5+\left[2y^2+1\right]^5+\left[3y-2y^2+2\right]^5+\left[2y^2-2\right]^5$$
$$=\left[-3y^2\right]^5+\left[1-y^2\right]^5+[3]^5+\left[2y^2+3y-2\right]^5. \tag{9.8}$$

This is a parametric solution with one variable, however, by changing the value t in equation (3.14), infinite parametric solutions can be had. On the basis of parametric solution given by equation (9.8), some solutions of Diophantine Equation 5.5.4 are given in Table 9.2**.** These solutions prove veracity of parametric solutions given by this equation.

***Table* 9.2:** Solution to Diophantine equation $A^5+B^5+C^5+D^5+E^5=G^5+H^5+I^5+J^5$

S.N.	y	$A^5+B^5+C^5+D^5+E^5=F^5+G^5+H^5+I^5$
1	3	$11^5+24^5+7^5+3^5+25^5=27^5+8^5+19^5+16^5$
2	4	$18^5+45^5+18^5+3^5+42^5=33^5+20^5+48^5+15^5$
3	5	$27^5+72^5+33^5+3^5+63^5=51^5+48^5+75^5+24^5$
4	6	$38^5+105^5+52^5+3^5+88^5=73^5+70^5+108^5+35^5$
5	7	$51^5+144^5+75^5+3^5+117^5=99^5+96^5+147^5+48^5$
6	8	$66^5+189^5+102^5+3^5+150^5=129^5+126^5+192^5+63^5$
7	9	$83^5+240^5+133^5+3^5+187^5=163^5+160^5+243^5+80^5$

9.5 Parametric solution to Diophantine Equations 5.6.6 and 5.6.5

Equation (5.7) can be written as $x=-2p(P/Q)$ where $P=a_1^3+a_3^3+a_5^3-a_2^3-a_4^3-a_6^3$ and $Q=a_1^4+a_3^4+a_5^4-a_2^4-a_4^4-a_6^4$. Putting this value of x in equation (5.1), it takes the form

$$(-2a_1P+Q)^5+(-2a_2P)^5+(-2a_3P+Q)^5+(-2a_4P)^5+(-2a_5P+Q)^5+(-2a_6P)^5$$
$$=(-2a_1P)^5+(-2a_2P+Q)^5+(-2a_3P)^5+(-2a_4P+Q)^5+(-2a_5P)^5+(-2a_6P+Q)^5, \tag{9.9}$$

where a_1 and a_2 are given by equation (5.6) and (5.5) and a_3, a_4, a_5 and a_6 are real rational quantities. By fixing the value of one variable, say a_3, and assigning different real rational values to a_4, a_5 and a_6, infinite numbers of parametric solutions to Diophantine equation 5.6.6 are obtained.

Following the same procedure and equating $a_1 = 0$, parametric solutions to Diophantine Equation 5.6.5 are obtained as follow.

$$
\begin{aligned}
&((Q)^5 + (-2a_2P)^5 + (-2a_3P + Q)^5 + (-2a_4P)^5 + (-2a_5P + Q)^5 + (-2a_6P)^5 \\
&= (-2a_2P + Q)^5 + (-2a_3P)^5 + (-2a_4P + Q)^5 + (-2a_5P)^5 + (-2a_6P + Q)^5 ,
\end{aligned} \tag{9.10}
$$

where$P = a_3^3 + a_5^3 - a_2^3 - a_4^3 - a_6^3$,$Q = a_3^4 + a_5^4 - a_2^4 - a_4^4 - a_6^4$. ,$a_3$ and a_4 are given by Equations (7.2), (7.3) and a_4, a_5 and a_6 are real rational quantities. By fixing the value of one variable say a_4 and assigning different real rational values to a_5 and a_6, infinite numbers of parametric solutions to Diophantine equation 5.6.5 are obtained. Based on this parametrisation, some solutions are given in Table 7.1. and may be perused.

9.6 Parametric solution to Diophantine Equation 5.n.n where $n = 2k$ and $k \geq 3$

Equation (8.19) can be written as $x = -2p(P/Q)$ where $P = \left(a_1^3 + a_3^3 + a_5^3 \ldots + a_{2k-1}^3\right) - \left(a_2^3 + a_4^3 + a_6^3 + \ldots + a_{2k}^3\right)$ and $Q = \left(a_1^4 + a_3^4 + \ldots + a_{2k-1}^4\right) - \left(a_2^4 + a_4^4 + a_6^4 + \ldots + a_{2k}^4\right)$. Putting this value of x in equation (8.1), it takes the form

$$
\begin{aligned}
&(-2a_1P + Q)^5 + (-2a_2P)^5 + (-2a_3P + Q)^5 + \ldots + (-2a_{2k-2}P)^5 + (-2a_{2k-1}P + Q)^5 + (-2a_{2k}P)^5 \\
&= (-2a_1P)^5 + (-2a_2P + Q)^5 + (-2a_3P)^5 + \ldots + (-a_{2k-2}P + Q)^5 + (-2a_{2k-1}P)^5 + (-2a_{2k}P + Q)^5 ,
\end{aligned} \tag{9.11}
$$

where a_1 and a_2 are given by equations (8.14), (8.13),A_{2k} and B_{2k} are given by Equations (8.15) and (8.16), P_{2k} and Q_{2k} are given by equations (8.17) and (8.18), a_3, a_4 and a_5 are real rational quantities. By fixing the value of one variable say a_3 and assigning different real rational values to a_4, a_5, $a_6, \ldots a_{2k}$infinite numbers of parametric solutions to Diophantine equation 5.n.n where $n = 2k$ are obtained. Based on this parametrisation, some solutions are given in Table 8.1 and may be perused.

9.7 Parametric solution to Diophantine Equation5.n.n where $\mathbf{n = 2k - 1}$ and $\infty > \mathbf{k} > 3$

Equation (8.31) can be written as $x = -2p(P/Q)$ where $P = \left(a_1^3 + a_3^3 + a_5^3 \ldots + a_{2k-3}^3\right) - \left(a_2^3 + a_4^3 + a_6^3 + \ldots + a_{2k-2}^3\right)$ and $Q = \left(a_1^4 + a_3^4 + \ldots + a_{2k-3}^4 - a_{2k-1}^4\right) - \left(a_2^4 + a_4^4 + a_6^4 + \ldots + a_{2k-4}^4 - a_{2k-2}^4\right)$. Putting this value of x in equation (8.21), it takes the form

$$
\begin{aligned}
&(-2a_1P + Q)^5 + (-2a_2P)^5 + (-2a_3P + Q)^5 + \ldots + (-2a_{2k-2}P)^5 + (-2a_{2k-1}P - Q)^5 \\
&= (-2a_1P)^5 + (-2a_2P + Q)^5 + (-2a_3P)^5 + \ldots + (-2a_{2k-2}P - Q)^5 + (-a_{2k-1}P + Q)^5 .
\end{aligned} \tag{9.12}
$$

where a_1 and a_2 are given by equations (8.32), (8.31), A_{2k-1} and B_{2k-1} are given by Equations (8.33) and (8.34),P_{2k-1} and Q_{2k-1} are given by equations (8.35) and (8.36), $a_3, a_4a_5, \ldots a_{2k-1}$ are real rational quantities. By fixing the value of one variable say a_3 and assigning different real rational values to a_4, a_6, $\ldots a_{2k-1}$ infinite numbers of parametric solutions to Diophantine equation 5.n.$n$where $n = 2k - 1$ are obtained. Based on this parametrisation, some solutions are given in Table 8.1 and may be perused.

10 Results and conclusions

On overviewing what have been derived in this paper, it can be concluded that a real rational number say n can be expressed in algebraic form as $a \cdot x + b$ where a and$b$ are real rational quantities as assigned and x is a real rational quantity which is a variable. On the basis of this representation, a Diophantine equation say 5.n.n

$$Y_1^5 + Y_2^5 + Y_3^5 + \ldots + Y_{n-2}^5 + Y_{n-1}^5 + Y_n^5 = Z_1^5 + Z_2^5 + Z_3^5 + \ldots + Z_{n-2}^5 + Z_{n-1}^5 + Z_n^5$$

where integer $n > 3$ can be written as algebraic equation

$$(a_1x + A_1)^5 + (a_2x + A_2)^5 + (a_3x + A_3)^5 + .. + (a_nx + A_n)^5 = (b_1x + B_1)^5 + (b_2x + B_2)^5 + (b_3x + B_3)^5 + .. + (b_nx + B_n)^5$$

where $a_1, a_2, a_3, \ldots a_n, b_1, b_2, b_3, \ldots b_n, A_1, A_2, A_3, \ldots A_n$ and $B_1, B_2, B_3, \ldots B_n$ are real rational quantities. Obviously, this is a fifth power equation, if R_1, R_2, R_3, R_4 and R_5 are its rational roots then substituting R_1, R_2, R_3, R_4 and R_5 in above would be its solutions. Stumbling block for these solutions is determination of roots R_1, R_2, R_3, R_4 and R_5 of fifth power algebraic equation. To tide over this difficulty, fifth degree equation is transformed into a linear equation. In the preliminary stage, coefficient of x^5 is equated with zero so that $a_1^5 + a_2^5 + a_3^5 + \ldots + a_{n-2}^5 + a_{n-1}^5 + a_n^5 = b_1^5 + b_2^5 + b_3^5 + \ldots + b_{n-2}^5 + b_{n-1}^5 + b_n^5$. This was achieved by assigning values so that $a_i = b_i$, where i varies from 1 to n. Next task was to get rid off constant term. That required

$$A_1^5 + A_2^5 + A_3^5 + \ldots + A_{n-2}^5 + A_{n-1}^5 + A_n^5 = B_1^5 + B_2^5 + B_3^5 + \ldots + B_{n-2}^5 + B_{n-1}^5 + B_n^5.$$

For that equation $5.n.n$ was written in the following manner
$(a_1x+A_1)^5+(a_2x+A_2)^5+(a_3x+A_3)^5+\ldots+(a_nx+A_n)^5++(a_nx+A_n)^5=(a_1x+A_2)^5+(a_2x+A_3)^5+(a_3x+A_4)^5+\ldots+(a_{n-1}x+A_n)^5+(a_nx+A_1)^5$.

To further simplify it, the equation with $n=2k$ was written as
$(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+\ldots+(a_{2k-1}x+p)^5+(a_{2k}x)^5=(a_1x)^5+(a_2x+p)^5+(a_3x)^5+\ldots+(a_{2k-1}x)^5+(a_{2k}x+p)^5$
by putting $A_2=A_4=A_6=\ldots=A_n=0$, and $A_1=A_3=A_5=\ldots=A_{n-1}=p$, where p is a rational number. This equation, in fact, is a cubic equation
$x^3\left\{a_1^4+a_3^4+a_5^4+\ldots+a_{2k-1}^4-a_2^4-a_4^4+a_6^4+\ldots+a_{2k}^4\right\}$
$+2x^2p\left\{a_1^3+a_3^3+a_5^3+\ldots+a_{2k-1}^3-a_2^3-a_4^3-a_6^3-\ldots-a_{2k}^3\right\}$
$+2xp^2\left\{a_1^2+a_3^2+a_5^2+\ldots+a_{np2k-1}^2-a_2^2-a_4^2-a_6^2-\ldots-a_{2k}^2\right\}$
$+p^3\left\{a_1+a_3+a_5-\ldots+a_{2k-1}-a_2-a_4-a_6-\ldots-a_{2k}\right\}=0.$

By equating coefficients of x and constant term to zero, this equation transforms into

$$x=-2p\frac{(a_1^3+a_3^3+a_5^3+\ldots+a_{2k-1}^3)-(a_2^3+a_4^3+a_6^3+\ldots+a_{2k}^3)}{(a_1^4+a_3^4+a_5^4+\ldots+a_{2k-1}^4)-(a_2^4+a_4^4+a_6^4+\ldots+a_{2k}^4)},$$

where $a_2, a_1, A_{2k}, B_{2k}, P_{2k}$ and Q_{2k} are given by Equations 8.13, 8.14, 8.15. 8.16, 8.17 and 8.18 respectively.

When $n=2k-1$ and $k>3$, Diophantine equation is written as
$(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+\ldots+(a_{2n-2}x)^5+(a_{2n-1}x-p)^5=(a_1x)^5+(a_2x+p)^5+(a_3x)^5+\ldots+(a_{2n-2}x-p)^5+(a_{2n-1}x+p)^5$.
when a_1 and a_2 are given by Equations (8.32), (8.31), A_{2k-1} and B_{2k-1} are given by Equations (8.33) and (8.34), P_{2k-1} and Q_{2k-1} are given by Equations (8.35) and (8.36),$a_3, a_4a_5,\ldots a_{2k-1}$ are real rational quantities, above said equation transforms into linear equation (8.31). For Diophantine equations, 5.6.6 and 5.5.5, above said methods were adopted. For Diophantine equations $5.m.n$ where $m<n$,terms $n-m$ in numbers can be eliminated by equating $a_1, a_3\ldots$ equal to zero.

Highlights of the paper are

a) Write summing numbers of Diophantine Equation in algebraic form as a_ix+b_i choosing a_i and b_i so that constant term and coefficient of fifth power of x vanishes. Put these algebraic numbers in Diophantine Equation and expand it.
b) Equate to zero coefficients of power two and power one of x and obtain two relations between various a_i and b_i.
c) Satisfy above two relations and obtain transformed linear equation. Substitute value of x obtained from transformed linear equation in algebraic numbers.
d) Note value of algebraic numbers of the form a_ix+b_i after multiplication with lowest common multiplier.

Data Availability Statement

All data generated or analysed during this study are included in this published article.

Conflict of interest

There is no conflict of interest of any nature involved in this paper.

Acknowledgement. I acknowledge the help provided by the websites `https://www.desmos.com`, `https://www.mathsisfun.com/calculator-precision.html` and `https://keisan.casio.com/calculator`, in calculating the values of tedious exponential terms.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 146-156**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

CERTAIN SUMMATION FORMULAE AND RELATIONS DUE TO DOUBLE SERIES ASSOCIATED WITH THE GENERAL HYPERGEOMETRIC TYPE HURWITZ-LERCH ZETA FUNCTIONS

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(Received : February 12, 2023; In format: February 22, 2023; Accepted: February 26, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53117

Abstract

In this paper, we exhibit certain double series associated with general hypergeometric type Hurwitz-Lerch Zeta functions and then derive their summation formulae and relations due to their series and integral identities. We also obtain various known and unknown results in terms of Hurwitz-Lerch Zeta functions and their generating relations.

2020 Mathematical Sciences Classification: 11M35, 33C65.
Keywords and Phrases: Double series associated with general hypergeometric type Hurwitz - Lerch Zeta functions, summation formulae, series and integral identities, Hurwitz-Lerch Zeta generating relations.

1 Introduction and preliminaries

Recently, the authors [10] studied the generalized hypergeometric type Hurwitz-Lerch Zeta function defined by

$$ {}_pH_q\left(\begin{matrix}(\alpha)_{1,p}; \\ (\gamma)_{1,q};\end{matrix}\quad z,s,a\right)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\gamma_i)_n n!}\frac{z^n}{(n+a)^s}, \tag{1.1}$$

where $p,q\in\mathbb{N}_0, \alpha_i\in\mathbb{C}, (i=1,2,3,\ldots,p); a,\gamma_i\in\mathbb{C}\backslash\mathbb{Z}_0^-, (i=1,2,3,\ldots,q); s,z\in\mathbb{C}$.

Here in (1.1) the notations denote
$\mathbb{C}=\{z:z=x+iy:x,y\in\mathbb{R}, i=\sqrt{(-1)}\}, \mathbb{Z}_0^-=\{0,-1,-2,\ldots\}$,
$\mathbb{R}=(-\infty,\infty)$, $\mathbb{R}^+=\mathbb{R}\backslash(-\infty,0]$ and $\mathbb{N}_0=\{0,1,2,3,\ldots\}$.

Again for $a\neq 0$, the Pochhammer symbol ([14, p.45] and [21, pp.21-22]) as generalized factorial function is given by

$$(a)_n=\begin{cases}a(a+1)(a+2)\ldots(a+n-1); n\geq 1,\\ 1; n=0,\end{cases}$$

and in general it is defined as

$$(a)_v=\frac{\Gamma(a+v)}{\Gamma(a)}\quad \forall v\in\mathbb{R}.$$

In (1.1) it is also claimed that due to [7,8,10], for fixed and large value of N and with the properties of Gaussian gamma function [21, p.20], we find that the function (1.1) is written as partial sum of hypergeometric type Hurwitz-Lerch Zeta series and the generalized Gaussian hypergeometric series ([14, p. 73] and [21, pp. 42-43]), as

$$\begin{aligned} {}_pH_q\left(\begin{matrix}(\alpha)_{1,p}; \\ (\gamma)_{1,q};\end{matrix}\quad z,s,a\right)&=\sum_{n=0}^{N-1}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\gamma_i)_n n!}\frac{z^n}{(n+a)^s}\\ &+\frac{\prod_{i=1}^{p}(\alpha_i)_N\Gamma(N+a)z^N}{\prod_{i=1}^{q}(\gamma_i)_N\Gamma(N+s+a)N!}{}_{p+2}F_{q+2}\left(\begin{matrix}(\alpha+N)_{1,p},N+a,1; \\ (\gamma+N)_{1,q},N+1,N+s+a;\end{matrix}\quad z\right)\end{aligned} \tag{1.2}$$

Since in formula (1.2) for fixed and large N, the first series is finite and the second series is the generalized Gaussian hypergeometric function ${}_pF_q\ (.)$ which follows the convergent conditions given by [21, p.43]

(i) converges for $|z| < \infty$, if $p \leq q$;
(ii) converges for $|z| < 1$, if $p = q + 1$;
(iii) diverges for all $z, z \neq 0$, if $p > q + 1$;
(iv) converges absolutely for $|z| = 1$, if $p = q + 1$ alongwith

$$\Re(\omega) = \Re\left(\sum_{i=1}^{q} \gamma_i + s - \sum_{i=1}^{p} \alpha_i\right) > 0;$$

(v) converges conditionally for $|z| = 1, z \neq 1$, if $p = q + 1$ and $-1 < \Re(\omega) \leq 0$;
(vi) diverges for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) < -1$.

Therefore, the series in (1.1) also satisfies same convergence conditions as given above in (i) to (vi).

In the formula (1.1) taking $p = 2$, $q = 1$, $\alpha_1 = \alpha, \alpha_2 = \beta$, and $\gamma_1 = \gamma$, we convert it specially into the extended hypergeometric type Hurwitz-Lerch Zeta function, used in the probability distributions due to Garg et al. [5], in the form

$$ {}_2H_1\left(\begin{array}{c}\alpha, \beta; \\ \gamma;\end{array} z, s, a\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} \frac{z^n}{(n+a)^s} = \phi_{\alpha,\beta;\gamma}(z, s, a), \tag{1.3}$$

where, $\alpha, \beta, s, z \in \mathbb{C}$ and a, $\gamma \in \mathbb{C}\backslash\mathbb{Z}_0^-$, converges if $\Re(s) > 0$, when $|z| < 1, (z \neq 1)$. But when $z = 1$, for $\Re(\gamma) > \frac{1}{2}\Re(\alpha + \beta + 1) > 0$, the series in (1.3) converges if

$$\Re(s) > \frac{1}{2}\Re(\alpha + \beta) - \frac{1}{2}, (\text{ see [10]}). \tag{1.4}$$

It is remarked that on combining both the conditions of $\Re(\gamma)$ and the $\Re(s)$ given in (1.3) and (1.4), we get

$$\Re(\gamma + s - \alpha - \beta) > 0,$$

which is identical to $\Re(\omega)$ given in (iv) of (1.2) for $p = 2$ and $q = 1$.

Further in the generalized hypergeometric type Hurwitz-Lerch Zeta function (1.1), if we set $q = p - 1, \gamma_1 = \alpha_1, \gamma_2 = \alpha_2, \ldots, \gamma_{p-1} = \alpha_{p-1}, \alpha_p = 1$, it becomes Hurwitz-Lerch Zeta function as

$$\begin{aligned} {}_pH_{p-1}\left(\begin{array}{c}(\alpha)_{1,p-1}, 1; \\ (\alpha)_{1,p-1};\end{array} z, s, a\right) &= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p-1}(\alpha_i)_n (1)_n}{\prod_{i=1}^{p-1}(\alpha_i)_n n!} \frac{z^n}{(n+a)^s} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} = \phi(z, s, a), \end{aligned} \tag{1.5}$$

which converges if $\Re(s) > 0$, when $|z| < 1, (z \neq 1)$, but when $z = 1$, the series (1.5) converges for $\Re(s) > 1$. We also verify it as setting $\gamma = \alpha, \beta = 1$ in remark of Eqn. (1.4) and $\Re(s) > 1$.

In extension of (1.1), we again define a general hypergeometric type Hurwitz -Lerch Zeta function in following form

$$ {}_pK_q\left(\begin{array}{c}(\alpha)_{1,p}; \\ (\gamma)_{1,q};\end{array} A; z, s, a\right) = \sum_{n=0}^{\infty} \frac{A_n \prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\gamma_i)_n} \frac{z^n}{(n+a)^s n!}, \tag{1.6}$$

where $p, q \in \mathbb{N}_0, \alpha_i \in \mathbb{C}, (i = 1, 2, 3, \ldots, p); a, \gamma_i \in \mathbb{C}\backslash\mathbb{Z}_0^-, (i = 1, 2, 3, \ldots, q)$; $s, z \in \mathbb{C}$. A symbolizes for a bounded real or complex A_n $\forall n \in \mathbb{N}_0$ and follows certain restrictions.

For a sequence $\langle A_n\rangle = \langle 1\rangle$ $\forall n \in \mathbb{N}_0$, by (1.1) and (1.6), we find an identity

$$ {}_pK_q\left(\begin{array}{c}(\alpha)_{1,p}; \\ (\gamma)_{1,q};\end{array} 1; z, s, a\right) = {}_pH_q\left(\begin{array}{c}(\alpha)_{1,p}; \\ (\gamma)_{1,q};\end{array} z, s, a\right).$$

Again, for a sequence $\langle A_n\rangle = \langle (1)_n\rangle$ $\forall n \in \mathbb{N}_0$, we have a relation with (1.1) and (1.6) as

$$ {}_pK_q\left(\begin{array}{c}(\alpha)_{1,p}; \\ (\gamma)_{1,q};\end{array} (1); z, s, a\right) = {}_{p+1}H_q\left(\begin{array}{c}(\alpha)_{1,p}, 1; \\ (\gamma)_{1,q};\end{array} z, s, a\right).$$

It is recalled that Exton [3] obtained some theorems on general hypergeometric generating relations, Srivastava [17] established certain generating relations of Hurwitz-Lerch Zeta functions and recently, Kumar and Chandel [10] derived various relations and identities for double series associated with general Hurwitz-Lerch type Zeta functions. In this motivation, we exhibit these researches for exploring new ideas in the theory of extended generalized hypergeometric type Hurwitz-Lerch Zeta functions and thus consider $x, y, s \in \mathbb{C}; a, 2d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and A_n, a bounded real or complex sequence $\forall n \in \mathbb{N}_0$, which follows certain restrictions to introduce following families of general hypergeometric type Hurwitz-Lerch Zeta functions in the form

$$\phi_1(A, d, d-1/2; 2d; x, y; s, a) = \sum_{m,n=0}^{\infty} \frac{A_n (d)_{m+n} \left(d-\frac{1}{2}\right)_{m+n}}{(2d)_{m+2n}} \frac{x^{m+n} y^n}{(n+a)^s m! n!}, \tag{1.7}$$

$$\phi_2(A, d, d+1/2; 2d; x, y; s, a) = \sum_{m,n=0}^{\infty} \frac{A_n (d)_{m+n} \left(d+\frac{1}{2}\right)_{m+n}}{(2d)_{m+2n}} \frac{x^{m+n} y^n}{(n+a)^s m! n!}. \tag{1.8}$$

Here in left hand sides of (1.7) and (1.8), A stands for bounded real or complex sequence A_n $\forall n \in \mathbb{N}_0$ as in right hand side of their series.

Again, to obtain summation formulae, series and integral identities of the functions defined in (1.7) and (1.8) in terms of (1.6), we make an appeal to following preliminary formulae:

For $z \in \mathbb{C}, |z| \leq 1, 2d \neq 0, -1, -2, \ldots$, (see Erdélyi et al. [2, Vol. I, p. 101], Srivastava and Manocha [21, p. 34])

$$ {}_2F_1\left(\begin{array}{c} d, d-\frac{1}{2}; \\ 2d; \end{array} z\right) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2d}, \tag{1.9}$$

but by (1.9), we immediately have

$$ {}_2F_1\left(\begin{array}{c} d, d-\frac{1}{2}; \\ 2d; \end{array} 1\right) = 2^{2d-1}. \tag{1.10}$$

Also there exists another result

$$ {}_2F_1\left(\begin{array}{c} d, d+\frac{1}{2}; \\ 2d; \end{array} z\right) = \frac{1}{\sqrt{1-z}} \left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2d}, \tag{1.11}$$

provided that, $z \in \mathbb{C}, |z| < 1, 2d \neq 0, -1, -2, \ldots$.

For all $0 \leq n \leq m$

$$\frac{1}{(m-n)!} = \frac{(-1)^n (-m)_n}{m!}, \tag{1.12}$$

and $\forall n \in \mathbb{N}_0$

$$(\lambda)_{2n} = 2^{2n} \left(\frac{\lambda}{2}\right)_n \left(\frac{\lambda}{2} + \frac{1}{2}\right)_n. \tag{1.13}$$

2 Eulerian Integral representations

In this section, we derive Eulerian integral representations of the general hypergeometric type Hurwitz-Lerch Zeta functions defined in the Eqns. (1.1), (1.6), (1.7) and (1.8) involving known and unknown hypergeometric functions.

Here $\forall n \in \mathbb{N}_0, a, s \in \mathbb{C}, \Re(s) > 0, \Re(a) > 0$, we apply the following Eulerian integral formula [4,11,12,13,18]

$$\frac{1}{\Gamma(s)} \int_0^{\infty} e^{-(a+n)t} t^{s-1} dt = \frac{1}{(n+a)^s},$$

in the Eqns. (1.1) and (1.6) and obtain their Eulerian integral representations

$$ {}_pH_q\left(\begin{array}{c} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{array} z, s, a\right) = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-at} t^{s-1} {}_pF_q\left(\begin{array}{c} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{array} ze^{-t}\right) dt, \tag{2.1}$$

$$ {}_pK_q\left(\begin{array}{c} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{array} A; z, s, a\right) = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-at} t^{s-1} {}_pG_q\left(\begin{array}{c} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{array} A; ze^{-t}\right) dt, \tag{2.2}$$

where, ${}_pG_q\left(\begin{array}{c} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{array} A; z\right) = \sum_{n=0}^{\infty} \frac{A_n \prod_{i=1}^{p} (\alpha_i)_n}{\prod_{i=1}^{q} (\gamma_i)_n} \frac{z^n}{n!}$ is a general hypergeometric function. A stands for a bounded real or complex sequence A_n $\forall n \in \mathbb{N}_0$. For example if $\langle A_n \rangle = \langle 1 \rangle$, $\forall n \in \mathbb{N}_0$, then there exists a relation

$$ {}_pG_q\left(\begin{array}{c} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{array} 1; z\right) = {}_pF_q\left(\begin{array}{c} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{array} z\right).$$

It is remarked that on specialization of the parameters in (2.1) and (2.2), we may obtain various Hurwitz-Lerch Zeta functions associated with the hypergeometric functions and hypergeometric polynomials found in the literature (for example, Rainville [14], Slater [15], Sneddon [16], Srivastava and Karlsson [20], Srivastava and Manocha [21] and others).

In above motivation of (2.1) and (2.2) $\forall x, y \in \mathbb{C}; 2d \in \mathbb{C}\backslash\mathbb{Z}_0^-; a, s \in \mathbb{C}, \Re(s) > 0, \Re(a) > 0$, we introduce following integral representations of the functions (1.7) and (1.8) defined by

$$\phi_1(A, d, d-1/2; 2d; x, y; s, a) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-at} t^{s-1} \phi_1{}^* \left(A, d, d-1/2; 2d; x, xye^{-t}\right) dt, \tag{2.3}$$

$$\phi_2(A, d, d+1/2; 2d; x, y; s, a) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-at} t^{s-1} \phi_2{}^* \left(A, d, d+1/2; 2d; x, xye^{-t}\right) dt, \tag{2.4}$$

where the general double functions $\phi_1^*(.)$ and $\phi_2^*(.)$ are defined in the double series

$$\phi_1{}^*(A, d, d-1/2; 2d; x, y) = \sum_{m,n=0}^\infty \frac{A_n (d)_{m+n} \left(d-\frac{1}{2}\right)_{m+n}}{(2d)_{m+2n}} \frac{x^m y^n}{m!n!}, \tag{2.5}$$

$$\phi_2{}^*(A, d, d+1/2; 2d; x, y) = \sum_{m,n=0}^\infty \frac{A_n (d)_{m+n} \left(d+\frac{1}{2}\right)_{m+n}}{(2d)_{m+2n}} \frac{x^m y^n}{m!n!}. \tag{2.6}$$

Here, A denotes for bounded real or complex sequence A_n $\forall n \in \mathbb{N}_0$ and follows certain restrictions.

It is noticed that on specialization of the parameters of (2.5) and (2.6) and making an appeal to the formulae (2.3) and (2.4), we may obtain various Hurwitz-Lerch Zeta functions associated with the hypergeometric functions of two variables like Appell's functions, Kampé de Fériet functions, Humbert functions and others found in the literature (see for example, Bailey [1], Exton [4], Srivastava and Panda [19], Srivastava and Karlsson [20], Srivastava and Manocha [21] and so on).

For example, under the conditions $\sum_{j=1}^Q \vartheta_j - \sum_{j=1}^P \theta_j > 0$, if we set the sequence $A_n = \frac{\prod_{j=1}^P \Gamma(\alpha_j + \theta_j n)}{\prod_{j=1}^Q \Gamma(\beta_j + \vartheta_j n)}$, $\forall n \in \mathbb{N}_0, \alpha_j \in \mathbb{C}, \theta_j \in \mathbb{R}^+, \ \forall (j = 1, 2, 3, \ldots, P); \beta_j \in \mathbb{C}, \vartheta_j \in \mathbb{R}^+, \ \forall (j = 1, 2, 3, \ldots, Q)$, then for $2d, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-, \ \forall (j = 1, 2, 3, \ldots, Q)$, the functions (2.5) and (2.6) become double Srivastava-Daoust functions [18] in the following form

$$\phi_1{}^*(d, d-1/2, [(\alpha):\theta]; 2d, [(\beta):\vartheta]; x, y) = \frac{2^{2d-1}\Gamma\left(d+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(d-\frac{1}{2}\right)} S^{2:0;P}_{1:0;Q}\left[\begin{matrix} [d:1,1], \left[d-\frac{1}{2}:1,1\right]:-;[(\alpha):\theta]; \\ [2d:1,2]:-;[(\beta):\vartheta]; \end{matrix}\ x, y\right], \tag{2.7}$$

provided that $|x| < \infty, |y| < 1$,

and

$$\phi_2{}^*(d, d+1/2, [(\alpha):\theta]; 2d, [(\beta):\vartheta]; x, y) = \frac{2^{2d-1}}{\Gamma\left(\frac{1}{2}\right)} S^{2:0;P}_{1:0;Q}\left[\begin{matrix} [d:1,1], \left[d+\frac{1}{2}:1,1\right]:-;[(\alpha):\theta]; \\ [2d:1,2]:-;[(\beta):\vartheta]; \end{matrix}\ x, y\right], \tag{2.8}$$

provided that $|x| < \infty, |y| < 1$, respectively.

Thus making an appeal to (2.7) and (2.8) in the Eqns. (2.3) and (2.4) respectively, for $a, s \in \mathbb{C}, \Re(s) > 0, \Re(a) > 0; 2d, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-, \ \forall (j = 1, 2, 3, \ldots, Q)$, we generate integral representation of the Hurwitz-Lerch double Zeta functions associated with the Srivastava-Daoust double series, given by

$$\begin{aligned}\psi_1\left(\begin{matrix} d, d-\frac{1}{2}:[(\alpha):\theta]; \\ 2d:[(\beta):\vartheta]; \end{matrix}\ x, y; s, a\right) &= \sum_{m,n=0}^\infty \frac{(d)_{m+n}(d-\frac{1}{2})_{m+n}}{(2d)_{m+2n}} \frac{\prod_{j=1}^P \Gamma(\alpha_j + \theta_j n)}{\prod_{j=1}^Q \Gamma(\beta_j + \vartheta_j n)} \frac{x^{m+n} y^n}{(m+2n+a)^s m!n!} \\ &= \frac{\Gamma(d+\frac{1}{2})}{\sqrt{\pi}\Gamma(d-\frac{1}{2})} \frac{2^{2d-1}}{\Gamma(s)} \int_0^\infty e^{-at} t^{s-1} S^{2:0;P}_{1:0;Q}\left[\begin{matrix} [d:1,1], [d-\frac{1}{2}:1,1]:-;[(\alpha):\theta]; \\ [2d:1,2]:-;[(\beta):\vartheta]; \end{matrix}\ xe^{-t}, xye^{-2t}\right] dt,\end{aligned}$$

provided that $|x| < \infty, |y| < 1$, (2.9)

and

$$\begin{aligned}\psi_2\left(\begin{matrix} d, d+1/2:[(\alpha):\theta]; \\ 2d:[(\beta):\vartheta]; \end{matrix}\ ; s, a\right) &= \sum_{m,n=0}^\infty \frac{(d)_{m+n}\left(d+\frac{1}{2}\right)_{m+n}}{(2d)_{m+2n}} \frac{\prod_{j=1}^P \Gamma(\alpha_j + \theta_j n)}{\prod_{j=1}^Q \Gamma(\beta_j + \vartheta_j n)} \frac{x^{m+n} y^n}{(m+2n+a)^s m!n!} \\ &= \frac{2^{2d-1}}{\sqrt{\pi}\Gamma(s)} \int_0^\infty e^{-at} t^{s-1} S^{2:0;P}_{1:0;Q}\left[\begin{matrix} [d:1,1], [d+\frac{1}{2}:1,1]:-;[(\alpha):\theta]; \\ [2d:1,2]:-;[(\beta):\vartheta]; \end{matrix}\ xe^{-t}, xye^{-2t}\right],\end{aligned} \tag{2.10}$$

provided that $|x| < \infty, |y| < 1$, respectively.

Recently, some general Hurwitz-Lerch type Zeta functions associated with the double and multiple Srivastava-Daoust hypergeometric functions are analyzed in [18] which are applied in different scientific problems for example see [6,9]. Therefore importance in further researches, we study analytic continuation properties of the double functions (2.9) and (2.10) through their integral representations.

3 Summation Formulae

In this section, we obtain summation formulae of the general hypergeometric type Hurwitz-Lerch Zeta functions of one and two variables defined by Eqns. (1.1), (1.6), (1.7) and (1.8). Again we show that the functions (1.7) and (1.8) are represented as the sum of functions (1.6).

Lemma 3.1. *If $p, q \in \mathbb{N}_0$, $\alpha_i \in \mathbb{C}, (i = 1, 2, 3, \ldots, p)$; $a, \gamma_i \in \mathbb{C}\backslash\mathbb{Z}_0^-, (i = 1, 2, 3, \ldots, q)$; $s, z \in \mathbb{C}$. Then under the conditions given in* (1.2)*, the summation formula of* (1.1) *exists as*

$$ {}_pH_q\left(\begin{array}{l}(\alpha)_{1,p};\\(\gamma)_{1,q};\end{array} z, s, a\right) = \exp[-\operatorname{s}\log a] + \frac{\prod_{i=1}^{p}(\alpha_i)}{\prod_{i=1}^{q}(\gamma_i)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r\,{}_pH_q\left(\begin{array}{l}(\alpha+1)_{1,p};\\(\gamma+1)_{1,q};\end{array} z, s+r+1, 1\right). \quad (3.1)$$

Proof. Considering the formula (1.1) and for $a \neq 0$, using the binomial theorem, we write it as

$$ {}_pH_q\left(\begin{array}{l}(\alpha)_{1,p};\\(\gamma)_{1,q};\end{array} z, s, a\right) = \frac{1}{a^s} + \sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\gamma_i)_n\, n!}\frac{z^n}{n^{s+r}}. \quad (3.2)$$

The Eqn. (3.2) on aid of (1.1) immediately gives the result (3.1).

Clearly, making an appeal to the formula (3.1), we get following summation formulae in terms of the hyperbolic functions:

$$\frac{\prod_{i=1}^{p}(\alpha_i)}{2\prod_{i=1}^{q}(\gamma_i)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r\,{}_pH_q\left(\begin{array}{l}(\alpha+1)_{1,p};\\(\gamma+1)_{1,q};\end{array} z, s+r+1, 1\right) - \frac{1}{2}{}_pH_q\left(\begin{array}{l}(\alpha)_{1,p};\\(\gamma)_{1,q};\end{array} z, s, a\right) + \frac{1}{2}{}_pH_q\left(\begin{array}{l}(\alpha)_{1,p};\\(\gamma)_{1,q};\end{array} z, s, a^{-1}\right)$$
$$- \frac{\prod_{i=1}^{p}(\alpha_i)}{2\prod_{i=1}^{q}(\gamma_i)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a^{-1})^r\,{}_pH_q\left(\begin{array}{l}(\alpha+1)_{1,p};\\(\gamma+1)_{1,q};\end{array} z, s+r+1, 1\right)$$
$$= \sinh[s\log a] \quad (3.3)$$

and

$$\frac{1}{2}{}_pH_q\left(\begin{array}{l}(\alpha)_{1,p};\\(\gamma)_{1,q};\end{array} z, s, a\right) - \frac{\prod_{i=1}^{p}(\alpha_i)}{2\prod_{i=1}^{q}(\gamma_i)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r\,{}_pH_q\left(\begin{array}{l}(\alpha+1)_{1,p};\\(\gamma+1)_{1,q};\end{array} z, s+r+1, 1\right) -$$
$$+ \frac{1}{2}{}_pH_q\left(\begin{array}{l}(\alpha)_{1,p};\\(\gamma)_{1,q};\end{array} z, s, a^{-1}\right) - \frac{\prod_{i=1}^{p}(\alpha_i)}{2\prod_{i=1}^{q}(\gamma_i)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a^{-1})^r\,{}_pH_q\left(\begin{array}{l}(\alpha+1)_{1,p};\\(\gamma+1)_{1,q};\end{array} z, s+r+1, 1\right)$$
$$= \cosh[s\log a]. \quad (3.4)$$

Similarly by the formula (1.6), we get

$$ {}_pK_q\left(\begin{array}{l}(\alpha)_{1,p};\\(\gamma)_{1,q};\end{array} A; z, s, a\right) = A_0\exp[-s\log a] + \frac{\prod_{i=1}^{p}(\alpha_i)}{\prod_{i=1}^{q}(\gamma_i)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r\,{}_pK_q\left(\begin{array}{l}(\alpha+1)_{1,p};\\(\gamma+1)_{1,q};\end{array} A^+; z, s+r+1, 1\right), \quad (3.5)$$

where $p, q \in \mathbb{N}_0, \alpha_i \in \mathbb{C}, (i = 1, 2, 3, \ldots, p); a, \gamma_i \in \mathbb{C}\backslash\mathbb{Z}_0^-, (i = 1, 2, 3, \ldots, q); s, z \in \mathbb{C}, A$ stands for a sequence A_n, a bounded real or complex sequences $\forall n \in \mathbb{N}_0$. Also A^+stands for a sequence A_{n+1}, $\forall n \in \mathbb{N}_0$. □

Theorem 3.1. *If all $x, y, s \in \mathbb{C}; a, 2d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and A_n be bounded real or complex sequences $\forall n \in \mathbb{N}_0$, then under the conditions $|x| \leq 1$, the double series* (1.7) *follows a summation formula*

$$\phi_1(A, d, d-1/2; 2d; x, y; s, a)$$
$$= \frac{A_0}{a^s}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d} + \frac{xy(2d-1)}{4(2d+1)}\left(\frac{1+\sqrt{1-x}}{2}\right)^{-2d-1}$$
$$\times\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r\,{}_1K_1\left(\begin{array}{l}d+\frac{1}{2};\\d+\frac{3}{2};\end{array} A^+; \frac{xy}{(1+\sqrt{1-x})^2}, s+r+1, 1\right), \quad (3.6)$$

where, ${}_1K_1$ (.) is a general hypergeometric type Hurwitz -Lerch Zeta function (1.6) *and A^+stands for the sequence A_{n+1}, a bounded real or complex sequences $\forall n \in \mathbb{N}_0$ that follows certain restrictions.*

Proof. We consider the double series (1.7) in the form

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{A_n(d)_{m+n}\left(d-\frac{1}{2}\right)_{m+n}}{(2d)_{m+2n}}\frac{x^m(xy)^n}{(n+a)^s m!n!},$$

and apply series rearrangement techniques to derive hypergeometric function

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\sum_{n=0}^{\infty}\frac{A_n(d)_n\left(d-\frac{1}{2}\right)_n}{(2d)_{2n}}\frac{(xy)^n}{(n+a)^s n!}{}_2F_1\left(\begin{array}{c}d+n,d+n-\frac{1}{2};\\ 2d+2n;\end{array}x\right). \tag{3.7}$$

Now in (3.7) under the conditions $|x|\leq 1$, using the formulae (1.9) and (1.13), we get

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}\sum_{n=0}^{\infty}\frac{A_n\left(d-\frac{1}{2}\right)_n}{\left(d+\frac{1}{2}\right)_n}\frac{\left(\frac{xy}{(1+\sqrt{1-x})^2}\right)^n}{(n+a)^s n!}. \tag{3.8}$$

In (3.8) applying the formula (1.6), we obtain the result

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}{}_1K_1\left(\begin{array}{c}d-\frac{1}{2};\\ d+\frac{1}{2};\end{array}A;\frac{xy}{(1+\sqrt{1-x})^2},s,a\right),$$

in which making an appeal to the techniques of Lemma 3.1, we obtain the summation formula (3.6). Hence the Theorem 3.1 is proved. □

Corollary 3.1. *If in the Theorem* 3.1 *put* $x=1$ *and all* $y,s\in\mathbb{C};a,2d\in\mathbb{C}\backslash\mathbb{Z}_0^-$ *and* A_n *be bounded real or complex sequences* $\forall n\in\mathbb{N}_0$*, then following summation formula exists*

$$2^{1-2d}\phi_1(A,d,d-1/2;2d;1,y;s,a)=\frac{A_0}{a^s}+\frac{y(2d-1)}{(2d+1)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r{}_1K_1\left(\begin{array}{c}d+\frac{1}{2};\\ d+\frac{3}{2};\end{array}A^+;y,s+r+1,1\right). \tag{3.9}$$

Also there exists an identity

$$2^{1-2d}\phi_1(A,d,d-1/2;2d;1,y;s,a)={}_1K_1\left(\begin{array}{c}d-\frac{1}{2};\\ d+\frac{1}{2};\end{array}A;y,s,a\right). \tag{3.10}$$

Proof. Considering the Eqn. (3.6) and putting $x=1$, we obtain the summation formula (3.9). Further making same process with an appeal to the Eqns. (1.6), (1.10) and (3.8), we find an identity (3.10). □

Corollary 3.2. *If* $\Re(s)>0,\Re(a)>0$*, then under the conditions of the Theorem 3.1 an Eulerian integral representation of the double series* (1.7) *exists in the following form*

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\frac{1}{\Gamma(s)}\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}\int_0^{\infty}e^{-at}t^{s-1}{}_1G_1\left(\begin{array}{c}d-\frac{1}{2};\\ d+\frac{1}{2};\end{array}A;\frac{xye^{-t}}{(1+\sqrt{1-x})^2}\right)dt \tag{3.11}$$

Proof. Consider the formula (3.8) and then here under the conditions of the Theorem 3.1, use an Eulerian integral formula and thus apply the techniques of Section 2, we get the formula

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}\sum_{n=0}^{\infty}\frac{A_n\left(d-\frac{1}{2}\right)_n}{\left(d+\frac{1}{2}\right)_n n!}\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-at}t^{s-1}\left(\frac{xye^{-t}}{(1+\sqrt{1-x})^2}\right)^n dt. \tag{3.12}$$

Now in right hand side of (3.12) using the function (2.2), we obtain the result (3.11). □

Theorem 3.2. *If all* $x,y,s\in\mathbb{C};a,2d\in\mathbb{C}\backslash\mathbb{Z}_0^-$ *such that If* $\Re(s)>0,\Re(a)>0$ *and* A_n *be bounded real or complex sequences* $\forall n\in\mathbb{N}_0$*, then by the function* (1.7) *under the conditions* $|x|\leq 1$*, following summation formula of* (1.7) *also exists*

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\sum_{n=0}^{\infty}\frac{(d)_n\left(d-\frac{1}{2}\right)_n}{(2d)_n n!}{}_1K_1\left(\begin{array}{c}-n;\\ 2d+n;\end{array}A;-y,s,a\right)x^n, \tag{3.13}$$

where, the partial sum of the extended general hypergeometric type Hurwitz -Lerch Zeta function (1.6) *is defined by*

$${}_1K_1\left(\begin{array}{c}-n;\\ 2d+n;\end{array}A;-y,s,a\right)=\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-at}t^{s-1}{}_1G_1\left(\begin{array}{c}-n;\\ 2d+n;\end{array}A;-ye^{-t}\right)dt,\Re(s)>0,\Re(a)>0,\ \forall n=0,1,2,3,\ldots, \tag{3.14}$$

${}_1G_1(\cdot)$ *is defined in (2.2).*

Proof. Considering the function (1.7) and applying the series rearrangement techniques, we find that

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{A_n(d)_m\left(d-\frac{1}{2}\right)_m}{(2d)_{m+n}}\frac{x^m y^n}{(n+a)^s(m-n)!n!}.$$

Now using the formula (1.12) and making an appeal to the Eulerian integral formula given in the Section 2 we find that

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\sum_{n=0}^{\infty}\frac{(d)_n\left(d-\frac{1}{2}\right)_n x^n}{(2d)_n n!}\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-at}t^{s-1}\left\{\sum_{m=0}^{n}\frac{A_m(-n)_m}{(2d+n)_m}\frac{(-ye^{-t})^m}{m!}\right\}dt. \tag{3.15}$$

In right hand side of the Eqn. (3.15) making an appeal to the formula (2.2), we derive (3.14) and from which, we finally obtain the result (3.13).

In the similar manner, we obtain following results: □

Theorem 3.3. *If all $x,y,s\in\mathbb{C}; a,2d\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and A_n be bounded real or complex sequences $\forall n\in\mathbb{N}_0$, then due to the function* (1.8) *under the conditions $|x|<1$, following summation formula exists*

$$\begin{aligned}\phi_2(A,d,d+1/2;2d;x,y;s,a)&=\frac{A_0}{(a)^s}\frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d}\\&+\frac{xy}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{-1-2d}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r{}_0K_0\left({}^{-;}_{-;}A^+;\frac{xy}{(1+\sqrt{1-x})^2},s+r+1,1\right)\end{aligned} \tag{3.16}$$

Here, A^+ stands for A_{n+1}, a bounded real or complex sequences $\forall n\in\mathbb{N}_0$, follows certain restrictions.

Proof. Under the conditions given in the Theorem 3.3, for the double series (1.8), we write

$$\phi_2(A,d,d+1/2;2d;x,y;s,a)=\sum_{n=0}^{\infty}\frac{A_n(d)_n\left(d+\frac{1}{2}\right)_n(xy)^n}{(2d)_{2n}n!(n+a)^s}{}_2F_1\left[\begin{matrix}d+n,d+n+\frac{1}{2};\\2d+2n;\end{matrix}\ x\right]. \tag{3.17}$$

Now in the Eqn. (3.17) using of the formulae (1.11)-(1.13) for $|x|<1$, we obtain

$$\phi_2(A,d,d+1/2;2d;x,y;s,a)=\frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{xy}{(1+\sqrt{1-x})^2}\right)^n}{n!(n+a)^s}. \tag{3.18}$$

Now in Eqn. (3.18) making an appeal to formula (1.6) and the theory given in Theorem 3.1, we derive the result (3.16). □

Corollary 3.3. *If $\Re(s)>0,\Re(a)>0$, then due to the function* (1.8) *following formula holds*

$$\phi_2(A,d,d+1/2;2d;x,y;s,a)=\frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d}\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-at}t^{s-1}{}_0K_0\left({}^{-;}_{-;}A;\frac{xye^{-t}}{(1+\sqrt{1-x})^2}\right)dt. \tag{3.19}$$

Proof. In the Eqn. (3.18) of the Theorem 3.3 applying the Eulerian formula given in (2.1), we derive

$$\phi_2(A,d,d+1/2;2d;x,y;s,a)=\frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d}\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-at}t^{s-1}\left\{\sum_{n=0}^{\infty}\frac{A_n}{n!}\left(\frac{xye^{-t}}{(1+\sqrt{1-x})^2}\right)^n\right\}dt. \tag{3.20}$$

Now in (3.20), applying the formula (1.6) and same technique of proof of the Theorem 3.1, we obtain the formula (3.19). □

Theorem 3.4. *If all $x,y,s\in\mathbb{C}; a,2d\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and A_n be bounded real or complex sequences $\forall n\in\mathbb{N}_0$, then due to the function* (1.8) *under the conditions $|x|<1$, following summation formula exists*

$$\varphi_2(A,d,d+\frac{1}{2};2d;x,y;s,a)=\sum_{n=0}^{\infty}\frac{(d)_n(d+\frac{1}{2})_n}{(2d)_n n!}{}_1K_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}A;-y,s,a\right)x^n, \tag{3.21}$$

where, the function ${}_1K_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}A;-y,s,a;\right)$ is defined by (3.14).

Proof. Making an appeal to the function (1.8), we get

$$\phi_2(A,d,d+1/2;2d;x,y;s,a) = \sum_{n=0}^{\infty} \frac{(d)_n\left(d+\frac{1}{2}\right)_n x^n}{(2d)_n n!} \sum_{m=0}^{n} \frac{A_m(-n)_m}{(2d+n)_m} \frac{(-y)^m}{(m+a)^{s s m}}$$

$$= \sum_{n=0}^{\infty} \frac{(d)_n\left(d+\frac{1}{2}\right)_n x^n}{(2d)_n n!} \frac{1}{\Gamma(\mathrm{s})} \int_0^{\infty} e^{-at} t^{s-1} \left\{ \sum_{m=0}^{n} \frac{A_m(-n)_m}{(2d+n)_m} \frac{(-ye^{-t})^m}{m!} \right\} dt. \quad (3.22)$$

Now in the second series of (3.22) making an appeal to the function (3.14), we obtain the summation formula (3.21).

We present following applications of our results derived in the Sections 2 and 3 : □

4 Applications

In this section, we make an application of the Theorems presented in the previous Sections 2 and 3 . Then we obtain generating relations and the integral identities.

Application 4.1. If all conditions of the Theorems 3.1 and 3.2 are satisfied, then there exists a generating relation of the extended general hypergeometric type Hurwitz -Lerch Zeta function

$$\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1} {}_1K_1\left(\begin{matrix} d-\frac{1}{2}; \\ d+\frac{1}{2}; \end{matrix}\ A; \frac{xy}{(1+\sqrt{1-x})^2}, s, a\right) = \sum_{n=0}^{\infty} \frac{(d)_n\left(d-\frac{1}{2}\right)_n x^n}{(2d)_n n!} {}_1K_1\left(\begin{matrix} -n; \\ 2d+n; \end{matrix}\ A; -y, s, a\right). \quad (4.1)$$

Solution. Considering the Eqn. (3.8) of the Theorem 3.1 and applying (3.15) of the Theorem 3.2, we derive the equality given by

$$\phi_1(A,d,d-1/2;2d;x,y;s,a) = \left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1} \sum_{n=0}^{\infty} \frac{A_n\left(d-\frac{1}{2}\right)_n}{\left(d+\frac{1}{2}\right)_n} \frac{\left(\frac{xy}{(1+\sqrt{1-x})^2}\right)^n}{(n+a)^s n!}$$

$$= \sum_{n=0}^{\infty} \frac{(d)_n\left(d-\frac{1}{2}\right)_n x^n}{(2d)_n n!} \sum_{m=0}^{n} \frac{A_m(-n)_m}{(2d+n)_m} \frac{(-y)^m}{(m+a)^s m!}. \quad (4.2)$$

Now in the relation (4.2) making an appeal to the extended general hypergeometric type Hurwitz -Lerch Zeta function (1.6) in the last two equalities, we obtain the generating relation (4.1).

Application 4.2. If all conditions of the Theorems 3.3 and 3.4 are satisfied, then there exists a generating relation of the extended general hypergeometric type Hurwitz -Lerch Zeta function as

$$\frac{1}{\sqrt{1-x}}\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1} {}_0K_0\left(\begin{matrix} -; \\ -; \end{matrix}\ A; \frac{xy}{(1+\sqrt{1-x})^2} z, s, a\right) = \sum_{n=0}^{\infty} \frac{(d)_n\left(d+\frac{1}{2}\right)_n}{(2d)_n n!} {}_1K_1\left(\begin{matrix} -n; \\ 2d+n; \end{matrix}\ A; -y, s, a\right) x^n. \quad (4.3)$$

Proof. Making an appeal to the Theorems 3.3 and 3.4, we get the equalities

$$\phi_2(A,d,d+1/2;2d;x,y;s,a) = \frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d} \sum_{n=0}^{\infty} \frac{A_n\left(\frac{xy}{(1+\sqrt{1-x})^2}\right)^n}{n!(n+a)^s}$$

$$= \sum_{n=0}^{\infty} \frac{(d)_n\left(d+\frac{1}{2}\right)_n x^n}{(2d)_n n!} \sum_{m=0}^{n} \frac{A_m(-n)_m}{(2d+n)_m} \frac{(-y)^m}{(m+a)^s m!}. \quad (4.4)$$

Then in the relation (4.4) making an appeal to the extended general hypergeometric type Hurwitz-Lerch Zeta function (1.6) in the last two equalities, we obtain the generating relation (4.3).

Application 4.3. If all conditions of the Theorems 3.1 and 3.2 are satisfied, then there exists an Eulerian integral identity for the extended general hypergeometric type Hurwitz -Lerch Zeta function (1.7), given by

$$\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1} \int_0^{\infty} e^{-at} t^{s-1} {}_1G_1\left(\begin{matrix} d-\frac{1}{2}; \\ d+\frac{1}{2}; \end{matrix}\ A; \frac{xy}{(1+\sqrt{1-x})^2} e^{-t}\right) dt$$

$$= \sum_{n=0}^{\infty} \frac{(d)_n(d-\frac{1}{2})_n x^n}{(2d)_n n!} \int_0^{\infty} e^{-at} t^{s-1} {}_1G_1\left(\begin{matrix} -n; \\ 2d+n; \end{matrix}\ A; -ye^{-t}\right) dt. \quad (4.5)$$

Solution. Making an appeal to the methods given in the Eqn. (2.1) and the formula (2.2) in the relation (4.1), we derive the Eulerian integral identity (4.5).

Application 4.4. If all conditions of the Theorems 3.3 and 3.4 are satisfied, then there exists an Eulerian integral identity for the extended general hypergeometric type Hurwitz -Lerch Zeta function (1.8), as

$$\frac{1}{\sqrt{1-x}}\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}\int_0^\infty e^{-at}t^{s-1}{}_0G_0\left(\begin{matrix}-;\\-;\end{matrix}\quad A;\frac{xy}{(1+\sqrt{1-x})^2}e^{-t}\right)dt$$
$$=\sum_{n=0}^{\infty}\frac{(d)_n(d+\frac{1}{2})_n x^n}{(2d)_n n!}\int_0^\infty e^{-at}t^{s-1}{}_1G_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}\quad A;-ye^{-t}\right)dt \quad (4.6)$$

Solution. Making an appeal to the same techniques given in the Eqn. (2.1) and the formula (2.2) in the relation (4.3), we derive the Eulerian integral identity (4.6).

Application 4.5. If all conditions of the Theorems 3.1 and 3.2 are satisfied, then for the extended general hypergeometric type Hurwitz -Lerch Zeta function (1.7), there exists a hypergeometric generating relation

$$\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}{}_1G_1\left(\begin{matrix}d-\frac{1}{2};\\d+\frac{1}{2};\end{matrix}\ A;\frac{xy}{(1+\sqrt{1-x})^2}e^{-t}\right)=\sum_{n=0}^{\infty}\frac{(d)_n\left(d-\frac{1}{2}\right)_n x^n}{(2d)_n n!}{}_1G_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}\ A;-ye^{-t}\right). \quad (4.7)$$

Solution. By the Eulerian integral identity (4.5), we find that

$$\int_0^\infty e^{-at}t^{s-1}R^{(1)}_{d,A}(x,y;t)dt=0, \quad (4.8)$$

where,

$$R_{d,A}{}^{(1)}(x,y;t)=\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}{}_1G_1\left(\begin{matrix}d-\frac{1}{2};\\d+\frac{1}{2};\end{matrix}\ A;\frac{xy}{(1+\sqrt{1-x})^2}e^{-t}\right)$$
$$-\sum_{n=0}^{\infty}\frac{(d)_n\left(d-\frac{1}{2}\right)_n x^n}{(2d)_n n!}{}_1G_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}\ A;-ye^{-t}\right).$$

Then, equating both sides of Eqn. (4.8), we obtain the result (4.7).

Application 4.6. If all conditions of the Theorems 3.3 and 3.4 are satisfied, then for the extended general hypergeometric type Hurwitz -Lerch Zeta function (1.8), there exists the hypergeometric generating relation

$$\frac{1}{\sqrt{1-x}}\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}{}_0G_0\left(\begin{matrix}-;\\-;\end{matrix}\ A;\frac{xy}{(1+\sqrt{1-x})^2}e^{-t}\right)=\sum_{n=0}^{\infty}\frac{(d)_n\left(d+\frac{1}{2}\right)_n x^n}{(2d)_n n!}{}_1G_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}\ A;-ye^{-t}\right). \quad (4.9)$$

Solution. By the Eulerian integral identity (4.6), we find that

$$\int_0^\infty e^{-at}t^{s-1}R_{d,A}{}^{(2)}(x,y;t)dt=0, \quad (4.10)$$

where,

$$R^{(2)}_{d,A}(x,y;t)=\frac{1}{\sqrt{1-x}}\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}{}_0G_0\left(\begin{matrix}-;\\-;\end{matrix}\quad A;\frac{xy}{(1+\sqrt{1-x})^2}e^{-t}\right)$$
$$-\sum_{n=0}^{\infty}\frac{(d)_n\left(d+\frac{1}{2}\right)_n x^n}{(2d)_n n!}{}_1G_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}\ A;-ye^{-t}\right).$$

Finally, equating both sides of Eqn. (4.10), we obtain the result (4.9).

In concluding remarks, we derive interesting summation formulae from our above obtained results.

5 Interesting Results as Special Cases

Particularly, in Eqn. (4.2) set $A_n=(2)_n\ \forall n\in\mathbb{N}_0, d=\frac{3}{2}$, there exists an interesting summation formula in terms of Hurwitz-Lerch Zeta function

$$\phi_1\left(2,\frac{3}{2},1;3;x,y;s,a\right)=\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n x^n}{(3)_n}\sum_{m=0}^{n}\frac{(2)_n(-n)_m}{(3+n)_m}\frac{(-y)^m}{(m+a)^s m!}$$
$$=\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n x^n}{(3)_n}{}_2H_1\left(\begin{matrix}2,-n;\\3+n;\end{matrix}\ -y,s,a\right)=\left(\frac{4}{(1+\sqrt{1-x})^2}\right)\phi\left(\frac{xy}{(1+\sqrt{1-x})^2},s,a\right), \quad (5.1)$$

where, all $x, y, s \in \mathbb{C}, |x| \leq 1; a \in \mathbb{C} \backslash \mathbb{Z}_0^-$.

In the result (5.1) for $x = 1$, we obtain the following identical formulae for Hurwitz-Lerch Zeta function

$$\frac{1}{4}\phi_1\left(2, \frac{3}{2}, 1; 3; 1, y; s, a\right) = \frac{1}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n}{(3)_n}\sum_{m=0}^{n}\frac{(2)_n(-n)_m}{(3+n)_m}\frac{(-y)^m}{(m+a)^s m!}$$
$$= \frac{1}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n}{(3)_n}{}_2H_1\left(\begin{matrix}2, -n;\\ 3+n;\end{matrix}\ -y, s, a\right) = \phi(y, s, a), \quad (5.2)$$

where, all $y, s \in \mathbb{C}; a \in \mathbb{C} \backslash \mathbb{Z}_0^-$.

Again, in Eqn. (4.4) choosing $A_n = (1)_n\ \forall n \in \mathbb{N}_0, 2d \neq 0, -1, -2, \ldots$, there exists another interesting summation formula in terms of Hurwitz-Lerch Zeta function

$$\phi_2(1, d, d+1/2; 2d; x, y; s, a) = \sum_{n=0}^{\infty}\frac{(d)_n\left(d+\frac{1}{2}\right)_n x^n}{(2d)_n n!}\sum_{m=0}^{n}\frac{(-n)_m}{(2d+n)_m}\frac{(-y)^m}{(m+a)^s}$$
$$= \sum_{n=0}^{\infty}\frac{(d)_n(d+\frac{1}{2})_n x^n}{(2d)_n n!}{}_1H_1\left(\begin{matrix}-n;\\ 2d+n;\end{matrix}\ -y, s, a\right)$$
$$= \frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d}\varphi\left(\frac{xy}{(1+\sqrt{1-x})^2}, s, a\right). \quad (5.3)$$

where, $x, y, s \in \mathbb{C}, |x| < 1;\ a, 2d \in Z_0^-$.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 157-164**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

INEQUALITIES VIA MEAN FUNCTIONS USING E - CONVEXITY

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(Received: October 10, 2022*; In format: February* 14, 2023*; Received: February* 22, 2023*; Accepted: February* 28, 2023*)*

DOI: https://doi.org/10.58250/jnanabha.2023.53118

Abstract

In this paper, we extend the concept of GG - convexity to GG- E - convexity and then we derived some new integral inequalities for GG- E - convex function using Holder's integral inequalities. Enough examples are given to verify the obtained results.

2020 Mathematical Sciences Classification: 26A51, 26D10, 26D15
Keywords and Phrases: Convex function, E - convex set, E - convex function, E - concave function, GG- E - convex function, GG- E - concave function, Holder integral inequality.

1 Introduction

Convexity contains a broad spectrum of significance in both applied and pure mathematics. Nowadays, the concept of convexity is not confined to convex functions only, but it has also extended to non - convex functions as well as convex programming.Minkowski [9] did the first methodical study of convexity. Convexity plays an important role in convex - optimization. So Many problems of quasi - convex optimization arise in spanning economics [8, 13], industrial organization [14]. Realted to quasi-convex optimization also available In offline case [10, 11, 12, 19]. Convexity is also useful in concept of special means like arithmetic mean, . geometric mean, harmonic mean, logarithmic mean and identric mean Anderson et al.[1] mentioned mean function. Anderson et al. [1] derived similar results for some power series, especially hypergeometric functions. Dragomir [5] gave inequalities at the same time Akdemir et al. [2] gave generalization in sense of convex functions. Then new integral inequalities arise via GG - convexity and GA - convexity [3].

Hanson and Mond [6] extended the class of convex functions to the class of invex functions and showed that programming problems that can be transformed in this way are a strict subset of invex programming problems, then Bector and Singh [4] introduced B - vex functions and discussed differentiable and non differentiable cases. Class of B - vex functions forms a subset of the sets of both semistrictly quasiconvex as well as quasiconvex functions. For the first time Youness [17] provided the concept of well known class of generalised convexity, namely E-convexity. Furthermore, he formulated some results from E - convex functions in programming problems [18]. Yang [15] refined few results of E - convex programming, which were obtained by Younness [17]. Over the past few years, many researchers have focused on the theory of inequalities. Because of the wide range of ideas and applications, the theory of inequalities has become a captivating, engrossing and gripping area for researchers.

In this paper, our aim is to establish some new inequalities for GG - E - convex function. Also we prove that not only the inequalities of the convex function are possible through the mean function, but also the composite functions in which one is convex and the other is non-convex obey the inequalities. The Interesting techniques and the useful ideas of this paper may encourage further research in this dynamic area.

2 Definitions and Preliminaries

Definition 2.1. *The function $f : I \subset R \to R$ is a convex function on I, if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. We say that g is concave if - f is convex [3].

Let $f : I \subset R \to R$ be a convex function where $a, b \in I$ with $a < b$. Then the following double inequality holds :

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

This inequality is well - known in the literature as Hermite - Hadamard inequality [5].

Definition 2.2. *A set $N \subset R^n$ is said to be E - convex iff there is a map $E : R^n \to R^n$ such that*

$$(1-t)E(x) + tE(y) \in N.$$

for each $x, y \in N$ and $0 \leq t \leq 1$[17].

Definition 2.3. *A function $f : R^n \to R^n$ is said to be E - convex on a set $N \subset R^n$, iff there is a map $E : R^n \to R^n$ such that N is an convex set and*

$$f(tE(x) + (1-t)E(y)) \leq tf(E(x)) + (1-t)f(E(y)).$$

for each $x, y \in N$ and $0 \leq t \leq 1$ on the other hand, if

$$f(tE(x) + (1-t)E(y)) \geq tf(E(x)) + (1-t)f(E(y)).$$

then f is called E - concave on N. If the inequality signs in the previous two inequalities are strict, then f is called strictly E convex and strictly E concave, respectively[17].

Definition 2.4. *Let $f : S \to (0,\infty)$ be continuous, where I is subinterval of $(0,\infty)$. Let N and P be any two mean functions, $T \subset R$ and there is a map $E : R \to R$ then we say f is NP -E-convex(concave) on T if*

$$f(N(E(x), E(y)) \leq (\geq) P(f(E(x), f(E(y)).$$

for all $x, y \in S$.

Definition 2.5. *The GG - E - convex functions are those functions $f : S \subseteq R_+ \to R$ and there is a map $E : R \to R$ such that $x, y \in S$ and*

$$t \in [0,1] \Rightarrow f((E(x))^{1-t}(E(y))^t) \leq (f(E(x))^{1-t}(f(E(y))^t.$$

Definition 2.6. *The GG - E - concave functions are those functions $f : S \subseteq R_+ \to R$ and there is a map $E : R \to R$ such that $x, y \in S$ and*

$$t \in [0,1] \Rightarrow f((E(x))^{1-t}(E(y))^t) \geq (f(E(x))^{1-t}(f(E(y))^t.$$

Lemma 2.1. *Let $g : S \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on S^o and $x, y \in S^o$, where S^o is the interior set of S with $\alpha < \beta$, and there is a map $E : R \to R$. If $g' \in L([E(\alpha)], [E(\beta)])$, then the following identity holds :*

$$\begin{aligned}
&E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \textstyle\int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v) \\
= \quad & \tfrac{lnE(\beta)-lnE(\alpha)}{2}\left[\textstyle\int_0^1 \left((E(\beta))^t(E(\alpha))^{(2-t)}\right) g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right) dt\right] \\
+ \quad & \tfrac{lnE(\beta)-lnE(\alpha)}{2}\left[\textstyle\int_0^1 \left((E(\alpha))^t(E(\beta))^{(2-t)}\right) g'\left((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-t)}{2}}\right) dt\right].
\end{aligned}$$

Proof. Let $J_1 = \int_0^1 \left((E(\beta))^t(E(\alpha))^{(2-t)}\right) g^{'}\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right)dt$ and $J_2 = \int_0^1 \left((E(\alpha))^t(E(\beta))^{(2-t)}\right) g^{'}\left((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-}{2}}\right.$

Then we notice that

$J_1 = \int_0^1 \left((E(\beta))^t(E(\alpha))^{(2-t)}\right) g^{'}\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right)dt$

$\frac{2}{lnE(\beta)-lnE(\alpha)}\int_0^1 \left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right) g^{'}\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right) d\left(E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right).$

Now by the change of variable $E(v) = (E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}})$ and integrating by parts, we have

$J_1 = \frac{2}{lnE(\beta)-lnE(\alpha)}\left[\sqrt{(E(\alpha)E(\beta))}g\sqrt{(E(\alpha)E(\beta))} - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{\sqrt{(E(\alpha)E(\beta))}} g(E(v))dE(v)\right].$

Conformably, we have

$J_2 = \frac{2}{lnE(\beta)-lnE(\alpha)}\left[E(\beta)g(E(\beta)) - \sqrt{(E(\alpha)E(\beta))}g\sqrt{(E(\alpha)E(\beta))} - \int_{\sqrt{(E(\alpha)E(\beta))}}^{E(\beta)} g(E(v))dE(v)\right].$

Multiplying J_1 and J_2 by $\frac{lnE(\beta)-lnE(\alpha)}{2}$ and adding the results we get the appealed identity.

Our first result is given in the following theorem. □

Example 2.1. Let $f : \left[0, \frac{\pi}{2}\right] \to [0, \infty)$ and $n \in \mathbb{N} - \{1\}$ such that

$$f(x) = -\int_0^{\sqrt[n]{x}} ln(cos(t))dt$$

is not $GG - convex$ on $\left(0, \frac{\pi}{2}\right)$ and there is a map $E : \left[0, \sqrt[n]{\frac{\pi}{2}}\right] \to \left[0, \frac{\pi}{2}\right]$ such that $E(x) = x^n$, then the function

$$f(E(x)) = -\int_0^{x} ln(cos(t))dt$$

is GG - E - convex function on $\left(0, \frac{\pi}{2}\right)$.

Example 2.2. Let $f : \left[0, \frac{(4n+1)\pi}{2}\right] \to [0, \infty)$ where $n \in \mathbb{Z}$ such that

$$f(x) = ln(sinx)$$

is not GG - convex on $\left(0, \frac{(4n+1)\pi}{2}\right)$ and there is a map $E : \left[0, \frac{(4n+1)\pi}{2}\right] \to \left[0, \frac{(4n+1)\pi}{2}\right]$ such that $E(x) = \frac{(4n+1)\pi}{2} - x$ then the function

$$f(E(x)) = ln\left(sin\left(\frac{(4n+1)\pi}{2} - x\right)\right)$$

$$f(E(x)) = ln\,(cosx)$$

is GG - E - convex function on $\left(0, \frac{(4n+1)\pi}{2}\right)$.

Example 2.3. Let $E : \Re \to \Re$ and $f : \Re \to \Re$ are defined as

$$E(x) = \begin{cases} 2 & x > 0 \\ -x & x \leq 0 \end{cases}$$

And

$$f(x) = x^2$$

Then the function $E(x)$ is not obeying our inequalities and the function $f(x)$ is obeying our inequalities. Also the composition foE will obey our inequalities.

Remark 2.1. Example 2.3 shows that a non convex function $E(x)$ does not obey our inequalities, but a convex function $f(x)$ obey our inequalities. And also an $E-$convex function $foE(x)$ obey our inequalities.

Example 2.4. Let the function $E : \Re \to \Re$ such that $E(x) = lnx$ which is non convex, so this function is not obeying our inequalities.

Example 2.5. Let the function $F : \Re \to \Re$ such that $F(x) = e^x$ which is convex, so this function is obeying our inequalities.

Example 2.6. Let the composition of two functions E, F be defined as $FoE : \Re \to \Re$ such that $F(E(x)) = x$ which is $E-$convex, so this function is obeying our inequalities.

3 Main Results

Theorem 3.1. *Let $g : S \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on S^o and $\alpha, \beta \in S^o$ with $\alpha < \beta$ and $E : R \to R$ is a non decreasing function so $E(\alpha) < E(\beta)$. If $g' \in L[E(\alpha), E(\beta)]$. If $|g'|$ is $GG - E -$ Convex on $[E(\alpha), E(\beta)]$, then the following inequality holds:*

$$|E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v)|$$
$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left(E(\alpha)\sqrt{|g'(E(\alpha))|} + E(\beta)\sqrt{|g'(E(\beta))|}\right) L\left(E(\alpha)\sqrt{|g'(E(\alpha))|}, E(\beta)\sqrt{|g'(E(\beta))|}\right).$$

Proof. From Lemma 2.1, using the property of the modulus and $GG \ E$ convexity of $|g'|$, we can write

$$|E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v)|$$
$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\int_0^1 \left((E(\beta))^t(E(\alpha))^{(2-t)}\right)\left|g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right)\right|dt\right]$$
$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\int_0^1 \left((E(\alpha))^t(E(\beta))^{(2-t)}\right)\left|g'\left((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-t)}{2}}\right)\right|dt\right]$$
$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\int_0^1 \left((E(\beta))^t(E(\alpha))^{(2-t)}\right)\left|g'(E(\beta))\right|^{\frac{t}{2}}\left|g'(E(\alpha))\right|^{\frac{(2-t)}{2}} dt\right]$$
$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\int_0^1 \left((E(\alpha))^t(E(\beta))^{(2-t)}\right)\left|g'(E(\alpha))\right|^{\frac{t}{2}}\left|g'(E(\beta))\right|^{\frac{(2-t)}{2}} dt\right]$$
$$= \frac{lnE(\beta)-lnE(\alpha)}{2}\left[(E(\alpha))^2\left|g'(E(\alpha))\right|\int_0^1\left(\frac{E(\beta)\sqrt{|g'(E(\beta))|}}{E(\alpha)\sqrt{|g'(E(\alpha))|}}\right)^t dt\right]$$
$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[(E(\beta))^2\left|g'(E(\beta))\right|\int_0^1\left(\frac{E(\alpha)\sqrt{|g'(E(\alpha))|}}{E(\beta)\sqrt{|g'(E(\beta))|}}\right)^t dt\right].$$

Then we get the desired result. □

Theorem 3.2. *Let $g : S \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on S^o and $\alpha, \beta \in S^o$ with $\alpha < \beta$ and $E : R \to R$ is a non decreasing function so $E(\alpha) < E(\beta)$. If $g' \in L[E(\alpha), E(\beta)]$. If $|g'|^n$ is GG - E - Convex on $[E(\alpha), E(\beta)]$, for all $E(\gamma) \in [E(\alpha), E(\beta)]$,then the following inequality*

$$|E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v)|$$
$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left(E(\alpha)\sqrt{|g'(E(\alpha))|} + E(\beta)\sqrt{|g'(E(\beta))|}\right)\left(L\left((E(\alpha))^m, (E(\beta))^m\right)\right)^{\frac{1}{m}}$$
$$\times\left[\left(L\left(\sqrt{|g'(E(\alpha))|^n}, \sqrt{|g'(E(\beta))|^n}\right)\right)^{\frac{1}{n}}\right]$$

holds, where $n > 1$ and $\frac{1}{m} + \frac{1}{n} = 1$.

Proof. From Lemma 2.1, using the property of the modulus, $GG \ E$ convexity of $|g'|^n$ and Holder integral inequality, we can write

$$|E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v)|$$
$$= \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\int_0^1 \left((E(\beta))^t(E(\alpha))^{(2-t)}\right)\left|g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right)\right|dt\right]$$
$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\int_0^1 \left((E(\alpha))^t(E(\beta))^{(2-t)}\right)\left|g'\left((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-t)}{2}}\right)\right|dt\right]$$
$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_{E(\alpha)}^{E(\beta)}(E(\beta))^{tm}(E(\alpha))^{(2-t)m}dt\right)^{\frac{1}{m}}\left(\int_0^1\left|g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right)\right|^n\right)^{\frac{1}{n}}\right]$$

$$+ \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_{E(\alpha)}^{E(\beta)}(E(\alpha))^{tm}(E(\beta))^{(2-t)m}dt\right)^{\frac{1}{m}}\left(\int_0^1\left|g'\left((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-t)}{2}}\right)\right|^n\right)^{\frac{1}{n}}\right]$$

$$\leq \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[(E(\alpha))^2\left(\int_0^1\left(\frac{(E(\beta))^m}{(E(\alpha))^m}\right)^t dt\right)^{\frac{1}{m}}\left(\int_0^1\left|g'(E(\beta))\right|^{\frac{tn}{2}}\left|g'(E(\alpha))\right|^{\frac{(2-t)n}{2}}dt\right)^{\frac{1}{n}}\right]$$

$$+ \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[(E(\beta))^2\left(\int_0^1\left(\frac{(E(\alpha))^m}{(E(\beta))^m}\right)^t dt\right)^{\frac{1}{m}}\left(\int_0^1\left|g'(E(\alpha))\right|^{\frac{tn}{2}}\left|g'(E(\beta))\right|^{\frac{(2-t)n}{2}}dt\right)^{\frac{1}{n}}\right].$$

Then we get the desired result. □

Theorem 3.3. *Under the assumptions of Theorem* 3.2*, the following inequality holds :*

$$|E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v)|$$

$$\leq \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left(L\left(\sqrt{|g'(E(\alpha))|^n},\sqrt{|g'(E(\beta))|^n}\right)\right)^{\frac{1}{n}}$$

$$\times \quad \left[\left(\frac{(E(\beta))^{m+1}-mE(\beta)-(E(\beta))}{m+1}\right)^{\frac{1}{m}}(E(\alpha))^2\sqrt{|g'(E(\alpha))|}+\left(\frac{(E(\alpha))^{m+1}-mE(\alpha)-(E(\alpha))}{m+1}\right)^{\frac{1}{m}}(E(\beta))^2\sqrt{|g'(E(\beta))|}\right].$$

Proof. From Lemma 2.1, using the property of the modulus, $GG\ \ E$ convexity of $|g'|^n$ and Holder integral inequality, we can write

$$|E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v)|$$

$$\leq \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1(E(\beta))^{tm}dt\right)^{\frac{1}{m}}\left(\int_0^1(E(\beta))^{(2-t)n}\left|g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right)\right|^n dt\right)^{\frac{1}{n}}\right]$$

$$+ \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1(E(\alpha))^{tm}dt\right)^{\frac{1}{m}}\left(\int_0^1(E(\alpha))^{(2-t)n}\left|g'\left((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-t)}{2}}\right)\right|^n dt\right)^{\frac{1}{n}}\right]$$

$$\leq \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1(E(\beta))^{tm}dt\right)^{\frac{1}{m}}\left(\int_0^1(E(\beta))^{(2-t)n}\left|g'\ (E(\beta))\right|^{\frac{t}{2}}\left|g'\ (E(\alpha))\right|^{\frac{(2-t)}{2}}dt\right)^{\frac{1}{n}}\right]$$

$$+ \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1(E(\alpha))^{tm}dt\right)^{\frac{1}{m}}\left(\int_0^1(E(\beta))^{(2-t)n}\left|g'\ (E(\alpha))\right|^{\frac{t}{2}}\left|g'\ (E(\beta))\right|^{\frac{(2-t)}{2}}dt\right)^{\frac{1}{n}}\right].$$

If we calculate the integral above, we get the desired result. □

Theorem 3.4. *Under the assumptions of Theorem* 3.2*, the following inequality holds :*

$$|E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v)|$$

$$\leq \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left(E(\alpha)\sqrt{|g'(E(\alpha))|}+E(\beta)\sqrt{|g'(E(\beta))|}\right)$$

$$\times\left(L\left((E(\alpha))^n\sqrt{|g'(E(\alpha))|^n},(E(\beta))^n\sqrt{|g'(E(\beta))|^n}\right)\right)^{\frac{1}{n}}.$$

Proof. From Lemma 2.1, using the property of the modulus, $GG\ \ E$ - convexity of $|g'|^n$ and Power mean integral inequality, we can write

$$\left|E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v)\right|$$

$$\leq \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1 dt\right)^{1-\frac{1}{n}}\left(\int_0^1(E(\beta))^{tn}(E(\alpha))^{(2-t)n}\left|g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right)\right|^n dt\right)^{\frac{1}{n}}\right]$$

$$+ \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1 dt\right)^{1-\frac{1}{n}}\left(\int_0^1(E(\alpha))^{tn}(E(\beta))^{(2-t)n}\left|g'\left((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-t)}{2}}\right)\right|^n dt\right)^{\frac{1}{n}}\right]$$

$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[(E(\alpha))^2\left|g^{'}(E(\alpha))\right|\left(\int_0^1\left(\frac{(E(\beta))^n|g^{'}(E(\beta))|^{\frac{n}{2}}}{(E(\alpha))^n|g^{'}(E(\alpha))|^{\frac{n}{2}}}\right)^t dt\right)^{\frac{1}{n}}\right]$$
$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[(E(\beta))^2\left|g^{'}(E(\beta))\right|\left(\int_0^1\left(\frac{(E(\alpha))^n|g^{'}(E(\alpha))|^{\frac{n}{2}}}{(E(\beta))^n|g^{'}(E(\beta))|^{\frac{n}{2}}}\right)^t dt\right)^{\frac{1}{n}}\right].$$

Then we get the desired result. □

Theorem 3.5. *Under the assumptions of Theorem 3.2, the following inequality holds :*

$$|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)|$$
$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\frac{E(\beta)-1}{ln(E(\beta))}\right)^{\frac{1}{m}}E(\alpha)\sqrt{|g^{'}(E(\alpha))|}L^{\frac{1}{n}}\left((E(\alpha))^n\sqrt{|g^{'}(E(\alpha))|^n},(E(\beta))\sqrt{|g^{'}(E(\beta))|^n}\right)\right]$$
$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\frac{E(\alpha)-1}{ln(E(\alpha))}\right)^{\frac{1}{m}}E(\beta)\sqrt{|g^{'}(E(\beta))|}L^{\frac{1}{n}}\left((E(\alpha))^n\sqrt{|g^{'}(E(\alpha))|},(E(\beta))^n\sqrt{|g^{'}(E(\beta))|^n}\right)\right].$$

Proof. From Lemma 2.1, using the property of the modulus, $GG\ \ E$ - convexity of $|g'|^n$ and Power mean integral inequality, we can write

$$|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)|$$
$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1(E(\beta))^t dt\right)^{\frac{1}{m}}\left(\int_0^1(E(\beta))^t(E(\alpha))^{(2-t)n}\left|g^{'}(E(\beta))\right|^{\frac{tn}{2}}\left|g^{'}(E(\alpha))\right|^{(1-\frac{t}{2})n}dt\right)^{\frac{1}{n}}\right]$$
$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1(E(\alpha))^t dt\right)^{\frac{1}{m}}\left(\int_0^1(E(\alpha))^t(E(\beta))^{(2-t)n}\left|g^{'}(E(\alpha))\right|^{\frac{tn}{2}}\left|g^{'}(E(\beta))\right|^{(1-\frac{t}{2})n}dt\right)^{\frac{1}{n}}\right].$$

Then we get the desired result. □

Theorem 3.6. *Under the assumptions of Theorem 3.2, the following inequality holds :*

$$|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)|$$
$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left((E(\alpha))^{\frac{m}{n}}\sqrt{g^{'}(E(\alpha))}+((E(\beta))^{\frac{m}{n}}\sqrt{g^{'}(E(\beta))}\right)$$
$$\times\left(L\left((E(\alpha))^{\frac{n-m}{n-1}},(E(\beta))^{\frac{n-m}{n-1}}\right)\right)^{1-\frac{1}{n}}\left(L\left((E(\alpha))^m\sqrt{|g^{'}(E(\alpha))|^n}+(E(\beta))^m\sqrt{|g^{'}(E(\beta))|^n}\right)\right)^{\frac{1}{n}}.$$

Proof. From Lemma 2.1, using the property of the modulus, $GG\ \ E$ convexity of $|g'|^n$ and Power mean integral inequality, we can write

$$|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)|$$
$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1\left((E(\beta))^t(E(\alpha))^{(2-t)}\right)^{\frac{n-m}{n-1}}dt\right)^{1-\frac{1}{n}}\right]$$
$$\times\left[\left(\int_0^1\left((E(\beta))^t(E(\alpha))^{(2-t)}\right)^m\left|g^{'}(E(\alpha))\right|^{(\frac{2-t}{2})n}\left|g^{'}(E(\beta))\right|^{\frac{tn}{2}}dt\right)^{\frac{1}{n}}\right]$$
$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1\left((E(\alpha))^t(E(\beta))^{(2-t)}\right)^{\frac{n-m}{n-1}}dt\right)^{1-\frac{1}{n}}\right]$$
$$\times\left[\left(\int_0^1\left((E(\alpha))^t(E(\beta))^{(2-t)}\right)^m\left|g^{'}(E(\beta))\right|^{(\frac{2-t}{2})n}\left|g^{'}(E(\alpha))\right|^{\frac{tn}{2}}dt\right)^{\frac{1}{n}}\right].$$

Then we get the desired result. □

Theorem 3.7. *Let $g : S \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on S^o and $\alpha,\beta \in S^o$ with $\alpha < \beta$ and $E : R \to R$ is a non decreasing function so $E(\alpha) < E(\beta)$. If $g^{'} \in L[E(\alpha), E(\beta)]$. If $|g'|$ is $GG - E -$ Convex on $[E(\alpha), E(\beta)]$, then the following inequality holds:*

$$|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)|$$
$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left(E(\alpha)\sqrt{|g^{'}(E(\alpha))|}+E(\beta)\sqrt{|g^{'}(E(\beta))|}\right)L\left(E(\alpha)\sqrt{|g^{'}(E(\alpha))|},E(\beta)\sqrt{|g^{'}(E(\beta))|}\right).$$

4 Application Area

The concept of E - convexity is the generalizations of convex sets and convex functions, in a respective manner. E - convexity is also used in the study of E-convex programming. We believe that our new class of functions will have a very profound research in this entrancing domain of inequalities and also in the pure and applied sciences. The interesting inequalities and successful ideas in this article can be extended to other mean functions like harmonic mean, Arithmetic mean etc. As we move forward, we aim to continue that research to find inequalities for non-convex functions as well.

5 Conclusions

In this paper, we derived the inequalities for GG - E - Convex function using Holder integral inequality. If we take $E(x)$ as an identity function then it shows the result of [16].

Acknowledgement. Authors are very much thankful to the Editor and Reviewer for their valuable suggestions to bring the paper in its present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 165-172**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

ON HOMOGENEOUS CUBIC EQUATION WITH FOUR UNKNOWNS $(\mathbf{x^3+y^3})=\mathbf{7zw^2}$

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(Received: February 26, 2023; Informat: March 09, 2023; Revised: March 22, 2023; Accepted: May 20, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53119

Abstract

This paper concerns with the problem of obtaining non-zero distinct integer solutions to homogeneous cubic equation with four unknowns given by $x^3+y^3=7zw^2$. A few interesting properties among the solutions are presented.
2020 Mathematical Sciences Classification: 11D25
Keywords and Phrases: homogeneous cubic, cubic with four unknowns, integer solutions

1 Introduction

The cubic diophantine equations are rich in variety and offer an unlimited field for research [1, 2]. In particular refer [3]- [23] for a few problems on cubic equation with 3 and 4 unknowns. This paper concerns with yet another interesting homogeneous cubic diophantine equation with four unknowns $x^3+y^3=7zw^2$ for determining its infinitely many non-zero distinct integral solutions through employing linear transformations. A few interesting relations among the solutions are presented.

Method of Analysis

The homogeneous cubic equation with four unknowns to be solved is represented by

$$x^3+y^3=7zw^2. \tag{1.1}$$

Introduction of the linear transformations

$$x=u+v, y=u-v, z=2u, \tag{1.2}$$

in (1.1) leads to

$$u^2+3v^2=7w^2. \tag{1.3}$$

Different methods of obtaining the patterns of integer solutions to (1.1) are illustrated below:

2 Patterns

Pattern 2.1. Let

$$w=a^2+3b^2, \tag{2.1}$$

where a and b are non-zero integers.

Write 7 as

$$7=(2+i\sqrt{3})(2-i\sqrt{3}). \tag{2.2}$$

Using (2.1), (2.2) in (1.3) and applying the method of factorization, define

$$(u+i\sqrt{3}v)=(2+i\sqrt{3})(a+i\sqrt{3}b)^2, \tag{2.3}$$

from which, we have

$$\left.\begin{aligned} u&=2a^2-6ab-6b^2\\ v&=a^2+4ab-3b^2\end{aligned}\right\}. \tag{2.4}$$

Using (2.4) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x=x(a,b)&=3a^2-2ab-9b^2\\ y=y(a,b)&=a^2-10ab-3b^2\\ z=z(a,b)&=4a^2-12ab-12b^2\end{aligned}\right\}. \tag{2.5}$$

Thus (2.1) and (2.5) represent the non-zero integer solutions to (1.1).

Observation 2.1. 1. $z(a,a+1)-4y(a,a+1)=56t_{3,a}$

2. $z(a,2a-1)-4y(a,2a-1)=28t_{6,a}$
3. $z(a,a)-4y(a,a)-t_{58,a}\equiv 0 \pmod 3$
4. $z(a,a)-4y(a,a)-t_{34,a}-t_{26,a}\equiv 0 \pmod{13}$
5. $42[z(a,a)-4y(a,a)]$ is a nasty number.

Pattern 2.2. Write 7 as

$$7=\frac{(5+i\sqrt{3})(5-i\sqrt{3})}{4}. \tag{2.6}$$

Using (2.1), (2.6) in (1.3) and applying the method of factorization, define

$$(u+i\sqrt{3}v)=\frac{(5+i\sqrt{3})}{2}(a+i\sqrt{3}b)^2, \tag{2.7}$$

from which, we have

$$\left.\begin{aligned} u&=\frac{1}{2}(5a^2-6ab-15b^2)\\ v&=\frac{1}{2}(a^2+10ab-3b^2)\end{aligned}\right\}, \tag{2.8}$$

Using (2.8) and (1.2), the values of x,y and z are given by

$$\left.\begin{aligned} x&=x(a,b)=3a^2+2ab-9b^2\\ y&=y(a,b)=2a^2-8ab-6b^2\\ z&=z(a,b)=5a^2-6ab-15b^2\end{aligned}\right\}, \tag{2.9}$$

Thus (2.1) and (2.9) represent the non-zero integer solutions to (1.1).

Observation 2.2. 1. $x(a,a)-z(a,a)-t_{38,a}\equiv 0 \pmod{17}$

2. $y(a,a)-z(a,a)=t_{4,2a}$

Pattern 2.3. Write (1.3) as

$$u^2+3v^2=7w^2*1. \tag{2.10}$$

Write 1 as

$$1=\frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}. \tag{2.11}$$

Using (2.1), (2.2), (2.11) in (1.3) and applying the method of factorization, define

$$(u+i\sqrt{3}v)=(\frac{1+i\sqrt{3}}{2})(2+i\sqrt{3})(a+i\sqrt{3}b)^2, \tag{2.12}$$

from which, we have

$$\left.\begin{aligned} u&=\frac{1}{2}(-a^2-18ab+3b^2)\\ v&=\frac{1}{2}(3a^2-2ab-9b^2)\end{aligned}\right\}, \tag{2.13}$$

Using (2.13) and (1.2), the values of x,y and z are given by

$$\left.\begin{aligned} x&=x(a,b)=a^2-10ab-3b^2\\ y&=y(a,b)=-2a^2-8ab+6b^2\\ z&=z(a,b)=-a^2-18ab+3b^2\end{aligned}\right\}. \tag{2.14}$$

Thus (2.1) and (2.14) represent the non-zero integer solutions to (1.1).

Observation 2.3. 1. $y(a,a)-x(a,a)-t_{18,a}\equiv 0 \pmod 7$

2. $y(b,b)-z(b,b)-t_{b,22}-t_{b,6}\equiv 0 \pmod 2$

Pattern 2.4. Consider 1 as

$$1=\frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49}. \tag{2.15}$$

Using (2.1), (2.2), (2.15) in (1.3) and applying the method of factorization, define

$$(u+i\sqrt{3}v)=(\frac{1+i4\sqrt{3}}{7})(2+i\sqrt{3})(a+i\sqrt{3}b)^2, \tag{2.16}$$

from which, we have

$$\left.\begin{aligned} u &= \frac{1}{7}(-10a^2 - 54ab + 30b^2) \\ v &= \frac{1}{7}(9a^2 - 20ab - 27b^2) \end{aligned}\right\}. \tag{2.17}$$

Using (2.17) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x = x(a,b) &= \frac{1}{7}(-a^2 - 74ab + 3b^2) \\ y = y(a,b) &= \frac{1}{7}(-19a^2 - 34ab + 57b^2) \\ z = z(a,b) &= \frac{1}{7}(-20a^2 - 10ab + 60b^2) \end{aligned}\right\}. \tag{2.18}$$

Since our interest is on finding integer solutions, replacing a by $7A$, b by $7B$ in (2.1) and (2.18), the corresponding integer solutions to to (1.1) are given by

$$\left.\begin{aligned} x = x(A,B) &= 7(-A^2 - 74AB + 3B^2) \\ y = y(A,B) &= 7(-19A^2 - 34AB + 57B^2) \\ z = z(A,B) &= 7(-20A^2 - 10AB + 60B^2) \\ w = w(A,B) &= 49(A^2 + 3B^2) \end{aligned}\right\}. \tag{2.19}$$

Observation 2.4. 1. $x(A,A) - z(A,A) + t_{38,A} + t_{22,A} \equiv 0 \pmod{13}$
2. $y(A,A) - z(A,A) - 7[t_{62,A} + t_{42,A} + t_{26,A} + t_{22,A}] \equiv 0 \pmod 7$

Pattern 2.5. Take 1 as

$$1 = \frac{(1 + i15\sqrt{3})(1 - i15\sqrt{3})}{676}. \tag{2.20}$$

Using (2.1), (2.2), (2.20) in (1.3) and applying the method of factorization, define

$$(u + i\sqrt{3}v) = (\frac{1 + i15\sqrt{3}}{26})(2 + i\sqrt{3})(a + i\sqrt{3}b)^2, \tag{2.21}$$

from which, we have

$$\left.\begin{aligned} u &= \frac{1}{26}(-43a^2 - 186ab + 129b^2) \\ v &= \frac{1}{26}(31a^2 - 86ab - 93b^2) \end{aligned}\right\}. \tag{2.22}$$

Using (2.22) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x = x(a,b) &= \frac{1}{13}(-6a^2 - 136ab + 18b^2) \\ y = y(a,b) &= \frac{1}{13}(-37a^2 - 50ab + 111b^2) \\ z = z(a,b) &= \frac{1}{13}(-43a^2 - 186ab + 129b^2) \end{aligned}\right\}. \tag{2.23}$$

Since our interest is on finding integer solutions, replacing a by $13A$, b by $13B$ in (2.1) and (2.23), the corresponding integer solutions to to (1.1) are given by

$$\left.\begin{aligned} x = x(A,B) &= 13(-6A^2 - 136AB + 18B^2) \\ y = y(A,B) &= 13(-37A^2 - 50AB + 111B^2) \\ z = z(A,B) &= 13(-43A^2 - 186AB + 129B^2) \\ w = w(A,B) &= 169(A^2 + 3B^2) \end{aligned}\right\}. \tag{2.24}$$

Observation 2.5. 1. $z(A,A) - x(A,A) - 13[t_{30,A} + t_{14,A} + t_{10,A}] \equiv 0 \pmod{13}$
2. $x(A,A) - y(A,A) + 15[t_{62,A} + t_{16,A}] = 35A$

Pattern 2.6. Assume 1 as

$$1 = \frac{(1 + i56\sqrt{3})(1 - i56\sqrt{3})}{9409}. \tag{2.25}$$

Using (2.1), (2.2), (2.25) in (1.3) and applying the method of factorization, define

$$(u+i\sqrt{3}v)=(\frac{1+i56\sqrt{3}}{97})(2+i\sqrt{3})(a+i\sqrt{3}b)^2, \tag{2.26}$$

from which, we have

$$\left.\begin{aligned} u&=\frac{1}{97}(-166a^2-678ab+498b^2)\\ v&=\frac{1}{97}(113a^2-332ab-339b^2)\end{aligned}\right\}. \tag{2.27}$$

Using (2.27) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x&=x(a,b)=\frac{1}{97}(-53a^2-1010ab+159b^2)\\ y&=y(a,b)=\frac{1}{97}(-279a^2-346ab+837b^2)\\ z&=z(a,b)=\frac{1}{97}(-332a^2-1356ab+996b^2)\end{aligned}\right\}. \tag{2.28}$$

Since our interest is on finding integer solutions, replacing a by $97A$, b by $97B$ in (2.1) and (2.28), the corresponding integer solutions to to (1.1) are given by

$$\left.\begin{aligned} x&=x(A,B)=97(-53A^2-1010AB+159B^2)\\ y&=y(A,B)=97(-279A^2-346AB+837B^2)\\ z&=z(A,B)=97(-332A^2-1356AB+996B^2)\\ w&=w(A,B)=9409(A^2+3B^2)\end{aligned}\right\}. \tag{2.29}$$

Pattern 2.7. Using (2.1), (2.6), (2.11) in (1.3) and applying the method of factorization, define

$$(u+i\sqrt{3}v)=(\frac{5+i\sqrt{3}}{2})(a+i\sqrt{3}b)^2(\frac{1+i\sqrt{3}}{2}), \tag{2.30}$$

from which, we have

$$\left.\begin{aligned} u&=\frac{1}{2}(a^2-18ab-3b^2)\\ v&=\frac{1}{2}(3a^2+2ab-9b^2)\end{aligned}\right\}. \tag{2.31}$$

Using (2.31) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x&=x(a,b)=2a^2-8ab-6b^2\\ y&=y(a,b)=-a^2-10ab+3b^2\\ z&=z(a,b)=a^2-18ab-3b^2\end{aligned}\right\}. \tag{2.32}$$

Thus (2.1) and (2.32) represent the non-zero integer solutions to to (1.1).

Observation 2.6.
1. $x(b,b)-z(b,b)=2t_{2b,4}$
2. $x(a,a)-y(a,a)+t_{4,2a}=0$

Pattern 2.8. Using (2.1), (2.6), (2.15) in (1.3) and applying the method of factorization, define

$$(u+i\sqrt{3}v)=(\frac{5+i\sqrt{3}}{2})(a+i\sqrt{3}b)^2(\frac{1+i4\sqrt{3}}{7}), \tag{2.33}$$

from which, we have

$$\left.\begin{aligned} u&=\frac{1}{2}(-a^2-18ab+3b^2)\\ v&=\frac{1}{2}(3a^2-2ab-9b^2)\end{aligned}\right\}. \tag{2.34}$$

Using (2.34) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x&=x(a,b)=a^2-10ab-3b^2\\ y&=y(a,b)=-2a^2-8ab+6b^2\\ z&=z(a,b)=-a^2-18ab+3b^2\end{aligned}\right\}. \tag{2.35}$$

Thus (2.1) and (2.35) represent the non-zero integer solutions to to (1.1).

Observation 2.7. 1. $x(b,b) - z(b,b) - t_{b,10} \equiv 0 \pmod 3$
2. $y(b,b) - z(b,b) - 2t_{b,14} \equiv 0 \pmod 5$

Pattern 2.9. Using (2.1), (2.6), (2.20) in (1.3) and applying the method of factorization, define

$$(u + i\sqrt{3}v) = (\frac{5 + i\sqrt{3}}{2})(a + i\sqrt{3}b)^2(\frac{1 + i15\sqrt{3}}{26}), \tag{2.36}$$

from which, we have

$$\left.\begin{aligned} u &= \frac{1}{13}(-10a^2 - 114ab + 30b^2) \\ v &= \frac{1}{13}(19a^2 - 20ab - 57b^2) \end{aligned}\right\}. \tag{2.37}$$

Using (2.37) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x = x(a,b) &= \frac{1}{13}(9a^2 - 134ab - 27b^2) \\ y = y(a,b) &= \frac{1}{13}(-29a^2 - 94ab + 87b^2) \\ z = z(a,b) &= \frac{1}{13}(-20a^2 - 228ab + 60b^2) \end{aligned}\right\}. \tag{2.38}$$

Since our interest is on finding integer solutions, replacing a by $13A$, b by $13B$ in (2.1) and (2.38), the corresponding integer solutions to (1.1) are given by

$$\left.\begin{aligned} x = x(A,B) &= 13(9A^2 - 134AB - 27B^2) \\ y = y(A,B) &= 13(-29A^2 - 94AB + 87B^2) \\ z = z(A,B) &= 13(-20A^2 - 228AB + 60B^2) \\ w = w(A,B) &= 169(A^2 + 3B^2) \end{aligned}\right\}. \tag{2.39}$$

Pattern 2.10. Using (2.1), (2.6), (2.25) in (1.3) and applying the method of factorization, define

$$(u + i\sqrt{3}v) = (\frac{5 + i\sqrt{3}}{2})(a + i\sqrt{3}b)^2(\frac{1 + i56\sqrt{3}}{97}), \tag{2.40}$$

from which, we have

$$\left.\begin{aligned} u &= \frac{1}{194}(-163a^2 - 1686ab + 489b^2) \\ v &= \frac{1}{194}(281a^2 - 326ab - 843b^2) \end{aligned}\right\}. \tag{2.41}$$

Using (2.41) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x = x(a,b) &= \frac{1}{97}(59a^2 - 1006ab - 177b^2) \\ y = y(a,b) &= \frac{1}{97}(-222a^2 - 680ab + 666b^2) \\ z = z(a,b) &= \frac{1}{97}(-163a^2 - 1686ab + 489b^2) \end{aligned}\right\}. \tag{2.42}$$

Since our interest is on finding integer solutions, replacing a by $97A$, b by $97B$ in (2.1) and (2.42), the corresponding integer solutions to (1.1) are given by

$$\left.\begin{aligned} x = x(A,B) &= 97(59A^2 - 1006AB - 177B^2) \\ y = y(A,B) &= 97(-222A^2 - 680AB + 666B^2) \\ z = z(A,B) &= 97(-163A^2 - 1686AB + 489B^2) \\ w = w(A,B) &= 9409(A^2 + 3B^2) \end{aligned}\right\}. \tag{2.43}$$

Pattern 2.11. (1.3) is rewritten as

$$u^2 = 7w^2 - 3v^2. \tag{2.44}$$

In (2.44), taking

$$\left.\begin{aligned} w &= X + 3T \\ v &= X + 7T \\ u &= 2U \end{aligned}\right\}, \tag{2.45}$$

it leads to

$$X^2 - U^2 = 21T^2, \tag{2.46}$$

which is written as the system of double equations as shown in Table 2.1:

***Table* 2.1:** System of Double Equations

$SYSTEM$	1.1	1.2	1.3	1.4
$X+U$	T^2	$3T^2$	$7T^2$	$7T$
$X-U$	21	7	3	$3T$

Consider system (1.1) in the Table 2.1: Solving the pair of equations, note that

$$X = \frac{T^2+21}{2}$$
$$U = \frac{T^2-21}{2}.$$

The choice

$$T = 2k+1, \tag{2.47}$$

gives

$$\left.\begin{aligned} X &= 2k^2+2k+11 \\ U &= 2k^2+2k-10 \end{aligned}\right\}. \tag{2.48}$$

The substitution of (2.47) and (2.48) in (2.45) gives

$$\left.\begin{aligned} u &= 4k^2+4k-20 \\ v &= 2k^2+16k+18 \\ w &= 2k^2+8k+14 \end{aligned}\right\}. \tag{2.49}$$

In view of (1.2), one obtains

$$\left.\begin{aligned} x &= 6k^2+20k-2 \\ y &= 2k^2-12k-38 \\ z &= 8k^2+8k-40 \end{aligned}\right\}. \tag{2.50}$$

Thus (2.49) and (2.50) represent the non-zero integer solutions to (1.1).

Consider system (1.2) in the Table 2.1: Solving the pair of equations, note that

$$X = \frac{3T^2+7}{2}$$
$$U = \frac{3T^2-7}{2}.$$

Using (2.47) the above equation become

$$\left.\begin{aligned} X &= 6k^2+6k+5 \\ U &= 6k^2+6k-2 \end{aligned}\right\}. \tag{2.51}$$

The substitution of (2.47) and (2.51) in (2.45) gives

$$\left.\begin{aligned} u &= 12k^2+12k-4 \\ v &= 6k^2+20k+12 \\ w &= 6k^2+12k+8 \end{aligned}\right\}. \tag{2.52}$$

In view of (1.2), one obtains

$$\left.\begin{aligned} x &= 18k^2+32k+8 \\ y &= 6k^2-8k-16 \\ z &= 24k^2+24k-8 \end{aligned}\right\}. \tag{2.53}$$

Thus (2.52) and (2.53) represent the non-zero integer solutions to (1.1).

Consider system (1.3) in the Table 2.1: Solving the pair of equations, note that

$$X = \frac{7T^2+3}{2}$$
$$U = \frac{7T^2-3}{2}$$

Using (2.47) the above equation become

$$\left.\begin{aligned} X &= 14k^2+14k+5 \\ U &= 14k^2+14k+2 \end{aligned}\right\}. \tag{2.54}$$

The substitution of (2.47) and (2.54) in (2.45) gives

$$\left.\begin{aligned} u &= 28k^2+28k+4 \\ v &= 14k^2+28k+12 \\ w &= 14k^2+20k+8 \end{aligned}\right\}. \tag{2.55}$$

In view of (1.2), one obtains

$$\left.\begin{aligned} x &= 42k^2+56k+16 \\ y &= 14k^2-8 \\ z &= 56k^2+56k+8 \end{aligned}\right\}. \tag{2.56}$$

Thus (2.55) and (2.56) represent the non-zero integer solutions to (1.1).

Consider system (1.4) in the Table 2.1: On solving, it is seen that $X = 5T, U = 2T$.
In view of (2.45), we have

$$\left.\begin{aligned} u &= 4T \\ v &= 12T \\ w &= 8T \end{aligned}\right\}. \tag{2.57}$$

Substituting the above values of u and v in (1.2), we get

$$\left.\begin{aligned} x &= 16T \\ y &= -8T \\ z &= 8T \end{aligned}\right\}. \tag{2.58}$$

Thus (2.57) and (2.58) represent the non-zero integer solutions to (1.1).

Pattern 2.12. It is seen that (2.46) is satisfied by

$$\left.\begin{aligned} T &= 2rs \\ U &= 21r^2-s^2 \\ X &= 21r^2+s \end{aligned}\right\}. \tag{2.59}$$

Substituting the values of T, U and X in (2.53), we get

$$\left.\begin{aligned} u &= 42r^2-2s^2 \\ v &= 21r^2+14rs+s^2 \\ w &= 21r^2+6rs+s^2 \end{aligned}\right\}. \tag{2.60}$$

Substituting the above values of u and v in (1.2), the non-zero distinct integral values of x, y and z are given by

$$\left.\begin{aligned} x &= x(r,s) = 63r^2+14rs-s^2 \\ y &= y(r,s) = 21r^2-14rs-3s^2 \\ z &= z(r,s) = 84r^2-4s^2 \end{aligned}\right\}. \tag{2.61}$$

Thus (2.60) and (2.61) represent the non-zero integer solutions to (1.1).

3 Conclusion

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutins to the homogeneous cubic equation with four unknowns. As the cubic equations are rich in variety, one may search for other forms of cubic equations with multivariables to obtain their corresponding solutions.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 173-180**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

ON STABILITY OF A-QUARTIC FUNCTIONAL EQUATIONS IN RANDOM NORMED SPACES

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(Received: August 30, 2022; In format: February 14, 2023; Accepted: March 24, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53120

Abstract

In this paper, we shall prove the generalized Hyers-Ulam stability of the additive-quartic functional equation introduced by C. Muthamilarasi et al. [11] in Random Normed spaces by using direct and fixed-point methods.
2020 Mathematical Sciences Classification: 39B52; 39B82
Keywords and Phrases: Additive-Quartic functional equation; Hyers-Ulam stability; fixed point methods; Random Normed spaces.

1 Introduction

In the field of stability of functional equations, a type of stability named after the Mathematician Ulam [15] is often considered. In 1940, Ulam [15], triggered the study of stability problems for various functional equations. He presented a number of unsolved problems. Since then, this question has attracted the attention of many researchers. In the next year, Hyers [9] gave answer of Ulams question in the case of approximately additive mappings. Thereafter, Hyers result was generalized by Aoki [3] and improved for additive mappings, and subsequently improved by Rassias [[6],[7]] for linear mappings by allowing the Cauchy difference to be unbounded.

Since then, stability of functional equation had been discussed in various spaces by researchers [[2],[4],[5]]. In 1963, Serstnev [13] introduced the theory of random normed spaces (briefly, RN-spaces) which is generalization of deterministic result of normed spaces and also in the study of random operator equations. A number of papers and research monographs have been published on generalizations of the stability of different functional equations in RN- spaces [12]. Recently, in 2017, Abdou et al. [1] discussed the stability of a quintic functional equations in random normed space. In this paper, we shall discuss about the stability of A-Quartic functional equation in random normed space.

To prove our main results, we need some notions and definitions from the literature as follows: A function $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$ is called a distribution function if it is nondecreasing and left -continuous with $F(0) = 0$ and $F(\infty) = 1$. The class of all probability distribution functions F with $F(0) = 0$ is denoted by $A.D^+$ is a subset of A consisting of all functions $F \in A$ for which $F(\infty) = 1$, where $l^-F(x) = \lim_{t\to x^-} F(t)$. For any $a \geq 0, \epsilon_a$ is the element of D^+ , which is defined as

$$\begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{otherwise.} \end{cases}$$

Definition 1.1 ([14]). *A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a t-norm) if T satisfies the following conditions:*

1. T is commutative and associative,
2. T is continuous,
3. $T(a, 1) = 1$ for all $a[0, 1]$,
4. $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

The examples of continuous t-norm are as follows:
$T_M(a, b) = min\{a, b\}$, $T_P(a, b) = minab, T_L(a, b) = max\{a + b - 1, 0\}$
Recall that, if T is a t-norm and $\{x_n\}$ is a sequence of number in $[0, 1]$, then $T^n_{i=1}x_i$ is defined recurrently by

$T^1_{i=1}x_i = x_1 and T^n_{i=1}x_i = T(T^{n-1}_{i=1}x_i, x_n) = T(x_1, x_2, , ..., x_n)$ for each $n \geq 2$ and $T^{\infty}_{i=n}x_n$ is defined as $T^{\infty}_{i=1}x_{n+i}$.

Definition 1.2 ([13]). *Let X be a real linear space, μ be a mapping from X into $D^+(for any x X, \mu(x)$ is denoted by μ_x and T be a continuous t norm. The triple (X, μ, t) is called a random normed space (briefly RN-space) if μ satisfies the following conditions:*

(RN1) $\mu_x = \epsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN2) $\mu_\alpha x(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X, \alpha \neq 0$ and all $t \geq 0$;

(RN3) $\mu_x + y(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s > 0$.

Example 1.3. Every normed space $(X, \|.\|)$ defines a RN-space (X, μ, T_M), where $\mu_x(t) = \frac{t}{t+\|x\|}$, for all $t > 0$ and T_M is the minimum t-norm. This space is called induced random normed space.

Definition 1.3 ([13]). *Let (X, μ, T) be a RN-space.*

1. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$ and $\lambda > 0$ there exists a positive integer N such that $\mu_{(x_n - x)}(t) > 1 - \lambda$, whenever $n \geq N$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} \mu_{x_n - x} = 1$.
2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for all $t > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(t) > 1 - \lambda$, whenever $n \geq m \geq N$.
3. The RN -space (X, μ, T) is said to be complete if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.1 ([14]). *If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence of X such that $x_n \to x$ then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Recently in 2021, Muthamilarasi et al. [11] proved the general solution and generalized Hyers-Ulam stability of additive quartic functional equation.

$$\begin{aligned} f(ax + a^2y + a^3z) + f(-ax + a^2y + a^3z) &+ f(ax - a^2y + a^3z) + f(ax + a^2y - a^3z) \\ &= 2[f(ax + a^2y) + f(a^2y + a^3z) + f(ax + a^3z) \\ &+ f(ax - a^2y) + f(a^2y - a^3z) + f(a^3z - ax)] \\ &- 2[a^4(f(x) + f(-x)) + a^8(f(y) + f(-y)) \\ &+ a^{12}(f(z) + f(-z))] - [a(f(x) - f(-x)) \\ &+ a^2(f(y) - f(-y)) + a^3(f(z) - f(-z))]. \end{aligned} \tag{1.1}$$

for fixed $a \in Z^+$ in Banach spaces.

Lemma 1.1. *Let W and X be real vector spaces. If an odd mapping $f : W \to X$ satisfies (1.1), then f is additive.*

Lemma 1.2. *Assume that W and X are real vector spaces. If an even mapping $f : W \to X$ satisfies the quartic functional equation*
$f(2w + x) + f(2w - x) = 4f(w + x) + 4f(w - x) + 24f(w) - 6f(x)$, if and only if $f : W \to X$ satisfies the functional equation (1.1) for all $x, y, z, w \in W$.Throughout this paper,let X be a real linear space, (Z, μ', T_M) be an RN-space and (Y, μ, T_M) be a complete RN-spaces. For mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x, y, z) &= f(ax + a^2y + a^3z) + f(-ax + a^2y + a^3z) + f(ax - a^2y + a^3z) \\ &+ f(ax + a^2y - a^3z) - 2[f(ax + a^2y) + (a^2y + a^3z) + f(ax + a^3z) \\ &+ f(ax - a^2y) + f(a^2y - a^3z) + f(a^3z - ax)] \\ &+ 2[a^4(f(x) + f(-x)) + a^8(f(y) + f(-y)) + a^12(f(z) + f(-z))] \\ &+ [a(f(x) - f(-x)) + a^2(f(y) - f(-y)) + a^3(f(z) - f(-z))], \end{aligned} \tag{1.2}$$

$$\tag{1.3}$$

for all $x, y, z \in X$.

In this paper, using the direct and fixed-point methods, we investigate the generalised Hyers -Ulam stability of the A-Quartic functional equation (1.1) in random normed spaces under the minimum t-norm.

2 Random stability of the functional equation

In this section, we investigate the generalized Hyers-Ulam stability problem of the A-Quartic functional equation (1.1) in RN-spaces.

Theorem 2.1. *. Let $\phi : X^3 \to Z$ be a function such that, for some $0 < \alpha < a$,*

$$\mu'_{\phi(ax,ay,az)}(t) \geq \mu'_{\alpha(\phi(x,y,z))}(t). \tag{2.1}$$

and $\lim_{n\to\infty} \mu'_{\phi(a^n x,a^n y,a^n z)}(a^n t) = 1$. For all $x, y, z \in X$ and $t > 0$.
If $f : X \to Y$ is an odd mapping with $f(0) = 0$ such that

$$\mu_{Df(x,y,z)}(t) \geq \mu'_{\phi(x,y,z)}(t) \tag{2.2}$$

for all $x, y, z \in X$ and $t > 0$.
Then there exists a unique additive mapping $A : X \to Y$ such that,

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\phi(x,0,0)}(2(a-\alpha)t) \tag{2.3}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = z = 0$ in equation (2.2), we get

$$\mu_{2af(x)-2f(ax)}(t) \geq \mu'_{\phi(x,0,0)}(t). \tag{2.4}$$
$$\mu_{(f(x)-\frac{f(ax)}{a})}(t) \geq \mu_{\phi(x,0,0)}(2at). \tag{2.5}$$

for all $x \in X$ *and* $t > 0$. Replacing x by ax in equation (2.4), we get

$$\begin{aligned} \mu_{(f(ax)-\frac{f(a^2x)}{a})}(t)\mu'_{\phi(ax,0,0)}(2at) &\geq \mu'_{\phi(x,0,0)}(\frac{2at}{\alpha}), \\ \mu_{(f(ax)-\frac{f(a^2x)}{a})}(t) &\geq \mu'_{\phi(ax,0,0)}(2at)\mu'_{\phi(x,0,0)}(\frac{2at}{\alpha}), \\ \mu_{(f(ax)/a-\frac{f(a^2x)}{a^2})}(t/a) &\geq \mu'_{\phi(x,0,0)}(\frac{2at}{\alpha})), \\ \mu_{(f(ax)/a-\frac{f(a^2x)}{a^2})}(t) &\geq \mu'_{\phi(x,0,0)}(t). \end{aligned} \tag{2.6}$$

for all $x \in X$ and $t > 0$.
Continuing like this, we have

$$\mu_{(\frac{f(a^n x)}{a^n}-f(\frac{f(a^{n+1}x)}{a^{n+1}}))}(t) \geq \mu'_{\phi(x,0,0)}(\frac{2a^{n+1}t}{a^n}). \tag{2.7}$$

Now, since

$$\begin{aligned} \frac{f(a^n x)}{a^n} - f(x) &= (\frac{f(a^n x)}{a^n} - \frac{f(a^{n-1}x))}{a^{n-1}}) \\ &+ (\frac{(f(a^{n-1}x)}{a^{n-1}} - \frac{f(a^{n-2}x)}{a^{n-2})} \\ &+ ... + (\frac{f(ax)}{a} - f(x)), \\ &= \sum_{j=0}^{n-1}(\frac{f(a^{j+1}x)}{a^{j+1}} - \frac{f(a^j x)}{a^j}) \\ \mu_{(\frac{f(a^n x)}{a^n}-f(x))}(\sum_{j=0}^{n-1}\frac{1}{2a}(\frac{\alpha}{a})^j t) &\geq T_M(\mu'_{\phi(x,0,0)}(t), \\ &= \mu'_{\phi(x,0,0)}(t). \end{aligned} \tag{2.8}$$

Now, replacing x by $a_m x$ in equation (2.8), we get

$$\mu_{(\frac{f(a^{n+m}x)}{a^n})-f(a^m x)}(\sum_{j=0}^{n-1}\frac{1}{2a}(\frac{\alpha}{a})^j t) \geq \mu'_{\phi(a^m x,0,0)}(t),$$

$$\mu_{(\frac{f(a^{n+m}x)}{a^{n+m}}-\frac{f(a^m x)}{a^m})}(\sum_{j=0}^{n-1}\frac{1}{2aa^m}(\frac{\alpha}{a})^j t) \geq \mu'_{\phi(x,0,0)}(\frac{t}{\alpha^m}),$$

$$\begin{aligned}
\mu_{(\frac{f(a^{n+m}x)}{a^{n+m}}-\frac{f(a^m x)}{a^m})}(t) &\geq \mu'_{\phi(x,0,0)}(\frac{t}{\alpha^m(\sum_{j=0}^{n-1}\frac{t}{2aa^m}(\frac{\alpha}{a})^j t)}),\\
&= \mu'_{\phi(x,0,0)}\frac{2at}{\sum_{j=0}^{n-1}(\frac{\alpha}{a})^j},\\
&\geq \mu'_{\phi(x,0,0)}\frac{2at}{\sum_{j=m}^{n+m-1}(\frac{\alpha}{a})^{j+m}}. \qquad (2.9)
\end{aligned}$$

for all $x \in X$ and $m, n \in Z$ with $n > m \geq 0$ since $< a$, the sequence $\{\frac{(f(a^n x))}{a^n}\}$ is a Cauchy sequence in the complete RN-spaces (Y, μ, T_M) and so it converges to some point $A(x) \in Y$. Fix $x \in X$ and put $m = 0$ in equation (2.9), we get

$$\mu_{(\frac{(f(a^n x))}{a^n}-f(x))}(t) \geq \mu'_{\phi(x,0,0)}\frac{2at}{\sum_{j=0}^{n-1}(\frac{\alpha}{a})^j},$$

So, for any $\delta > 0$,

$$\begin{aligned}
\mu_{(A(x)-f(x))}(\delta + t) &\geq T_M(\mu_{A(x)-\frac{f(a^n x)}{a^n}}(\delta), \mu_{\frac{f(a^n x)}{a^n}-f(x)}(t) \qquad (2.10)\\
&\geq T_M(\mu_{A(x)-\frac{f(a^n x)}{a^n}}(\delta), \mu'_{\phi(x,0,0)}\frac{2at}{\sum_{j=0}^{n-1}(\frac{\alpha}{a})^j}).
\end{aligned}$$

for all $x \in X$ and $t > 0$.
Taking the limit in (2.10) as $n \to \infty$, we get

$$\mu_{(A(x)-f(x))}(\delta + t) \geq \mu'_{\phi(x,0,0)}(\frac{2at}{\frac{1}{1-\frac{\alpha}{a}}}) = \mu'_{\phi(x,0,0)}(2t(a-\alpha)) \qquad (2.11)$$

Since δ is arbitrary, by taking $\delta \to 0$ in equation (2.11), we have

$$\mu_{(A(x)-f(x))}(t) \geq \mu'_{\phi(x,0,0)}(2(a-\alpha)(t), \qquad (2.12)$$

for all $x \in X$ and $t > 0$.
Therefore, we conclude that the condition of equation (2.3) holds.
Also, by replacing x, y and z by $a^n x, a^n y$ and $a^n z$ in equation (2.2), we have

$$\mu_{\frac{Df(a^n x, a^n y, a^n z}{a^n}}(t) \geq \mu_{\phi(a^n x, a^n y, a^n z)}(a^n)(t) = \mu'_{\phi(x,y,z)}(\frac{a}{\alpha})^n(t),$$

for all $x, y, z \in X$ and $t > 0$.
It follows from $\lim_{n\to\infty} \mu'_{\phi(a^n x, a^n y, a^n z)}(a^n t) = 1$, that A satisfies the equation (1.1), which implies that A is an additive mapping.
To prove the uniqueness of the quartic mapping A, let us assume that there exists another mapping $A'X \to Y$ which satisfies equation (2.3). Fix $x \in X$, then $A(a^n x) = a^n A(x)$ and $A'(a^n x) = a^n A'(x)$ for all $n \in Z^+$. Thus it follows from the equation (2.3) that

$$\begin{aligned}
\mu_{(A(x)-A'(x))}(t) &= \mu_{(\frac{A(a^n x)}{a^n}-\frac{A'(a^n x)}{a^n})}(t)\\
&\geq T_M(\mu_{\frac{A(a^n x)}{a^n}-\frac{f(a^n x)}{a^n})}(\frac{t}{2}), \mu_{\frac{f(a^n x)}{a^n}-\frac{A'(a^n x)}{a^n})}(\frac{t}{2}) \qquad (2.13)\\
&\geq \mu'_{\phi(x,0,0)}((a-\alpha)(\frac{a}{\alpha})^n t).
\end{aligned}$$

Since, $\lim_{n\to\infty}(a-\alpha)(\frac{a}{\alpha})^n t = \infty$, we have $\mu_{(A(x)-A'(x))}(t) = 1$ for all $t > 0$.
Thus the additive mapping is unique.
This completes the proof. □

Theorem 2.2. *Let $\phi : X^3 \to Z$ be a function such that, for some $0 < \alpha < a^4$,*

$$\mu'_{(\phi(ax,ay,az))}(t) \geq \mu_{\alpha\phi(x,y,z)}(t) \qquad (2.14)$$

and $\lim_{n\to\infty} \mu'_{a^n\phi(a^n x, a^n y, a^n z)}(t) = 1$ for all $x, y, z \in X$ and $t > 0$. If $f : X \to Y$ is an even mapping with $f(0) = 0$ which satisfies equation (2.2), then there exists a unique additive mapping $Q : X \to Y$ such that

$$\mu_{(f(x)-A(x))}(t) \geq \mu'_{\phi(x,0,0)}(4(a^4-\alpha)t), \qquad (2.15)$$

for all $x \in X$ and $t > 0$.

Replace x, y, z by $x, 0, 0$ respectively in equation (2.14), we obtain

$$\mu_{(4f(ax)-4a^4f(x))}(t) \geq \mu'_{\phi(x,0,0)}(t),$$
$$\mu_{4a^4(\frac{f(ax)}{a^4}-f(x))}(t) \geq \mu'_{\phi(x,0,0)}(t),$$
$$\mu_{(\frac{f(ax)}{a^4}-f(x))}(\frac{t}{4a^4}) \geq \mu'_{\phi(x,0,0)}(t),$$
$$\mu_{\frac{f(ax)}{a^4}-f(x)}(t) \geq \mu'_{\phi(x,0,0)}(4a^4t). \tag{2.16}$$

for all $x \in X$ and $t > 0$.
Replacing x by ax in equation (2.16), we get

$$\mu_{\frac{f(a^2x)}{a^4}-f(ax)}(t) \geq \mu'_{\phi(ax,0,0)}(4a^4t),$$
$$\geq \mu'_{\phi(x,0,0)}(\frac{4a^4t}{a}),$$
$$\mu_{\frac{f(a^2x)}{a^8}-\frac{f(ax)}{a^4}}(t) \geq \mu'_{\phi(x,0,0)}(\frac{4a^4t}{a}),$$
$$\mu_{(\frac{f(a^2x)}{a^8})-(\frac{f(ax)}{a^4})}(t) \geq \mu'_{\phi}(x,0,0)(\frac{4a^8t}{a}), \tag{2.17}$$

for all $x \in X$ and $t > 0$.
Now again, replacing x by ax in equation (2.17), we have

$$\mu_{\frac{f(a^3x)}{a^8}-\frac{f(a^2x)}{a^4}}(t) \geq \mu_{\phi(ax,0,0)}(\frac{4a^8t}{a}),$$
$$\mu_{\frac{f(a^3x)}{a^{12}}-\frac{f(a^2x)}{a^8}}(t) \geq \mu'_{\phi(x,0,0)}(\frac{4a^{12}t}{2}),$$

Continuing this process, we get

$$\mu_{\frac{f(a^nx)}{a^{4n}}-\frac{f(a^{n-1}x)}{a^{4(n-1)}}}(t) \geq \mu'_{\phi(x,0,0)}\frac{(4a^{4n}t)}{(n-1)},$$

Now, since

$$\frac{f(a^nx)}{a^{4n}} - f(x) = \sum_{j=0}^{n-1} \frac{f(a^{j+1}x)}{a^{4(j+1)}} - \frac{f(a^jx)}{a^{4j}},$$

Now,

$$\mu_{\frac{f(a^nx)}{a^{4n}}-f(x)}(\sum_{j=0}^{n-1} \frac{1}{(4a^4)}(\frac{\alpha}{a^4})^j t) \geq T_M(\mu'_{\phi(x,0,0)}(t))$$
$$= \mu'_{\phi(x,0,0)}(t). \tag{2.18}$$

Now replacing x by a^mx in equation (2.18), we get

$$\mu_{\frac{f(a^{n+m}x)}{a^{4n}}}(\sum_{j=0}^{n-1} \frac{1}{4a^4}(\frac{\alpha}{a^4})^j t) \geq \mu'_{\phi(a^mx,0,0)}(t),$$
$$\mu_{\frac{f(a^{n+m}x)}{a^{4n+4m}}-\frac{f(a^mx)}{a^{4m}}}(\sum_{j=0}^{n-1} \frac{1}{4a^4a^{4m}}(\frac{\alpha}{a^4})^j t) \geq \mu'_{\phi(x,0,0)}(\frac{t}{\alpha^m}),$$
$$\mu_{\frac{f(a^{n+m}x)}{a^{4(n+m)}}-\frac{f(a^mx)}{a^4m}}(t) \geq \mu'_{\phi(x,0,0)(\frac{4a^4t}{\sum_{j=0}^{n-1}(\frac{\alpha}{a^4})^{j+m}})} \tag{2.19}$$

for all x and $m, n \in Z^+$ with $n > m \geq 0$. Since $< a^4$, the sequence $(\frac{f(a^n x)}{a^{4n}})$ is a Cauchy sequence in the complete RN-space (Y, μ, T_M) and it converge to a point $Q(x) \in Y$.
Fix $x \in X$ and $m = 0$ in equation (2.19), we get

$$\mu_{\frac{f(a^n x)}{a^4 n} - f(x)}(t) \geq \mu'_{\phi(x,0,0)} \frac{2a^4 t}{\sum_{j=0}^{n+m-1} (\frac{\alpha}{a^4})^j},$$

and so, for any $\delta > 0$,

$$\begin{aligned} \mu_{(Q(x)-f(x))}(\delta + t) &\geq T_M \mu_{(Q(x) - \frac{f(a^n x)}{a^{4n}})}(\delta), \mu_{(\frac{f(a^n x)}{a^{4n}} - f(x))}(t), \\ &\geq T_M \mu_{(Q(x) - \frac{f(a^n x)}{a^{4n}})}(\delta), \mu'_{\phi(x,0,0)}(\frac{4a^4 t}{\sum_{j=0}^{n+m-1} (\frac{\alpha}{a^4})^j}), \end{aligned} \tag{2.20}$$

for all $x \in X$ and $t > 0$. Taking the limit $n \to \infty$ in equation (2.20), we get

$$\begin{aligned} \mu_{(Q(x)-f(x))}(\delta + t) &=\geq \mu'_{\phi(x,0,0)}(\frac{4a^4 t}{\frac{1}{1-\frac{\alpha}{a^4}}}) \\ &= \mu'_{\phi(x,0,0)}(4t(a^4 - \alpha)). \end{aligned} \tag{2.21}$$

Since δ is arbitrary, by taking $\delta \to 0$ in equation (2.21), we have

$$\mu_{(Q(x)-f(x))}(t) \geq \mu'_{\phi(x,0,0)}(4t(a^4 - \alpha)). \tag{2.22}$$

for all $x \in X, t > 0$.
Therefore, we conclude that the condition of equation (2.15) holds.
Also replacing x, y, z by $a^n x, a^n y, a^z$ respectively in equation (2.15), we have

$$\begin{aligned} \mu_{\frac{a^n x, a^n y, a^z}{a^n}}(t) &\geq \mu'_{\phi(a^n x, a^n y, a^z)}(a^n t), \\ &\geq \mu'_{\phi(x,y,z)}((\frac{a^4}{\alpha})^n t). \end{aligned}$$

It follows from $\lim_{n\to\infty} \mu'_{\phi(a^n x, a^n y, a^z)}(a^{4n} t) = 1$ that Q satisfies the equation (1.1), which implies Q is a quartic mapping.

Lemma 2.1 ([8]). *Suppose that (ω, d) is a complete generalized metric space and $J : \omega \to \omega$ is astrictly contractive mapping with Lipschitz constant $L < 1$. Then for each $x \in \omega$, either $d(J^n x, J^{n+1} x) = \infty$. for all non negative integers $n \geq 0$ or there exists a natural numbaer n_0 such that*

1. $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
2. The sequence $J^n x$ is convergent to a fixed point $y*$ og J;
3. $y*$ is the unique fixed point of J in the set $A = \{y \in \omega : d(J^{n_0} x, y) < \infty\}$;
4. $d(y, y*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in A$.

Theorem 2.3. *Let $\phi : X^3 \to D^+$ be a function such that, for some $0 < \alpha < a^4$,*

$$\mu'_{\phi(x,y,z)}(t) \leq \mu'_{\phi(ax,ay,az)}(\alpha t) \tag{2.23}$$

for all $x, y, z \in X$ and $t > 0$. If $f : X \to Y$ is an even mapping with $f(0) = 0$ such that

$$\mu_{D(x,y,z)}(t) \geq \mu'_{\phi(x,y,z)}(t). \tag{2.24}$$

for all $x, y, z \in X$ and $t > 0$.
Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\mu_{(f(x)-Q(x))}(t) \geq \mu'_{\phi(x,y,z)}(2(a^4 - \alpha)t), \tag{2.25}$$

for all $x \in X, t > 0$.

Proof. It follows from equation (2.24) that

$$\mu_{(f(x) - \frac{f(ax)}{a^4})}(t) \geq \mu'_{\phi(x,0,0)}(4a^4 t), \tag{2.26}$$

for all $x \in X, t > 0$.
Let $\omega = \{g : X \to Y, g(x) = 0\}$ and mapping d defined on ω by

$$d(g, h) = \inf\{c \in [0, \infty) : \mu_{g(x)-h(x)}\}(ct) \geq \mu'_{\phi(x,0)}(t), \forall x \in X\}$$

where as usual $\inf \phi = -\infty$. Then (ω, d) is a generalized complete metric space. Now let us consider the mapping $J : \omega \to \omega$ defined by
$Jg(x) = \frac{1}{a^4} g(ax)$, for all $g \in \omega$ and $x \in X$.
Let $g, h \in \omega$ and $c \in [0, \infty)$ be any arbitrary constant with $d(g, h) < c$.
Then $\mu_{(g(x)-h(x))}(ct) \geq \mu'_{\phi(x,0,0)}$ for all $x \in X, t > 0$ and so,

$$\mu_{(Jg(x)-Jh(x))}(\frac{\alpha ct}{a^4}) = \mu_{g(ax)-h(ax)}(\alpha ct) \geq \mu'_{\phi(x,0,0)}(t) = \mu'_{\phi(\alpha x,0,0)}, \tag{2.27}$$

for all $x \in X, t > 0$. Hence we have $d(Jg, Jh) \leq \frac{\alpha c}{a^4} \leq \frac{\alpha c}{a^4} d(g, h)$.
for all $g, h \in \omega$.
Then J is a contractive mapping on ω with the Lipschitz constant $L = \frac{\alpha}{a^4} < 1$.
Thus it follows from Lemma 2.1, that there exists a mapping $Q : X \to Y$ which is a unique fixed point of J in the set $\omega_1 = \{g \in \omega : d(g, h) < \infty\}$, such that
$Q(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^4 n}$ for all $x \in X$ since $\lim_{n\to\infty} d(J^f, Q) = 0$. Also, using $\mu_{(f(x)-\frac{f(ax)}{a^4})}(t) \geq \mu'_{\phi(x,0,0)}(4(a^4 - \alpha)t)$, we have $d(f, Jf) \leq \frac{1}{4(a^4-\alpha)}$.
Therefore using Lemma 2.1, we get
$d(f, Q) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{4(a^4-\alpha)}$.
This means that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}(4(a^4 - \alpha)t),$$

for all $x \in X, t > 0$. Also by replacing x, y, z by $2^n x, 2^n y, 2^n z$ in equation (2.4) respectively, we have

$$\mu_{DQ(x,y,z)}(t) \geq \lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y, 2^z)}(a^{4n} t) = \lim_{n\to\infty} \mu'_{\phi(x,y,z)}((\frac{a^4}{\alpha})^n t) = 1,$$

for all $x, y, z \in X$ and $t > 0$. By (RN1), the mapping is quartic.
To prove the uniqueness let us assume that there exists a quartic mapping $Q' : X \to Y$, which satisfies equation (2.25). Then Q' is a fixed point of J in ω_1.
However it follows from the Lemma 2.3, that J has only one fixed point in ω_1.
Hence $Q = Q'$. □

Acknowledgement. Authors are very much thankful to the Editor and Reviewer for their valuable suggestions to bring the paper in its present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 181-195**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

GENERALIZATION OF CONVEX FUNCTION

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(Received: March 01, 2023; Informat: March 04, 2023; Revised: March 25, 2023; Accepted: April 08, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53121

Abstract

In this paper, We established a new class of convex function $(\phi_1, \phi_2) - \beta$-convex, which includes many well-known classes as its subclasses. We defined $(\phi_1, \phi_2) - \beta$-convex function and discussed various properties with non-differentiable and differentiable cases.

2010 Mathematics Subject Classification: 26B25, 90C30, 52A01, 52A40
Keywords and Phrases: Non linear programming, $(\phi_1, \phi_2) - \beta-$ convex function, First order conditions, Second order conditions, $\phi_1 - \beta-$quasi-convex

1 Introduction

Convexity has been of great importance in both applied and pure mathematics for the purpose of generalizing existing results in work over the past 60 years. In recent years, several extensions of the concept of convexity of a set and a function have been introduced. There are several inequalities introduced by Minkowski [14], Dragomir [10] and Ardic et al. [4] etc. using convexity. A class of convex functions introduced by Bector and Singh [5] called b-vex functions with differentiable and nondifferentiable cases were presented.

Hanson [11] introduced mathematical Programming Problem for invex functions with inequality constraints. He considered differentiable functions and then proved that instead of assumption of convexity, the objective function and each of the constraints function involved satisfy inequality with respect to the same function. Then Craven [8], inspired by Hanson's work, first systematically introduced the term "invariant convex". After that Craven and Glover [9], Ben Israel and Mond [6] and Martin [13] showed that the class of invex functions is equivalent to the class of functions whose stationary points are points of global minima.

Mishra [15] obtained optimality and duality results by combining the concepts of type I, type II, pseudo-type I, quasi-type I, quasi-pseudo-type I, pseudo-quasi-type I, strictly pseudo-quasi-type I, and univex functions. Mishra and Rueda [16] also introduced and discussed SFJ-univex programming problems then Ojha [18] extended the SFJ-univex programming problems in complex spaces. Antczak [1] introduced several nonlinear programming problems of *(p, r)*-invexity type. Antczak [2] elongated the idea of p-invex set and defined *(p, r)*-pre-invex function (non-differentiable) and *(p, r)*-invex function (differentiable) and obtained optimality conditions for nonlinear programming problem under the idea of those functions. Antczak [3] defined r-preinvexity, r-invexity and obtained optimality criteria and duality relations for these functions in programming problem. Antczak [3] also designed duality theorems for modified r-invex functions based on function η.

Weir, Mond and Craven [25] and also [26, 27] showed that how and where pre-invex functions can replace convex functions in multiple objective optimization. Then Bector - singh [5] and Suneja, Singh and Bector [24] introduced a class of functions, called b-vex functions which forms a subset of the sets of both semi-strictly quasi-convex and quasi-convex functions also.

Pini [19] introduced relations between invexity and generalized properties of convexity also gave a new class of generalized convex sets. Pini and Singh [21, 22] established duality results also defined $(\phi_1, \phi_2)-$

convexity which is an extremely powerful principle for characterization of generalized convexity from an integrated point of view. Where ϕ_1 is a continuous deformation of straight line segments and ϕ_2 identifies generalized convex combinations of values. So after that other type of convexity can be introduced.

$$\phi_1 : D \times D \times [0,1] \to \Re^n, \qquad \phi_1 = \phi_1(x,y,\lambda),$$
$$\phi_1(x,y,0) = y, \;\; \phi_1(x,x,\lambda) = x, \qquad \forall x,y \in D, \;\; \lambda \in [0,1],$$
$$\phi_2 : D \times D \times [0,1] \times F \to Re, \quad \phi_2 = \phi_2(x,y,\lambda,f),$$
$$\phi_2(x,y,0,f) = f(y), \;\; \phi_2(x,x,\lambda,f) = f(x), \quad \forall x,y \in D, \lambda \in [0,1], \; f \in F.$$

In this paper, $(\phi_1, \phi_2) - \beta-$ convexity is defined. It is a very powerful new principle for characterizing the generalized convexity of sets and functions from a unified perspective.

In section 2, the definition of $(\phi_1, \phi_2) - \beta-$convex function is given; we show that to appropriate selection of functions ϕ_1 and ϕ_2, some of the well-known classes of generalized convex functions are particular cases of this new class. An example of a $(\phi_1, \phi_2) - \beta-$convex function is also provided that does not belong to any of the known classes. We present some properties of nondifferentiable $(\phi_1, \phi_2) - \beta-$convex functions. In this section, we also examine some properties of the solution of a mathematical programming problem involving.$(\phi_1, \phi_2) - \beta-$convex functions; moreover, we state a senstivity result.

In section 3, we consider the differentible case. Here we state a natural necessary condition for differentiable $(\phi_1, \phi_2) - \beta-$convex functions; in particular, we provided criteria under which the differentiable and the nondifferentiable conditions are equivalent. we state a second order sufficient condition for $(\phi_1, \phi_2) - \beta-$convexity.

2 The Nondifferentiable Case

Let G be a vector space of real valued functions defined on a set $D \subseteq \Re^n$. We are assuming two maps ϕ_1, ϕ_2 which satisfy the following assumptions:

$$\phi_1 : D \times D \times [0,1] \to \Re^n, \qquad \phi_1 = \phi_1(x,y,\lambda),$$
$$\phi_1(p,q,0) = q, \;\; \phi_1(p,p,\lambda) = p, \qquad \forall p,q \in D, \;\; \lambda \in [0,1],$$
$$\phi_2 : D \times D \times [0,1] \times F \to \Re, \quad \phi_2 = \phi_2(p,q,\lambda,g),$$
$$\phi_2(p,q,0,g) = g(y), \;\; \phi_2(p,p,\lambda,g) = g(p), \quad \forall p,q \in D, \lambda \in [0,1], \; g \in G,$$
$$\phi_2(p,q,\lambda,g) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta}, \quad if \quad \beta \neq 0,$$
$$\phi_2(p,q,\lambda,g) = \lambda g\phi_1(p,p,\lambda) + (1-\lambda)g\phi_1(p,q,0), \qquad if \quad \beta = 0,$$
$$\phi_2(p,q,\lambda,g) = \lambda g(p) + (1-\lambda)g(q), \qquad if \quad \beta = 0. \tag{2.1}$$

We will also assume that ϕ_1 is continuous with respect to λ. We give the following definitions and preliminaries:

Definition 2.1 (ϕ_1-convex set)**.** *A set D is said to be ϕ_1-convex if $\phi_1(p,q,\lambda) \in D$ for all $p,q \in D$, $\lambda \in [0,1]$. The intersection of ϕ_1-convex sets is also ϕ_1-convex.*
From now onwards, we take D as a ϕ_1-convex set [21].

Definition 2.2 ((ϕ_1, ϕ_2)-convex(concave) function)**.** *A function $g \in G$ is (ϕ_1, ϕ_2)-convex(concave) if*

$$f(\phi_1(p,q,\lambda)) \leq \phi_2(p,q,\lambda,f) \qquad (\geq),$$

for all $p,q \in D$ and $0 \leq \lambda \leq 1$.
If $g = (g_1, g_2, ..., g_k) : D \to \Re^k, g_i \in G$, and g_i is (ϕ_1, ϕ_2)-convex(concave) for i=1,2,...,k, then the vector valued function g is said to be (ϕ_1, ϕ_2)-convex(concave) [21].

Definition 2.3 ($(\phi_1, \phi_2) - \beta$-convex(concave) function)**.** *A function $g \in G$ is said to be $(\phi_1, \phi_2) - \beta$-convex(concave) if*

$$g(\phi_1(p,q,\lambda)) \leqq \phi_2(p,q,\lambda,g) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta}$$
$$if \quad \beta \neq 0 \quad (\geq),$$
$$f(\phi_1(p,q,\lambda)) \leqq \phi_2(p,q,\lambda,g) = \lambda g(p) + (1-\lambda)g(q)$$
$$if \quad \beta = 0 \quad (\geq), \tag{2.2}$$

for all $p, q \in D$ and $0 \leq \lambda \leq 1$.
If $g = (g_1, g_2, ..., g_k) : D \to \Re^k, g_i \in G$, and g_i is $(\phi_1, \phi_2) - \beta$-convex(concave) for i=1,2,...,k, then the vector valued function g is said to be $(\phi_1, \phi_2) - \beta$-convex(concave).

Definition 2.4 (ϕ_1-quasi-convex function)**.** *A function $g \in G$ is said to be ϕ_1-quasi-convex [21] on D if for every $p, q \in D, \lambda \in [0,1]$*

$$g(\phi_1(p,q,\lambda)) \leqq max\{g(p), g(q)\}.$$

Definition 2.5 ($\phi_1 - \beta$-quasi-convex function)**.** *A function $g \in G$ is said to be $\phi_1 - \beta$-quasi-convex on D if for every $p, q \in D, \lambda \in [0,1]$*

$$g(\phi_1(p,q,\lambda)) \leqq \phi_2(p,q,\lambda,g) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta}$$
$$\leqq max\{g(p), g(p)\} \quad if \quad \beta \neq 0,$$

$$g(\phi_1(p,q,\lambda)) \leqq \phi_2(p,q,\lambda,g) = \lambda g(p) + (1-\lambda)g(q) \leqq max\{g(p), g(p)\} \quad if \quad \beta = 0. \tag{2.3}$$

Remark 2.1. We say that this definition is independent on the vector or topological structure on D; In fact, D can be any set.

Remark 2.2. If ϕ_1, ϕ_2 satisfy the assumptions of (2.1), then every $(\phi_1, \phi_2) - \beta$- convex function is $\phi_1 - \beta$-quasi-convex. We give some examples below.

Example 2.1. Let D be a convex subset of $\Re^n$, and define $\phi_1(p,q,\lambda) = \lambda p + (1-\lambda)q$, $\phi_2(p,q,\lambda,g) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta} \quad if \quad \beta \neq 0$ and $\phi_2(p,q,\lambda,g) = \lambda g(p) + (1-\lambda)g(q) \quad if \quad \beta = 0$, then the convex function on D is $(\phi_1, \phi_2) - \beta$-convex.

Example 2.2. If $\eta : \Re^n \times \Re^n \to \Re^n$, D is a pre-invex set with respect to η, then an η-pre-invex function $g : D \to \Re$ is $(\phi_1, \phi_2) - \beta$-convex with $\phi_1(p,q,\lambda) = \lambda\eta(p,q) + q$ and $\phi_2(p,q,\lambda,g) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta} \quad if \quad \beta \neq 0$ also $\phi_2(p,q,\lambda,g) = \lambda g(p) + (1-\lambda)g(q) \quad if \quad \beta = 0$, where $\eta(p,q) = p - q$ [27].

Example 2.3. Let $D \subseteq N$ where N is an Euclidean manifold and D is geodesically convex. A geodesically convex function on D is $(\phi_1, \phi_2) - \beta$-convex, with $\phi_1(p,q,\lambda) = \gamma_{p,q}(\lambda)$ and $\phi_2(p,q,\lambda,g) \leq ln(\lambda e^{\beta g(\gamma_{p,p}(\lambda)} + (1-\lambda)e^{\beta g(\gamma_{p,q}(0)})^{1/\beta} \quad if \quad \beta \neq 0$ also $\phi_2(p,q,\lambda,g) = \lambda g(\gamma_{p,p}(\lambda)) + (1-\lambda)g(\gamma_{p,q}(0)) \quad if \quad \beta = 0$, where $\gamma_{p,q}$ is the geodesic from q to p [23].

Example 2.4. Let D be a convex subset of $\Re^n$, $\phi_1(p,q,\lambda) = \lambda\eta(p,q) + q$ and
$\phi_2(p,q,\lambda,g) \leq ln(a_1(p,q,\lambda)e^{\beta g(\phi_1(p,p,a_1(p,q,\lambda)} + (1-a_1(p,q,\lambda))e^{\beta g(\phi_1(p,q,0)})^{1/\beta} \quad if \quad \beta \neq 0$
also
$\phi_2(p,q,\lambda,g) = a_1(p,q,\lambda)g(p) + (1-a_1(p,q,\lambda))g(q) \quad if \quad \beta = 0.$
Then every B-vex function on D (with respect to a_1) is $(\phi_1, \phi_2) - \beta$-convex [5, 24].

Example 2.5. Let I be a one - to - one mapping from $D \subseteq \Re^n$ to $\Re^n$, and Φ a strictly monotone increasing function mapping a subset $\sum$ of $\Re$ onto $\Re$. A function $g : D \to \sum$ is called $(I, \Phi) - \beta$-convex if, for any $p, q \in D$ and $\lambda \in [0,1]$

$$f(N_I([p,q],\lambda)) \leqq n_\Phi[(g(p), g(q)), \lambda],$$

provided that $rangeg \subset dom\Phi$. Here

$$N_I([p,q],\lambda) \leq I^{-1}(ln(\lambda e^{\beta I(\phi(p,p,\lambda)} + (1-\lambda)e^{\beta I(\phi(p,q,0)})^{1/\beta} \quad if \quad \beta \neq 0,$$

also

$$N_I([p,q],\lambda) = \lambda I(p) + (1-\lambda)I(q) \quad if \quad \beta = 0,$$

$$n_\Phi[(g(p), g(q)), \lambda] \leq \Phi^{-1}(ln(\lambda e^{\beta\Phi(g(\phi(p,p,\lambda)} + (1-\lambda)e^{\beta\Phi(g(\phi(p,q,0)})^{1/\beta} \quad if \quad \beta \neq 0,$$

$$n_\Phi[(g(p), g(q)), \lambda] = \Phi^{-1}(\lambda\Phi(g(p) + (1-\lambda)\Phi(g(q)) \quad if \quad \beta = 0.$$

Choosing $\phi_1(p,q,\lambda) = N_I([p,q],\lambda)$ and $\phi_2(p,q,\lambda,g) = n_\Phi[(g(p),g(q)),\lambda]$, we see that an $(I,\Phi) - \beta$-convex function is a particular $(\phi_1, \phi_2) - \beta$-convex function [7].

Remark 2.3. The functions ϕ_1, ϕ_2 of the Examples 1 - 5 satisfy (2.1).

Example 2.6. A function $g : \Re^n \to \bar{\Re}^1 = \Re \cup \{-\infty\}$ is called G-convex on the convex set D if, for every $p, q \in D$, $p \neq q$, $\lambda \in (0,1)$,

$$g((1-\lambda)q + \lambda p) \leqq G(g(p), g(q), \|p-q\|, \lambda),$$

where $G(m_1, m_2, \delta, \alpha) : \bar{\Re}^1 \times \bar{\Re}^1 \times \Re_+ \times \Re_+ \to \bar{\Re}^1$ is continuous and non-decreasing in (m_1, m_2) and $\|.\|$ is an arbitrary norm on $\Re^n$. If we take $\phi_1(p,q,\lambda) = \lambda p + (1-\lambda)q$ and $\phi_2(p,q,\lambda,g) = G(g(p),g(q),\|p-q\|,\lambda) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta}$, we get that a G-convex function is an example of $(\phi_1, \phi_2) - \beta$-convex function [12].

Now we will give some examples of $(\phi_1, \phi_2) - \beta$-convex function that justify our results.

Example 2.7. Let $D \subset \Re$ be the set $D = (-\infty, \infty)$, and $g : D \to \Re$ be the function defined as follows:

$$g(p) = \begin{cases} 4p, & if p > 0 \\ p^2 - p + 1, & if p < 0 \end{cases}.$$

Define the functions $\phi_1 : D \times D \times [0,1]$ and $\phi_2 : D \times D \times [0,1] \times G$ as follows:

$$\phi_1(p,q,\lambda) = \begin{cases} (1-\lambda)q + \lambda p, & if pq > 0 \\ q, & if pq < 0 \end{cases},$$

$$\phi_2(p,q,\lambda,g) = \begin{cases} g(q), & if \lambda = 0 \\ max\{g(p), g(q)\}, & if 0 < \lambda \leq 1 \end{cases}.$$

This function is $(\phi_1, \phi_2) - \beta-$ convex and also justify our results.

Example 2.8. Let $D \subset \Re$ be the set $D = (-\infty, -1) \cup (1, \infty)$, and $g : D \to \Re$ be the function defined as follows:

$$g(p) = \begin{cases} |p| - 1, & if |p| < 1 \\ 1, & if |p| > 1 \end{cases}.$$

Define the functions $\phi_1 : D \times D \times [0,1]$ and $\phi_2 : D \times D \times [0,1] \times G$ as follows:

$$\phi_1(p,q,\lambda) = \begin{cases} (1-\lambda)q + \lambda p, & if pq > 0 \\ q, & if pq < 0 \end{cases},$$

$$\phi_2(p,q,\lambda,g) = \begin{cases} g(q), & if \lambda = 0 \\ max\{g(p), g(q)\}, & if 0 < \lambda \leq 1 \end{cases}.$$

This function is $(\phi_1, \phi_2) - \beta-$ convex and also justify our results.

Now for suitable assumptions on ϕ_1 and/or ϕ_2, we will discuss some properties of the class of $(\phi_1, \phi_2) - \beta$-convex functions.

Observation (a). We are assuming that, ϕ_2 issuperlinear with respect to $g \in G$, that is ϕ_2 is superadditive and positively homogeneous. Then the class of $(\phi_1, \phi_2) - \beta$-convex functions is a convex cone. (Practically, if g,h are $(\phi_1, \phi_2) - \beta$-convex, and $\alpha > 0$.

$$(g+h)(\phi_1(p,q,\lambda)) \leqq \phi_2(p,q,\lambda,g) + \phi_2(p,q,\lambda,h) \leqq \phi_2(p,q,\lambda,g+h)$$

$$(g+h)(\phi_1(p,q,\lambda)) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0))})^{1/\beta}$$

$$+ln(\lambda e^{\beta h(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta h(\phi_1(p,q,0))})^{1/\beta}$$

$$\leq ln(\lambda e^{\beta (g+h)(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta (g+h)(\phi_1(p,q,0))})^{1/\beta} \quad if \quad \beta \neq 0$$

$$(g+h)(\phi_1(p,q,\lambda)) \leq \phi_2(p,q,\lambda,g) + \phi_2(p,q,\lambda,h)$$

$$= \lambda g(p) + (1-\lambda)h(q) + \lambda g(p) + (1-\lambda)h(q) = \lambda(g+h)(p) + (1-\lambda)(g+h)(q)$$

$$if \quad \beta = 0$$

$$(\alpha g)(\phi_1(p,q,\lambda)) = \alpha(g(\phi_1(p,q,\lambda)) \leqq \alpha\phi_2(p,q,\lambda,g) = \phi_2(p,q,\lambda,\alpha g)$$

$$\leq ln(\lambda e^{\beta \alpha g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta \alpha g(\phi_1(p,q,0))})^{1/\beta} \quad if \quad \beta \neq 0$$

$$(\alpha g)(\phi_1(p,q,\lambda)) = \alpha(g(\phi_1(p,q,\lambda)) \leqq \alpha\phi_2(p,q,\lambda,g)$$

$$= \phi_2(p,q,\lambda,\alpha g) = \lambda(\alpha g)(p) + (1-\lambda)(\alpha g)(q) \quad if \quad \beta = 0.$$

Observation (b). We are also assuming that, $g : D \to \Re$ is $(\phi_1, \phi_2) - \beta$-convex, $h : \Re \to \Re$ is an increasing function and $(\phi_3, \phi_4) - \beta$-convex and $hog \in G$. Then, if

$$\phi_2(p,q,\lambda,g) \leqq \phi_3(g(p), g(q), \lambda),$$

$$\phi_4(g(p), g(q), \lambda, h) \leqq \phi_2(p,q,\lambda,hog),$$

the function hog is $(\phi_1, \phi_2) - \beta$-pre convex.(Practically, we have that)

$$(hog)(\phi_1(p,q,\lambda)) \leqq h(\phi_2(p,q,\lambda,g))$$

$$\leqq h(\phi_3(g(p), g(q), \lambda))$$

$$\leqq \phi_4(g(p), g(q), \lambda, h)$$

$$\leqq \phi_2(p,q,\lambda,hog)$$

$$\leq ln(\lambda e^{\beta \alpha (hog)(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta \alpha (hog)(\phi_1(p,q,0))})^{1/\beta} \quad if \quad \beta \neq 0$$

$$= \lambda(hog)(p) + (1-\lambda)(hog)(q) \quad if \quad \beta = 0.$$

Remark 2.4. In Examples 2.1 - 2.4, we see that ϕ_2 is linear with respect to g. In Example 2.5, if Φ is superlinear then ϕ_2 will be superlinear.

Now we will consider a scalar value optimization problem, which can be expressed as

$$(P)\ \ ming(p)\ \ s.t.\ \ h(p) \leq 0,$$

where $g : D \to \Re$, $h : D \to \Re^k$ Denote the feasible set by D_0, where

$$D_0 = \{p \in D : h(p) \leq 0\}.$$

Then the following holds:

Proposition 2.1. *Suppose that*

(i) $h = (h_1, h_2, ..., h_k)$ *is* (ϕ_1, ϕ_2) - β- *convex(see Definition* (2.3)*:*

(ii) g *is* (ϕ_1, ϕ_2) - β- *convex.*

Then the set of solutions of problem (P) will be ϕ_1- β- convex.

Proof. The feasible set D_0 is ϕ_1- β-convex; Practically, if $p_1, p_2 \in D_0$, from (i) and (2.1) we have

$$\begin{aligned} h_i(\phi_1(p_1,p_2,\lambda)) &\leq \phi_2(p_1,p_2,\lambda,h_i) \leq ln(\lambda e^{\beta h_i(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta h_i(\phi_1(p,q,0)})^{1/\beta} \\ &\leq max\{h_i(\phi_1(p_1,p_2,\lambda)), h_i(\phi_1(p_1,p_2,0))\} \leq 0 \end{aligned}$$

for any $i = 1, 2, ..., k$. Next, let $\min_{p \in D_0} g(p)$ be attained at p_1^0 and p_2^0. By the hypothesis (ii) and (2.1)

$$\begin{aligned} f(\phi_1(p_1^0,p_2^0,\lambda)) &\leq \phi_2(p_1^0,p_2^0,\lambda,g) \leq ln(\lambda e^{\beta g(\phi_1(p_1^0,p_1^0,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p_1^0,p_2^0,0)})^{1/\beta} \\ &\leq max\{g(\phi_1(p_1^0,p_1^0,\lambda)), g(\phi_1(p_1^0,p_2^0,0))\} = g(p_1^0) \end{aligned}$$

But $g(p_1^0) = g(p_2^0) = min_{p \in D_0}$ g(p), hence $g(\phi_1(p_1^0, p_1^0, \lambda)) = g(p_1^0)$ which completes the proof. □

Definition 2.6 (($(\phi_1, \phi_2)-\beta$ - pre-strictly convex(concave) function)**.** *Let* $p^0 \in D$. *We say that* g *is* $(\phi_1, \phi_2)-\beta$ *strictly convex(concave) at* p_0 *if*

$$\begin{aligned} g(\phi_1(q,p_0,\lambda)) &< \phi_2(q,p_0,\lambda,g) \qquad (>) \\ &\leq ln(\lambda e^{\beta g(\phi_1(q,q,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(q,p_0,0)})^{1/\beta} \qquad (>), \end{aligned} \tag{2.4}$$

we say that g *is weakly* $(\phi_1, \phi_2) - \beta$ *strictly convex(concave) at* p_0 *if* (4) *holds for some* $\lambda \in (0, 1)$. *If* (4) *is satisfied at any* $p_0 \in D$*, then* g *is* $(\phi_1, \phi_2) - \beta$ *strictly convex(concave) on D.*

Proposition 2.2. *Suppose that* D_0 *is a* $\phi_1 - \beta$ *convex set, and*

(i) g *is* $(\phi_1, \phi_2) - \beta$ *strictly convex at* $p_0 \in D_0$.

(ii) p_0 *is a solution of problem (P).*

Then p_0 *is the unique solution of problem (P).*

Proof. Let p^* be another solution of (P). $p^* \neq p_0$. Then, for all $\lambda \in (0, 1)$

$$\begin{aligned} g(\phi_1(p^*,p_0,\lambda)) &\leq \phi_2(p^*,p_0,\lambda,g) \leq ln(\lambda e^{\beta g(\phi_1(p^*,p^*,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p^*,p_0,0)})^{1/\beta} \\ &\leq max\{g(\phi_1(p^*,p^*,\lambda)), g(\phi_1(p^*,p_0,0))\} = g(p_0) \end{aligned}$$

which contradicts hypothesis(ii).

In case of $(\phi_1, \phi_2) - \beta$ concave functions(see Definition *3*), we have the following: □

Theorem 2.1. *Suppose that*

(i) g *is* $(\phi_1, \phi_2) - \beta$ *strictly concave in D;*

(ii) $\forall p_0 \in int(D_0) \exists p, q \in D_0, p \neq q, \lambda \in (0,1]$ *such that* $\phi_1(p,q,\lambda) = p_0$*;*

(iii) D_0 *is* $\phi_1 - \beta$ *convex;*

(iv) $\phi_2(p,q,\lambda,g) \geq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta} \geq min\{g(\phi_1(p,p,\lambda)), g(\phi_1(p,q,0))\}$ *for every* $p, q \in D_0,\ \lambda \in [0,1]$.

Then there are no interior points of D_0 *which are solution of (P), i.e. if* p_0 *is a solution of (P), then* p_0 *is a boundary point of* D_0.

Proof. If the solution set of (P) is empty, or int(D_0) is empty, there is nothing to prove. Let p_0 is a solution of (P), and $p_0 \in int(D_0)$. Then by (ii) there exist $p, q \in D_0, p \neq q$ and $\lambda \in (0,1]$ such that $p_0 = \phi_1(p,q,\lambda)$. By (i) we have that

$$g(p_0) = g(\phi_1(p,q,\lambda)) > \phi_2(p,q,\lambda,g)$$
$$\geq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta} \geq min\{g(\phi_1(p,p,\lambda)), g(\phi_1(p,q,0))\} \geq g(p_0).$$

Contradiction, so it is concluded that p_0 is not a solution of (P). Let $\mu_\delta(p_0)$ denote a neighbourhood of p_0 of radius δ. □

Theorem 2.2. *Suppose that*

(i) g *is* $(\phi_1, \phi_2) - \beta$ *strictly convex;*

(ii) $p_0 \in D_0$ *is a local minimum of (P);*

(iii) $\forall \delta_1 > 0,$ *and* $\forall p \in D_0, \exists \lambda \in (0,1]$ *such that* $\phi_1(p_0,p,\lambda) \in \mu_{\delta_1}(p_0)$*;*

(iv) D_0 *is* $\phi_1 - \beta$ *convex;*

Then p_0 *is a strict global minimum of (P).*

Proof. By hypothesis (iv), for every $p \in D_0$, and for every $\lambda \in [0,1]$, $\phi_1(p_0,p,\lambda) \in D_0$. Since p_0 is a local minimum of (P), there exists $\mu_{\delta_2}(p_0)$ such that for every $p \in \mu_{\delta_2}(p_0) \cap D_0$, $g(p_0) \leq g(p)$. Now let $p \in D_0$, $p \neq p_0$. Then, by hypothesis (ii) and (iii), with $\delta_1 = \delta_2$ we have that $g(p_0) \leq \phi_1(p_0,p,\lambda))$ for some $\lambda \in (0,1]$. Therefore, using (i) and (2.1), we have

$$g(p_0) \leq g(\phi_1(p_0,p,\lambda)) < \phi_2(p_0,p,\lambda,g)$$
$$\leq ln(\lambda e^{\beta g(\phi_1(p_0,p_0,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p_0,p,0)})^{1/\beta} \leq max\{g(\phi_1(p_0,p_0,\lambda)), g(\phi_1(p_0,p,0))\}$$

Obviously, $max\{g(\phi_1(p_0,p_0,\lambda)), g(\phi_1(p_0,p,0))\} \neq g(p_0)$ since $g(p_0) \not< g(p_0)$. Therefore $g(p_0) < g(p)$. Since p is an arbitrary member of D_0, the proof is complete.

On the basis of Theorem 2.2, the following results can be obtained. □

Theorem 2.3. *Suppose that*

(i) g *is* $(\phi_1, \phi_2) - \beta$ *convex;*

(ii) $p_0 \in D_0$ *is a strict local minimum of (P);*

(iii) $\forall \delta_1 > 0,$ *and* $\forall p \in D_0, \exists \lambda \in (0,1]$ *such that* $\phi_1(p_0,p,\lambda) \in \mu_{\delta_1}(p_0) \setminus \{p_0\}$*;*

(iv) D_0 *is* $\phi_1 - \beta$ *convex;*

Then p_0 *is a strict global minimum of (P).*

Theorem 2.4. *Suppose that*

(i) g *is* $(\phi_1, \phi_2) - \beta$ *convex;*

(ii) $p_0 \in D_0$ *is a local minimum of (P);*

(iii) $\forall \delta_1 > 0,$ *and* $\forall p \in D_0, \exists \lambda \in (0,1]$ *such that* $\phi_1(p_0,p,\lambda) \in \mu_{\delta_1}(p_0)$*;*

(iv) D_0 *is* $\phi_1 - \beta$ *convex;*

(v) $\phi_2(p_0,p,\lambda,g) \leq ln(\lambda e^{\beta g(\phi_1(p_0,p_0,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p_0,p,0))})^{1/\beta} \leqq max\{g(\phi_1(p_0,p_0,\lambda)), g(\phi_1(p_0,p,0))\}$ *for every* $p \in D_0$ *with* $g(p) \neq g(p_0)$, *and for all* $\lambda \in (0,1)$.

Then p_0 *is a global minimum of (P).*

Example 2.9. Let $D \subset \Re$ be the set $D = (-\infty,-3) \cup (3,\infty)$, and $g : D \to \Re$ be the function defined as follows:

$$g(p) = \begin{cases} |p| - 3, & if|p| < 3 \\ 1, & if|p| \geqq 3 \end{cases}.$$

Define the functions $\phi_1 : D \times D \times [0,1]$ and $\phi_2 : D \times D \times [0,1] \times G$ as follows:

$$\phi_1(p,q,\lambda) = \begin{cases} (1-\lambda)q + \lambda p, & ifpq > 0 \\ q, & ifpq < 0 \end{cases},$$

$$\phi_2(p,q,\lambda,g) = \begin{cases} g(q), & if\lambda = 0 \\ max\{g(p), g(q)\}, & if0 < \lambda \leq 1 \end{cases}.$$

This function is $(\phi_1,\phi_2) - \beta-$ convex which verifies our results.

Now we will study a regularity property of the product of $(\phi_1,\phi_2) - \beta$ convex functions (i=2,3). First, we say the following

Lemma 2.1. *Suppose that g,h are satisfying the conditions and also real valued functions defined on D,*

(i) $g(p) \geqq 0$, $h(p) \geqq 0$

(ii) $g(p) - g(q))(h(p) - h(q)) \geqq 0 \ \forall \ p,q \in D$.

Then for every $p,q \in D$, *either*

$$g(p)h(p) \geqq g(q)h(p) \ \ and \ \ g(p)h(p) \geqq g(p)h(q)$$

or

$$g(q)h(q) \geqq g(p)h(q) \ \ and \ \ g(q)h(q) \geqq g(q)h(p).$$

Proof. Since, by (ii),

$$g(p) - g(q))(h(p) - h(q)) \geqq 0 \ \ \forall p,q \in D$$

it follows that either

$$g(p) \geqq g(q) \ \ and \ \ h(p) \geqq h(q)$$

or

$$g(q) \geqq g(p) \ \ and \ \ h(q) \geqq h(p)$$

which further implies(in view of (i)) that either

$$g(p)h(p) \geqq g(q)h(p) \ \ and \ \ g(p)h(p) \geqq g(p)h(q)$$

or

$$g(q)h(q) \geqq g(p)h(q) \ \ and \ \ g(q)h(q) \geqq g(q)h(p).$$

□

Proposition 2.3. *Suppose that*

(i) g,h are nonnegative functions defined on D and satisfying the inequality

$$(g(p) - g(q))(h(p) - h(q)) \geqq 0, \ \ \forall p,q \in D;$$

(ii) g *is* $(\phi_1,\phi_2) - \beta$ *convex,* h *is* $(\phi_1,\phi_2) - \beta$ *convex.*

Then pq is $\phi_1 - \beta$ quasi-convex.

Proof. For any $p, q \in D$ and $\lambda \in [0,1]$,

$$(gh)(\phi_1(p,q,\lambda)) = g(\phi_1(p,q,\lambda))h(\phi_1(p,q,\lambda))$$
$$\leqq \phi_2(p,q,\lambda,g)\phi_2(p,q,\lambda,h)$$
$$\leq \{ln(\lambda e^{\beta f(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta f(\phi_1(p,q,0)})^{1/\beta}\}$$
$$\times\{ln(\lambda e^{\beta h(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta h(\phi_1(p,q,0)})^{1/\beta}\}$$
$$\leqq max\{g(p), g(q)\}.max\{h(p), h(q)\}.$$

Now $max\{g(p), g(q)\}.max\{h(p), h(q)\}$, in view of lemma, is less than or equal to $max\{g(p), g(q)\}.max\{h(p), h(q)\}$; hence it follows that

$$(gh)(\phi_1(p,q,\lambda)) \leq ln(\lambda e^{\beta(gh)(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta(gh)(\phi_1(p,q,0)})^{1/\beta}$$
$$\leq max\{g(p)h(p), g(q)h(q)\}$$
$$= max\{(gh)(p), (gh)(q)\}.$$

Therefore gh is $\phi_1 - \beta$-quasi-convex. □

Now we will consider the following family of problems:

$$\min g(p) \;\; s.t. \;\; h(p) \leqq \epsilon,$$

where $g : \Re^n \to \Re$, $h : \Re^n \to \Re^k$, $\epsilon \in \Re^k$. Denote by $g^*(\epsilon)$ the function

$$g^* : \Re^k \to \Re, \;\; g^*(\epsilon) = inf\{g(p) : h(p) \leqq \epsilon\} \text{ ([25])}.$$

Assume that g is $(\phi_1, \phi_2) - \beta$ convex, where $\phi_2(p_1, p_2, \lambda, g) = \phi_4(g(p_1), g(p_2), \lambda)$ and the vector function h is $(\phi_1, \phi_2) - \beta$ convex, where

$$\bar{\phi_2} : \Re^n \times \Re^n \times [0,1] \times G^k \to \Re^k, \;\; \bar{\phi_2}(p,q,\lambda,h) = \phi_3(h(p), h(q), \lambda),$$

and $\phi_3(b_1, b_2, \lambda)$ is nondecreasing in (b_1, b_2) with respect to the component wise order (if $b_1^i \leqq c_1^i$ and $b_2^j \leqq c_2^j$, $\forall i, j$, then $\phi_3(b_1, b_2, \lambda) \leqq \phi_3(c_1, c_2, \lambda)$, for every $\lambda \in [0,1]$)
We have the following

Theorem 2.5. *The function g^* is a $(\phi_3, \phi_4) - \beta$convex on $\Re^k$ (i.e. $g^*(\phi_3(\epsilon_1, \epsilon_2, \lambda)) \leqq \phi_4(g^*(\epsilon_1), g^*(\epsilon_2), \lambda)$).*

Proof. Notice that if $h(p_1) \leqq \epsilon_1$, $h(p_2) \leqq \epsilon_2$ then

$$h(\phi_1(p_1, p_2, \lambda)) \leqq \phi_3(h(p_1), h(p_2), \lambda) \leqq \phi_3(\epsilon_1, \epsilon_2, \lambda);$$

in particular,

$$\{(p_1, p_2) : h(p_1) \leqq \epsilon_1, h(p_2) \leqq \epsilon_2\} \subseteq \{(p_1, p_2) : h(\phi_1(p_1, p_2, \lambda)) \leqq \phi_3(\epsilon_1, \epsilon_2, \lambda)\}$$

Hence

$g^*(\phi_3(\epsilon_1, \epsilon_2, \lambda)) = inf\{g(p) : h(p) \leqq \phi_3(\epsilon_1, \epsilon_2, \lambda)\}$
$\leqq inf\{g(\phi_1(p_1, p_2, \lambda)) : h(\phi_1(p_1, p_2, \lambda)) \leqq \phi_3(\epsilon_1, \epsilon_2, \lambda)\}$
$\leqq inf\{\phi_2(p_1, p_2, \lambda, g) : h(p_1) \leqq \epsilon_1, h(p_2) \leqq \epsilon_2\} \; (from(2.6))$
$\leq inf\{ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta} : h(p_1) \leqq \epsilon_1, h(p_2) \leqq \epsilon_2\}$
$= \phi_4(g^*(\epsilon_1), g^*(\epsilon_2), \lambda).$ □

3 The Differentiable Case

Let us assume that ϕ_1, ϕ_2 have right partial derivative with respect to λ at $\lambda = 0$, for all $p, q \in D$, for all $g \in G$. If we consider a differentiable $(\phi_1, \phi_2) - \beta$ convex function g, defined on $D \subseteq \Re^n$, taking into account (1), for $p, q \in D$ and $\lambda \in (0, 1]$ we get that

$g(\phi_1(p, q, \lambda) \leqq \phi_2(p, q, \lambda, g)$
$\Longrightarrow\ g(\phi_1(p, q, \lambda) - g(q) \leqq \phi_2(p, q, \lambda, g) - g(q)$
$\Longrightarrow\ g(\phi_1(p, q, \lambda) - g(\phi_1(p, q, 0) \leqq \phi_2(p, q, \lambda, g) - \phi_2(p, q, 0, g)$
$\Longrightarrow\ \frac{1}{\lambda}(g(\phi_1(p, q, \lambda) - g(\phi_1(p, q, 0)) \leqq \frac{1}{\lambda}(\phi_2(p, q, \lambda, g) - \phi_2(p, q, 0, g))$
$\Longrightarrow\ \frac{1}{\lambda}(g(\phi_1(p, q, \lambda) - g(\phi_1(p, q, 0))$
$\Longrightarrow \leqq \frac{1}{\lambda}(ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta} g(\phi_1(p, q, 0))^{1/\beta} - \phi_2(p, q, 0, g))$

and, taking the limit of both sided for $\lambda \to 0^+$ (and since $\phi_1(p, q, 0) = q$), we have

$$\nabla_q g(\phi_1(p, q, 0)) \frac{\partial^+ \phi_1}{\partial \lambda}(p, q, \lambda)\bigg|_{\lambda=0} \leqq \frac{\partial^+ \phi_2}{\partial \lambda}(p, q, \lambda, g)\bigg|_{\lambda=0} \nabla_q g(\phi_1(p, q, 0)) \frac{\partial^+ \phi_1}{\partial \lambda}(p, q, \lambda)\bigg|_{\lambda=0}$$
$$\leqq \frac{\partial^+(ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta})}{\partial \lambda}\bigg|_{\lambda=0}$$

We therefore have the following.

Proposition 3.1. *We are assuming that ϕ_1, ϕ_2 have right partial derivative with respect to λ at $\lambda = 0$. Then a differentiable $(\phi_1, \phi_2) - \beta$ convex function g satisfies the inequality*

$$\phi_2(p, q, g) \geqq \nabla_q g(q) \phi_1(p, q),$$

for every $p, q \in D$, where

$$\phi_1(p, q) = \frac{\partial^+ \phi_1}{\partial \lambda}(p, q, \lambda)\bigg|_{\lambda=0}, \phi_2(p, q, f) = \frac{\partial^+ \phi_2}{\partial \lambda}(p, q, \lambda, g)\bigg|_{\lambda=0} \phi_1(p, q) = \frac{\partial^+ \phi_1}{\partial \lambda}(p, q, \lambda)\bigg|_{\lambda=0},$$
$$\phi_2(p, q, g) = \frac{\partial^+(ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta})}{\partial \lambda}\bigg|_{\lambda=0}.$$

Remark 3.1. The same result holds in a more general setting, where D is a subset of a Riemannian manifold, and the r.h.s. of (2.4) is defined as $dg_q(\phi_1(p, q))$.

It is easy to verify that a $\phi_1 - \beta$ quasi-convex function h satisfies the condition

$$h(p) \leqq h(q) \implies \nabla_q h(q) \phi_1(p, q) \leqq 0,$$

for every $p, q \in D$.

Definition 3.1. *Let $\psi : D \times D \to D$. We say that ψ is skew-symmetric on $D \times D$ if $\psi(p, q) = -\psi(q, p)$ for every (p,q) $\in D \times D$.*

Corollary 3.1. *(To Proposition* 3.1*) Suppose that g is differentiable and $(\phi_1, \phi_2) - \beta$ convex; if ϕ_1, ϕ_2 are related to ϕ_1, ϕ_2 as in (Proposition* 3.1*), and skew-symmetric for any (p,q) $\in D \times D$, then ∇g is $\phi_1 - \beta-$monotone on D, i.e.*

$$(\nabla_p g(p) - \nabla_q g(q)) \phi_1(p, q) \geqq 0 \quad \forall (p, q) \in D \times D.$$

Proof. By (Proposition 3.1) we have that

$$\phi_2(p, q, g) \geqq \nabla_q g(q) \phi_1(p, q) \quad \phi_2(q, p, g) \geqq \nabla_p g(p) \phi_1(q, p)$$

and the conclusion follows from the skew-symmetry.

The local condition expressed by (Proposition 3.1) is usually not sufficient to guarantee the $(\phi_1, \phi_2) - \beta-$ convexity of g, unless we specify some more restrictive and global properties of the functions ϕ_1 and ϕ_2. Indeed, consider $\phi_1(p, q, \lambda) = q + \lambda \eta(p, q)$, $\phi_2(p, q, \lambda, g) = (1 - \lambda) g(q) + \lambda g(p)$ □

In Mohan and Neogy provided a counterexample, showing that the condition

$$f(x) - f(y) \geqq \nabla_y f(y)\eta(x, y)$$

does not imply in general that

$$g(q + \lambda\eta(p, q)) \leqq (1 - \lambda)g(q) + \lambda g(p) \quad \forall\lambda \in [0, 1].$$

We will assume that the function g is differentiable on D. The following results relate the necessary condition for a differentiable $\phi_1, \phi_2)-\beta$-convex function g, and the definition of $\phi_1, \phi_2)-\beta-$convexity. In the first result we assume that a "regularity condition" is satisfied by ϕ_1, whereas ϕ_2 is the usual r.h.s. of the definition of convexity, providing a slight extension of the ordinary convex case.

Proposition 3.2. *Assume that ϕ_1 is differentiable with respect to λ in $[0, 1]$: if the following are satisfied*

(i) $\phi_1(p, q, 0) = q, \phi_1(p, q, 1) = p$;

(ii) $\frac{\partial\phi_1}{\partial v}(p, q, v)(t' - v) = \phi_1(\phi_1(p, q, t'), (\phi_1(p, q, v))$;

(iii) $\phi_2(p, q, \lambda, g) = (1 - \lambda)g(q) + \lambda g(p)$

for every $p, q \in D$, $v, t', \lambda \in [0, 1]$, then a function f satisfying (Proposition 3.1) is $\phi_1, \phi_2 - \beta$-convex.

Proof. By Proposition 3.1 and condition (i), it follows that $g(p) - g(q) \geqq \nabla_q g(q)\phi_1(p, q)$, and for every $p, q \in D$, we get that the function $h(w) = f(\phi_1(p, q, w))$ is convex ; indeed

$$\begin{aligned} h(t') - h(v) &= g(\phi_1(p, q, t')) - g(\phi_1(p, q, v)) \\ &\geqq \nabla_{\phi_1} g(\phi_1(p, q, v))\phi_1(\phi_1(p, q, t'), (\phi_1(p, q, v)) \text{ (by Proposition 3.1)} \\ &= \nabla_{\phi_1} g(\phi_1(p, q, v))\frac{\partial\phi_1}{\partial v}(p, q, v)(t' - v) \text{ (by (ii))} \\ &\quad h'(v)(t' - v). \end{aligned}$$

It follows that h is convex. Hence, $h(\lambda) \leqq (1 - \lambda)h(0) + \lambda h(1)$. Now by hypotheses (i) and (ii) we get that

$$g(\phi_1(p, q, \lambda) \leqq (1 - \lambda)g(q) + \lambda g(p) = \phi_2(p, q, \lambda, g),$$

(see [26], where a special case of the Proposition 3.2 is proved).

More generally, the following result relating Proposition 3.1 and $\phi_1, \phi_2 - \beta-$convexity holds. □

Theorem 3.1. *We are assuming that, g is a differentiable function on D, where D is a ϕ_1- convex subset of $\Re^n$. Let $\phi_i (i = 1, 2)$ be the function associated with ϕ_i as in (Proposition 3.1). Then we are assuming that there exists a function $H' : \Re \times \Re \times [0, 1] \to \Re$, $H' = H'(w, t', \lambda)$, and the following conditions are satisfied:*

(i) $H'(\phi_2(p, \phi_1(p, q, \lambda), g), \phi_2(q, \phi_1(p, q, \lambda), g), \lambda) \leqq \phi_2(p, q, \lambda, g) - g(\phi_1(p, q, \lambda))$;

(ii) *H' is non decreasing in (w,t'), for every λ fixed (if $w_1 \leqq w_2, t'_1 \leqq t'_2$. we have that $H'(w_1, w_1, \lambda) \leqq H'(w_2, t'_2, \lambda)$);*

(iii) $H'(\nabla_{\phi_1}(g(\phi_1(p, q, \lambda))\phi_1(p, \phi_1(p, q, \lambda)), \nabla_{\phi_1}(g(\phi_1(p, q, \lambda))\phi_1(q, \phi_1(p, q, \lambda)) = 0$ *for every* $\lambda \in [0, 1]$, $g \in G$, $p, q \in D$;

(iv) $\phi_2(p, r, g) \geqq \nabla_r g(r)\phi_1(p, r)$, $\forall p, r \in D$.

Then g is $(\phi_1, \phi_2) - \beta-$convex on D.

Proof. From (iv), with $r = \phi_1(p, q, \lambda)$ we have that

$$\phi_2(p, \phi_1(p, q, \lambda), g) \geqq \nabla_{\phi_1}(g(\phi_1(p, q, \lambda))\phi_1(p, \phi_1(p, q, \lambda)),$$
$$\phi_2(q, \phi_1(p, q, \lambda), g) \geqq \nabla_{\phi_1}(g(\phi_1(p, q, \lambda))\phi_1(q, \phi_1(p, q, \lambda)).$$

Let $w = \phi_2(p, \phi_1(p, q, \lambda), g), t' = \phi_2(q, \phi_1(p, q, \lambda), g)$; from (ii) and (iii), we get that

$$H'(w, t', \lambda) \geqq H'(\nabla_{\phi_1} g(\phi_1(p, q, \lambda))\phi_1(p, \phi_1(p, q, \lambda)),$$

$$\nabla_{\phi_1} g(\phi_1(p,q,\lambda))\phi_1(q,\phi_1(p,q,\lambda)),\lambda) = 0,$$

Finally, by (i), we have that

$$\phi_2(p,q,\lambda,g) - g(\phi_1(p,q,\lambda)) \geqq 0, \qquad \forall p,q \in D, \lambda \in [0,1],$$

that is g is $(\phi_1,\phi_2)-\beta-$convex. Notice that Condition C in [27] is a particular case of Theorem 3.1 where $\phi_1(p,q,\lambda) = q + \lambda\eta(p,q), \phi_2(p,q,\lambda,g) = (1-\lambda)g(q) + \lambda g(p)$, and $H'(w,t',\lambda) = \lambda w + (1-\lambda)t'$. □

Example 3.1. Let $D \subset \Re$ be the set $D = [0,2]$, and $f : D \to [0,1]$ be the function defined as follows:

$$f(x) = \begin{cases} \sqrt{x}, & if x \in [0,1] \\ \sqrt{2-x}, & if x \in (1,2] \end{cases}.$$

Define the functions $\phi_1 : D \times D \times [0,1]$ and $\phi_2 : D \times D \times [0,1] \times F$ as follows:

$$\phi_1(x,y,\lambda) = \begin{cases} (1-\lambda)(2(2-\sqrt{y})) + \lambda(2-\sqrt{x}), & if xy > 0 \\ 2(2-\sqrt{y}), & if xy < 0 \end{cases},$$

$$\phi_2(x,y,\lambda,f) = \begin{cases} f(2(2-\sqrt{y})), & if \lambda = 0 \\ max\{f(2-\sqrt{x}), f(2(2-\sqrt{y}))\}, & if 0 < \lambda \leq 1 \end{cases}.$$

This function is $(\phi_1,\phi_2)-\beta-$ convex which verifies our results.

Proposition 3.3. *If $g : D \to \Re$ is differentiable, and Φ_1 satisfies assumptions (ii) and (iii) in Theorem* 3.1*, then g is Φ_1-quasi-convex if and only if (Remark* 3.1*) holds.*

Proof. Similar to the proof given in [27].

Under appropriate assumptions on Φ_2, a differentiable $(\phi_1,\phi_2)-\beta-$convex function, turns out to be invex, and we can guarantee that a stationary point is a global minimum. Here is a sufficient condition. Assume that ϕ_2 satisfies the inequality

$$c(p,q,\lambda,g)\phi_2(p,q,\lambda,g) \leqq (1-\lambda)g(q) + \lambda g(p), \tag{3.1}$$

for all $p,q \in D, \lambda \in [0,1], g \in G$, and for some function $c = c(p,q,\lambda,g) : D \times D \times [0,1] \times G \to \Re$, with $c(p,q,0,g) = 1$, $\frac{\partial c}{\partial \lambda}(p,q,\lambda,g)\big|_{\lambda=0} = 0$. Then we have the following. □

Proposition 3.4. *Let g be a differentiable $(\phi_1,\phi_2)-\beta-$convex function, where ϕ_1 and ϕ_2 are differentiable with respect to λ at $\lambda = 0$, for every $p,q \in D$. Assume that condition* (3.1) *holds. Then g is invex with respect to $\eta(p,q) = \phi_1(p,q)$. In particular, every stationary point of g is a global minimum.*

Proof. From (3.1), we have that

$$(1-\lambda)g(q) + \lambda g(p) - g(q) \geqq c(p,q,\lambda,g)\phi_2(p,q,\lambda,g) - c(p,q,0,g)\phi_2(p,q,0,g).$$

Adding and subtracting $c(p,q,0,g)\phi_2(p,q,\lambda,g)$ to the right hand side of the above inequality and then dividing both sides by λ and taking the limit $\lambda \to 0^+$, we get

$$g(p) - g(q) \geqq \frac{\partial c}{\partial \lambda}(p,q,\lambda,t')\bigg|_{\lambda=0} \phi_2(p,q,0,g) + c(p,q,0,g)\phi_2(p,q,g) = \phi_2(p,q,g)$$

Since, by Proposition 3.1, $\phi_2(p,q,g) \geqq \nabla_q g(q)\phi_1(p,q)$, we have that

$$g(p) - g(q) \geqq \phi_2(p,q,g) \geqq \nabla_q g(q)\phi_1(p,q).$$

This proves that g is $\phi_1(p,q)-$ invex and hence every stationary point is a global minimum point.

Now we will assume that $g : D \to R$ and $\phi_1 : D \times D \times [0,1] \to D$ satisfy the assumptions

(i) $g \in C^2(D)$;

(ii) $\phi_1(p,q) \in C^2([0,1])$.

After that we have the following sufficient condition for $(\phi_1, \phi_2) - \beta-$ convexity: $\square$

Proposition 3.5. *In our previous assumptions, g is $(\phi_1, \phi_2) - \beta-$ convex for every $\phi_2(p, q, t', g) = \int_0^{t'} g(p, q, w)dw + g(q)$, where h is any solution of the differential inequality*

$$\frac{\partial h}{\partial t'}(p, q, t') \geqq \left(\frac{\partial \phi_1}{\partial t'}\right)^{T'}(p, q, t')H'_{\phi_1} g(\phi_1(p, q, t'))\frac{\partial \phi_1}{\partial t'}(p, q, t')$$
$$+\nabla_{\phi_1} g(\phi_1(p, q, t'))\frac{\partial^2 \phi_1}{\partial t'}(p, q, t')$$
$$h(p, q, 0) = \nabla_q g(q)\phi_1(p, q) \tag{3.2}$$

$\left(H'_{\phi_1}\right.$ *denotes the Hessian of the function* ϕ_1*, and* $\left(\frac{\partial \phi_1}{\partial t'}\right)^{T'}$ *the transpose of* $\left.\left(\frac{\partial \phi_1}{\partial t'}\right)\right)$

Proof. Consider, for every $p, q \in D$,

$$s(t') = g(\phi_1(p, q, t')) - \phi_2(p, q, t', g),$$

where $\phi_2(p, q, t', g) = \int_0^{t'} h(p, q, w)dw + g(q)$, and h satisfies (Proposition 3.4). We prove that $s(t') \leqq 0$ for every $t' \in [0, 1]$. We have that

$$s(0) = g(q) - g(q) = 0,$$
$$s^{'}(0) = \nabla_q g(q)\phi_1(p, q) - \nabla_q g(q)\phi_1(p, q) = 0,$$
$$s^{''}(t') = \left(\frac{\partial \phi_1}{\partial t'}\right)^{T'}(p, q, t')H'_{\phi_1} f(\phi_1(p, q, t'))\frac{\partial \phi_1}{\partial t'}$$
$$+\nabla_{\phi_1} g(\phi_1(p, q, t'))\frac{\partial^2 \phi_1}{\partial t'}(p, q, t') - h^{'}(t') \leqq 0.$$

Therefore, $s(t) \leqq 0$ for every $t' \in [0, 1]$, and $g(\phi_1(p, q, t') \leqq \phi_2(p, q, t', g)$ for every $p, q \in D$, $t' \in [0, 1]$. $\square$

4 Conclusions

In this paper, we established a new class of convexity named $(\phi_1, \phi_2) - \beta$-convexity. Our new class is a super class of many well-known classes.

- When we take $\phi_1(p, q, \lambda) = \lambda\eta(p, q) + q$ and $\phi_2(p, q, \lambda, g) = \lambda g(p) + (1 - \lambda)g(q)$ then it shows the result of [16]
- When we take $\phi_1(p, q, \lambda) = \gamma_{p,q}(\lambda)$ and $\phi_2(p, q, \lambda, g) = \lambda g(\gamma_{p,p}(\lambda)) + (1 - \lambda)g(\gamma_{p,q}(0))$ then it shows the result of [23]
- If $a_1(p, q, \lambda) = \lambda$ then it shows the result of [4, 19]
- When we take $\phi_1(p, q, \lambda) = N_I([p, q], \lambda)$ and $\phi_2(p, q, \lambda, g) = n_\Phi[(g(p), g(q)), \lambda]$, we see that an $(I, \Phi) - \beta$-convex function is a particular $(\phi_1, \phi_2) - \beta$-convex function, which shows the result of [24]
- If we take $\phi_1(p, q, \lambda) = \lambda p + (1 - \lambda)q$ and $\phi_2(p, q, \lambda, g) = G(g(p), g(q), \|p - q\|, \lambda) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1 - \lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta}$, we get that a G-convex function is an example of $(\phi_1, \phi_2) - \beta$-convex function, which shows the result of [25]
- If we take $\beta = 0$ then this function is convert into (ϕ_1, ϕ_2)-convex function, which shows the result of [22]

We can extend results of our paper for interval-valued function under the assumptions of $(\phi_1, \phi_2) - \beta$-convexity.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 196-201**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

A COMMON FIXED POINT THEOREM FOR FOUR LIMIT COINCIDENTLY COMMUTING SELFMAPS OF A S-METRIC SPACE

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(Received : May 19, 2022; In format: June 02, 2022; Revised: March 11, 2023; Accepted: March 16, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53122

Abstract

In this paper we prove a common fixed point theorem for four selfmaps of a S -metric space. Also we deduce a common fixed point theorem for four selfmaps of a complete S-metric space. Moreover we show that a common fixed point theorem for four selfmaps of a metric space proved by Brian Fisher follows as a particular case.
2020 Mathematical Sciences Classification: 47H10; 54H25.
Keywords and Phrases: S-metric space, Fixed point, Contractive modulus, Associated sequence for four selfmaps.

1 Introduction

Fixed point theorems are extensively studied in the literature for several reasons and one of the reason is that there are a quite number of problems in integral and differential equations, for which solutions can be equivalently formulated as a fixed point of some operator on a suitable space. In an attempt to generalize fixed point theorems proved for selfmaps of metric spaces, Dhage [2,3] has introduced generalized metric spaces called D-metric space in his *Ph.D.* thesis [1] in the year 1984 which is a landmark in the history of metric fixed point theory in higher dimensional metric spaces. As a probable modification to D-metric spaces, Sedghi, Shobe and Zhou [11] introduced D^*-metric spaces. In 2006, Mustafa and Sims [10] initiated G-metric spaces; while Sedghi, Shobe and Aliouche [12] considered S-metric spaces in 2012. Hereafter we consider, in this paper, only S-metric spaces and common fixed point theorems on such spaces.

The notion of commutativity of self maps on a metric space has been generalized to weakly commuting by Sessa [13], which is further generalized to compatibility by Jungck [9]. These common fixed point theorems on the lines of Sessa [13] and Jungck[9] are further extended to D-metric spaces by Dhage [4,5] and Dhage et al. [6] under the meaningful terminology "coincidently commuting mappings" and "limit coincidently commuting mappings."

In this paper, we establish a common fixed point theorem for four limit coincidently commuting selfmaps of a S-metric space. Further we generalize a common fixed point theorem of Fisher [8].

2 Preliminaries

Definition 2.1 ([12]). *Let X be a non empty set. By S-metric we mean a function $S : X^3 \to [0, \infty)$ which satisfies the following conditions for $x, y, z, w \in X$*
(a) $S(x, y, z) \geq 0$.
(b) $S(x, y, z) = 0$ *if and only if* $x = y = z$.
(c) $S(x, y, z) \leq S(x, x, w) + S(y, y, w) + S(z, z, w)$.
An ordered pair (X, S) is called a S-metric space.

Remark 2.1. It was shown in ([12], Lemma 2.5) that $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.

Definition 2.2 ([12]). *Let (X, S) be a S-metric space. A sequence $\{x_n\}$ in X is said to converge, if there is a $x \in X$ such that $S(x_n, x_n, x) \to 0$ as $n \to \infty$; that is, for $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(x_n, x_n, x) < \epsilon$ and in this case we write $\lim\limits_{n\to\infty} x_n = x$.*

Definition 2.3 ([12]). *Let (X, S) be a S-metric space. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.*

Definition 2.4 ([12]). *A S-metric space (X, S) is said to be complete if every Cauchy sequence in it converges to some point in X.*

Definition 2.5 ([1]). *Let (X, d) be any metric space then $S_d(x, y, z) = d(x, y)+d(y, z)+d(z, x)$ is a S-metric on X. We call this S-metric as the S-metric induced by d (we denote this by S_d).*

Remark 2.2. Let (X, d) be any metric space and S_d be the S-metric induced by d. For any sequence $\{x_n\}$ in (X, S_d) is a Cauchy sequence if and only if $\{x_n\}$ is a Cauchy sequence in (X, d). Thus (X, S_d) is complete if and only if (X, d) is complete.

Definition 2.6. *Let (X, S) be a S-metric space. If there exists sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$, then we say that $S(x, y, z)$ is continuous in x and y.*

Definition 2.7. *If g and f are self maps of a S-metric space (X, S) such that for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in X$ we have $\lim_{n\to\infty} S(gfx_n, gfx_n, fgx_n) = 0$ then g and f are said to be limit coincidently commuting.*

Trivially commuting self maps of a S-metric space are limit coincidently commuting but not conversely.

Definition 2.8 ([7]). *An upper semi-continuous nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ is called D-function if $\phi(0) = 0.\phi$ is called contractive if $\phi(t) < t$ for $t > 0$.*

Definition 2.9. *Let g, f, h and p be self maps of a S-metric space such that $g(X) \subseteq p(X)$ and $f(X) \subseteq h(X)$. Then for any $x_0 \in X$, if $\{x_n\}$ is a sequence in X such that $gx_{2n} = px_{2n+1}$ and $fx_{2n+1} = hx_{2n+2}$ for $n \geq 0$, then $\{x_n\}$ is called an associated sequence of x_0 relative to self maps g, f, h and p.*

3 Main Result

Theorem 3.1. *Let g, f, h and p be self maps of a S- metric space (X, S) satisfying the following conditions*

(i) $g(X) \subseteq p(X)$ *and* $f(X) \subseteq h(X)$,

(ii) $S(gx, gx, fy) \leq \phi(\mu(x, y))$ *for all* $x, y \in X$ *where ϕ is a contractive D-function and* $\mu(x, y) = \max\{S(hx, hx, py), S(hx, hx, gx), S(py, py, fy)\}$ *for* $x, y \in X$,

(iii) one of g, f, h and p is continuous
and

(iv) the pairs (g, h) and (f, p) are limit coincidently commuting.
Further if

(v) there exists a point $x_0 \in X$ and an associated sequence $\{x_n\}$ relative to selfmaps such that $gx_0, fx_1, gx_2, fx_3, \cdots, gx_{2n}, fx_{2n+1} \cdots$ converges to some $z \in X$,

then g, f, h and p have a unique common fixed point $z \in X$. Also there is no other common fixed point for g and h; and that there is no other common fixed point for f and p.

Before proving the theorem, we establish some Lemmas which are noteworthy.

Lemma 3.1. *Suppose that g, f, h and p are self maps of a S-metric space satisfying the conditions (i),(ii) and (v) of* Theorem 3.1 *with the pair (g, h) is limit coincidently commuting. Then*
(a) $\lim_{n\to\infty} \mu(hx_{2n}, x_{2n+1}) = S(hz, hz, z)$ *whenever h is continuous,*
(b) $\lim_{n\to\infty} \mu(gx_{2n}, x_{2n+1}) = S(gz, gz, z)$ *whenever g is continuous.*

Proof. In view of (v), the sequences $\{gx_{2n}\}$ and $\{fx_{2n+1}\}$ converge to some $z \in X$ and since $gx_{2n} = px_{2n+1}$ and $fx_{2n+1} = hx_{2n+2}$, we have

$$gx_{2n}, fx_{2n+1}, hx_{2n}, p_{2n+1} \to z \text{ as } n \to \infty. \tag{3.1}$$

(a) If h is continuous, then we have

$$h^2x_{2n} \to hz, hgx_{2n} \to hz \text{ as } n \to \infty. \tag{3.2}$$

Also the limit coincidently commutativity of the pair (g, h) implies

$$\lim_{n\to\infty} S(ghx_{2n}, ghx_{2n}, hgx_{2n}) = 0. \tag{3.3}$$

From (3.2) and (3.3) we get

$$ghx_{2n} \to hz \text{ as } n \to \infty. \tag{3.4}$$

Now from (ii), we have

$$\begin{aligned}\mu(hx_{2n}, x_{2n+1}) = \max\{&S(h^2x_{2n}, h^2x_{2n}, px_{2n+1}),\\ &S(h^2x_{2n}, h^2x_{2n}, ghx_{2n}), S(px_{2n+1}, px_{2n+1}, fx_{2n+1})\}.\end{aligned} \tag{3.5}$$

Letting $n \to \infty$ in (3.5) and using the continuity of S(x, y, z) in x and y and (3.1), (3.2) and (3.4), we get $\lim\limits_{n\to\infty} \mu(hx_{2n}, x_{2n+1}) = \max\{S(hz, hz, z), S(hz, hz, hz), S(z, z, z)\} = S(hz, hz, z)$
This proves (a).

(b) If g is continuous, by (3.1) we have

$$g^2x_{2n} \to gz, ghx_{2n} \to gz \text{ as } n \to \infty. \tag{3.6}$$

Therefore in view of (3.3), we get

$$hgx_{2n} \to gz. \tag{3.7}$$

Now we have

$$\begin{aligned}\mu(gx_{2n}, x_{2n+1}) &= \max\{S(hgx_{2n}, hgx_{2n}, px_{2n+1}), S(hgx_{2n}, hgx_{2n}, gx_{2n+1}), S(px_{2n+1}, px_{2n+1}, fx_{2n+1})\}\\ &= \max\{S(gz, gz, z), S(gz, gz, z), S(z, z, z)\}\\ &= S(gz, gz, z).\end{aligned} \tag{3.8}$$

This proves (b). ☐

Lemma 3.2. *Suppose that g, f, h and p are self maps of a S-metric space (X,S) such that the pair (f,p) is limit coincidently commuting and the conditions (i),(ii) and (v) of* Theorem 3.1, *then*
(a) $\lim\limits_{n\to\infty} \mu(x_{2n}, px_{2n+1}) = S(z, z, pz)$ *whenever p is continuous,*
(b) $\lim\limits_{n\to\infty} \mu(x_{2n}, fx_{2n+1}) = S(z, z, fz)$ *whenever f is continuous.*

Proof. The proof of Lemma 3.2 is similar to the proof of Lemma 3.1 with appropriate changes. ☐

Proof of Theorem 3.1.

We first establish the existence of a common fixed point in case if h is continuous.
The proof is similar in other cases of condition (iii) of Theorem 3.1 with suitable changes.
Suppose that h is continuous.
Taking $x = hx_{2n}$ and $y = x_{2n+1}$ in condition (ii) of Theorem 3.1, we have

$$S(ghx_{2n}, ghx_{2n}, fx_{2n+1}) \leq \phi(\mu(hx_{2n}, x_{2n+1})). \tag{3.9}$$

Also the continuity of S(x, y, z) in x and y gives

$$S(hz, hz, z) = \lim_{n\to\infty} S(ghx_{2n}, ghx_{2n}, fx_{2n+1}).$$

Therefore by Lemma 3.1, we get

$$\begin{aligned}S(hz, hz, z) &= \limsup_{n\to\infty} S(ghx_{2n}, ghx_{2n}, fx_{2n+1})\\ &\leq \limsup_{n\to\infty} \phi(\mu(hx_{2n}, x_{2n+1}))\\ &= \phi(\limsup_{n\to\infty} \mu(hx_{2n}, x_{2n+1}))\\ &= \phi(\lim_{n\to\infty} \mu(hx_{2n}, x_{2n+1}))\\ &= \phi(S(hz, hz, z)).\end{aligned} \tag{3.10}$$

Hence

$$S(hz, hz, z) \leq \phi(S(hz, hz, z). \tag{3.11}$$

We now claim that $hz = z$.
In fact, if $hz \neq z$, then $S(hz, hz, z) > 0$ so that $\phi(S(hz, hz, z)) < S(hz, hz, z)$ and this contradicts (3.11),

therefore $hz = z$.
Now the continuity of S(x, y, z) in x and y gives

$$\begin{aligned} S(gz, gz, z) &= \lim_{n\to\infty} S(gz, gz, fx_{2n+1}) \\ &= \limsup_{n\to\infty} S(gz, gz, fx_{2n+1}). \end{aligned}$$

Using condition (ii) of Theorem 3.1 and the upper semicontinuity of ϕ in the above, we get

$$\begin{aligned} S(gz, gz, z) &\le \limsup_{n\to\infty} \phi(\mu(z, x_{2n+1})) \\ &= \phi(\limsup_{n\to\infty} \mu(z, x_{2n+1})). \end{aligned} \tag{3.12}$$

But

$$\begin{aligned} \lim_{n\to\infty} \mu(z, x_{2n+1}) &= \lim_{n\to\infty} \max\{S(hz, hz, px_{2n+1}, S(hz, hz, gz), S(px_{2n+1}, px_{2n+1}, fx_{2n+1})\} \\ &= \max\{S(hz, hz, z), S(z, z, gz), S(z, z, z)\} \\ &= S(z, z, gz) = S(gz, gz, z), \text{ since } hz = z, px_{2n+1} \to z \text{ and } fx_{2n+1} \to z \text{ as } n \to \infty. \end{aligned}$$

Therefore we get

$$S(gz, gz, z) \le \phi(S(gz, gz, z)). \tag{3.13}$$

If $gz \ne z$ then $S(gz, gz, z) > 0$ and by the definition of ϕ we get $\phi(S(gz, gz, z) < S(gz, gz, z)$, contradicting (3.13),hence $gz = z$
Thus we have $gz = hz = z$.
Now since $g(X) \subseteq p(X)$, there is a $u \in X$ with $z = gz = pu$ and we have $gz = hz = pu = z$.
We now claim that $fu = z$.
In fact if $fu \ne z$, then $S(z, z, fu) > 0$ and therefore by (ii) of Theorem 3.1 we get

$$\begin{aligned} S(z, z, fu) = S(gz, gz, fu) &\le \phi(\mu(z, u)) \\ &= \phi(\max\{S(gz, gz, pu), S(hz, hz, gz), S(pu, pu, fu)\}) \\ &= \phi(S(z, z, fu)), \end{aligned}$$

since $gz = hz = pu = z$ and above result implies $S(z, z, fu) \le \phi(S(z, z, fu)) < S(z, z, fu)$ which is contradiction. Therefore $fu = z$.
Hence we have $gz = hz = pu = fu = z$.
Now taking $y_n = u$ for all $n \ge 1$, it follows that $fy_n \to fu = z$ and $py_n \to pu = z$ as $n \to \infty$.
Also since the pair (f, p) is limit coincidently commuting,
we have $\lim_{n\to\infty} S(fpy_n, fpy_n, pfy_n) = 0$ which gives $S(fpu, fpu, pfu) = 0$ implies $fpu = pfu$ so that $fz = pz$.
Now by condition (ii) of Theorem 3.1, we have

$$\begin{aligned} S(z, z, fz) = S(gz, gz, fz) &\le \phi(\mu(z, z)) \\ &= \phi(\max\{S(hz, hz, pz), S(hz, hz, gz), S(pz, pz, fz)\} \\ &= \phi(S(hz, hz, fz)) = \phi(S(z, z, fz)), \end{aligned} \tag{3.14}$$

since $gz = hz = z$ and $pz = fz$.
Therefore we get $S(z, z, fz) \le \phi(S(z, z, fz))$ which yields $fz = z$.
Hence $gz = hz = pz = fz = z$, proving z is a common fixed point of g, f, h and p.
Now we prove uniqueness of common fixed point.
If possible let $z'(\ne z)$ be another common fixed point of g, f, h and p.
Then from condition (ii) of Theorem 3.1 we have

$$S(z, z, z') = S(gz, gz, gz') \le \phi(\mu(z, z')). \tag{3.15}$$

Since $\mu(z, z') = S(z, z, z')$ from (ii) of Theorem 3.1, (3.15) gives $S(z, z, z') \le \phi(S(z, z, z'))$ and this will be a contradiction if $z \ne z'$.
Hence z is unique common fixed point of f, g, h and p.
Now we prove that z is unique common fixed point of g and h and of f and p.
Let w be another fixed point of g and h. Then $z = hz = fz = gz = pz$ and $w = hw = gw$.
Now from condition (ii) of Theorem 3.1 we have

$$S(z, z, w) = S(w, w, z) = S(gw, gw, fz) \le \phi(\mu(w, z)), \tag{3.16}$$

since $\mu(w,z) = S(w,w,z)$. Therefore (3.16) gives $S(w,w,z) \leq \phi(S(w,w,z))$ and this will be a contradiction if $w \neq z$.
Hence $w = z$
Therefore z is unique common fixed point of g and h. Similarly we can show that z is unique common fixed point of f and p.
Hence Theorem 3.1 is completely proved.

4 Common fixed point Theorem for four self maps of a complete S-metric space

Before proving the main result in this section, first we establish a preparatory Lemma.

Lemma 4.1. *Let (X,S) be a S- metric space and g, f, h and p be self maps of X such that*

(i) $g(X) \subseteq p(X)$ *and* $f(X) \subseteq h(X)$,

(ii) $S(gx,gx,fy) \leq c.\mu(x,y)$ *for all* $x,y \in X$ *where* $0 \leq c < 1$ *and*
$\mu(x,y) = \max\{S(hx,hx,py), S(hx,hx,gx), S(py,py,fy)\}$.
Further if

(iii) (X,S) *is complete, then for any* $x_0 \in X$ *and for any associated sequence* $\{x_n\}$ *relative to four self maps the sequence* $gx_0, fx_1, gx_2, fx_3, \cdots, gx_{2n}, fx_{2n+1} \cdots$ *converges to some* $z \in X$.

Proof. Suppose that g, f, h and p are self maps of a S-metric space (X,S) for which conditions (i) and (ii) holds. Let a point $x_0 \in X$ and $\{x_n\}$ be any associated sequence of x_0 relative to four selfmaps. Then since $gx_{2n} = px_{2n+1}$ and $fx_{2n+1} = hx_{2n+2}$ for all $n \geq 0$.
Note that

$$\begin{aligned}
\mu(x_{2n}, x_{2n+1}) &= \max\{S(hx_{2n}, hx_{2n}, px_{2n+1}), S(hx_{2n}, hx_{2n}, gx_{2n}), S(px_{2n+1}, px_{2n+1}, fx_{2n+1})\} \\
&= \max\{S(hx_{2n}, hx_{2n}, gx_{2n}), S(hx_{2n}, hx_{2n}, gx_{2n}), S(px_{2n+1}, px_{2n+1}, fx_{2n+1})\} \\
&= \max\{S(hx_{2n}, hx_{2n}, gx_{2n}), S(px_{2n+1}, px_{2n+1}, fx_{2n+1})\} \\
&= \max\{S(hx_{2n}, hx_{2n}, gx_{2n}), S(gx_{2n}, gx_{2n}, fx_{2n+1})\} \\
&= \max\{S(gx_{2n}, gx_{2n}, fx_{2n-1}), S(gx_{2n}, gx_{2n}, fx_{2n+1})\}.
\end{aligned}$$

This together with (ii) of Lemma 4.1 gives

$$\begin{aligned}
S(gx_{2n}, gx_{2n}, fx_{2n+1}) &\leq c\mu(x_{2n}, x_{2n+1}) \\
&\leq c \max\{S(gx_{2n}, gx_{2n}, fx_{2n-1}), S(gx_{2n}, gx_{2n}, fx_{2n+1})\}
\end{aligned}$$

and since $0 \leq c < 1$, it follows from the above inequality that

$$\max\{S(gx_{2n}, gx_{2n}, fx_{2n-1}), S(gx_{2n}, gx_{2n}, fx_{2n+1})\} = S(gx_{2n}, gx_{2n}, fx_{2n-1}).$$

Therefore

$$S(gx_{2n}, gx_{2n}, fx_{2n+1}) \leq cS(gx_{2n}, gx_{2n}, fx_{2n-1}). \tag{4.1}$$

Similarly

$$S(gx_{2n}, gx_{2n}, fx_{2n-1}) \leq cS(gx_{2n-2}, gx_{2n-2}, fx_{2n-3}). \tag{4.2}$$

From (4.1) and (4.2), we get

$$\begin{aligned}
S(gx_{2n}, gx_{2n}, fx_{2n+1}) &\leq c^2 S(gx_{2n-2}, gx_{2n-2}, fx_{2n-1}) \\
&\leq c^4 S(gx_{2n-4}, gx_{2n-4}, fx_{2n-3}) \\
&\quad \cdots \quad \cdots \quad \cdots \\
&\quad \cdots \quad \cdots \quad \cdots \\
&\leq c^{2n} S(gx_0, gx_0, fx_1) \to 0,
\end{aligned}$$

as $c^{2n} \to 0$ as $n \to \infty$ (because $c < 1$), therefore the sequence $gx_0, fx_1, gx_2, fx_3, \cdots, gx_{2n}, fx_{2n+1} \cdots$ is a Cauchy sequence in (X,S) and since X is complete, it converges to a point say $z \in X$, proving lemma. □

Theorem 4.1. *Suppose that (X,S) is a S-metric space satisfying conditions (i) to (v) of* Theorem 3.1. *Further if*
(v)' (X,S) is complete,
then g, f, h and p have a unique common fixed point $z \in X$. Also there is no other common fixed point for g and h and that there is no other common fixed point for f and p.

Proof. In view of Lemma 4.1, the condition (v) of the Theorem follows from Theorem 3.1 because of (v)', hence Theorem 4.1 follows from Theorem 3.1. □

Corollary 4.1 ([8] Theorem 2)**.** *Let g, f, h and p be self maps of a metric space (X, d) satisfying the conditions*

(i) $g(X) \subseteq p(X)$ *and* $f(X) \subseteq h(X)$,

(ii) $d(gx, fy) \leq c.\mu_0(x, y)$ *for all* $x, y \in X$ *where*
$\mu_0(x, y) = \max\{d(hx, py), d(hx, gx), d(py, fy)\}$ *for all* $x, y \in X$ *and* $0 \leq c < 1$,

(iii) one of g, f, h *and* p *is continuous and*

(iv) $gh = hg$ *and* $fp = pf$.
Further if

(v) X *is complete.*

Then the four self maps g, f, h and p have a unique common fixed point. Also there is no other common fixed point for g and h and that there is no other common fixed for f and p.

Proof. Given that (X, d) is a metric space satisfying conditions (i) to (v) of Corollary 4.1.
Defining $S(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ for $x, y, z \in X$, it follows that (X, S) is a S-metric space. Also condition (ii) can be written as $S(gx, gx, fy) \leq c\mu(x, y)$ for all $x, y \in X$,
where $\mu(x, y) = \max\{S(hx, hx, py), S(hx, hx, gx), S(py, py, fy)\}$ which is the same as condition(ii) of Theorem 4.1.
Since (X, d) is complete, we have (X, S) is complete by Remark 2.2.
Now g, f, h and p are self maps of S-metric space (X, S) satisfying conditions of Theorem 4.1 and hence Corollary 4.1 follows from Theorem 4.1. □

5 Conclusion

we proved a common fixed point theorem for four limit coincidently commuting selfmaps of a S-metric space. Also we deduced a common fixed point theorem for four limit coincidently commuting selfmaps of a complete S-metric space. Moreover a common fixed point theorem for four self maps of a metric space proved by Brian Fisher follows as a particular case of our theorem.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 202-206**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

COMBINATORIAL PROOFS OF SOME IDENTITIES INVOLVING FIBONACCI AND LUCAS NUMBERS

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(Received: February 21, 2023; In format: March 02, 2023; Accepted: April 02, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53123

Abstract

Using combinatorial methods, we obtain some identities, involving binomial coefficients, for Fibonacci and Lucas numbers. We define a set and show that the cardinality of this set equals Fibonacci number. We discuss some properties of this set. Technique has been extended to obtain results for Lucas numbers.
2020 Mathematical Sciences Classification: 11B39, 11B65, 11B75
Keywords and Phrases: Fibonacci numbers; Lucas numbers; Binomial coefficients; Cardinality of a set

1 Introduction

A recursively defined sequence of positive integers that has been extensively studied is the well-known Fibonacci sequence $\{F_n\}$. Fibonacci sequence has been extended in many directions depending upon its recurrence relation as well as seed values[6, 8]. This sequence has wonderful and amazing properties and has found to be useful in different fields of knowledge[2, 4, 5, 7]. In this paper we look at the following application of Fibonacci numbers in a different manner.

Let us suppose that there are six steps with ground being first step and top being sixth. A person standing on the top (sixth step) wants to come down on the ground (first step) with the restriction that at a time he can take either one or two steps only. *In how many ways he can come to the ground* ? It is known that this can be done in F_6 ways. In [1], this has been established by the method of tilng. We shall arrive at the answer by using a novel approach.

We first introduce some terms and notations to be used.

Terms and Notations.

1.1 For a positive integer n, let Ω_n denotes set of tuples $(u_1, u_2, \cdots, u_k)$ of natural numbers with the property that $u_1 = n, u_k = 1$ and $0 < u_i - u_{i+1} \leq 2, 1 \leq i \leq k-1$.

1.2 Let $|\Omega_n|$ denotes the cardinality of the set Ω_n.

1.3 Let Rank Ω_n denotes the number of tuples $(u_1, u_2, \cdots, u_k)$ in Ω_n such that exactly even number of u_i's are odd.

1.4 For $\lambda = (u_1, u_2, \cdots, u_k) \in \Omega_n$, let Sign $\lambda = (-1)^{(u_1+u_2+\cdots+u_k)}$.

1.5 Let $\wedge_n$ denotes a set of all elements η which is obtained by replacing 1 by 0 in elements of the type $(n, \cdots, 2, 1) \in \Omega_n$.

We illustrate the above defined terms by following example.

Example 1.1 Let $n = 6$. Then
$\Omega_6 = \{(6, 5, 4, 3, 2, 1),\ (6, 5, 4, 3, 1),\ (6, 5, 3, 1),\ (6, 5, 3, 2, 1),\ (6, 4, 2, 1),\ (6, 4, 3, 1),$
$(6, 4, 3, 2, 1),\ (6, 5, 4, 2, 1)\}$. Thus $|\Omega_6| = 8$ and Rank $\Omega_6 = 3$.
For $\lambda = (6, 5, 4, 3, 1) \in \Omega_6$, Sign $\lambda = -1$.
$\wedge_6 = \{(6, 5, 4, 3, 2, 0), (6, 5, 3, 2, 0), (6, 4, 2, 0), (6, 4, 3, 2, 0), (6, 5, 4, 2, 0)\}$;
$|\wedge_6| = 5$, Rank $\wedge_6 = 2$ and for $\eta = (6, 5, 3, 2, 0) \in \wedge_6$, Sign $\eta = 1$.

2 Identities involving Fibonacci numbers

In this section, we shall obtain some identities for Fibonacci numbers. The well-known Fibonacci sequence $\{F_n\}$ is defined by $F_0 = 0,\ F_1 = 1$ and for $n \geq 2,\ F_n = F_{n-1} + F_{n-2}$. F_n is called the n^{th} Fibonacci number. We first have the following proposition.

Proposition 2.1. *For $n \geq 1, |\Omega_n| = F_n$.*

Proof. Let $i_n = 1, \forall\ n \geq 0$. For $\lambda = (u_1, u_2, \cdots, u_k) \in \Omega_n$, let i_λ denote the product $i_{u_1} i_{u_2} \cdots i_{u_k}$.Using Fibonacci recurrence relation, we have for $n \geq 2$,

$$F_n = i_{n-1}F_{n-1} + i_{n-2}F_{n-2}. \tag{2.1}$$

So that $i_nF_n = i_ni_{n-1}F_{n-1} + i_ni_{n-2}F_{n-2}$. Using (2.1) with n replaced by $n-1$ and $n-2$ on the right hand side, we get
$i_nF_n = i_ni_{n-1}i_{n-2}F_{n-2} + i_ni_{n-1}i_{n-3}F_{n-3} + i_ni_{n-2}i_{n-3}F_{n-3} + i_ni_{n-2}i_{n-4}F_{n-4}$.

Continuing this way, using (2.1) repeatedly, we get

$$i_nF_n = \sum_{(\lambda \in \Omega_n)} i_\lambda F_1 + \sum_{(\lambda \in \Omega_n)} i_\lambda F_0. \tag{2.2}$$

From definitions and seed values, it follows that $F_n = \sum_{\lambda \in \Omega_n} 1 = |\Omega_n|$.

This completes the proof. □

Remark 2.1 Observe that Ω_6 is the set of all possible ways in which the task, given in our question, can be carried out. Hence the number of ways is equal to $|\Omega_6| = 8 = F_6$.

Let $\binom{n}{r}$ denote the binomial coefficient, that is $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. We give an alternative proof of the following result in ([1, 3, 8]) by using above arguments.

Proposition 2.2. *For $n \geq 1,\ F_n = \sum_{(s=0)}^{[\frac{n-1}{2}]} \binom{n-1-s}{s}$.*

Proof. For $n \geq 1$ and $\lambda = (u_1, u_2, \cdots, u_k) \in \Omega_n,$ let $\epsilon_i = u_i - u_{i+1},\ (1 \leq i \leq k-1)$.

From the construction of Ω_n, it is clear that $\epsilon_i = 1$ or 2 and that

$$n - 1 = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k-1}.$$

First let us consider the case when all ϵ_i's are equal to 1. Here we have

$$n - 1 = 1 + 1 + \cdots + 1, \ \ (n-1 \text{ summands}) \tag{2.3}$$

and there is exactly $1 (= \binom{n-1}{0})$ way to write this. Next suppose exactly one of ϵ_i is 2. Now in this case, we have $n-2$ positions with one 2 and so there are $\binom{n-2}{1}$ ways to choose position of that 2. Next, there will be $(n-3)$ positions with two 2 's. This can be achieved in $\binom{n-3}{2}$ ways.

Proceeding this way we get, in general, that exactly s number of positions will be there with $(n-1-s)$ 2's and is obtained in $\binom{n-1-s}{s}$ ways. Also $\binom{n-1-s}{s}$ will be non zero for $(n-1-s) \geq s$; that is $(n-1) \geq 2s$.

Thus, we have $|\Omega_n| = \sum_{s=0}^{[\frac{n-1}{2}]} \binom{n-1-s}{s}$. Now the result follows from Proposition 2.1. □

Next we have the following result.

Proposition 2.3. *For $n \geq 1,\ |\wedge_n| = F_{n-1}$.*

Proof. For $n \geq 1$ and $\lambda = (u_1, u_2, \cdots, u_k) \in \wedge_n$, let $\epsilon_i = u_i - u_{i+1},\ (1 \leq i \leq k-1)$. From the construction of $\wedge_n$, it is clear that $\epsilon_i = 1$ or 2 and that

$$n - 2 = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k-1}.$$

First consider the case when all ϵ_i's are equal to 1. We shall have

$$n - 2 = 1 + 1 + \cdots + 1, \ \ (n-2 \text{ summands}) \tag{2.4}$$

and there is exactly 1 $(= \binom{n-2}{0})$ way to write this. Next suppose exactly one of ϵ_i is 2. Now in this case we have $n-3$ positions with one 2 and so there are $\binom{n-3}{1}$ ways to choose position of that 2. Next there will be $(n-4)$ positions with two 2's. This can be achieved in $\binom{n-4}{2}$ ways.

Proceeding this way we get, in general, that exactly s number of positions will be there with $(n-2-s)$ 2's and is obtained in $\binom{n-2-s}{s}$ ways. Also $\binom{n-2-s}{s}$ will be non zero for $(n-2-s) \geq s$; that is $(n-2) \geq 2s$.

Thus we have $|\wedge_n| = \sum_{s=0}^{[\frac{n-2}{2}]} \binom{n-2-s}{s} = F_{n-1}$ (by Proposition 2.3). □

3 Identities involving Lucas numbers

In this section, we shall obtain some identities involving Lucas numbers. Lucas sequence $\{L_n\}$ is defined by $L_0 = 2, L_1 = 1$ and for $n \geq 2, L_n = L_{n-1} + L_{n-2}$. L_n is called the n^{th} Lucas number. We first give the following result proved alternatively in [6, 8].

Proposition 3.1. *For $n \geq 1$, $L_n = F_n + 2F_{n-1}$.*

Proof. Let $i_n = 1$ for all $n \geq 0$. For $\lambda = (u_1, u_2, \cdots, u_k) \in \Omega_n$ or $\wedge_n$, let i denote the product $i_{u_1} i_{u_2} \cdots i_{u_k}$. Now we have $L_n = L_{n-1} + L_{n-2}$, $(n \geq 2)$, which may be written as

$$L_n = i_{n-1} L_{n-1} + (n-2) L_{n-2}, \tag{3.1}$$

so that, using (3.1) with n replaced by $n-1$ and $n-2$, we get

$$\begin{aligned} i_n L_n &= i_n i_{n-1} L_{n-1} + i_n i_{n-2} L_{n-2} \\ &= i_n i_{n-1} i_{n-2} L_{n-2} + i_n i_{n-1} i_{n-3} L_{n-3} + i_n i_{n-2} i_{n-3} L_{n-3} + i_n i_{n-2} i_{n-4} L_{n-4}. \end{aligned}$$

Continuing this way, using (3.1) repeatedly, we get

$$i_n L_n = \sum_{\lambda \in \Omega_n} i_\lambda L_1 + \sum_{\lambda \in \wedge_n} i_\lambda L_0, \tag{3.2}$$

Using seed values for Lucas sequence, we get

$$\begin{aligned} i_n L_n &= \sum_{\lambda \in \Omega_n} 1 + 2 \sum_{\lambda \in \wedge_n} 1, \\ &= |\Omega_n| + 2| \wedge_n | \\ &= F_n + 2F_{n-1} \text{ (Using Propositions 2.1 and 2.4).} \end{aligned} \tag{3.3}$$

Hence the result. □

Next if G_n is the nth generalized Fibonacci or Gibonacci number satisfying the relation $G_n = G_{n-1} + G_{n-2}$, $(n \geq 2)$ with $G_0 = a$ and $G_1 = b$, then arguing as in Proposition 3.1, we get

Proposition 3.2. *For $n \geq 1$, $G_n = bF_n + aF_{n-1}$.*

4 Some Properties of Ω_n and $\wedge_n$

In this section we discuss some properties of Ω_n and $\wedge_n$. First we define a Fibonacci type sequence $\{S_n\}$.

Let $j_n = (-1)^n$, $\forall\, n \geq 0$. For $\lambda = (u_1, u_2, \cdots, u_k) \in \Omega_n$ or $\wedge_n$, let j_λ denote the product $j_{u_1} j_{u_2} \cdots j_{u_k}$. Define a sequence

$$S_n = j_{n-1} S_{n-1} + j_{n-2} S_{n-2},\ (n \geq 2) \text{ with } S_0 = 2 \text{ and } S_1 = 1, \tag{4.1}$$

which implies

$$\begin{aligned} j_n S_n &= j_n j_{n-1} S_{n-1} + j_n j_{n-2} S_{n-2} \\ &= j_n j_{n-1} j_{n-2} S_{n-2} + j_n j_{n-1} j_{n-3} S_{n-3} + j_n j_{n-2} j_{n-3} S_{n-3} + j_n j_{n-2} j_{n-4} S_{n-4}, \end{aligned}$$

where last expression is obtained by using (4.1) with n replaced by $n-1$ and $n-2$.

Continuing this way, using (4.1) repeatedly, we get

$$j_n S_n = \sum_{\lambda \in \Omega_n} j_\lambda S_1 + \sum_{\lambda \in \wedge_n} j_\lambda S_0. \tag{4.2}$$

Using seed values, we get

$$j_n S_n = \sum_{\lambda \in \Omega_n} (Sign\ \lambda) + 2 \sum_{\lambda \in \wedge_n} (Sign\ \lambda). \tag{4.3}$$

In view of (3.3), this gives the following:

Proposition 4.1. *For $n \geq 1$, $L_n + (-1)^n S_n = 2(Rank\ \Omega_n) + 4(Rank\ \wedge_n)$.*

Next we have ,

Proposition 4.2. *For $m \geq 0$, $S_{2m+1} = S_{2m+4}$.*

Proof. Note that equation (4.1) can be rewritten as

$$S_n = (-1)^n (S_{n-2} - S_{n-1}). \tag{4.4}$$

So that $S_{2m+4} = S_{2m+2} - S_{2m+3}$ and $S_{2m+3} = -S_{2m+1} + S_{2m+2}$, which in turn gives $S_{2m+1} = S_{2m+4}$.

□

Proposition 4.3. *For* $m \geq 0$, $S_{2m+1} = (-1)^m F_{m-1}$ *and* $S_{2m+4} = (-1)^m F_{m-1}$.

Proof. First note that if $S_{2m+1} = (-1)^m F_{m-1}$ is true then, by Proposition 4.2, $S_{2m+4} = (-1)^m F_{m-1}$.

For $m = 0$, since $F_{-1} = 1$, $S_1 = 1$ which is true.

Suppose $S_{2m+1} = (-1)^m F_{m-1}, \forall\, m < n$.Then

$$\begin{aligned} S_{2n+1} &= S_{2n} - S_{2n-1} \\ &= S_{2(n-2)+4} - S_{2(n-1)+1} \\ &= (-1)^{n-2} F_{n-3} - (-1)^{n-1} F_{n-2} \\ &= (-1)^n [F_{n-3} + F_{n-2}] = (-1)^n F_{n-1}. \end{aligned}$$

This completes the proof. □

5 Computation of Rank Ω_n and Rank $\wedge_n$

In this section, we shall obtain some recurrence relations for Rank Ω_n and Rank $\wedge_n$.

Proposition 5.1. *For* $m \geq 2$,
(a) Rank $\Omega_{2m} =$ *Rank* $\Omega_{2m-1}+$ *Rank* Ω_{2m-2}.
(b) Rank $\Omega_{2m-1} = F_{2m-1} - ($ *Rank* $\Omega_{2m-2}+$ *Rank* $\Omega_{2m-3})$.

Proof. Define $A_n = \{(u_1, u_2, \cdots, u_k) \in \Omega_n \mid u_1 = n \text{ and } u_2 = n-1\}$ and $B_n = \{(u_1, u_2, \cdots, u_k) \in \Omega_n \mid u_1 = n \text{ and } u_2 = n-2\}$.
Note that Ω_n is a disjoint union of A_n and B_n.

(a) If $n = 2m$, then Rank $A_n =$ Rank Ω_{n-1} and Rank $B_n =$ Rank Ω_{n-2}.
Hence Rank $\Omega_n =$ Rank A_n+ Rank $B_n =$ Rank $\Omega_{n-1}+$ Rank Ω_{n-2} as required.

(b) If $n = 2m-1$, then Rank $A_n = |\Omega_{n-1}|-$ Rank Ω_{n-1} and
Rank $B_n = |\Omega_{n-2}|-$ Rank Ω_{n-2}. Then

$$\begin{aligned} \text{Rank } |\Omega_n| &= \text{Rank } A_n + \text{Rank } B_n \\ &= (F_{n-1} - \text{Rank } \Omega_{n-1} + (F_{n-2} - \text{Rank } \Omega_{n-2}) \\ &= F_n - (\text{Rank } \Omega_{n-1} + \text{Rank } \Omega_{n-2}). \end{aligned}$$

as required. □

Proceeding in the same way as above, we can prove the following relations for Rank $\wedge_n$.

Proposition 5.2. *For* $m \geq 2$,
(a) Rank $\wedge_{2m} =$ *Rank* $\wedge_{2m-1}+$ *Rank* $\wedge_{2m-2}$.
(b) Rank $\wedge_{2m-1} = F_{2m-2} - ($ *Rank* $\wedge_{2m-2}+$ *Rank* $\wedge_{2m-3})$.

Next we have following representation for Rank Ω_n.

Proposition 5.3. *For* $m \geq 2$,
(a) Rank $\Omega_{2m} = \sum_{s=0}^{[\frac{(2m-1)}{4}]} \binom{2m-2-2s}{2s+1}$.
(b) Rank $\Omega_{2m-1} = \sum_{s=0}^{[\frac{(2m-2)}{4}]} \binom{2m-1-2s}{2s}$.

Proof. (a) If $m \geq 2$, and $\lambda = (u_1, u_2, \cdots, u_k) \in \Omega_{2m}$, let $\epsilon_i = u_i - u_{i+1}$, $(1 \leq i \leq k-1)$. From the construction of Ω_{2m} it is clear that $\epsilon_i = 1$ or 2 and that

$$2m - 1 = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k-1}.$$

First consider the case when all ϵ_i's are equal to 1. In this case we will have

$$2m - 1 = 1 + 1 + \cdots + 1, \ (n - 2 \text{ summands}) \tag{5.1}$$

and there is exactly 1 $(= \binom{2m-1}{0})$ way to write this. In this case there are odd number of odd entries. So we do not count this case. Next suppose exactly one of ϵ_i is 2. Now in this case we have $n-2$ positions with one 2 and so there are $\binom{n-2}{1}$ ways to choose position of that 2. Here there are even number of odd entries. Counting this we have the required result.
Similarly we can prove (b). □

Arguing as in above proposition, we can prove the following:

Proposition 5.4. *For $m \geq 2$,*
(a) Rank $\wedge_{2m} = \sum_{s=0}^{[\frac{(2m-2)}{4}]} \binom{2m-3-2s}{2s+1}$.
(b) Rank $\wedge_{2m-1} = \sum_{s=0}^{[\frac{(2m-3)}{4}]} \binom{2m-2-2s}{2s}$.

6 Conclusion

In this paper we have used simple combinatorial arguments to prove some known results. For this purpose we have defined two sets and some properties of these sets are discussed. The technique can be extended to other Fibonacci like numbers to obtain the known results in a simple way.

Acknowledgement. Authors are thankful to the Editor for suggestions made to improve the presentation of the paper.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 207-211**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

ON THE DIOPHANTINE EQUATIONS $x^2 + 139^m = y^n$ AND $x^2 + 499^m = y^n$

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(Received: April 17, 2022; In format July 23, 2022; Revised; April 02, 2023; Accepted : April 10, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53124

Abstract

In this paper, we consider the Diophantine equations $x^2 + 139^m = y^n$ and $x^2 + 499^m = y^n$ $n \geq 3$, $m > 0$ and determine solutions of the equations.

2020 Mathematical Sciences Classification: 11D41, 11D61.
Keywords and Phrases: Exponential Diophantine equations, integer solutions.

1 Introduction

The problem of solving the equation $x^2+7 = 2^n$ was proposed by Ramanujan [13] in 1913. This equation was solved perfectly by Nagell [11] in 1960 using techniques from algebraic number theory. In the generalized form, this equation is called generalized Ramanujan- Nagell equation $x^2 + k = y^n$, k, x, y, n belongs to integers, $n \geq 3$, a kind of exponent type equation. This equation has been studied extensively. When $n = 3$, it is an elliptic curve. Mordell studied this type of equation carefully and collected most of the important results in his book [10]. However, when $n \geq 3$, it is a hyperelliptic curve which seems to be more difficult to study, but there is now a vast body of literature on it also.

For some small positive integers k, the solutions have been determined. Lebesgue [8] and Nagell [12] showed that there are no non-trivial solutions when $k = 1$ and $k = 3, 5$, respectively.

Ljungrren [7] proved in the case of $k = 2$ that the equation has only one positive solution. Several special case of the Diophantine equation $x^2 + q^m = y^n$ where q is a prime and m, n, x and y are positive integers have been studied in the last few years. When $q = 2$ and m is an odd integer, it was proved by Cohn [5] that this equation has exactly three families of solutions. When $q = 3$, and m is an odd integer, the equation has three families of solution as proved by Arif and Abu Muriefah [1]. It was shown by Luca [9] that there exists only one family of solution when $q = 3$ and m is an even integer. Tao [14] solved the equation when $q = 5$ and showed that there is no solution. J. H. E. Cohn [6] refined the earlier elementary approaches and solved the equation for 77 values of q under 100. Using advanced methods, Bugeaud et al. [4] solved this kind of equation for $1 \leq k \leq 100$.

In this short communication, we consider the Diophantine equations $x^2+139^m = y^n$ and $x^2+499^m = y^n$, $n \geq 3$, $m > 0$ and determine solutions of the equations.

2 Main Results

Theorem 2.1. *Let m be odd. Then the Diophantine equation*

$$x^2 + 139^m = y^n, \tag{2.1}$$

has only one solution in positive integers x, y, m and the unique solution is given by $x = 322$, $y = 47$, $m = 1$ and $n = 3$.

We start by stating the following lemma which will be used further.

Lemma 2.1. *The equation $139x^2 + 1 = y^n$ where n is an odd integer ≥ 3 has no solution in integers x and y for y odd and ≥ 1.*

The proof of the Theorem 2.1 is divided into two cases $(139, x) = 1$ and $139|x$. It is sufficient to consider x a positive integer.

Proof. Suppose $m = 2k+1$, $k \geq 0$. Then equation (2.1) becomes

$$x^2 + 139^{2k+1} = y^n, \quad n \geq 3. \tag{2.2}$$

If x is odd and y is even, then $x^2 + 139^{2k+1} \equiv 4(mod\ 8)$, but $y^n \equiv 0(mod\ 8)$, which is not possible. Thus x is even and y is odd. □

Case (i) Let $(139, x) = 1$. Let n be odd , then there is no loss of generality in considering $n = p$, an odd prime.Then from [Theorem 6, [6]] we have only two possibilities and they are

$$x + 139^k\sqrt{-139} = (s + t\sqrt{-139})^p, \tag{2.3}$$

where $y = s^2 + 139t^2$, for some rational integers s and t and

$$x + 139^k\sqrt{-139} = (\frac{s + t\sqrt{-139}}{2})^3, \tag{2.4}$$

because $139 \equiv 3(mod\ 8)$, $s \equiv t \equiv 1(mod\ 2)$ where $y = (s^2 + 139t^2)/4$ for some rational integers s and t and $x = |\frac{s^3 - 417st^2}{8}|$.

In (2.3), since $y = s^2 + 139t^2$ and y is odd and so only one of s or t is odd and other is even. Equating imaginary parts of 2.3, we get

$$139^k = t\sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r+1} s^{p-2r-1}\left(-139t^2\right)^r. \tag{2.5}$$

So t is odd and s is even. Since 139 does not divide the term inside summation, we get $t = \pm 139^k$.

$$\pm 1 = \sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r+1} s^{p-2r-1}\left(-139^{2k+1}\right)^r. \tag{2.6}$$

This is equation (1) in [6] and Lemmas 4 and 5 in [6] show that both the signs are impossible. Hence (2.3) gives rise to no solution.

Now let us consider equation (2.4). By equating imaginary parts , we obtain,

$$8 \cdot 139^k = t(3s^2 - 139t^2). \tag{2.7}$$

If $t = \pm 1$ in (2.7), we have

$$\pm 8 \cdot 139^k = 3s^2 - 139. \tag{2.8}$$

When we consider $k = 0$, then $\pm 8 = 3s^2 - 139$.

First we consider negative sign,

$$-8 = 3s^2 - 139.$$

Then

$$3s^2 = 131,$$

which is not possible.

Now we consider the positive sign,

$$8 = 3s^2 - 139. \tag{2.9}$$

This implies that

$$3s^2 = 147$$

or,

$$s = \pm 7.$$

This equation has only solution when

$$t = \pm 1,\ s = \pm 7,\ k = 0 \text{ and } y = \tfrac{s^2 + 139t^2}{4} = 47.$$

Hence from (2.4), we have $x = |\frac{s^3-417st^2}{8}| = 322$.

Finally if $t = \pm 139^k$, then from equation (2.7), we have

$$\pm 8 = 3s^2 - 139^{2k+1}, \tag{2.10}$$

where $k > 0$, which is impossible modulo 139. Hence there is no solution of this equation.

Now if n is even, then it is sufficient to consider $n = 4$, hence the equation (2.2) becomes

$$x^2 + 139^{2k+1} = y^4$$

or,

$$y^4 - x^2 = 139^{2k+1}$$

or,

$$(y^2 - x)(y^2 + x) = 139^{2k+1}.$$

Since $(139, x) = 1$, we have

$$y^2 + x = 139^{2k+1} \tag{2.11}$$

and

$$y^2 - x = 1. \tag{2.12}$$

Eliminating x from equations (2.11) and (2.12), we get

$$2y^2 = 139^{2k+1} + 1.$$

Then $2y^2 \equiv 4 \ (mod\ 8)$ i.e.$y^2 \equiv 2 \ (mod\ 4)$ as y is odd, which is impossible.

Case (ii) Suppose that $139|x$, then $x = 139^u \cdot X$; so that, $139|y$, then $y = 139^v \cdot Y$, where $u > 0, v > 0$ and $(139, X) = (139, Y) = 1$. Then

$$139^{2u}X^2 + 139^{2k+1} = 139^{nv}Y^n.$$

There are following possibilities for solving this equation as discussed below:

1) $2u = min(2u, 2k+1, nv)$. Then by cancelling 139^{2u}, we get

$$X^2 + 139^{2(k-u)+1} = 139^{nv-2u}Y^n.$$

If $nv - 2u = 0$, then we get $X^2 + 139^{2(k-u)+1} = Y^n$ with $(139, X) = 1$. If $k - u = 0$, this equation has the only solution $x = 322$ and $n = 3$. If $k - u > 0$, then it has no solution.

2) $2k+1 = min(2u, 2k+1, nv)$. Then $139^{2u-2k-1} \cdot X^2 + 1 = 139^{nv-2k-1}Y^n$ and considering this equation modulo 139, which is not possible. Hence this equation has no solution.

3) $nv = min(2u, 2k+1, nv)$. Then $139^{2u-nv} \cdot X^2 + 139^{2k+1-nv} = Y^n$. This is possible modulo 139 only if $2u - nv = 0$ or $2k + 1 - nv = 0$ and both cases are not possible. This completes the proof of the theorem.

Theorem 2.2. *The equation*

$$x^2 + 499^m = y^n, \qquad n \geq 3,\ m > 0 \tag{2.13}$$

has only one solution in positive integers (x, y, m) and the solution is given by

$$x = 2158,\ y = 167,\ m = 1,\ n = 3.$$

Lemma 2.2. *The equation $499x^2 + 1 = y^n$ where n is an odd integer ≥ 3 has no solution in integers x and y for y odd and ≥ 1*

The proof of the theorem is divided into two cases $(499, x) = 1$ and $499|x$. It is sufficient to consider x a positive integer.

Proof. Let us suppose that $m = 2k + 1$. We shall assume that $k > 0$, $n > 3$.

If x is odd and y even, we get $x^2 + 499^{2k+1} \equiv 4 \ (mod\ 8)$, but $y^n \equiv 0 \ (mod\ 8)$. Hence, we suppose that x is even and y is odd. □

Case (i) Let $(499, x) = 1$. Let n be odd , then there is no loss of generality in considering $n = p$, an odd prime.Then from [[6], Theorem 6] we have only two possibilities and they are

$$x + 499^k\sqrt{-499} = (a + b\sqrt{-499})^p, \tag{2.14}$$

where

$$y = a^2 + 499b^2,$$

and

$$x + 499^k\sqrt{-499} = (a + b\sqrt{-499}/2)^3, \tag{2.15}$$

because $499 \equiv 3(mod\ 8)$, $a \equiv b \equiv 1(mod\ 2)$, where $y = \frac{a^2+499b^2}{4}$ for some rational integers a and b and $x = |\frac{a^3-1497ab^2}{8}|$.

In (2.14), since $y = a^2 + 499b^2$ and y is odd and so only one of a or b is odd and other is even. Equating imaginary parts, we get

$$499^k = b\sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r+1} a^{p-2r-1}(-499b^2)^r. \tag{2.16}$$

So b is odd and a is even. Since 499 does not divide the term inside summation, we get $b = \pm 499^k$.

$$\pm 1 = \sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r+1} a^{p-2r-1}(-499b^{2k+1})^r. \tag{2.17}$$

This is Cohn [6, eqn (1)]. Therefore, Lemmas 4 and 5 due to Cohn [6] show that both the signs are impossible. Hence (2.14) gives rise no solution.

Now let us consider equation (2.15). By equating imaginary parts, we obtain,

$$8 \cdot 499^k = b(3a^2 - 499b^2). \tag{2.18}$$

If $b = \pm 1$ in (2.18), we have

$$\pm 8 \cdot 499^k = 3a^2 - 499. \tag{2.19}$$

When we consider $k = 0$, we get $\pm 8 = 3a^2 - 499$. We consider negative sign

$$-8 = 3a^2 - 499$$

or,

$$3a^2 = -8 + 499$$

or,

$$3a^2 = 491,$$

which is not possible.

Now we consider positive sign

$$8 = 3a^2 - 499 \tag{2.20}$$

or,

$$3a^2 = 507$$

or,

$$a = \pm 13.$$

This equation has only solution when $b = \pm 1$, $a = \pm 13$, $k = 0$ and $y = \frac{a^2+499b^2}{4} = 167$. Hence from (2.15), we have $x = |\frac{a^3-1497ab^2}{8}| = 2158$. Hence $x = 2158$.

Finally if $b = \pm 499^k$, then we have

$$\pm 8 = 3a^2 - 499^{2k+1}, \tag{2.21}$$

where $k > 0$, which is impossible modulo 499. Hence there is no solution of this equation.

Now if x is even, then from the equation (2.13), it is sufficient to consider $n = 4$, hence

$$(y^2 + x)(y^2 - x) = 499^{2k+1}.$$

Since $(499, x) = 1$, we have

$$y^2 + x = 499^{2k+1}, \tag{2.22}$$

and

$$y^2 - x = 1. \tag{2.23}$$

Eliminating x from both equations (2.22) and (2.23), we get

$$2y^2 = 499^{2k+1} + 1.$$

Then $2y^2 \equiv 4\ (mod\ 8)$ i.e. $y^2 \equiv 2\ (mod\ 4)$, which is impossible.

Case (ii) Let $499|x$. Then, of course, $499|y$. Suppose that $x = 499^u \cdot X$, $y = 499^v \cdot Y$, where $u > 0, v > 0$ and $(499, X) = (499, Y) = 1$. Then

$$499^{2u} \cdot X^2 + 499^{2k+1} = 499^{nv} \cdot Y^n,$$

i) $2u = min(2u, 2k+1, nv)$. Then by cancelling 499^{2u}, we get

$$X^2 + 499^{2k+1-2u} = 499^{nv-2u} Y^n$$

If $nv - 2u = 0$, then we get $X^2 + 499^{2(k-u)+1} = Y^n$, with $(499, X) = 1$. If $k - u = 0$, this equation has only solution $x = 2158$ and $n = 3$. If $k - u > 0$, then it has no solution.

ii) $2k+1 = min(2u, 2k+1, nv)$. Then $499^{2u-2k-1} \cdot X^2 + 1 = 499^{nv-2k-1} Y^n$ and considering this equation modulo 499, we get $nv - 2k - 1 = 0$, so n is odd, $499(499^{k-u-1}X)^2 + 1 = Y^n$. By the Lemma **2.2** this equation has no solution.

iii $nv = min(2u, 2k+1, nv)$. Then $499^{2u-nv} \cdot X^2 + 499^{2k+1-nv} = Y^n$. This is possible modulo 499 only if $2u - nv = 0$ or $2k + 1 - nv = 0$ and both cases are not possible. Hence this completes the proof of the ***Theorem* 2.2**.

Acknowledgement. The authors thank to the Editor and anonymous referees for their valuable suggestions and comments to improve the paper.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 212-218**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

TRIPLE SERIES EQUATIONS INVOLVING GENERALIZED LAGUERRE POLYNOMIALS

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(Received: November 05, 2022*; In format: November* 10, 2022*' Revised: April* 04, 2023*; Accepted: April* 08, 2023*)*

DOI: https://doi.org/10.58250/jnanabha.2023.53125

Abstract

In this paper, the solutions of two sets of triple series equations involving generalized Laguerre polynomials have been obtained by reducing them to a Fredholm integral equation of second kind. In each case, the problem is reduced to the solution of a Fredholm integral equation of the second kind. We consider certain triple series equations involving generalized Laguerre polynomials which are generalization of those considerd by Lawndes-Srivastava .Connected to this work solutions have been considered by Sneddon, Lowndes and Srivastava ,Dwivedi and Trivedi, Singh *et al.*, Narain, Srivastava - Panda etc.

2020 Mathematical Sciences Classification: 45XX, 45F10, 33C45.
Keywords and Phrases: Triple Series equations, generalized Laguerre polynomials, Fredholm integral equation.

1 Introduction

The problem of dual and triple series equations arises during solving many boundary value problems in Sneddon [17, Chap.5] and Srivastava [20] of Mathematical physics. Earlier Lowndes [5-8] has also obtained solutions for some dual and triple series equations involving Jacobi and Laguerre polynomials. Chandel [3] discussed a problem on Heat conduction employing dual series equation involving Legendre polynomials. Lowndes and Srivastava [9] have shown that a certain class of triple series equations involving the generalized Laguerre polynomials can be reduced to some triple integral equations. Srivastava [18-24] and Srivastava - Panda [25] have investigated the solutions of some dual and triple series equations involving the generalized Laguerre polynomials, Bateman-k functions and the Konhauser biorthogonal polynomials. Ashour , Ismail and Mansour [1] have solved dual and triple series equations involving q-orthogonal polynomials with some examples. Recently, Mudaliar and Narain [11] have solved certain dual and quadruple series equations involving generalized Laguerre polynomials and also Narain [13] has solved triple series equations involving Laguerre polynomials with Matrix Augument. Certain quadruple series equations involving Laguerre polynomials are solved by Shrivastava and Narain [15] recently.Closed-form solutions of triple series equations involving Laguerre polynomials are recently obtained by Singh, Rokne and Dhaliwal [16]. Dwivedi and Trivedi [4] have obtained the solution of triple series equations involving Jacobi and Laguerre polynomials by reducing them to a Fredholm integral equation of second kind. We consider certain triple series equations involving generalized Laguerre polynomials which are generalization of those considered by Sneddon, Lowndes and Srivastava , Dwivedi and Trivedi, Singh *et al.*, Narain, Srivastava - Panda etc. connected to this work. In present paper, the solutions of two sets of triple series equations involving generalized Laguerre polynomials have been obtained. The triple series equations of the first kind

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)} L_n^{(\sigma)}(x) = g_1(x), 0 \leq x < a, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{A_n\,(1+H_n)}{\Gamma(\alpha+n+1)} L_n^{(\nu)}(x) = f(x), a < x < b, \tag{1.2}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{(\beta+n+1)} L_n^{(\sigma)}(x) = h_1(x), b < x < \infty, \tag{1.3}$$

and the triple series equations of the second kind

$$\sum_{n=0}^{\infty} \frac{A_n\,(1+H_n)}{\Gamma(\alpha+n+1)} L_n^{(\nu)}(x) = g(x), 0 \leq x < a, \tag{1.4}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)} L_n^{(\sigma)}(x) = f_1(x), a < x < b, \tag{1.5}$$

$$\sum_{n=0}^{\infty} \frac{A_n\,(1+H_n)}{\Gamma(\alpha+n+1)} L_n^{(\nu)}(x) = h(x), b < x < \infty, \tag{1.6}$$

where, A_n is an unknown coefficient, $L_n^{(\alpha)}(x)$ is the generalized Laguerre polynomial $f(x), f_1(x), g(x), g_1(x), h(x)$ and $h_1(x)$ are known functions of x and the parameters $\alpha, \beta, \nu, \sigma$ all are > -1; can be reduced to that of solving a Fredholm integral equation of second kind. It is assumed that the series (1.1) to (1.6) are uniformly convergent and the known functions f, f_1, g, g_1, h, h_1 and their derivatives are continuous bounded and integrable in the interval of their definition.

The analysis throughout is formal and no attempt has been made to justify the various limiting processes.

2 Some Useful Results

Here are some useful results for ready reference:

The orthogonality relation for the Laguerre polynomials is

$$\int_0^{\infty} x^{\alpha} \quad e^{-x} L_m(\alpha;x) L_n(\alpha;x) dx = \frac{\Gamma(\alpha+1+n)}{\Gamma(n+1)} \delta_{m,n}, \alpha > -1, \tag{2.1}$$

where $\delta_{m,n}$ is Kronecker delta.

From equations (2.6) and (3. 7) due to Srivastava ([18],p.589 and p.591]) it is easily shown that

$$(\lambda)\Gamma(1-\lambda)S(r,x) = \Gamma(\lambda)\Gamma(1-\lambda) r^{\sigma} x^{\nu} \sum_{n=0}^{\infty} \frac{\Gamma(\beta+n+1)}{\Gamma(\alpha+n+1)} . \frac{\Gamma(n+1)}{\Gamma(\sigma+1+n)} L_n^{(\sigma)}(r) L_n^{(\nu)}(x), \tag{2.2}$$

$$= a_n^* \int_0^t n(\xi)(r-\xi)^{\lambda-1}(x-\xi)^{\lambda+\nu-\sigma-1} d\xi = a_n^* S_t(r,x), \tag{2.3}$$

where $n(\xi) = e^{\xi}.\xi^{\sigma-\lambda}, t = min(r,\ x)$

$\alpha, \beta, \sigma < -1, \lambda + \nu > \sigma$ and

$$a_n^* = \frac{\Gamma(1-\lambda)\Gamma(\beta+n+1)\Gamma(\nu+n+1)}{\Gamma(\lambda+\nu-\sigma)\Gamma(\alpha+n+1)\Gamma(\sigma-\lambda+n-1)}.$$

It is further assumed that the parameters $\alpha, \beta, \lambda, \nu$ and σ are so constrained that a_n^* is independent of n. This of course is possible when, for instance $\alpha = \nu, \lambda = \sigma - \beta$, the parameter β and σ remains free.

3 Solution of the Equations of First Kind

Let us assume that

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)} L_n^{(\sigma)}(x) = \varphi(x), a < x < b, \tag{3.1}$$

where, $\beta > -1$ and $\varphi(x)$ is bounded and integrable in the interval (a, b). On making use of the orthogonality relation (2.1), we find that

$$A_n = \frac{\Gamma(\beta+n+1)\Gamma(n+1)}{\Gamma(\alpha+n+1)} \int_a^b r^{\sigma} e^{-r} L_n^{(\sigma)}(r)\phi(r) dr,\ n = 0,\ 1,\ 2,\ \ldots; \tag{3.2}$$

provided $\beta > -1$ and $\sigma > -1$. Substituting for A_n in eqn. (1.2) and since the series in eqn. (1.2) is uniformly convergent, we can change the order of the summation and integration and thus we have

$$\int_a^b e^{-r}\phi(r)S(r,x)dr \ + \int_a^b e^{-r}\phi(r)T(r,x)dr = x^{\nu} f(x), a < x < b, \tag{3.3}$$

where, $S(r,x)$ is defined by eqn. (2.2) and

$$T(r,x) = r^{\sigma} x^{v} \sum_{n=0}^{\infty} \frac{\Gamma(\beta+n+1)}{\Gamma(\sigma+1+n)} \cdot \frac{\Gamma(n+1)}{\Gamma(\sigma+1+n)} H_{n} L_n^{(\sigma)}(r) L_n^{(\nu)}(x). \tag{3.4}$$

Using the notation of eqn. (2.3) this can be written as:

$$\int_a^x e^{-r}\phi(r) S_r(r,x)dr + \int_x^b e^{-r}\phi(r) S_x(r,x)dr + \int_a^b e^{-r}\phi(r) T(r,x)dr = \frac{x^{\nu} f(x)\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^*}, \quad a < x < b \tag{3.5}$$

provided $\alpha, \beta, \sigma < -1, 0 < \lambda < 1, \nu + \lambda > \sigma$.

Inverting the order of integration in Carslow [2, eqn. (3.5)], we get

$$\int_a^x \frac{n(\xi)}{(x-\xi)^{1+\sigma-\lambda-\nu}} \overline{\phi}(\xi) d\xi + \int_a^b e^{-r}\phi(r) T(r,x)dr =$$
$$\frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^*} x^{\nu} f(x) - \int_0^a \frac{n(\xi)}{(x-\xi)^{1+\sigma-\lambda-\nu}} d\xi \int_a^b \frac{e^{-r}\phi(r)}{(r-\xi)^{1-\lambda}} dr, a < x < b, \tag{3.6}$$

where

$$\phi(\xi) = \int_{\xi}^b \frac{e^{-r}\phi(r)}{(r-\xi)^{1-\lambda}} dr, a \le \xi < b, \tag{3.7}$$

provided $\alpha, \beta, \sigma \underline{>} -1, 0 < \lambda < 1, 0 < 1 - \lambda - \nu + \sigma < 1$ and $\phi(r)$ being continuous and integrable in (a,b).

If $\phi(\xi)$ and $\overline{\phi(\xi)}$ are continuous in $a \le \xi \le b$ and $0 < \lambda < 1$, then (3.7) is an Abel integral equation and its solution is given by

$$e^{-r}\phi(r) = -\frac{\sin(1-\lambda)\pi}{\pi} \frac{d}{dr} \int_r^b \frac{\overline{\phi}(\xi)}{(\xi-r)^{\lambda}} d\xi. \tag{3.8}$$

Similarly, when $\sigma, \beta, \sigma > -1, 0 < \lambda < 1, 0 < 1+\sigma-\lambda-\nu < 1$ and $f(x), f'(x)$ are continuous in $a \le x \le b$, then from eqn. (3.1) and (3.3), we have

$$n(\xi)\overline{\phi(\xi)} + \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi} \int_a^b e^{-r}\phi(r)dr \frac{d}{d\xi} \int_a^{\xi} \frac{T(r,x)dx}{(\xi-x)^{\lambda+\nu-\sigma}}$$
$$= F(\xi) - \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi} \int_0^a n(\xi) l(\xi,n) d\eta. \int_a^b \frac{e^{-r}\phi(r)dr}{(r-\eta)^{1-\lambda}}, a < \xi < b, \tag{3.9}$$

where

$$F(\xi) = \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi} . \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^*} \frac{d}{d\xi} \int_a^{\xi} \frac{x^{\nu} f(x)}{(\xi-x)^{\lambda+\nu-\sigma}} \quad dx, \tag{3.10}$$

is a known function and

$$l(\xi,n) = \frac{d}{d\xi} \int_a^{\xi} \frac{dx}{(\xi-x)^{\lambda+\nu-\sigma}.(x-\eta)^{1+\sigma-\lambda-\nu}}. \tag{3.11}$$

By Lowndes ([8],p.276, eqn.26)

$$l(\xi,n) = \frac{(a-\eta)^{\lambda+\nu-\sigma}}{(\xi-\eta)(\xi-a)^{\lambda+\nu-\sigma}}, \ 0 < 1+\sigma-\lambda-\nu < 1, \tag{3.12}$$

eqn.(3.9) becomes

$$n(\xi)\overline{\phi(\xi)} + \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi} \int_a^b e^{-r}\phi(r)dr \frac{d}{d\xi} \int_a^{\xi} \frac{T(r,x)dx}{(\xi-x)^{\lambda+\nu-\sigma}}$$
$$= F(\xi) - \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi(\xi-a)^{\lambda+\nu-\sigma}} \int_0^a \frac{(a-\eta)^{\lambda+\nu-\sigma}\eta(\xi)d\eta}{(\xi-\eta)} . \int_a^b \frac{e^{-r}\phi(r)dr}{(r-\eta)^{1-\lambda}}. \tag{3.13}$$

Using (3.8), we can write

$$\int_a^b \frac{e^{-r}\phi(r)dr}{(r-\eta)^{1-\lambda}} = -\frac{\sin(1-\lambda)\pi}{\pi} \int_a^b \frac{dr}{(r-\eta)^{1-\lambda}} \frac{d}{dr} \int_r^b \frac{\overline{\phi(\xi)}d\xi}{(\xi-r)^{\lambda}}$$

$$= \frac{\sin(1-\lambda)\pi}{\pi} . \frac{1}{(a-\eta)^{1-\lambda}} . \int_a^b \frac{\overline{\phi(\xi)} d\xi}{(\xi-a)^\lambda} - (1-\lambda) \int_a^b \frac{dr}{(r-\eta)^{2-\lambda}} . \int_r^b \frac{\overline{\phi(\xi)}}{(\xi-r)^\lambda} d\xi. \quad (3.14)$$

Inverting the order of integration in the last term of eqn. (3.14) and using the result of Lowndes ([9], p.276, eqn. 27)

$$\beta \int_a^y \frac{dr}{(r-\xi)^{1+\beta}(y-r)^{1-\beta}} = \frac{(y-a)^\beta}{(y-\xi)(a-\xi)^\beta}, 0 < \beta < 1, \quad (3.15)$$

we get

$$\int_a^b \frac{e^{-r}\phi(r)dr}{(r-\eta)^{1-\lambda}} = \frac{\sin(1-\lambda)\pi(a-\eta)^\lambda}{\pi} \int_a^b \frac{\phi(\xi)d\xi}{(\xi-\eta)(\xi-a)^\lambda}, \quad (3.16)$$

provided $0 < \lambda < 1$ and $\overline{\phi(\xi)}$ is bounded and integrable.

Substituting the expression in eqn. (3.13), $\overline{\phi(\xi)}$ is given by

$$n(\xi)\overline{\phi(\xi)} + \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi} \int_a^b e^{-r}\phi(r)dr . \frac{d}{d\xi} \int_a^\xi \frac{T(r,x)dx}{(\xi-x)^{\lambda+\nu-\sigma}}$$
$$+ \int_a^b \overline{\phi(\xi)} M(x,\xi)dx = F(\xi), \; a < \xi < b, \quad (3.17)$$

where

$$M(x,\xi) = \frac{\sin(1-\lambda)\pi sin(1+\sigma-\lambda-\nu)\pi}{\pi^2(x-a)^\lambda(\xi-a)^{\lambda+\nu-\sigma}} \int_0^a \frac{n(\xi)(a-n)^{\lambda+\nu-\sigma}}{(x-n)} . \frac{(a-n)^\lambda}{(\xi-n)} d\eta. \quad (3.18)$$

Eqn. (3.17) is a Fredholm integral equation which determines, $\overline{\phi(\xi)}$. Thus $\varphi(r)$ is then obtained from eqn. (3.8) and the coefficients A_n, which satisfy eqns. (1.1), (1.2) and (1.3) can be found from eqn. (3.2).

4 Solutions of the Equations of Second Kind

To solve the triple series equations (1.4), (1.5) and (1.6), we put

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)} L_n^{(\sigma)}(x) = \psi_1(x), 0 \le x < a$$
$$= \psi_2(x), b < x < \infty \quad (4.1)$$

where $\psi_1(x)$ and $\psi_2(x)$ are bounded and integrable in the interval of their definitions. Using the orthogonality relation, we get from eqn. (1.5) and eqn. (4.1).

$$A_n = \frac{\Gamma(\beta+n+1)\Gamma(n+1)}{\Gamma(\sigma+n+1)} \left\{ \int_0^a \psi_1(r) \;\; + \int_b^\infty \psi_2(r) . r^\sigma e^{-r} L_n(\sigma;r)dr, \; n = 0,1,2,\ldots \right. \quad (4.2)$$

provided $\beta > -1, \sigma > -1$.

Substituting A_n in eqns. (1.4) and (1.6) and since these series are uniformly convergent, we get on interchanging the order of summation and integration, that

$$\left\{ \int_b^a \psi_1(r) + \int_b^\infty \psi_2(r) \right\} e^{-r} \left\{ S(r,x) + T(r,x) \right\} dr = x^\nu g(x) . 0 \le x < a$$
$$= x^\nu h(x), \; b < x < \infty, \quad (4.3)$$

where $S(r,x)$ is given by eqn. (2.3) and $T(r,x)$ is given by eqn. (3.4).

These equations may be written as:

$$\int_0^x e^{-r}\psi_1(r)S_r(r,x)dr + \int_x^a e^{-r}\psi_2(r)S_r(r,x)dr + \int_b^\infty e^{-r}\psi_2(r)S_x(r,x)dr$$
$$+ \int_0^a e^{-r}\psi_1(r)T(r,x)dr + \int_b^\infty e^{-r}\psi_2(r)T(r,x)dr = \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^x} . x^\nu g(x), \; 0 \le x < a, \quad (4.4)$$

$$\int_0^a e^{-r}\psi_1(r)S_r(r,x)dr + \int_b^x e^{-r}\psi_2(r)S_r(r,x)dr + \int_x^\infty e^{-r}\psi_2(r)S_x(r,x)dr + \int_0^a e^{-r}\psi_1(r)T(r,x)dr$$
$$+ \int_b^\infty e^{-r}\psi_2(r)T(r,x)dr = \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^x} x^\nu h(x), \; b < x < \infty, \quad (4.5)$$

provided $\alpha, \beta, \sigma > -1, 0 < \lambda < 1, \lambda+\nu > \sigma$. Since $\psi_1(x)$ and $\psi_2(x)$ are bounded and integrable in their interval of definitions, we get on interchanging the order of integration that

$$\int_a^x \frac{n(\xi)}{(x-\xi)^{1+\sigma-\lambda-\nu}} \left\{\overline{\psi_1}(\xi) + \int_b^\infty \frac{e^{-r}\psi_2(r)}{(r-\xi)^{1-\lambda}}dr\right\} d\xi + \int_0^a e^{-r}\psi_1(r)T(r,x)dr + \int_b^\infty e^{-r}\psi_2(r)T(r,x)$$
$$= \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a^x} x^\nu g(x), 0 \le x < a, \quad (4.6)$$

$$\int_b^x \frac{n(\xi)\overline{\psi_2}(\xi)d\xi}{(x-\xi)^{1+\sigma-\lambda-\nu}} + \int_0^a e^{-r}\psi_1(r)T(r,x)dr + \int_b^\infty e^{-r}\psi_2(r)T(r,x)dr = \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^x}, x^\nu h(x)$$
$$-\int_0^a \frac{n(\xi)\overline{\psi_1(\xi)}d\xi}{(x-\xi)^{1+\sigma-\lambda-\nu}} - \int_0^b \frac{n(\xi)d\xi}{(x-\xi)^{1+\sigma-\lambda-\nu}}.\int_b^\infty \frac{e^{-r}\psi_2(r)dr}{(r-\xi)^{1-\lambda}}, \ b < x < \infty, \quad (4.7)$$

where,

$$\begin{cases} (i)\ \overline{\psi}_1(\xi) = \int_\xi^a \frac{e^{-r}\psi_1(r)dr}{(r-\xi)^{1-\lambda}} \\ (\text{ii })\ \overline{\psi}_2(\xi) = \int_\xi^\infty \frac{e^{-r}\psi_2(r)dr}{(r-\xi)^{1-\lambda}} \end{cases} \quad (4.8)$$

provided $\alpha, \beta, \sigma > -1, 0 < \lambda < 1, 0 < 1-\lambda-\nu+\sigma < 1$, when $0 < 1+\sigma-\lambda-\nu < 1$. On using equations (10) to (14) of Lowndes ([8],p.168) with the help of eqns. (4.6), (4.7), (4.8) in a similar manner as to obtain eqns. (3.8) and (3.13) , we find that

$$n(\xi)\overline{\psi_1(\xi)} + \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi}\frac{d}{d\Psi}\int_a^\xi \left\{\int_0^a e^{-r}\psi_1(r)dr \int_b^\infty e^{-r}\psi_2(r)dr\right\} T(r,x)\frac{dx}{(\xi-x)^{\lambda+\nu-\sigma}}$$
$$= \quad G(\xi) - n(\xi)\int_b^\infty \frac{e^{-r}\psi_2(r)dr}{(r-\xi)^{1-\lambda}}, \quad (4.9)$$

$$n(\xi)\psi_2(\xi) + \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi}\frac{d}{d\xi}\int_a^\xi \left\{\int_0^a e^{-r}\psi_1(r)dr + \int_b^\infty e^{-r}\psi_2(r)dr\right\} T(r,x)\frac{dx}{(\xi-x)^{\lambda+\nu-\sigma}}$$
$$= H(\xi) - \frac{\sin(1+\sigma-\lambda-\nu)}{\pi(\xi-b)^{\lambda+\nu-\sigma}}\int_0^a (b-\eta)^{\lambda+\nu-\sigma}n(\eta)\overline{\psi_1}(\xi)d\eta - \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi(\xi-b)^{\lambda+\nu-\sigma}}$$
$$\times \int_a^b \frac{(b-\eta)^{\lambda+\nu-\sigma}}{(\xi-\eta)}.n(\eta)d\eta\int_0^\infty \frac{e^{-r}\psi_2(r)dr}{(\xi-n)^{1-\lambda}}, \quad (4.10)$$

$$e^{-r}\overline{\psi_1}(r) = -\frac{\sin(1-\lambda)}{\pi}\frac{d}{dr}\int_r^a \frac{\overline{\psi_1}(\xi)d\xi}{(\xi-r)^\lambda}, \ 0 < r < a, \quad (4.11)$$

$$e^{-r}\overline{\psi_2}(r) = -\frac{\sin(1-\lambda)}{\pi}\frac{d}{dr}\int_r^\infty \frac{\overline{\psi_2}(\xi)}{(\xi-r)^\lambda}, b < r < \infty, \quad (4.12)$$

where, $G(\xi)$ and $H(\xi)$ are known functions, defined as

$$G(\xi) = \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi}.\left\{\frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^x}\right\}.\frac{d}{d\xi}\int_0^\xi \frac{x^v g(x)dx}{(\xi-x)^{\lambda+\nu-\sigma}}, 0 < \xi < a, \quad (4.13)$$

$$H(\xi) = \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi}.\frac{\{\Gamma(\lambda)\Gamma(1-\lambda)\}}{a_n^x}.\frac{d}{d\xi}\int_b^\xi \frac{x^v h(x)dx}{(\xi-x)^{\lambda+\nu-\sigma}}, b < \xi < \infty, \quad (4.14)$$

By a method similar to that used to obtain eqn. (3.16), we can show that

$$\int_b^\infty \frac{e^{-r}\psi_2(r)dr}{(r-\xi)^{1-\lambda}} = \frac{\sin(1-\lambda)\pi}{\pi(b-\xi)^{-\lambda}}\int_b^\infty \frac{(\eta-b)^{-\lambda}\psi_2(\eta)d\eta}{(\eta-\xi)}. \quad (4.15)$$

Using this result and eqn. (4.9), it can be shown after some manipulation, that eqn. (4.10) can be written as

$$n(\xi)\psi_2(\xi) + \int_b^\infty \psi_2(x)N(x,\xi)\ dx = H(\xi) - \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi(\xi-b)^{\sigma+\nu-\lambda}}\int_0^a \frac{(b-\eta)^{\lambda+\nu-\sigma}}{(\xi-\eta)}G(\eta)d\eta, \quad (4.16)$$

where $N(x,\xi)$ is the kernel

$$N(x,\xi) = \frac{\sin(1+\sigma-\lambda-\nu)\pi.sin(1-\lambda)\pi}{\pi^2(x-a)^\lambda(\xi-a)^{\lambda+\nu-\sigma}}.\int_a^b \frac{n(\eta)(a-\eta)^\lambda(a-\eta)^{\lambda+\nu-\sigma}}{(x-\eta)(\xi-\eta)}d\eta, \ b < \xi < \infty, \quad (4.17)$$

provided $\alpha, \beta, \sigma < 1, 0 < \lambda < 1$ and $0 < 1-\lambda-\nu+\sigma < 1$.

Equation (4.16) is a Fredhelm integral equation of second kind which determines $\psi_2(\xi)$, $\psi_2(r)$, can be found from eqn. (4.12) and $\psi_1(r)$ from

$$e^{-r}\psi(r) = -\frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi}\frac{d}{dr}\int_r^a \frac{G(\xi)d\xi}{n(\xi)(\xi-r)^\lambda} + \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi(a-r)^\lambda} \times \int_b^\infty \frac{e^{-n}(n-a)^\lambda\psi_2(\xi)}{(r-\eta)}d\xi, 0 < r < a. \quad (4.18)$$

Finally the coefficients A_n which satisfy the triple series equations of second kind when $\alpha, \beta, \sigma > -1, 0 < \lambda < 1, 0 < 1-\lambda-\nu+\sigma < 1$ are given by eqn. (4.2).

5 Conclusion

The generalized Laguerre polynomials have been applied by many authors like Lowndes [7,8], Srivastava [18,19,21], Srivastava- Panda[25] and Mudaliar-Narain [11] to solve dual, triple and quadruple series equations. The solutions presented in this paper are obtained by employing the techniques of Sneddon[17], Lowndes[8,9]and Srivastava[19]. Method of this paper, involving different boundary conditions, has a distinct advantage over that by the multiplying factor technique. These solutions are useful in Mathematical Physics, Mixed Boundary Problems in Potential Theory, Quantum Physics etc.We have obtained the solution of two sets of triple series equations involving generalized Laguerre polynomials by reducing them to the solution of a Fredholm integral equation of the second kind.

Acknowledgement. The authors express their sincere gratitude to the editors and referees for carefully reading the manuscript and for their valuable comments and suggestions which greatly improved this paper.

Conflict of interest: We declare that authors have no conflict of interest.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 219-225**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

ON A UNIFIED OBERHETTINGER-TYPE INTEGRAL INVOLVING THE PRODUCT OF BESSEL FUNCTIONS AND SRIVASTAVA POLYNOMIALS

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(Received : October 07, 2022; In format: November 14, 2022; Revised: April 06, 2023; Accepted: April 08, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53126

Abstract

The present paper is devoted to derive a generalized Oberhettinger-type integral formula. The derived form of the integral involves a finite product of the Srivastava polynomials with the first-kind Bessel functions. The outcomes are obtained in terms of the Srivastava and Daoust functions. Some of the significant particular cases are also determined.
2020 Mathematical Sciences Classification: 33C10, 33C20, 44A20.
Keywords and Phrases: Bessel function, Hypergeometric functions, Integral transforms.

1 Introduction

The Bessel function frequently appears in a wide variety of problems pertaining to applied Sciences. The theory of the Bessel function is extensively used to solve several problems including radio physics, nuclear physics, atomic, acoustics, information theory and hydrodynamics. These functions can also be used to solve problems in the fields of mechanics and elasticity. In some recent investigations [7, 8, 9, 10, 13, 14, 15, 17, 18], several authors have proposed a number of interesting integral formulas associated with Bessel functions.

Srivastava [22] introduced a general class of polynomials defined by

$$S_l^m[x] = \sum_{k=0}^{\left[\frac{l}{m}\right]} \frac{(-l)_{mk}}{k!} A_{l,k} x^k, \qquad l = 0, 1, 2, ..., \tag{1.1}$$

where m is an arbitrary positive integer and the coefficients $A_{l,k}$ $(l, k \geq 0)$ are arbitrary constants may be real or complex. Also $(\varrho)_l$ represents the Pochhammer's symbol or rising factorial [23] defined by

$$(\varrho)_l = \frac{\Gamma(\varrho + l)}{\Gamma(\varrho)} = \begin{cases} 1 & \text{if } l = 0, \\ \varrho(\varrho+1)...(\varrho+l-1) & \text{if } l \in \mathbb{N}. \end{cases}$$

For applications of generalized polynomials of Srivastava [22], among others we may also refer to Chaurasia and Pandey [6], Chandel and Sengar [3, 4] and Chandel and Chauhan [5]. On suitably specializing the coefficients $A_{l,k}$ in the definition of $S_l^m[x]$ one can yields several known polynomials as its special cases including, the Jacobi polynomials, the Hermite polynomials, the Legendre polynomials, the Chebyshev polynomials of the first kind and the Chebyshev polynomials of the second kind, the Ultraspherical polynomials, the Gould-Hopper polynomials, the Laguerre polynomials and the Bessel polynomials. For more detail we refer [26].

The classical Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ can be presented in the following series form (see [20, 25])

$$P_n^{(\tau,\varsigma)}(x) = \sum_{k=0}^{n} \frac{(1+\tau)_n (1+\tau+\varsigma)_{n+k}}{(n-k)!k!(1+\tau)_k(1+\tau+\varsigma)_n} \left(\frac{x-1}{2}\right)^k, \tag{1.2}$$

which equivalently can be expressed in terms of the Gauss function as follows

$$P_n^{(\tau,\varsigma)}(x) = \frac{(1+\tau)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, & (n+\tau+\varsigma+1); \\ (1+\tau); & \end{matrix} \left(\frac{1-x}{2}\right)\right]. \tag{1.3}$$

The generalized Wright function ${}_p\Psi_q(x)$ defined by [11, 24]

$$ {}_p\Psi_q(x) = {}_p\Psi_q \left[\begin{matrix} (\alpha_i, a_i)_{1,p}; \\ (\beta_i, b_i)_{1,q}; \end{matrix} \; x \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + a_i k)}{\prod_{j=1}^{q} \Gamma(\beta_j + b_j k)} \frac{x^k}{k!}, \tag{1.4} $$

the coefficients $a_1, ..., a_p$ and $b_1, ..., b_q$, involved in (1.4), are positive real numbers such that

$$ \sum_{i=1}^{p} a_i \le 1 + \sum_{j=1}^{q} b_j, $$

and $\Gamma(.)$ is the standard Gamma function (see, for more details, [16, 25]).

The Bessel function $J_\nu(x)$ of first kind is defined by (see [2, 20, 27])

$$ J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+\nu+k)} \frac{\left(\frac{x}{2}\right)^{\nu+2k}}{k!}, \tag{1.5} $$

where $Re(\nu) > -1$, $\nu \in \mathbb{C}$ and $x \in \mathbb{C} \setminus \{0\}$.

Srivastava and Daoust [24] proposed multivariable generalized hypergeometric function, given as

$$ F^{A:B';...;B^{(n)}}_{C:D';...;D^{(n)}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = F^{A:B';...;B^{(n)}}_{C:D';...;D^{(n)}} \left[\begin{matrix} [(a) : \theta', ..., \theta^{(n)}] : [(b') : \phi']; ...; \\ [(c) : \psi', ..., \psi^{(n)}] : [(d') : \delta']; ...; \end{matrix} \right. $$

$$ \left. \begin{matrix} [(b^{(n)}) : \phi^{(n)}]; \\ [(d^{(n)}) : \delta^{(n)}]; \end{matrix} x_1, ..., x_n \right] = \sum_{m_1, \dots, m_n = 0}^{\infty} \Omega(m_1, ..., m_n) \frac{x^{m_1}}{m_1!} ... \frac{x^{m_n}}{m_n!}, \tag{1.6} $$

where, for convenience

$$ \Omega(m_1, ..., m_n) = \frac{\prod_{j=1}^{A} (a_j)_{m_1\theta'_j + ... + m_n\theta_j^{(n)}} \prod_{j=1}^{B'} (b'_j)_{m_1\phi'_j} \cdots \prod_{j=1}^{B^{(n)}} (b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C} (c_j)_{m_1\psi'_j + ... + m_n\psi_j^{(n)}} \prod_{j=1}^{D'} (d'_j)_{m_1\delta'_j} \cdots \prod_{j=1}^{D^{(n)}} (d_j^{(n)})_{m_n\delta_j^{(n)}}}, $$

the coefficients $\theta_j^{(k)}$, $j = 1, ..., A$; $\phi_j^{(k)}$, $j = 1, ..., B^{(k)}$; $\psi_j^{(k)}$, $j = 1, ..., C$; $\delta_j^{(k)}$, $j = 1, ..., D^{(k)}$ are real and positive, and (a) abbreviates the array of A parameters $a_1, ..., a_A$, $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}$, $j = 1, ..., B^{(k)}$; $\forall k \in \{1, ..., n\}$, with similar interpretations for (c) and $(d^{(k)})$, $\forall k \in \{1, ..., n\}$; etcetera.

In the present work, we recall the following integral mentioned in the classical monograph by Oberhettinger (see [19], p. 22)

$$ \int_0^\infty x^{\delta-1} \left(x + h + \sqrt{x^2 + 2hx} \right)^{-\eta} dx = 2\eta h^{-\eta} \left(\frac{h}{2} \right)^{\delta} \frac{\Gamma(2\delta)\Gamma(\eta - \delta)}{\Gamma(1 + \delta + \eta)}, \tag{1.7} $$

provided $0 < Re(\delta) < Re(\eta)$.

The intention of this paper is to propose a unified integral involving the Oberhettinger-type that includes a finite product of the Bessel functions and Srivastava polynomials. The main result in the current investigation is presented in terms of a Theorem. Further, two Corollaries of the main result are derived. Some other interesting well-known special cases of the main result are also determined.

2. Main results

In this section, we derive an integral formula involving a finite product of general class of polynomials and Bessel functions. The outcome is expressed in terms of an elegant form Srivastava and Daoust function, defined above in (1.6).

Theorem 2.1. *For $\mu, \lambda_i, \nu_q \in \mathbb{C}$, $l_p, s_p \ge 0$, $Re(\nu_q) > -1$, $Re(\sigma_j^{(p)}) > Re(\eta_p) > 0$ and $Re\left(\lambda_i + \sum_{q=1}^{w} \lambda_k^{(q)} \nu_q\right) > Re\left(\mu + \sum_{q=1}^{w} \delta_q \nu_q\right) > 0$ $(i = 1, 2, ..., n_1, j = 1, 2, ..., n_2,$ $k = 1, 2, ..., n_3,$ $p = 1, 2, ..., v,$ $q = 1, 2, ..., w)$ the following integral formula holds true:*

$$ \int_0^\infty x^{\mu-1} \prod_{i=1}^{n_1} \left(x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda_i} \prod_{p=1}^{v} S_{l_p}^{m_p} \left[\xi_p x^{\eta_p} \prod_{j=1}^{n_2} \left(x + a + \sqrt{x^2 + 2ax} \right)^{-\sigma_j^{(p)}} \right] $$

$$\times \prod_{q=1}^{w} J_{\nu_q}\left[\zeta_q x^{\delta_q} \prod_{k=1}^{n_3}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_k^{(q)}}\right] dx$$

$$= \frac{\zeta_1^{\nu_1} \dots \zeta_w^{\nu_w}}{\Gamma(1+\nu_1)\dots\Gamma(1+\nu_w)} 2^{\left[1-\mu-\sum_{q=1}^{w}(1+\delta_q)\nu_q\right]} \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\prod_{k=1}^{n_3}(a)^{\left[\mu-\lambda_i-\sum_{q=1}^{w}\nu_q\left(\lambda_k^{(q)}-\delta_q\right)\right]} \sum_{s_1=0}^{\left[\frac{l_1}{m_1}\right]} \dots \sum_{s_v=0}^{\left[\frac{l_v}{m_v}\right]}$$

$$\times \frac{(-l_1)_{m_1 s_1}(A)_{l_1,s_1}}{s_1!}\left(\frac{\xi_1}{2^{\eta_1} a^{\left(\sigma_j^{(1)}-\eta_1\right)}}\right)^{s_1} \dots \frac{(-l_v)_{m_v s_v}(A)_{l_v,s_v}}{s_v!}\left(\frac{\xi_v}{2^{\eta_v} a^{\left(\sigma_j^{(v)}-\eta_v\right)}}\right)^{s_v}$$

$$\times \frac{\Gamma\left\{1+\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right\}\Gamma\left\{2\mu+2\sum_{p=1}^{v}\eta_p s_p+2\sum_{q=1}^{w}\delta_q\nu_q\right\}}{\Gamma\left\{\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right\}}$$

$$\times \frac{\Gamma\left\{-\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}-\eta_p\right)s_p+\sum_{q=1}^{w}(\lambda_k^{(q)}-\delta_q)\nu_q\right\}}{\Gamma\left\{1+\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}+\eta_p\right)s_p+\sum_{q=1}^{w}(\lambda_k^{(q)}+\delta_q)\nu_q\right\}}$$

$$\times F^{3:0;\dots;0}_{2:1;\dots;1}\left[\begin{array}{l}\left[-\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}-\eta_p\right)s_p+\sum_{q=1}^{w}(\lambda_k^{(q)}-\delta_q)\nu_q : 2(\lambda_k^{(1)}-\delta_1),\right.\\ \left[1+\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}+\eta_p\right)s_p+\sum_{q=1}^{w}(\lambda_k^{(q)}+\delta_q)\nu_q : 2(\lambda_k^{(1)}+\delta_1),\right.\end{array}\right.$$

$$\left.\dots, 2(\lambda_k^{(w)}-\delta_w)\right], \left[1+\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q : 2\lambda_k^{(1)},\dots,2\lambda_k^{(w)}\right],$$

$$\left.\dots, 2(\lambda_k^{(w)}+\delta_w)\right], \left[\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q : 2\lambda_k^{(1)},\dots,2\lambda_k^{(w)}\right] :$$

$$\begin{array}{l}\left[2\mu+2\sum_{p=1}^{v}\eta_p s_p+2\sum_{q=1}^{w}\delta_q\nu_q : 4\delta_1,\dots,4\delta_w\right] : \_;\dots;\_;\\ \left[1+\nu_1:1\right],\dots,\left[1+\nu_w:1\right];\end{array} \quad \frac{-\zeta_1^2}{4^{(1+\delta_1)}(a)^{2\left(\lambda_k^{(1)}-\delta_1\right)}},$$

$$\left.\dots, \frac{-\zeta_w^2}{4^{(1+\delta_w)}(a)^{2\left(\lambda_k^{(w)}-\delta_w\right)}}\right]. \tag{2.1}$$

Proof. To prove Theorem 2.1, we first express Srivastava polynomials and Bessel functions in series forms given by (1.1) and (1.5) respectively, we have

$$\text{L.H.S. of (2.1)} = \int_0^\infty x^{\mu-1}\left\{\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_1}\dots\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_{n_1}}\right\}$$

$$\times \sum_{s_1=0}^{\left[\frac{l_1}{m_1}\right]}\dots\sum_{s_v=0}^{\left[\frac{l_v}{m_v}\right]} \frac{(-l_1)_{m_1 s_1}(A)_{l_1,s_1}}{s_1!}(\xi_1 x^{\eta_1})^{s_1}\left\{\left(x+a+\sqrt{x^2+2ax}\right)^{-\sigma_1^{(1)}s_1}\dots\right.$$

$$\left.\left(x+a+\sqrt{x^2+2ax}\right)^{-\sigma_{n_2}^{(1)}s_1}\right\}\dots\frac{(-l_v)_{m_v s_v}(A)_{l_v,s_v}}{s_v!}(\xi_v x^{\eta_v})^{s_v}\left\{\left(x+a+\right.\right.$$

$$\left.\left.\sqrt{x^2+2ax}\right)^{-\sigma_1^{(v)}s_v}\dots\left(x+a+\sqrt{x^2+2ax}\right)^{-\sigma_{n_2}^{(v)}s_v}\right\}\sum_{r_1=0}^{\infty}\dots\sum_{r_w=0}^{\infty}$$

$$\frac{(-1)^{r_1}}{r_1!\Gamma(1+\nu_1+r_1)}\left(\frac{\zeta_1 x^{\delta_1}}{2}\right)^{\nu_1+2r_1}\left\{\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_1^{(1)}(\nu_1+2r_1)}\dots\right.$$

$$\left.\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_{n_3}^{(1)}(\nu_1+2r_1)}\right\}\dots\frac{(-1)^{r_w}}{r_w!\Gamma(1+\nu_w+r_w)}\left(\frac{\zeta_w x^{\delta_w}}{2}\right)^{\nu_w+2r_w}$$

$$\left\{\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_1^{(w)}(\nu_w+2r_w)} \ldots \left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_{n_3}^{(w)}(\nu_w+2r_w)}\right\} dx.$$

Now, we interchange the order of summations and integration (permissible with the uniform convergence of the series forms under the given conditions), we obtain

$$= \frac{\left(\frac{\zeta_1}{2}\right)^{\nu_1} \ldots \left(\frac{\zeta_w}{2}\right)^{\nu_w}}{\Gamma(1+\nu_1)\ldots\Gamma(1+\nu_w)} \sum_{s_1=0}^{\left[\frac{l_1}{m_1}\right]} \ldots \sum_{s_v=0}^{\left[\frac{l_v}{m_v}\right]} \frac{(-l_1)_{m_1 s_1} A_{l_1,s_1}}{s_1!} \xi_1^{s_1} \ldots \frac{(-l_v)_{m_v s_v} A_{l_v,s_v}}{s_v!} \xi_v^{s_v}$$

$$\times \sum_{r_1=0}^{\infty} \ldots \sum_{r_w=0}^{\infty} \frac{\left(\frac{-\zeta_1^2}{4}\right)^{r_1}}{r_1!(1+\nu_1)_{r_1}} \ldots \frac{\left(\frac{-\zeta_w^2}{4}\right)^{r_w}}{r_w!(1+\nu_w)_{r_w}} \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\prod_{k=1}^{n_3} \int_0^{\infty} x^{\left[\mu+\sum_{p=1}^{v}\eta_p s_p+\sum_{q=1}^{w}\delta_q(\nu_q+2r_q)-1\right]}$$

$$\times \left(x+a+\sqrt{x^2+2ax}\right)^{-\left[\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}(\nu_q+2r_q)\right]} dx.$$

Now, using the Oberhettinger's integral Eq.(1.7) formula, we get

$$= \frac{\left(\frac{\zeta_1}{2}\right)^{\nu_1} \ldots \left(\frac{\zeta_w}{2}\right)^{\nu_w}}{\Gamma(1+\nu_1)\ldots\Gamma(1+\nu_w)} \sum_{s_1=0}^{\left[\frac{l_1}{m_1}\right]} \ldots \sum_{s_v=0}^{\left[\frac{l_v}{m_v}\right]} \frac{(-l_1)_{m_1 s_1} (A)_{l_1,s_1}}{s_1!} \xi_1^{s_1} \ldots \frac{(-l_v)_{m_v s_v} (A)_{l_v,s_v}}{s_v!} \xi_v^{s_v}$$

$$\times \sum_{r_1=0}^{\infty} \ldots \sum_{r_w=0}^{\infty} \frac{\left(\frac{-\zeta_1^2}{4}\right)^{r_1}}{r_1!(1+\nu_1)_{r_1}} \ldots \frac{\left(\frac{-\zeta_w^2}{4}\right)^{r_w}}{r_w!(1+\nu_w)_{r_w}} 2^{\left[1-\mu-\sum_{p=1}^{v}\eta_p s_p-\sum_{q=1}^{w}\delta_q(\nu_q+2r_q)\right]}$$

$$\times \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\prod_{k=1}^{n_3} (a)^{\left[\mu+\sum_{p=1}^{v}\eta_p s_p+\sum_{q=1}^{w}\delta_q(\nu_q+2r_q)-\lambda_i-\sum_{p=1}^{v}\sigma_j^{(p)}s_p-\sum_{q=1}^{w}\lambda_k^{(q)}(\nu_q+2r_q)\right]}$$

$$\Bigg[\frac{\Gamma\left\{1+\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}(\nu_q+2r_q)\right\}\Gamma\left\{2\sum_{q=1}^{w}\delta_q(\nu_q+2r_q)\right.}{\Gamma\left\{\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}(\nu_q+2r_q)\right\}}$$

$$\frac{\left.+2\mu+2\sum_{p=1}^{v}\eta_p s_p\right\}\Gamma\left\{\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}-\eta_p\right)s_p-\sum_{q=1}^{w}\delta_q(\nu_q+2r_q)\right.}{\Gamma\left\{1+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}+\eta_p\right)s_p+\sum_{q=1}^{w}\delta_q(\nu_q+2r_q)\right.}$$

$$\left.\frac{\left.-\mu+\sum_{q=1}^{w}\lambda_k^{(q)}(\nu_q+2r_q)\right\}}{\left.+\mu+\sum_{q=1}^{w}\lambda_k^{(q)}(\nu_q+2r_q)\right\}}\right].$$

$$= \frac{\zeta_1^{\nu_1}\ldots\zeta_w^{\nu_w}}{\Gamma(1+\nu_1)\ldots\Gamma(1+\nu_w)} 2^{\left[1-\mu-\sum_{q=1}^{w}(1+\delta_q)\nu_q\right]} \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\prod_{k=1}^{n_3} (a)^{\left[\mu-\lambda_i-\sum_{q=1}^{w}\nu_q\left(\lambda_k^{(q)}-\delta_q\right)\right]}$$

$$\times \sum_{s_1=0}^{\left[\frac{l_1}{m_1}\right]} \ldots \sum_{s_v=0}^{\left[\frac{l_v}{m_v}\right]} \frac{(-l_1)_{m_1 s_1}(A)_{l_1,s_1}}{s_1!} \left(\frac{\xi_1}{2^{\eta_1} a^{\left(\sigma_j^{(1)}-\eta_1\right)}}\right)^{s_1} \ldots \frac{(-l_v)_{m_v s_v}(A)_{l_v,s_v}}{s_v!} \left(\frac{\xi_v}{2^{\eta_v} a^{\left(\sigma_j^{(v)}-\eta_v\right)}}\right)^{s_v}$$

$$\times \sum_{r_1=0}^{\infty} \ldots \sum_{r_w=0}^{\infty} \frac{1}{r_1!(1+\nu_1)_{r_1}} \left(\frac{-\zeta_1^2}{4^{(1+\delta_1)} a^{2\left(\lambda_k^{(1)}-\delta_1\right)}}\right)^{r_1} \ldots \frac{1}{r_w!(1+\nu_w)_{r_w}} \left(\frac{-\zeta_w^2}{4^{(1+\delta_w)} a^{2\left(\lambda_k^{(w)}-\delta_w\right)}}\right)^{r_w}$$

$$\times \frac{\Gamma\left\{1+\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right\}\Gamma\left\{2\mu+2\sum_{p=1}^{v}\eta_p s_p+2\sum_{q=1}^{w}\delta_q\nu_q\right\}}{\Gamma\left\{\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right\}}$$

$$\times \frac{\Gamma\left\{-\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}-\eta_p\right)s_p+\sum_{q=1}^{w}(\lambda_k^{(q)}-\delta_q)\nu_q\right\}}{\Gamma\left\{1+\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}+\eta_p\right)s_p+\sum_{q=1}^{w}(\lambda_k^{(q)}+\delta_q)\nu_q\right\}}$$

$$\left[\frac{\left(1+\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right)_{2\sum_{q=1}^{w}\lambda_k^{(q)}r_q}\left(2\sum_{p=1}^{v}\eta_p s_p+2\sum_{q=1}^{w}\delta_q\nu_q\right.}{\left(\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right)_{2\sum_{q=1}^{w}\lambda_k^{(q)}r_q}}\right.$$

$$\left.\frac{\left.+2\mu\right)_{4\sum_{q=1}^{w}\delta_q r_q}\left(-\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}-\eta_p\right)s_p+\sum_{q=1}^{w}\left(\lambda_k^{(q)}-\delta_q\right)\nu_q\right)_{2\sum_{q=1}^{w}\left(\lambda_k^{(q)}-\delta_q\right)r_q}}{\left(1+\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}+\eta_p\right)s_p+\sum_{q=1}^{w}\left(\lambda_k^{(q)}+\delta_q\right)\nu_q\right)_{2\sum_{q=1}^{w}\left(\lambda_k^{(q)}+\delta_q\right)r_q}}\right].$$

Now, using the Srivastava and Daoust function Eq.(1.6), we arrive at the desire form given in *RHS* of (2.1).

On substituting $l_p = 0$ (for $p = 1, 2, ..., v$) in Theorem 2.1 the Srivastava polynomial $S_{l_p}^{m_p}(x)$ reduces to unity, i.e., $S_0^{m_p}(x) = 1$ and we can deduce the following Corollary 2.1 based on the main integral presented in Theorem 2.1.

Corollary 2.1. *For* $\mu, \lambda_i, \nu_q \in \mathbb{C}, Re(\nu_q) > -1$ *and* $Re\left(\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q+\lambda_i\right)$ $> Re\left(\sum_{q=1}^{w}\delta_q\nu_q+\mu\right) > 0$ $(i = 1, 2, ..., n_1,\ k = 1, 2, ..., n_3,\ q = 1, 2, ..., w)$, *the following integral formula holds true:*

$$\int_0^\infty x^{\mu-1}\prod_{i=1}^{n_1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_i}\prod_{q=1}^{w}J_{\nu_q}\left[\zeta_q x^{\delta_q}\prod_{k=1}^{n_3}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_k^{(q)}}\right]dx$$

$$=\frac{\zeta_1^{\nu_1}\ldots\zeta_w^{\nu_w}}{\Gamma(1+\nu_1)\ldots\Gamma(1+\nu_w)}2^{\left[1-\mu-\sum_{q=1}^{w}(1+\delta_q)\nu_q\right]}\prod_{i=1}^{n_1}\prod_{k=1}^{n_3}(a)^{\left[\mu-\lambda_i+\sum_{q=1}^{w}\nu_q\left(\delta_q-\lambda_k^{(q)}\right)\right]}$$

$$\times\frac{\Gamma\left\{2\mu+2\sum_{q=1}^{w}\delta_q\nu_q\right\}\Gamma\left\{1+\lambda_i+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right\}\Gamma\left\{-\mu+\lambda_i+\sum_{q=1}^{w}(\lambda_k^{(q)}-\delta_q)\nu_q\right\}}{\Gamma\left\{\lambda_i+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right\}\Gamma\left\{1+\mu+\lambda_i+\sum_{q=1}^{w}(\lambda_k^{(q)}+\delta_q)\nu_q\right\}}$$

$$\times F_{2:1;\ldots;1}^{3:0;\ldots;0}\left[\begin{array}{l}\left[-\mu+\lambda_i+\sum_{q=1}^{w}(\lambda_k^{(q)}-\delta_q)\nu_q : 2(\lambda_k^{(1)}-\delta_1),\ldots,2(\lambda_k^{(w)}-\delta_w)\right],\\ \left[1+\mu+\lambda_i+\sum_{q=1}^{w}(\lambda_k^{(q)}+\delta_q)\nu_q : 2(\lambda_k^{(1)}+\delta_1),\ldots,2(\lambda_k^{(w)}+\delta_w)\right],\end{array}\right.$$

$$\begin{array}{l}\left[1+\lambda_i+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q : 2\lambda_k^{(1)},\ldots,2\lambda_k^{(w)}\right],\left[2\mu+2\sum_{q=1}^{w}\delta_q\nu_q :\right.\\ \left[\lambda_i+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q : 2\lambda_k^{(1)},\ldots,2\lambda_k^{(w)}\right] : \left[1+\nu_1 : 1\right],\ldots,\end{array}$$

$$\left.\begin{array}{l}\left.4\delta_1,\ldots,4\delta_v\right] : \_;\ldots;\_;\\ \left[1+\nu_w : 1\right];\end{array}\quad\frac{-\zeta_1^2}{4^{(1+\delta_1)}(a)^{2\left(\lambda_k^{(1)}-\delta_1\right)}},\ldots,\frac{-\zeta_w^2}{4^{(1+\delta_w)}(a)^{2\left(\lambda_k^{(w)}-\delta_w\right)}}\right]. \tag{2.2}$$

On setting $n_1 = 1, n_2 = 1, n_3 = 1, v = 1, w = 1$ in Theorem 2.1, we arrive at the following Corollary 2.2, where the outcome is computed in the form of generalized Wright function.

Corollary 2.2. *For* $\eta, \lambda_1, \nu_1 \in \mathbb{C}, l_1, s_1 \geq 0, Re(\nu_1) > -1, Re(\sigma_1^{(1)}) > Re(\eta_1) > 0,$ *and* $Re\left(\lambda_1+\lambda_1^{(1)}\nu_1\right) >$ $Re\left(\mu+\delta_1\nu_1\right) > 0$, *the following integral formula holds true:*

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_1}S_{l_1}^{m_1}\left[\xi_1 x^{\eta_1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\sigma_1^{(1)}}\right]$$

$$\times J_{\nu_1}\left[\zeta_1 x^{\delta_1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_1^{(1)}}\right]dx=\frac{\zeta_1^{\nu_1}}{\Gamma(1+\nu_1)}2^{[1-\mu-(1+\delta_1)\nu_1]}$$

$$\times (a)^{\left[\mu-\lambda_1+\nu_1\left(\delta_1-\lambda_1^{(1)}\right)\right]} \sum_{s_1=0}^{\left[\frac{l_1}{m_1}\right]} \frac{(-l_1)_{m_1 s_1} A_{l_1,s_1}}{s_1!} \left(\frac{\xi_1}{2^{\eta_1} a^{\left(\sigma_1^{(1)}-\eta_1\right)}}\right)^{s_1}$$

$$\times {}_3\Psi_3 \left[\begin{matrix} \left[-\mu+\lambda_1+(\sigma_1^{(1)}-\eta_1)s_1+(\lambda_1^{(1)}-\delta_1)\nu_1, 2(\lambda_1^{(1)}-\delta_1)\right], \\ \left[1+\mu+\lambda_1+(\sigma_1^{(1)}+\eta_1)s_1+(\lambda_1^{(1)}+\delta_1)\nu_1, 2(\lambda_1^{(1)}+\delta_1)\right], \end{matrix}\right.$$

$$\left.\begin{matrix} \left[1+\lambda_1+\sigma_1^{(1)}s_1+\lambda_1^{(1)}\nu_1, 2\lambda_1^{(1)}\right], \left[2\mu+2\eta_1 s_1+2\delta_1\nu_1, 4\delta_1\right]; \\ \left[\lambda_1+\sigma_1^{(1)}s_1+\lambda_1^{(1)}\nu_1, 2\lambda_1^{(1)}\right], \left[1+\nu_1, 1\right]; \end{matrix} \frac{-\zeta_1^2}{4^{(1+\delta_1)} a^{2\left(\lambda_1^{(1)}-\delta_1\right)}}\right]. \tag{2.3}$$

3. Special Cases

In this section, we present some of the well-known and interesting special cases which can be determined by specializing the parameters of the Corollaries 2.1 and 2.2.

(i) For $n_1 = 1, n_3 = 1, \delta_q = 0$ and $\lambda_1^{(q)} = 1$ $(q = 1, 2, ..., w)$, Corollary 2.1 reduces to an interesting result given by Choi and Agarwal [7, Theorem 1, p. 671, (2.1)].

(ii) Assuming $n_1 = 1, \delta_q = 1$ and $\lambda_1^{(q)} = 1$ $(q = 1, 2, ..., w)$, Corollary 2.1 produces another known result of Choi and Agarwal [7, Theorem 2, p. 671, (2.2)].

(iii) Substituting $l_1 = 0, S_{l_1}^{m_1}[x] = 1, \delta_1 = 0$ and $\lambda_1^{(1)} = 1$, Corollary 2.2 reduces in to [8, Theorem 1, p. 3, (2.1)] investigated by Choi and Agarwal.

(iv) Taking $l_1 = 0, S_{l_1}^{m_1}[x] = 1, \delta_1 = 1$ and $\lambda_1^{(1)} = 1$, Corollary 2.2 reduces to another result due to Choi and Agarwal [8, Theorem 2, p. 3, (2.2)].

(v) Also, for $\eta_1 = 0, \delta_1 = 0, \xi_1 = \frac{\xi}{2}, m_1 = 1, \lambda_1^{(1)} = 1, \sigma_1^{(1)} = 1, A_{l_1,s_1} = \binom{l_1+\tau}{l_1}\frac{(l_1+\tau+\varsigma+1)_{s_1}}{(\tau+1)_{s_1}}, S_{l_1}^{1}(x) = P_{l_1}^{(\tau,\varsigma)}(1-2x)$ and using equation (1.3) and (1.2), the Corollary 2.2 reduces to [13, Theorem 1, p. 341, (2.1)] and [13, Theorem 2, p. 343, (2.5)] respectively, presented by Khan et al.

Remark: For the appropriate settings of parameters in the proposed integral, i.e., Theorem 2.1, one can derive a number of interesting integrals. Further, an interesting form involving the product of several Bessel functions established by Exton and Srivastava [12, *p.* 4, (2.8)] can be deduced as a particular case of the Theorem 2.1. Furthermore, one can also obtain the legendary and classical integral formulas investigated by Bailey [1, *p.* 38, (1.2)] and Rice [21, *p.* 60, (2.6), (2.8)] for proper choice of parameter in the Theorem 2.1.

4. Conclusion

By the use of Oberhettinger type integral formula, in the current investigation by the applications of the Oberhettinger integral formula we have establised some of the results involving the Bessel functions and general class of polynomial whose outcomes in terms of the Srivastava and Daoust functions and also the Bessel function of the first kind is a special case of Fox H-function [8, *p.* 9, (4.1)]. Consequently, all the result of this paper can easily converted in terms of the Fox H-function for the appropriate settings of parameters. We can find some other results in terms of the Srivastava and Daoust functions for the proper settings of parameters in the general class of polynomial.

Acknowledgement

The authors are very thankful to the editor and referees for their valuable suggestions to improve the paper in its present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 226-237**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

AN EXTENDED GENERALIZED FIBONACCI POLYNOMIAL BASED CODING METHOD WITH ERROR DETECTION AND CORRECTION

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(Received: March 15, 2023; In format: March 24, 2023; Revised April 13, 2023; Accepted April 24, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53127

Abstract

A Fibonacci coding method is introduced using Extended Generalized Fibonacci Polynomials in this paper. A new square matrix $Q_m^n(a,b)$, the n^{th} power of $Q_m(a,b)$ of order $m \times m$ is defined whose elements are based on extended Generalized Fibonacci Polynomial. Matrix $Q_m^n(a,b)$ for integer $x \geq 1$, $a \geq 1$ and $b \geq 1$ is considered as the encoding matrix and a matrix $Q_m^{-n}(a,b)$ is considered as decoding matrix. An error-detection and error-correction method is also defined in Extended Generalized Fibonacci polynomials.
2020 Mathematical Sciences Classification: 11C08, 11C20, 11H71.
Keywords and Phrases: Extended Generalized Fibonacci Polynomial, Extended Generalized Fibonacci Polynomial matrices, Error detection and Error correction.

1 Introduction

The Fibonacci sequence is one of the most well-known sequences, with numerous intriguing aspects and major applications in a variety of fields including Mathematics,Statistics, Biology, Physics, Finance, Architecture and Computer Sciences. The Fibonacci sequences and golden ratio have rich history, features and uses. This sequence has been modified in a variety of ways.
The Fibonacci Polynomial [5] and the Extended Generalized Fibonacci Polynomial [8] are two such extensions that will be used in this paper. The Fibonacci Polynomial $F_n(x)$ is defined by the recurrence relation shown below,

$$f_n(x) = \begin{cases} 1 & n = 1; \\ x & n = 2; \\ xf_{n-1}(x) + f_{n-2}(x) & n \geq 3. \end{cases} \tag{1.1}$$

There is no restriction on Fibonacci Polynomials for $n \leq 0$.
one such extension of Fibonacci Polynomial is the Extended Generalized Fibonacci Polynomial which is defined by the recurrence relation

$$g_n(x) = \begin{cases} 1 & n = 1; \\ a(x) & n = 2; \\ a(x)g_{n-1}(x) + b(x)g_{n-2}(x) & n \geq 3. \end{cases} \tag{1.2}$$

where $a(x)$, $b(x)$, $g_0(x)$ and $g_1(x)$ are arbitrary real Polynomials and $n \geq 0$. A non-recursive expression for $g_n(x)$, given below is introduced in [14].

$$g_{n+1}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} a^{n-2i} b^i \qquad n \geq 0. \tag{1.3}$$

This expression will appear several times in this paper. The first five Extended Generalized Fibonacci Polynomials are shown below.

$$g_n(x) = \begin{cases} 1 & n = 1; \\ a(x) & n = 2; \\ [a(x)]^2 + b(x) & n = 3; \\ [a(x)]^3 + 2a(x)b(x) & n = 4; \\ [a(x)]^4 + 3[a(x)]^2 b(x) + [b(x)]^2 & n = 5. \end{cases} \tag{1.4}$$

There is no restriction on Extended Generalized Fibonacci Polynomial for $n \leq 0$. In this paper, we set $g_0(x) = 0$ and $g_n(x) = 1$ for $n \leq -1$. It's worth nothing that the classical Fibonacci Polynomial can be created by substituting $a(x) = x$ and $b(x) = 1$ in the Extended Generalized Fibonacci Polynomial and the classical Jacobsthal Polynomial can be created by substituting $a(x) = 1$ and $b(x) = x$ in the Extended Generalized Fibonacci Polynomial. A Square matrices Q_m^n of order $m \times m$, $n \geq 1$, Properties, coding and decoding method, relation between code elements of message matrix and error-detection error-correction method has been introduced in Extended Generalized Fibonacci polynomials. This result,s is an extension of the result's [5]. For simplicity, we denote $g_n(x)$,$a(x)$, $b(x)$, $g_0(x)$ and $g_1(x)$ by g_n, a, b, g_0 and g_1 respectively.

2 Main Results

2.1 Extended Generalized Fibonacci Polynomial matrices of order m

The Extended Generalized Fibonacci Polynomials generated by the matrix given below.

$$Q_2(a,b) = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

For any a and b, we have $\det(Q_2(a,b)) = -b$. Setting $g_0 = 0$ and applying induction on $n \geq 1$, it is easily verified that

$$Q_2^n(a,b) = \begin{pmatrix} g_{n+1} & bg_n \\ g_n & bg_{n-1} \end{pmatrix}. \tag{2.2}$$

By using the determinant theorem, we see that $\det(Q_2^n(a,b)) = (-b)^n$. The following defines the $m \times m$ matrix $Q_m(a,b)$.

$$Q_m(a,b) = \begin{pmatrix} a & b & 0 & 0 & \cdots & 0 \\ 0 & a & b & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a & b \\ 0 & 0 & \vdots & 0 & 1 & 0 \end{pmatrix}_{m \times m}.$$

Thus $Q_m(a,b)$ has a recursive expression and $\det(Q_m(a,b)) = -a^{m-2}b$. The n^{th}, $n \geq 2$, power of $Q_m(a,b)$ is given by the following theorem.

Theorem 2.1. *For $n \geq 2$ and $m \geq 2$, we have*
$Q_m^n(a,b)$

$$= \begin{pmatrix} \binom{n}{0}a^n & \binom{n}{1}a^{n-1}b & \cdots & \binom{n}{m-3}a^{n-m+3}b^{m-3} & \sum_{i=0}^{\lfloor \frac{n-m+2}{2} \rfloor} \binom{n-i}{i+m-2} a^{n-m+2-2i} b^{i+m-2} & \sum_{i=0}^{\lfloor \frac{n-m+1}{2} \rfloor} \binom{n-1-i}{i+m-2} a^{n-m+1-2i} b^{i+m-1} \\ 0 & \binom{n}{0}a^n & \cdots & \binom{n}{m-4}a^{n-m+4}b^{m-4} & \sum_{i=0}^{\lfloor \frac{n-m+3}{2} \rfloor} \binom{n-i}{i+m-3} a^{n-m+3-2i} b^{i+m-3} & \sum_{i=0}^{\lfloor \frac{n-m+2}{2} \rfloor} \binom{n-1-i}{i+m-3} a^{n-m+2-2i} b^{i+m-2} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{n}{0}a^n & \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i}{i+1} a^{n-1-2i} b^{i+1} & \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1-i}{i+1} a^{n-2-2i} b^{i+2} \\ 0 & 0 & \cdots & 0 & \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} a^{n-2i} b^{i} & \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} a^{n-1-2i} b^{i+1} \\ 0 & 0 & \cdots & 0 & \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} a^{n-1-2i} b^{i} & \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-i}{i} a^{n-2-2i} b^{i+1} \end{pmatrix}. \tag{2.3}$$

Proof. For the sake of simplicity, assume $m = 4$. The proof is based on induction. The following equality shows that eq.(2.3) holds for $n = 1$.

$$Q_4^1(a,b) = \begin{pmatrix} \binom{1}{0}a & \binom{1}{1}b & \sum\limits_{i=0}^{-1}\binom{1-i}{i+2}a^{-1-2i}b^{i+2} & \sum\limits_{i=0}^{-1}\binom{-i}{i+2}a^{-2-2i}b^{i+3} \\ 0 & \binom{1}{0}a & \sum\limits_{i=0}^{0}\binom{1-i}{i+1}a^{2i}b^{i+1} & \sum\limits_{i=0}^{-1}\binom{-i}{i+1}a^{-1-2i}b^{i+2} \\ 0 & 0 & g_2(x) & bg_1(x) \\ 0 & 0 & g_1(x) & bg_0(x) \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Suppose the statement holds for $n = k$. Therefore, for $n = k+1$ we have,

$$Q_4^{k+1}(a,b) = \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \binom{k}{0}a^k & \binom{k}{1}a^{k-1}b & \sum\limits_{i=0}^{\lfloor\frac{k-2}{2}\rfloor}\binom{k-i}{i+2}a^{k-2-2i}b^{i+2} & \sum\limits_{i=0}^{\lfloor\frac{k-3}{2}\rfloor}\binom{k-1-i}{i+2}a^{k-3-2i}b^{i+3} \\ 0 & \binom{k}{0}a^k & \sum\limits_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k-i}{i+1}a^{k-1-2i}b^{i+1} & \sum\limits_{i=0}^{\lfloor\frac{k-2}{2}\rfloor}\binom{k-1-i}{i+1}a^{k-2-2i}b^{i+2} \\ 0 & 0 & g_{k+1} & bg_k \\ 0 & 0 & g_k & bg_{k-1} \end{pmatrix}$$

$$= \begin{pmatrix} \binom{k}{0}a^{k+1} & \binom{k}{1}a^k b + \binom{k}{0}a^k b & q_{1,3} & q_{1,4} \\ 0 & \binom{k}{0}a^{k+1} & a\sum\limits_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k-i}{i+1}a^{k-1-2i} + bg_{k+1} & a\sum\limits_{i=0}^{\lfloor\frac{k-2}{2}\rfloor}\binom{k-1-i}{i+1}a^{k-2-2i}b^{i+2} + b^2 g_k \\ 0 & 0 & ag_{k+1} + bg_k & b(ag_k + bg_{k-1}) \\ 0 & 0 & g_{k+1} & bg_k \end{pmatrix},$$

where,

$$\begin{cases} q_{1,3} = a\sum\limits_{i=0}^{\lfloor\frac{k-2}{2}\rfloor}\binom{k-i}{i+2}a^{k-2-2i}b^{i+2} + b\sum\limits_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k-i}{i+1}a^{k-1-2i}b^{i+1} \\ q_{1,4} = a\sum\limits_{i=0}^{\lfloor\frac{k-3}{2}\rfloor}\binom{k-1-i}{i+2}a^{k-3-2i}b^{i+3} + b\sum\limits_{i=0}^{\lfloor\frac{k-2}{2}\rfloor}\binom{k-1-i}{i+1}a^{k-2-2i}b^{i+2}. \end{cases}$$

Consider the first row of the last matrix. We need to show that the following four cases

$$\begin{cases} \binom{k}{0}a^{k+1} = \binom{k+1}{0}a^{k+1} \\ \binom{k}{1}a^k b + \binom{k}{0}a^k b = ka^k b + a^k b = \binom{k+1}{1}a^k b \\ q_{1,3} = \sum\limits_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k+1-i}{i+2}a^{k-1-2i}b^{i+2} \\ q_{1,4} = \sum\limits_{i=0}^{\lfloor\frac{k-2}{2}\rfloor}\binom{k-i}{i+2}a^{k-2-2i}b^{i+3} \end{cases}, \tag{2.4}$$

hold.

The first two cases of eq.(2.4)are easily verified. We will prove the third case of equation (2.4); fourth case is proved in a similar way. For the third case, there are two cases arise.

Case 1. Suppose k is even, so that $k = 2l$.Therefore

$$\left\lfloor\frac{k-2}{2}\right\rfloor = \left\lfloor\frac{2l-2}{2}\right\rfloor = l-1,$$
$$\left\lfloor\frac{k-1}{2}\right\rfloor = \left\lfloor\frac{2l-1}{2}\right\rfloor = l-1. \tag{2.5}$$

By substituting these relations in the L.H.S. of third case of equation (2.4), we get

$$\begin{aligned} q_{1,3} &= a\sum_{i=0}^{l-1}\binom{2l-i}{i+2}a^{2l-2-2i}b^{i+2} + b\sum_{i=0}^{l-1}\binom{2l-i}{i+1}a^{2l-1-2i}b^{i+1} \\ &= \sum_{i=0}^{l-1}\left[\binom{2l-i}{i+2} + \binom{2l-i}{i+1}\right]a^{2l-1-2i}b^{i+2} \\ &= \sum_{i=0}^{l-1}\binom{2l+1-i}{i+2}a^{2l-1-2i}b^{i+2} \end{aligned}$$

$$= \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k+1-i}{i+2} a^{k-1-2i} b^{i+2}.$$

Case 2. Now assuming $k = 2l+1$, we have

$$\left\lfloor \frac{k-2}{2} \right\rfloor = \left\lfloor \frac{2l-1}{2} \right\rfloor = l-1,$$
$$\left\lfloor \frac{k-1}{2} \right\rfloor = \left\lfloor \frac{2l}{2} \right\rfloor = l. \tag{2.6}$$

By substituting these relations in the L.H.S. of third case of equation (2.4), we get

$$\begin{aligned}
q_{1,3} &= a \sum_{i=0}^{l-1} \binom{2l+1-i}{i+2} a^{2l-1-2i} b^{i+2} + b \sum_{i=0}^{l} \binom{2l+1-i}{i+1} a^{2l-2i} b^{i+1} \\
&= \sum_{i=0}^{l-1} \binom{2l+1-i}{i+2} a^{2l-2i} b^{i+2} + \sum_{i=0}^{l} \binom{2l+1-i}{i+1} a^{2l-2i} b^{i+2} \\
&= \sum_{i=0}^{l-1} \left[\binom{2l+1-i}{i+2} + \binom{2l+1-i}{i+1} \right] a^{2l-2i} b^{i+2} + b^{l+2} \\
&= \sum_{i=0}^{l-1} \binom{2l+2-i}{i+2} a^{2l-2i} b^{i+2} + b^{l+2} \\
&= \sum_{i=0}^{l} \binom{2l+2-i}{i+2} a^{2l-2i} b^{i+2} \\
&= \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k+1-i}{i+2} a^{k-1-2i} b^{i+2}.
\end{aligned}$$

Further, other rows of $Q_4^{k+1}(a,b)$ can also be solved using above process.
This completes the proof. $\square$

Example 2.1. For $m = 6$ and n=5 we have

$$Q_6^5(a,b) = \begin{pmatrix}
a^5 & 5a^4b & 10a^3b^2 & 10a^2b^3 & 5ab^4 & b^5 \\
0 & a^5 & 5a^4b & 10a^3b^2 & 10a^2b^3 + b^4 & 4ab^4 \\
0 & 0 & a^5 & 5a^4b & 10a^3b^2 + 4ab^3 & 6a^2b^3 + b^4 \\
0 & 0 & 0 & a^5 & 5a^4b + 6a^2b^2 + b^3 & 4a^3b^2 + 3ab^3 \\
0 & 0 & 0 & 0 & a^5 + 3a^3b + a^2b + 3b^2 & a^4b + 3a^2b^2 + b^2 \\
0 & 0 & 0 & 0 & a^4 + 3a^2b^2 + b^2 & a^3b + 2ab^2
\end{pmatrix}.$$

2.2 Properties of Extended Generalized Fibonacci Polynomial

Lemma 2.1. *For $n \geq k$ and $k \geq 1$ we have*

$$g_n = \frac{1}{a^{n-k}} \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} g_{2n-k-2i} b^i, \tag{2.7}$$

where g_n is the n^{th} Extended Generalized Fibonacci Polynomial.

Proof. Let k be a fixed number. The proof is by induction on $n \geq k$. Suppose that the equation holds for $k \leq n \leq l$. we show that (2.7) holds for $n = l+1$.
Then by the recurrence relation, we have

$$\begin{aligned}
g_{l+1} &= a g_l + b g_{l-1} \\
&= a \left(\frac{1}{a^{l-k}} \sum_{i=0}^{l-k} (-1)^i \binom{l-k}{i} g_{2l-k-2i} b^i \right) + b \left(\frac{1}{a^{l-k-1}} \sum_{i=0}^{l-1-k} (-1)^i \binom{l-1-k}{i} g_{2l-2-k-2i} b^i \right)
\end{aligned}$$

$$= \frac{1}{a^{l-k-1}}\left(\sum_{i=1}^{l-k}(-1)^i\binom{l-k}{i}g_{2l-k-2i}b^i + \sum_{i=0}^{l-1-k}(-1)^i\binom{l-1-k}{i}g_{2l-2-k-2i}b^{i+1} + g_{2l-k}\right)$$
$$= \frac{1}{a^{l-k-1}}\left(\sum_{i=0}^{l-k-1}(-1)^{i+1}\binom{l-k}{i+1}g_{2l-2-k-2i}b^{i+1} + \sum_{i=0}^{l-1-k}(-1)^i\binom{l-1-k}{i}g_{2l-2-k-2i}b^{i+1} + g_{2l-k}\right)$$
$$= \frac{1}{a^{l-k-1}}\left(\sum_{i=0}^{l-k-1}(-1)^i\left(\binom{l-1-k}{i} - \binom{l-k}{i+1}\right)g_{2l-2-k-2i}b^{i+1} + g_{2l-k}\right)$$
$$= \frac{1}{a^{l-k-1}}\left(\sum_{i=0}^{l-k-2}(-1)^{i+1}\binom{l-1-k}{i+1}g_{2l-2-k-2i}b^{i+1} + g_{2l-k}\right)$$
$$= \frac{1}{a^{l-k-1}}\left(\sum_{i=1}^{l-k-1}(-1)^i\binom{l-1-k}{i}g_{2l-k-2i}b^i + g_{2l-k}\right)$$
$$= \frac{1}{a^{l-k-1}}\sum_{i=0}^{l-k-1}(-1)^i\binom{l-1-k}{i}g_{2l-k-2i}b^i.$$

Expanding this relation using (1.2), we get

$$= \frac{a^{k+1}}{a^l}\sum_{i=0}^{l-k-1}(-1)^i\binom{l-1-k}{i}g_{2l-k-2i}b^i$$
$$= \frac{a^{k-1}}{a^l}\sum_{i=0}^{l-k-1}(-1)^i\binom{l-k-1}{i}\left(g_{2l-k-2i+2}b^i - 2g_{2l-k-2i}b^{i+1} + g_{2l-k-2-2i}b^{i+2}\right)$$
$$= \frac{a^{k-1}}{a^l}\left(\sum_{i=2}^{l-k-1}(-1)^i\left(\binom{l-k-1}{i} + 2\binom{l-k-1}{i-1} + \binom{l-k-1}{i-2}\right)g_{2l+2-k-2i}b^i + g_{2l+2-k}\right.$$
$$- (l-k-1)g_{2l-k}b - 2g_{2l-k}b - 2(-1)^{l-k+1}g_{k+2}b^{l-k} - (-1)^{l-k+1}(l-k-1)g_{k+2}b^{l-k}$$
$$\left. + (-1)^{l-k+1}g_kb^{l+1-k}\right)$$
$$= \frac{1}{a^{l+1-k}}\sum_{i=0}^{l+1-k}(-1)^i\binom{l+1-k}{i}g_{2l+2-k-2i}b^i.$$

This completed the proof. □

Lemma 2.2. *Binet formula:- The n^{th} Extended generalized Fibonacci polynomial is given by*

$$g_n = \frac{z_1^n - z_2^n}{z_1 - z_2},$$

where z_1, z_2 are the roots of the characteristic equation (1.2) *and $z_1 > z_2$.*

Proof. We can express the recurrence relation (1.2) into the function of roots of z_1 and z_2 and the characteristic equation of recurrence relation (1.2) is $z^2 = az + b$. The roots of the characteristic equation are $z_1 = \frac{a+\sqrt{a^2+4b}}{2}$ and $z_2 = \frac{a-\sqrt{a^2+4b}}{2}$.
Note that $z_2 < 0 < z_1$ and $|z_2| < |z_1|$.Also $z_1 + z_2 = a$, $z_1z_2 = -b$ and $z_1 - z_2 = \sqrt{a^2+4b}$.
Therefore, the general terms of Extended Generalized Fibonacci Polynomial may be expressed in the form $g_n = P_1z_1^n + P_2z_2^n$, for some coefficient P_1 and P_2, for the value $n = 0$ and $n = 1$, we have $P_1 = \frac{1}{z_1-z_2} = -P_2$, so that $g_n = \frac{z_1^n-z_2^n}{z_1-z_2}$. □

Lemma 2.3. $\lim_{n\to\infty}\frac{g_n}{g_{n-1}} = z_1$
Where z_1 is the positive root of characteristic equation (1.2).

Proof. By using Binet formula (see, Lemma 2.2), we have

$$\lim_{n\to\infty}\frac{g_n}{g_{n-1}}=\lim_{n\to\infty}\left(\frac{z_1^n-z_2^n}{z_1-z_2}\times\frac{z_1-z_2}{z_1^{n-1}-z_2^{n-1}}\right)=\lim_{n\to\infty}\frac{1-(\frac{z_2}{z_1})^n}{\frac{1}{z_1}-(\frac{z_2}{z_1})^n(\frac{1}{z_2})}$$

and taking into account that $\lim\limits_{n\to\infty}(\frac{z_2}{z_1})^n=0$, Since $|z_2|<|z_1|$ then we get our result. □

2.3 The inverse Extended Generalized Fibonacci polynomial matrices

Now, by use of lemma 2.1, the next theorem establishes the structure of the inverse Extended Generalized Fibonacci Polynomial Matrix $Q_m^{-n}(a,b)$.

Theorem 2.2. *For $m\geq 2$, $n\geq 1$,$a\neq 0$and $b\neq 0$, the matrix $Q_m^{-n}(a,b)$ is in the form*
$Q_m^{-n}(a,b)=(AB)_{m\times m}$
where

$$A=\begin{pmatrix}\frac{\binom{n-1}{0}}{a^n} & -\frac{\binom{n}{1}b}{a^{n+1}} & -\frac{\binom{n+1}{2}b^2}{a^{n+2}} & \cdots & \frac{\binom{n+m-4}{m-3}b^{m-3}}{a^{n-m+3}}\\ 0 & \frac{\binom{n-1}{0}}{a^n} & -\frac{\binom{n}{1}b}{a^{n+1}} & \cdots & \frac{\binom{n+m-5}{m-4}b^{m-4}}{a^{n-m+4}}\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \cdots & \frac{\binom{n-1}{0}}{a^n}\\ 0 & 0 & \cdots & \cdots & 0\\ 0 & 0 & \cdots & \cdots & 0\end{pmatrix}_{m\times m-2},$$

$$B=\begin{pmatrix}\frac{(-a)^{m-2}a^{(m-3)(n-2)}}{(-a^{m-2})^n}\sum\limits_{i=0}^{n-2}(-1)^i\binom{n+m-4}{i}g_{2n-3-2i}b^{i+m-n-1} & \frac{(a^{m-3})^{n-1}}{(-a^{m-2})^n}\sum\limits_{i=0}^{n-1}(-1)^i\binom{n+m-3}{i}g_{2n-1-2i}b^{i+m-n-1}\\ \frac{(-a)^{m-3}a^{(m-4)(n-2)}}{(-a^{m-3})^n}\sum\limits_{i=0}^{n-2}(-1)^i\binom{n+m-5}{i}g_{2n-3-2i}b^{i+m-n-2} & \frac{(a^{m-4})^{n-1}}{(-a^{m-3})^n}\sum\limits_{i=0}^{n-1}(-1)^i\binom{n+m-4}{i}g_{2n-1-2i}b^{i+m-n-2}\\ \vdots & \vdots\\ \frac{-a}{(-a)^n}\sum\limits_{i=0}^{n-2}(-1)^i\binom{n-1}{i}g_{2n-3-2i}b^{i+2-n} & \frac{1}{(-a)^n}\sum\limits_{i=0}^{n-1}(-1)^i\binom{n}{i}g_{2n-1-2i}b^{i+2-n}\\ (-1)^n g_{n-1}b^{1-n} & (-1)^{n-1}g_nb^{1-n}\\ (-1)^{n-1}g_nb^{-n} & (-1)^n g_{n+1}b^{-n}\end{pmatrix}_{m\times 2}.$$

Proof. For the simplicity we prove the statement for $m=3$. We show that $Q_3^n(a,b)\times Q_3^{-n}(a,b)=I_{3\times 3}$ holds for any n, where $I_{3\times 3}$ is the identity matrix of order 3.

$$Q_3^n(a,b)\times Q_3^{-n}(a,b)=\begin{pmatrix}\binom{n}{0}a^n & \sum\limits_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-i}{i+1}a^{n-1-2i}b^{i+1} & \sum\limits_{i=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1-i}{i+1}a^{n-2-2i}b^{i+2}\\ 0 & g_{n+1} & bg_n\\ 0 & g_n & bg_{n-1}\end{pmatrix}$$

$$\times\begin{pmatrix}\frac{\binom{n-1}{0}}{a^n} & \frac{-a}{(-a)^n}\sum\limits_{i=0}^{n-2}(-1)^i\binom{n-1}{i}g_{2n-3-2i}b^{i+2-n} & \frac{1}{(-a)^n}\sum\limits_{i=0}^{n-1}(-1)^i\binom{n}{i}g_{2n-1-2i}b^{i+2-n}\\ 0 & \frac{(-1)^n}{b^{n-1}}g_{n-1} & \frac{(-1)^{n-1}}{b^{n-1}}g_n\\ 0 & \frac{(-1)^{n-1}}{b^n}g_n & \frac{(-1)^n}{b^n}g_{n+1}\end{pmatrix}.$$

Using the relation $g_{n+1}g_{n-1}-g_n^2=(-1)^nb^{n-1}$, it is easily verified that all the diagonal entries of this matrix are one. Now we have to show that all the other entries of this matrix are zero.for this, consider the elements of first row and second column.

$$q_{(1,2)}=\binom{n}{0}a^n\frac{-a}{(-a)^n}\sum_{i=0}^{n-2}(-1)^i\binom{n-1}{i}g_{2n-3-2i}b^{i+2-n}+\frac{(-1)^n}{b^{n-1}}g_{n-1}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-i}{i+1}a^{n-1-2i}b^{i+1}$$
$$+\frac{(-1)^{n-1}}{b^n}g_n\sum_{i=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1-i}{i+1}a^{n-2-2i}b^{i+2}.$$

For an even integer $n = 2l$, using (2.5) and (2.7), we have

$$\begin{aligned}
q_{(1,2)} &= (-a)\sum_{i=0}^{2l-2}(-1)^i\binom{2l-1}{i}g_{4l-3-2i}b^{i+2-2l} + \frac{g_{2l-1}}{b^{2l-1}}\sum_{i=0}^{l-1}\binom{2l-i}{i+1}a^{2l-1-2i}b^{i+1}\\
&\quad - \frac{g_{2l}}{b^{2l}}\sum_{i=0}^{l-1}\binom{2l-1-i}{i+1}a^{2l-2-2i}b^{i+2}\\
&= (-a)\sum_{i=0}^{2l-2}(-1)^i\binom{2l-1}{i}g_{4l-3-2i}b^{i+2-2l} + \frac{g_{2l-1}}{b^{2l-1}}\sum_{i=0}^{l-1}\binom{2l-i}{i+1}a^{2l-1-2i}b^{i+1}\\
&\quad - \frac{g_{2l-1}}{b^{2l}}\sum_{i=0}^{l-1}\binom{2l-1-i}{i+1}a^{2l-1-2i}b^{i+2} - \frac{g_{2l-2}}{b^{2l}}\sum_{i=0}^{l-1}\binom{2l-1-i}{i+1}a^{2l-2-2i}b^{i+3}\\
&= (-a)\sum_{i=0}^{2l-2}(-1)^i\binom{2l-1}{i}g_{4l-3-2i}b^{i+2-2l} + \frac{g_{2l-1}}{b^{2l}}\sum_{i=0}^{l-1}\binom{2l-1-i}{i}a^{2l-1-2i}b^{i+2}\\
&\quad - \frac{g_{2l-2}}{b^{2l}}\sum_{i=0}^{l-1}\binom{2l-1-i}{i+1}a^{2l-2-2i}b^{i+3}\\
&= (-a)\sum_{i=0}^{2l-2}(-1)^i\binom{2l-1}{i}g_{4l-3-2i}b^{i+2-2l} + \frac{g_{2l-1}}{b^{2l-2}}\sum_{i=0}^{l-1}\binom{2l-1-i}{i}a^{2l-1-2i}b^{i}\\
&\quad - \frac{g_{2l-2}}{b^{2l-2}}\left(g_{2l+1} - a^{2l}\right)\\
&= (-a)\sum_{i=0}^{2l-2}(-1)^i\binom{2l-1}{i}g_{4l-3-2i}b^{i+2-2l} + \frac{g_{2l-1}g_{2l}}{b^{2l-2}} - \frac{g_{2l-2}g_{2l+1}}{b^{2l-2}} + \frac{g_{2l-2}a^{2l}}{b^{2l-2}}\\
&= (-a)\sum_{i=0}^{2l-2}(-1)^i\binom{2l-1}{i}g_{4l-3-2i}b^{i+2-2l} + a + \frac{g_{2l-2}a^{2l}}{b^{2l-2}}\\
&= a\left(1 - \frac{1}{b^{2l-2}}\sum_{i=0}^{2l-2}(-1)^i\binom{2l-1}{i}g_{4l-3-2i}b^{i}\right) + \frac{g_{2l-2}a^{2l}}{b^{2l-2}}\\
&= a\left(1 - \frac{a^{2l-1}g_{2l-2}}{b^{2l-2}} - 1\right) + \frac{g_{2l-2}a^{2l}}{b^{2l-2}}\\
&= 0.
\end{aligned}$$

Now for odd number $n = 2l+1$, using the equations (2.6) and (2.7), we have

$$\begin{aligned}
q_{(1,2)} &= (a)\sum_{i=0}^{2l-1}(-1)^i\binom{2l}{i}g_{4l-1-2i}b^{i+1-2l} - \frac{g_{2l}}{b^{2l}}\sum_{i=0}^{l}\binom{2l+1-i}{i+1}a^{2l-2i}b^{i+1}\\
&\quad + \frac{g_{2l+1}}{b^{2l+1}}\sum_{i=0}^{l-1}\binom{2l-i}{i+1}a^{2l-1-2i}b^{i+2}\\
&= (a)\sum_{i=0}^{2l-1}(-1)^i\binom{2l}{i}g_{4l-1-2i}b^{i+1-2l} - \frac{g_{2l}}{b^{2l}}\sum_{i=0}^{l}\binom{2l+1-i}{i+1}a^{2l-2i}b^{i+1}\\
&\quad + \frac{g_{2l}}{b^{2l+1}}\sum_{i=0}^{l-1}\binom{2l-i}{i+1}a^{2l-2i}b^{i+2} + \frac{g_{2l-1}}{b^{2l+1}}\sum_{i=0}^{l-1}\binom{2l-i}{i+1}a^{2l-1-2i}b^{i+3}\\
&= (a)\sum_{i=0}^{2l-1}(-1)^i\binom{2l}{i}g_{4l-1-2i}b^{i+1-2l} - \frac{g_{2l}}{b^{2l-1}}\sum_{i=0}^{l}\binom{2l-i}{i}a^{2l-2i}b^{i} + \frac{g_{2l-1}}{b^{2l-1}}\sum_{i=0}^{l-1}\binom{2l-i}{i+1}a^{2l-1-2i}b^{i+1}\\
&= (a)\sum_{i=0}^{2l-1}(-1)^i\binom{2l}{i}g_{4l-1-2i}b^{i+1-2l} - \frac{g_{2l}g_{2l+1}}{b^{2l-1}} + \frac{g_{2l-1}}{b^{2l-1}}\left(g_{2l+2} - a^{2l+1}\right)
\end{aligned}$$

$$= (a)\sum_{i=0}^{2l-1}(-1)^i\binom{2l}{i}g_{4l-1-2i}b^{i+1-2l} + a - \frac{g_{2l-1}a^{2l+1}}{b^{2l-1}}$$
$$= (a)\left(1+\sum_{i=0}^{2l-1}(-1)^i\binom{2l}{i}g_{4l-1-2i}b^{i+1-2l}\right) - \frac{g_{2l-1}a^{2l+1}}{b^{2l-1}}$$
$$= a\left(1+\frac{a^{2l}g_{2l-1}}{b^{2l-2}}-1\right) - \frac{g_{2l-1}a^{2l+1}}{b^{2l-1}}$$
$$= 0.$$

Similarly, we can shown that any other non-diagonal entries of the matrix is also zero.
This completes the proof. □

2.4 The Extended Generalized Fibonacci Polynomial based coding algorithm

The Extended Generalized Fibonacci Polynomial coding algorithm is described in detail in this section.For coding and decoding algorithm Extended Generalized Fibonacci polynomials is converted into integer, for that we choose $a \neq 0$ and $b \neq 0$ for integer x such that a and b also gives non zero integer values. The initial message needs to be represented in the form of a square matrix M of order m, referred as the message-matrix, in order to employ this type of coding. This representation has no constraints and the user is free to arrange it how they want. For instance, the message 283954267 can be represented by the message matrix of order 2:

$$M = \begin{pmatrix} 283 & 95 \\ 42 & 67 \end{pmatrix}.$$

The encoding matrix $Q_m^n(a,b)$ is obtained from (2.3). Once the sender and receiver agree on above parameters and an integer n. To get the message matrix E, multiply the encoding matrix by the message matrix M from right side. For example, for $m = 3$ and $n = 2$ we have

$$E = Q_3^2(a,b)M_{3\times 3}$$
$$= \begin{pmatrix} a^2 & 2ab & b^2 \\ 0 & a^2+b & ab \\ 0 & a & b \end{pmatrix}\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}.$$

The elements of E are delivered by the channel in the following order $e_{11}, e_{12}, e_{13}, \cdots, e_{33}$, followed by the value of $det(M)$. Assuming that the send sequence is received without error, the original message matrix is produced by multiplying E and $Q_3^{-2}(a,b)$:

$$M = Q_3^{-2}(a,b)E$$
$$= \begin{pmatrix} \frac{1}{a^2} & -\frac{1}{a} & 1-\frac{b}{a^2} \\ 0 & \frac{1}{b} & -\frac{a}{b} \\ 0 & -\frac{a}{b^2} & \frac{a^2+b}{b^2} \end{pmatrix}\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}.$$

2.5 A relation among the elements of a code message-matrix

Inside this part, we develop a fascinating relationship between the components of a code message matrix E, which plays an important role in the error-correction process. Let m = 3 for the sake of simplicity. Assume that all values of M are positive and $a, b \geq 1$ for $(x \geq 1)$. Therefore,

$$M = Q_3^{-n}(a,b) \times E$$
$$= \begin{pmatrix} \frac{\binom{n-1}{0}}{a^n} & \frac{-a}{(-a)^n}\sum_{i=0}^{n-2}(-1)^i\binom{n-1}{i}g_{2n-3-2i}b^{i+2-n} & \frac{1}{(-a)^n}\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}g_{2n-1-2i}b^{i+2-n} \\ 0 & \frac{(-1)^n}{b^{n-1}}g_{n-1} & \frac{(-1)^{n-1}}{b^{n-1}}g_n \\ 0 & \frac{(-1)^{n-1}}{b^n}g_n & \frac{(-1)^n}{b^n}g_{n+1} \end{pmatrix} \times \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$$
$$= \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}.$$

For the elements of the first columns of M, we have

$$\begin{cases} m_{11} = e_{11} + (-1)^{n-1}ae_{21}\sum_{i=0}^{n-2}(-1)^i\binom{n-1}{i}g_{2n-3-2i}b^{i+2-n} + (-1)^n e_{31}\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}g_{2n-1-2i}b^{i+2-n} \geq 0; \\ m_{21} = \frac{(-1)^n}{b^{n-1}}e_{21}g_{n-1} + \frac{(-1)^{n-1}}{b^{n-1}}e_{31}g_n \geq 0; \\ m_{31} = \frac{(-1)^{n-1}}{b^n}e_{21}g_n + \frac{(-1)^n}{b^n}e_{31}g_{n+1} \geq 0. \end{cases}$$

Using (2.7) for an even integer $n = 2l$, we obtain the following inequalities.

$$\begin{cases} e_{11} - ae_{21}\left(\frac{a^{2l-1}g_{2l-2}}{b^{2l-2}} + 1\right) + be_{31}\left(\frac{a^{2l}g_{2l-1}}{b^{2l-1}} - 1\right) \geq 01 & (a) \\ \frac{e_{21}g_{2l-1}}{b^{2l-1}} - \frac{e_{31}g_{2l}}{b^{2l-1}} \geq 0 & (b) \\ -\frac{e_{21}g_{2l}}{b^{2l}} + \frac{e_{31}g_{2l+1}}{b^{2l}} \geq 0 & (c) \end{cases}. \tag{2.8}$$

From (2.8)(b) and (2.8)(c), we have

$$\frac{g_{2l}}{g_{2l-1}} \leq \frac{e_{21}}{e_{31}} \leq \frac{g_{2l+1}}{g_{2l}}. \tag{2.9}$$

It follows from 2.8(a) that

$$\frac{e_{11}}{e_{31}} \geq a\frac{e_{21}}{e_{31}}\Big(\frac{a^{2l-1}g_{2l-2}}{b^{2l-2}} + 1\Big) - b\Big(\frac{a^{2l}g_{2l-1}}{b^{2l-1}} - 1\Big).$$

This together with (2.9) gives

$$\begin{aligned} \frac{e_{11}}{e_{31}} &\geq a\frac{g_{2l}}{g_{2l-1}}\Big(\frac{a^{2l-1}g_{2l-2}}{b^{2l-2}} + 1\Big) - b\Big(\frac{a^{2l}g_{2l-1}}{b^{2l-1}} - 1\Big) \\ &\geq \frac{a^{2l}}{b^{2l-2}g_{2l-1}}\Big(g_{2l}g_{2l-2} - g_{2l-1}^2\Big) + \frac{ag_{2l}}{g_{2l-1}} + b \\ &\geq -\frac{a^{2l}b}{g_{2l-1}} + \frac{ag_{2l} + bg_{2l-1}}{g_{2l-1}} \\ \frac{e_{11}}{e_{31}} &\geq \frac{g_{2l+1} - a^{2l}b}{g_{2l-1}}. \end{aligned} \tag{2.10}$$

Similarly, dividing (2.8)(a) by e_{11} results in

$$b\frac{e_{31}}{e_{11}}\Big(\frac{a^{2l}g_{2l-1}}{b^{2l-1}} - 1\Big) \geq a\frac{e_{21}}{e_{11}}\Big(\frac{a^{2l-1}g_{2l-2}}{b^{2l-2}} + 1\Big) - 1.$$

It follows from this and (2.9) that

$$b\frac{e_{31}}{e_{11}}\Big(\frac{a^{2l}g_{2l-1}}{b^{2l-1}} - 1\Big) \geq a\frac{g_{2l}}{g_{2l-1}}\Big(\frac{a^{2l-1}g_{2l-2}}{b^{2l-2}} + 1\Big) - 1 \tag{2.11}$$

and hence

$$\frac{e_{11}}{e_{31}} \leq \frac{g_{2l+1} - a^{2l}b}{g_{2l-1}}. \tag{2.12}$$

Using (**??**) and (2.12), we get

$$\frac{e_{11}}{e_{31}} = \frac{g_{2l+1} - a^{2l}b}{g_{2l-1}}. \tag{2.13}$$

For l large enough, we have from equation (2.9) and (2.13)

$$\frac{e_{11}}{e_{31}} \approx \sigma^2, \qquad \frac{e_{21}}{e_{31}} \approx \sigma,$$

where,

$$\sigma = \frac{a + \sqrt{a^2 + 4b}}{2}.$$

Therefore,

$$\frac{e_{11}}{e_{21}} \approx \sigma.$$

Similarly, assuming that (2.8) for $n = 2l + 1$, we obtain,

$$\frac{g_{2l+2}}{g_{2l+1}} \leq \frac{e_{21}}{e_{31}} \leq \frac{g_{2l+1}}{g_{2l}}$$

$$\frac{g_{2l+2} - a^{2l+1}b}{g_{2l}} \leq \frac{e_{11}}{e_{31}} \leq \frac{g_{2l+2} - a^{2l+1}b}{g_{2l}}$$

For l large enough, we have

$$\frac{e_{11}}{e_{31}} \approx \sigma^2, \qquad \frac{e_{21}}{e_{31}} \approx \sigma.$$

Therefore,

$$\frac{e_{11}}{e_{21}} \approx \sigma.$$

The result is that for large values of n, the following equation holds.

$$\frac{e_{11}}{e_{21}} \approx \frac{e_{21}}{e_{31}} \approx \sigma.$$

In general, for $1 \leq i \leq m$ we get

$$\frac{e_{1,i}}{e_{2,i}} \approx \frac{e_{2,i}}{e_{3,i}} \approx \cdots \approx \frac{e_{m-1,i}}{e_{m,i}} \approx \sigma, \tag{2.14}$$

where $e_{i,j}$ is the element of i^{th} row and j^{th} column of message matrix.

2.6 Error-detection and error-correction

The Fibonacci Polynomial based coding error-correction technique has been developed in [5]. This approach is used in Extended Generalized Fibonacci Polynomial based coding method described. We will start with error detection.From $E = Q_m^n(a,b)M$ we have,

$$det(E) = det(M) \times (-a^{m-2}b)^n \tag{2.15}$$

Using determinant theorem, we have $\det(Q_m^n(a,b)) = (-a^{m-2}b)^n$. Relation (2.15) is controlled when an estimation matrix $\hat{E}$ is rebuilt using the received elements. If the relation is satisfied, we claim there was no error. otherwise, either the components of E or $det(M)$ are incorrect. We may presume that the number $det(M)$ was received correctly after sending it many times and utilising majority logic decoding. As a result, relation (2.15) is regarded as criterion for detecting errors. Assume that some of elements of E are incorrect. Of course, this matrix might have one-fold, two-fold,$\cdots$, or m^2-fold errors.

For simplicity consider a 2×2 receiving matrix to demonstrate how to remedy these problem. Three cases are examined.

Case1. Assume that one of the elements was delivered incorrectly. Then one of the four cases below is feasible, where p, q, r and s are the incorrect elements.

$$\begin{pmatrix} p & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \qquad \begin{pmatrix} e_{11} & q \\ e_{21} & e_{22} \end{pmatrix} \qquad \begin{pmatrix} e_{11} & e_{12} \\ r & e_{22} \end{pmatrix} \qquad \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & s \end{pmatrix}.$$

It follows from (2.15) and $det(Q_2^n(a,b)) = (-b)^n$ that

$pe_{22} - e_{12}e_{21} = (-b)^n det(M)$,

$e_{11}e_{22} - qe_{21} = (-b)^n det(M)$,

$e_{11}e_{22} - re_{12} = (-b)^n det(M)$,

$se_{11} - e_{12}e_{21} = (-b)^n det(M)$,

or equivalently

$p = \dfrac{(-b)^n det(M) + e_{12}e_{21}}{e_{22}}$,

$q = \dfrac{-(-b)^n det(M) + e_{11}e_{22}}{e_{21}}$,

$r = \dfrac{-(-b)^n det(M) + e_{11}e_{22}}{e_{12}}$,

$s = \dfrac{(-b)^n det(M) + e_{12}e_{21}}{e_{11}}$.

The above equation provides four alternative single-error variations, but we must select the right variant only from the instance of integer solutions p, q, r and s; moreover, we must select solutions that satisfy the relation (2.14). Note that only numbers that are integers and satisfy (2.14) are the elements of E.If no such elements is obtained from these equations, we must conclude that our single-error hypothesis is false and we have to consider multiple-fold error cases.

Case 2. Suppose that two elements of E was delivered incorrectly as shown below:

$\begin{pmatrix} p & e_{12} \\ q & e_{22} \end{pmatrix}.$

From (2.15) we have $pe_{22} - e_{12}q = (-b)^n det(M)$. Since above equation has many solutions, we have to choose solutions of p and q, which satisfy (2.14). Again only integer solutions are acceptable. It's worth nothing that if the two errors occur in the same row or in one of the matrix's two diagonals, they may be readily fixed by just applying (2.14). Two-fold error do not arise if no integer solution is discovered. If none of the cases above produce solutions that fulfil the criteria, then all of the elements of E have been received incorrectly. Errors cannot be remedied in this case.

Case 3. Assume that three elements of E was delivered incorrectly as shown below

$\begin{pmatrix} p & q \\ r & e_{22} \end{pmatrix}.$

From (2.14), q can be obtained. Now remaining errors can be corrected by case2 solution.

If none of the cases above produce solutions that fulfil the criteria, then all of the elements of E have been received incorrectly. Errors cannot be remedied in this case.

According to the method described in [11], there are consequently 15 error conditions in the elements of E. Since 14 cases between them can be corrected, the approach's correctable probability is equal to $\frac{14}{15} = 0.9333 = 93.33\%$. Capability to fix errors: Because only m^2-fold faults may notbe rectified, As in [2], the method's error-correction capacity is $\frac{2^{m^2}-2}{2^{m^2-1}}$, where m is the message-matrix order. As a result, the probability of decoding mistake is nearly nil for large values of m.

3 Conclusion

We presented a coding scheme based on Extended Generalized Fibonacci polynomials. The encoder matrix for integers $m \geq 2$, $a, b \geq 1$ and $n \geq 1$ is a matrix $Q_m^n(a, b)$, the nth power of $Q_m(a, b)$ with Extended Generalized Fibonacci polynomial elements.Further, established some properties of Extended Generalized Fibonacci polynomial. Each source word is represented by a matrix M that has been encoded into a code message matrix $E = MQ_m^n(a, b)$. The suggested coding scheme was given a basic error-correcting algorithm. We demonstrated that this approach can correct up to $\frac{2^{m^2}-2}{2^{m^2}-1}$ mistakes, implying that the chance of decoding error is nearly nil for large values of m.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 238-242**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

GROUP ANALYSIS FOR KLEIN-GORDON EQUATION VIA THEIR SYMMETRIES

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(Received: March 29, 2023*; In format : April* 04, 2023*; Revised : May* 24. 2023*; Accepted : June* 15, 2023*)*

DOI: https://doi.org/10.58250/jnanabha.2023.53128

Abstract

In the present paper we have obtained the one parameter groups and symmetry transformations associated to the classical symmetries of the Klein-Gordon (KG) equation, we have also constructed an optimal system of two dimensional sub-algebras of the KG equation which provides the preliminary classification of group invariant solutions and yield the most general group invariant solution.
2020 Mathematical Sciences Classification: 17B66, 22E70, 35Q75, 46N50, 70G65, 81R05
Keywords and Phrases: Space Invariance, Translation, Rotation, Hyperbolic Rotation.

1 Introduction

Symmetry method for differential equations, was originally developed by Lie [8] these methods without any doubt are very useful and algorithmic for analyzing and solving linear and nonlinear differential equations. Classification of differential equation as well as linearization of them are some other important applications of symmetry transformation approach. Symmetry groups of a system of partial differential equation is a group of transformations G on the space of independent and dependent variables which has the property that the elements of G transform solution of the system to other solution of the system. The general prolongation formula for computing the symmetry groups for infinitesimal generators of a group of transformations was given by Olver [11, 12, 13] for obtaining the group invariance solution of linear and non-linear differential equation. The group invariant solution of complex modified KdV equation has been studied by Hyzel [3] and for the differential equation describing the radial jet having finite fluid velocity at orifice has been studied by Naeem & Naz [9]. Pal et al. [14, 15] obtained the group invariant solution with the help of infinitesimal parameter. Hereman et al. [5], obtained the exact travelling wave solutions of KG equation with cubic nonlinearity by using direct algebraic method. Ye and Zhang [16], obtained exact travelling wave solutions of KG equation with cubic nonlinearity by using the bifurcation method and qualitative theory of dynamical systems. Dehghan and Ali [2], obtained numerical solutions of KG equation with quadratic and cubic nonlinearity by using radial basis function and analyze the accuracy of their results with the analytical solutions. Jang [6], obtained the travelling wave solutions of nonlinear KG equations. Gupta and Sharma [4] obtained the exact travelling wave solutions for the KG equation with cubic nonlinearity by using First Integral Method. Other researcher also applied the different approaches to obtain the invariant solution of KG equation [7, 10].

In this paper we extend the application of general prolongation formula to find the most general solution of the KG equation

$$\left(1/c^2\right)u_{tt} = u_{xx} + u_{yy} - \left(m^2c^2/\hbar^2\right)u$$

which use to model the two dimensional motion of free particle with mass m (Bates[1]), where u is wave function, c denotes the velocity of light and $\hbar$ is plank constant.

2 Solution of Klein-Gordon Equation

We find the group invariant solution by calculating the symmetries for two-dimensional KG equation for the motion of free particle with mass m. The equation

$$\left(1/c^2\right)u_{tt} - u_{xx} - u_{yy} + \left(m^2c^2/\hbar^2\right)u = 0 \tag{2.1}$$

which is the second order differential equation with three independent variables and one dependent variable. A vector field on X × U takes the form

$$v = \xi(x,y,t,u)\partial_x + \eta(x,y,t,u)\partial_y + \tau(x,y,t,u)\partial_t + \phi(x,y,t,u)\partial_u, \tag{2.2}$$

where ξ, η, τ and ϕ are the smooth coefficient functions. Using General Prolongation formula (Olver[13], equation (2.38), page 110) the second prolongation of v

$$pr^{(2)}v = v + \phi^x\left(\partial/\partial u_x\right) + \phi^y\left(\partial/\partial u_y\right) + \phi^t\left(\partial/\partial u_t\right) + \phi^{xx}\left(\partial/\partial u_{xx}\right) + \phi^{xy}\left(\partial/\partial u_{xy}\right) + \phi^{xt}\left(\partial/\partial u_{xt}\right) + \phi^{yy}\left(\partial/\partial u_{yy}\right) + \phi^{yt}\left(\partial/\partial u_{yt}\right) + \phi^{tt}\left(\partial/\partial u_{tt}\right), \tag{2.3}$$

and the coefficients present in (2.3) are calculated by using (Olver[13], equation (2.39), page 110), and by the use of infinitesimal criterion of invariance (Olver[13], equation (2.26), page 104) the two-dimensional KG equation takes the form

$$\left(\phi^{tt}/c^2\right) - \phi^{xx} - \phi^{yy} + \left(m^2c^2/\hbar^2\right)\phi = Q\left(\left(u_{tt}/c^2\right) - u_{xx} - u_{yy} + \left(m^2c^2/\hbar^2\right)u\right), \tag{2.4}$$

in which $Q\left(x,y,t,u^{(2)}\right)$ depend up-to second order derivatives of u. By substituting the values of $\phi^{tt}, \phi^{xx}, \phi^{yy}$ and ϕ in equation (2.4) and equating the coefficients of the terms in the first and second order partial derivatives of u, the determining equations for the symmetry group of the two-dimensional KG equation for a free particle are found as follows

***Table* 2.1**

Monomial	Coefficient	Equation Number	Monomial	Coefficient	Equation Number
1	$\frac{\phi_{tt}}{c^2} - \phi_{xx} - \phi_{yy} + \frac{m^2c^2\phi}{\hbar^2} - \frac{m^2c^2Q\,u}{\hbar^2} = 0$	(40)	u_{yt}	$-\frac{2\eta_t}{c^2} + 2\tau_y = 0$	(20)
u_x	$-\frac{\xi_{tt}}{c^2} - \left(2\phi_{xu} - \xi_{xx} - \xi_{yy}\right) = 0$	(39)	u_{xy}	$2\eta_x + 2\xi_y = 0$	(19)
u_y	$-\frac{\eta_{tt}}{c^2} - \left(2\phi_{yu} - \eta_{xx} - \eta_{yy}\right) = 0$	(38)	u_{tt}	$\frac{\left(\phi_u - 2\tau_t\right)}{c^2} = \frac{Q}{c^2}$	(18)
u_t	$\frac{\left(2\phi_{tu} - \tau_{tt}\right)}{c^2} + \tau_{xx} + \tau_{yy} = 0$	(37)	u_{xx}	$-\left(\phi_u - 2\xi_x\right) = -Q$	(17)
u_x^2	$-\left(\phi_{uu} - 2\xi_{xu}\right) = 0$	(36)	u_{yy}	$-\left(\phi_u - 2\eta_y\right) = -Q$	(16)
u_y^2	$-\left(\phi_{uu} - 2\eta_{yu}\right) = 0$	(35)	$u_x u_{tt}$	$-\frac{\xi_u}{c^2} = 0$	(15)
u_t^2	$\frac{\left(\phi_{uu} - 2\tau_{tu}\right)}{c^2} = 0$	(34)	$u_y u_{tt}$	$-\frac{\eta_u}{c^2} = 0$	(14)
u_x^3	$\xi_{uu} = 0$	(33)	$u_t u_{tt}$	$-\frac{3\tau_u}{c^2} = 0$	(13)
u_y^3	$\eta_{uu} = 0$	(32)	$u_x u_{xx}$	$3\xi_u = 0$	(12)
u_t^3	$-\frac{\tau_{uu}}{c^2} = 0$	(31)	$u_y u_{xx}$	$\eta_u = 0$	(11)
$u_x u_t$	$-\frac{2\xi_{tu}}{c^2} + 2\tau_{xu} = 0$	(30)	$u_t u_{xx}$	$\tau_u = 0$	(10)
$u_y u_t$	$-\frac{2\eta_{tu}}{c^2} + 2\tau_{yu} = 0$	(29)	$u_x u_{yy}$	$\xi_u = 0$	(9)
$u_x u_y$	$2\xi_{yu} + 2\eta_{xu} = 0$	(28)	$u_y u_{yy}$	$3\eta_u = 0$	(8)
$u_x u_t^2$	$-\frac{\xi_{uu}}{c^2} = 0$	(27)	$u_t u_{yy}$	$\tau_u = 0$	(7)
$u_y u_t^2$	$-\frac{\eta_{uu}}{c^2} = 0$	(26)	$u_t u_{xt}$	$-\frac{2\xi_u}{c^2} = 0$	(6)
$u_x^2 u_t$	$\tau_{uu} = 0$	(25)	$u_t u_{yt}$	$-\frac{2\eta_u}{c^2} = 0$	(5)
$u_y^2 u_t$	$\tau_{uu} = 0$	(24)	$u_x u_{xy}$	$2\eta_u = 0$	(4)
$u_x^2 u_y$	$\eta_{uu} = 0$	(23)	$u_x u_{xt}$	$2\tau_u = 0$	(3)
$u_y^2 u_x$	$\xi_{uu} = 0$	(22)	$u_y u_{xy}$	$2\xi_u = 0$	(2)
u_{xt}	$-\frac{2\xi_t}{c^2} + 2\tau_x = 0$	(21)	$u_y u_{yt}$	$2\eta_u = 0$	(1)

The requirement for equations (1) to (15) is that ξ, η and τ are independent of u, equations (16), (17) and (18) give $\xi_x = \tau_t = \eta_y$, equations (19), (20) and (21) give $\left(\xi_t/c^2\right) = \tau_x$, $\left(\eta_t/c^2\right) = \tau_y$ and $\eta_x = -\xi_y$, equations (34), (35) and (36) give $\phi = \beta u + \alpha$ where $\alpha = \alpha(x, y, t)$ and $\beta = \beta(x, y, t)$ are functions. From the equation (37), (38) and (39) we get $\beta_x = 0, \beta_y = 0$ and $\beta_t = 0$, from (40) we find $\beta = Q = c_4\left(\hbar^2/\mathrm{m}^2c^2\right)$. The most general infinitesimal symmetry of the two-dimensional KG equation in ideal fluid has coefficient function of the form $\xi = c_5 y + (c_6 t + c_2)/c^2, \eta = -c_5\, x + (c_7 t + c_2)/c^2, \tau = c_6 x + c_7 y + c_1 c^2$ and $\phi = c_4\left(\hbar^2/m^2c^2\right)u + \alpha$ where $c_1, \ldots, c_7$ are arbitrary constant and α is an arbitrary solution of the KG equation. The Lie algebras of infinitesimal symmetries of two-dimensional KG equation for a free particle with mass m is spanned by the seven vector fields $v_1 = c^2\partial_t, v_2 = c^2\partial_x, v_3 = c^2\partial_y, v_4 = \left(\hbar^2/m^2c^2\right)u\partial_u, v_5 = y\partial_x - x\partial_y, v_6 = c^2\, t\partial_x + x\partial t, v_7 = c^2 t\partial_x + x\partial_t$, and the infinite-dimensional sub-algebra $v_\alpha = \alpha\partial_u$ where α is an arbitrary solution of two-dimensional KG equation. The commutation relation between these vector fields are given by the following

***Table* 2.2:** Commutation-Relation

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_α
v_1	0	0	0	0	0	c^2v_2	c^2v_3	$c^2v_{\alpha_t}$
v_2	0	0	0	0	$-v_3$	v_1	0	$c^2v_{\alpha_x}$
v_3	0	0	0	0	v_2	0	v_1	$c^2v_{\alpha_y}$
v_4	0	0	0	0	0	0	0	$-\left(\frac{\hbar^2}{m^2c^2}\right)v_\alpha$
v_5	0	v_3	$-v_2$	0	0	v_7	$-v_6$	v_{α_1}
v_6	$-c^2v_2$	$-v_1$	0	0	$-v_7$	0	$-c^2v_5$	v_{α_2}
v_7	$-c^2v_3$	0	$-v_1$	0	v_6	c^2v_5	0	v_{α_3}
v_α	$-c^2v_{\alpha_t}$	$-c^2v_{\alpha_x}$	$-c^2v_{\alpha_y}$	$\left(\frac{\hbar^2}{m^2c^2}\right)v_\alpha$	$-v_{\alpha_1}$	$-v_{\alpha_2}$	$-v_{\alpha_3}$	0

where $\alpha_1 = y\alpha_x - x\alpha_y, \alpha_2 = tc^2\alpha_x + x\alpha_t, \alpha_3 = tc^2\alpha_y + y\alpha_t$.

The one-parameter groups G_i generated by the v_i are given as follows

$G_1 : \left(x, y, t + c^2\varepsilon, u\right), G_2 : \left(x + c^2\varepsilon, y, t, u\right), G_3 : \left(x, y + c^2\varepsilon, t, u\right)$,

$G_4 : \left(x, y, t, e^{\left(\frac{\hbar^2\varepsilon}{m^2c^2}\right)}u\right), G_5 : (x\cos\varepsilon + y\sin\varepsilon, y\cos\varepsilon - x\sin\varepsilon, t, u)$,

$G_6 : (x\cosh c\varepsilon + tc\sinh c\varepsilon, y, t\cosh c\varepsilon + (x/c)\sinh c\varepsilon, u)$,

$G_7 : (x, y\cosh c\varepsilon + tc\sinh\varepsilon, t\cosh c\varepsilon + (y/c)\sinh c, \varepsilon)$,

$G_\alpha : (x, y, t, u + \varepsilon\alpha)$ where each G_i is a symmetry group.

If we take $u = f(x, y, t)$ be a solution of the KG equation then the functions

$u^{(1)} = f\left(x, y, t - c^2\varepsilon\right), u^{(2)} = f\left(x - c^2\varepsilon, y, t\right), u^{(3)} = f\left(x, y - c^2\varepsilon, t\right)$,

$u^{(4)} = e^{\left(\frac{\hbar^2\varepsilon}{m^2c^2}\right)}f(x, y, t), u^{(5)} = f(x\cos\varepsilon - y\sin\varepsilon, y\cos\varepsilon + x\sin\varepsilon, t)$,

$u^{(6)} = f(x\cosh c\varepsilon - tc\sinh c\varepsilon, y, t\cosh c\varepsilon - (x/c)\sinh c\varepsilon)$,

$u^{(7)} = f(x, y\cosh c\varepsilon - tc\sinh c\varepsilon, t\cosh c\varepsilon - (y/c)\sinh\varepsilon)$,

$u^{(\alpha)} = f(x, y, t) + \varepsilon\alpha(x, y, t)$ where ε is any real number and $\alpha(x, y, t)$ is any other solution to two dimensional KG equation for a free particle with mass m. At the end the most general solution that we can obtain from a given solution $u = f(x, y, t)$, by group transformations is in the form given below

$$\begin{aligned} u = e^{\left(\frac{\hbar^2\varepsilon_4}{m^2c^2}\right)} f\Big(&x\cosh c\varepsilon_6\cdot\cos\varepsilon_5 - tc\sinh c\varepsilon_6 - y\sin\varepsilon_5 - c^2\varepsilon_2,\\ &y\cosh c\varepsilon_7\cdot\cos\varepsilon_5 - tc\sinh c\varepsilon_7 + x\sin\varepsilon_5 - c^2\varepsilon_3,\\ &t\cosh c\varepsilon_7\cdot\cosh c\varepsilon_6 - (y/c)\sinh c\varepsilon_7 - (x/c)\sinh c\varepsilon_6 - c^2\varepsilon_1\Big) + \alpha(x, y, t), \end{aligned} \quad (2.5)$$

where $\varepsilon_1, \ldots, \varepsilon_7$ are real constant and α be an arbitrary solution to two-dimensional KG equation for free particle with mass m.

3 Special Case

If we take $\hbar = c = 1$ then equation (2.5) reduces to

$$
\begin{aligned}
u = e^{\left(\frac{\varepsilon_4}{m^2}\right)} f\,(& x\cosh\varepsilon_6\cdot\cos\varepsilon_5 - t\sinh\varepsilon_6 - y\sin\varepsilon_5-\varepsilon_2,\\
& y\cosh\varepsilon_7\cdot\cos\varepsilon_5 - t\sinh\varepsilon_7 + x\sin\varepsilon_5-\varepsilon_3,\\
& t\cosh\varepsilon_7\cdot\cosh\varepsilon_6 - y\sinh\varepsilon_7 - x\sinh\varepsilon_6-\varepsilon_1)\\
& + \alpha(x,y,t),
\end{aligned}
$$

where $\varepsilon_1,\ldots\varepsilon_7$ are real constant and α be an arbitrary solution to two-dimensional KG equation for free particle with $\hbar = c = 1$ and mass m.

If we take y = 0 and $\hbar = c = 1$ then equation (2.5) reduces to

$$
u = e^{\left(\frac{\varepsilon_4}{m^2}\right)} f\,(x\cosh\varepsilon_6 - t\sinh\varepsilon_6-\varepsilon_2, t\cosh c\varepsilon_6 - x\sinh\varepsilon_6-\varepsilon_1) + \alpha(x,t),
$$

where $\varepsilon_1,\ldots\varepsilon_7$ are real constant and α be an arbitrary solution to one-dimensional KG equation for free particle with $\hbar = c = 1$ and mass m.

4 Conclusion

In our investigation the symmetry group G_4 and G_α reflects the linearity of two-dimensional KG equation for free particle with mass m. The group G_1 is space invariance symmetry group. The group G_2 and G_3 are time invariance group. The group G_5 represent rotational symmetry group. The group G_6 and G_7 are well-known hyperbolic rotational symmetry group.

Acknowledgement. We are very much thankful to the Editor and Reviewer for their valuable suggestions to bring the paper in its present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 243-252**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

SHORTEST PATH ON INTERVAL-VALUED INTUITIONISTIC TRAPEZOIDAL NEUTROSOPHIC FUZZY GRAPHS WITH APPLICATION

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(Received: July 10, 2021; Informat: August 2021; Received April 26, 2023, Accepted: June 15, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53129

Abstract

In this article, stretch esteemed Interval Valued Intuitionistic Trapezoidal Neutrosophic Fuzzy Graph (IVITrNFG) of SPP, which is drew on trapezoidal numbers and IVITrNFG. Hear a genuine application is given an illustrative model for IVITrNFG. Additionally Shortest way is determined for this model. This present Dijkstra's Algorithm briefest way was checked.
2010 Mathematics Subject Classification: 05C85
Keywords and Phrases: Interval-Valued Intuitionistic Fuzzy Number (IVIFN), Trapezoidal Fuzzy Number (TrFN), Shortest Path (SP), Interval-Valued Intuitionistic Trapezoidal Neutrosophic Fuzzy Graph(IVITrNFG).

1 Introduction

The creators Ahuja et al. [1] examined systematic execution of Dijkstra's calculation. Arsham [2] introduced another crucial arrangement calculation which permits affectability examination without utilizing any counterfeit, slack or surplus factors. Anusuya et al. [3] apply positioning capacity for briefest way issue. Broumi et al. [4] proposed for extend esteemed neutrosophic number. Broumi [5] presented neutrosophic charts with most limited way issues. Broumi [6] proposed calculation gives Shortest way issue on single esteemed neutrosophic charts. Broumi [7] proposed the Shortest way under Bipolar Neutrosophic setting. Broumi [8] gave the Shortest way issue under span esteemed neutrosophic setting. Chiranjibe Jana et al. [9] Presented Trapezoidal neutrosophic aggregation operators and its application in multiple attribute decision making process. De et al. [10] Computation of Shortest Path in a fuzzy organization. De et al. [11] study on ranking of trapezoidal intuitionistic fuzzy numbers. Enayattabar [12] introduced Dijkstra calculation for briefest way issue under Pythagorean fuzzy climate. Jana et al. [14] presented stretch esteemed trapezoidal neutrosophic set. Jayagowri et al. [15] discover Optimized Path in a Network utilizing trapezoidal intuitionistic fuzzy numbers. Kalaiarasi et al. [16] determine fuzzy optimal total cost and fuzzy optimal order quantity obtained by ranking function method and Kuhn-tucker method for the proposed inventory model. Kalaiarasi et al. [17] constructed inventory parameters that are fuzzy using trapezoidal fuzzy numbers. Kumar et al. [18] proposed to tackling briefest way issue with edge weight. Kumar et al. [19] introduced Algorithm for most limited way issue in an organization with span esteemed intuitionistic trapezoidal fuzzy number. Kumar et al. [20] presented the SPP from an underlying hub to an objective hub on neutrosophic chart . Majumdar et al. [21] introduced an intuitionistic fuzzy most brief way organization. Gani et al. [13] looking intuitionistic fuzzy most brief organization. Ojekudo et al. [22] tended to the most brief way utilizing Dijkstra's calculation. Said Broumi [23] processing the most brief way Neutrosophic Information. Smarandache et al. [24] summed up the fuzzy rationale and presented two neutrosophic ideas. Victor Christianto et al. [25] gave a neutrosophic approach to futurology. Wang et al. [26] contributed neutrosophic sets with their properties. Xu [27] introduced a strategies for amassing span esteemed intuitionistic fuzzy data, Yang et al. [28] introduced rectangular hindrance subject to various improvement capacities regarding the quantity of curves. Ye [29] proposed a Trapezoidal fuzzy Neural Computing and Applications. Ye [30] developed of the Multi models dynamic strategy utilizing shape liking measure, Ye [31] presented a Prioritized aggregation operators of trapezoidal intuitionistic fuzzy sets and their application.

Here, in this paper disclosed the briefest way to India famous seven tourist places utilized the proposed calculation.

Section 2, introduces some basic concepts related to definitions. Section 3, introduces *IVITrNFG* proposed algorithm and find *SPP* using that proposed algorithm. Section 4, we apply real life application. The application has India famous seven tourist place and find its *SPP* using *IVITrNFG* proposed algorithm . Section 5, verifies shortest path on India famous seven tourist place with Dijkstras algorithm. Conclusion is given in Section 6.

2 Methodology

In this section we explain some important definitions.

Definition 2.1. *Let*

$$\bar{n}_1 = \Big\langle ([(t_a^L, t_b^L, t_c^L, t_d^L), (t_e^L, t_f^L, t_g^L, t_h^L)], [(t_a^U, t_b^U, t_c^U, t_d^U), (t_e^U, t_f^U, t_g^U, t_h^U)]),$$
$$([(i_a^L, i_b^L, i_c^L, i_d^L), (i_e^L, i_f^L, i_g^L, i_h^L)], [(i_a^U, i_b^U, i_c^U, i_d^U), (i_e^U, i_f^U, i_g^U, i_h^U)]),$$
$$([(f_a^L, f_b^L, f_c^L, f_d^L), (f_a^L, f_b^L, f_c^L, f_d^L)], [(f_e^U, f_f^U, f_g^U, f_h^U), (f_e^U, f_f^U, f_g^U, f_h^U)])\Big\rangle$$

and

$$\bar{n}_2 = \Big\langle ([(T_a^L, T_b^L, T_c^L, T_d^L), (T_e^L, T_f^L, T_g^L, T_h^L)], [(T_a^U, T_b^U, T_c^U, T_d^U), (T_e^U, T_f^U, T_g^U, T_h^U)]),$$
$$([(I_a^L, I_b^L, I_c^L, I_d^L), (I_e^L, I_f^L, I_g^L, I_h^L)], [(I_a^U, I_b^U, I_c^U, I_d^U), (I_e^U, I_f^U, I_g^U, I_h^U)]),$$
$$([(F_a^L, F_b^L, F_c^L, F_d^L), (F_e^L, F_f^L, F_g^L, F_h^L)], [(F_a^U, F_b^U, F_c^U, F_d^U), (F_e^U, F_f^U, F_g^U, F_h^U)])\Big\rangle$$

both Interval-Valued Trapezoidal Neutrosophic Numbers. Therefore following procedure holds:

$$\bar{n}_2 = \Big\langle ([(T_a^L, T_b^L, T_c^L, T_d^L), (T_e^L, T_f^L, T_g^L, T_h^L)], [(T_a^U, T_b^U, T_c^U, T_d^U), (T_e^U, T_f^U, T_g^U, T_h^U)]),$$
$$([(I_a^L, I_b^L, I_c^L, I_d^L), (I_e^L, I_f^L, I_g^L, I_h^L)], [(I_a^U, I_b^U, I_c^U, I_d^U), (I_e^U, I_f^U, I_g^U, I_h^U)]),$$
$$([(F_a^L, F_b^L, F_c^L, F_d^L), (F_e^L, F_f^L, F_g^L, F_h^L)], [(F_a^U, F_b^U, F_c^U, F_d^U), (F_e^U, F_f^U, F_g^U, F_h^U)])\Big\rangle$$

We propose definition of score and accuracy functions for an Interval-Valued Trapezoidal Neutrosophic Number.

Definition 2.2. *Let*

$$\bar{n}_1 = \Big\langle ([(t_a^L, t_b^L, t_c^L, t_d^L), (t_e^L, t_f^L, t_g^L, t_h^L)], [(t_a^U, t_b^U, t_c^U, t_d^U), (t_e^U, t_f^U, t_g^U, t_h^U)]),$$
$$([(i_a^L, i_b^L, i_c^L, i_d^L), (i_e^L, i_f^L, i_g^L, i_h^L)], [(i_a^U, i_b^U, i_c^U, i_d^U), (i_e^U, i_f^U, i_g^U, i_h^U)]),$$
$$([(f_a^L, f_b^L, f_c^L, f_d^L), (f_a^L, f_b^L, f_c^L, f_d^L)], [(f_e^U, f_f^U, f_g^U, f_h^U), (f_e^U, f_f^U, f_g^U, f_h^U)])\Big\rangle$$

and be an Interval-Valued, Intuitionistic Trapezoidal Neutrosophic Number, then their score functions are defined as

$$S(\bar{n}) = \frac{1}{3}\left[\begin{array}{r} 2 + \frac{[(t_a^U + t_b^U + t_c^U + t_d^U + t_e^L + t_f^L + t_g^L + t_h^L) - (t_a^L + t_b^L + t_c^L + t_d^L + t_e^U + t_f^U + t_g^U + t_h^U)]}{8} \\ -\frac{[(i_a^U + i_b^U + i_c^U + i_d^U + i_e^L + i_f^L + i_g^L + i_h^L) - (i_a^L + i_b^L + i_c^L + i_d^L + i_e^U + i_f^U + i_g^U + i_h^U)]}{8} \\ -\frac{[(f_a^U + f_b^U + f_c^U + f_d^U + f_e^L + f_f^L + f_g^L + f_h^L) - (f_a^L + f_b^L + f_c^L + f_d^L + f_e^U + f_f^U + f_g^U + f_h^U)]}{8} \end{array}\right], S(\bar{n}) \in [-1, 1]$$

where the higher value of $S(\bar{n})$, larger the Interval-Valued Intuitionistic Trapezoidal Number $\bar{n}$.

3 Interval-Valued Intuitionistic Trapezoidal Neutrosophic Fuzzy Graph

In this research, we using proposed algorithm for finding shortest path.

Step 3.1 Let

$$d_1 = \langle [(0,0,0,0),(0,0,0,0)],[(0,0,0,0),(0,0,0,0)],[(1,1,1,1),(1,1,1,1)],\\ [(1,1,1,1),(1,1,1,1)],[(1,1,1,1),(1,1,1,1)],[(1,1,1,1),(1,1,1,1)]\rangle$$

and the source node as

$$d_1 = \langle [(0,0,0,0),(0,0,0,0)],[(0,0,0,0),(0,0,0,0)],[(1,1,1,1),(1,1,1,1)],\\ [(1,1,1,1),(1,1,1,1)],[(1,1,1,1),(1,1,1,1)],[(1,1,1,1),(1,1,1,1)]\rangle$$

and the source node as

Step 3.2 Find $d_j = \text{minimum}\{d_i \oplus d_{ij}\};\ j = 2,3,\ldots,n.$

Step 3.3 If the minimum value of i, ie., $i = r$ then the lable node j as $[d_j, r]$. If minimum arise related to more than one values of i. Their position we choose minimum value of i.

Step 3.4 Let the destination node be $[d_n, l]$. Here source node is d_n. We conclude a score function and we finds minimum value of Interval-Valued Trapezoidal Neutrosophic Number.

Step 3.5 We calculate shortest path problem between source and destination node. Review the label of node 1. Let it be as $[d_n, A]$. Now review the label of node A and so on. Replicate the same procedure until node 1 is procured.

Step 3.6 The shortest path can be procured by combined all the nodes by the **Step 3.5**.

4 Data Analysis

To find shortest path on India famous seven tourist place using Interval-Valued Intuitionistic Trapezoidal Neutrosophic Fuzzy Graph.

***Figure* 4.1:** The Beaches of Goa

***Figure* 4.2:** Gate way of India

***Figure* 4.3:** Mecca Masjid

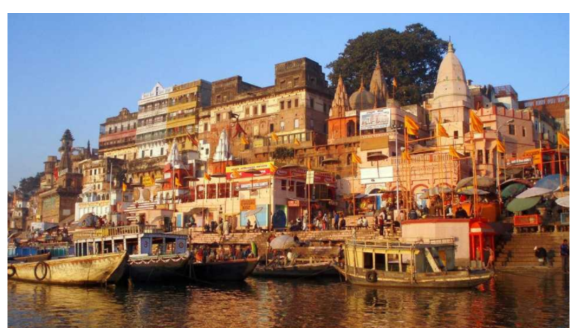


***Figure* 4.4:** Holy City of Varanasi

***Figure* 4.5:** Taj mahal

***Figure* 4.6:** The Golden City (Jaisalmer)

***Figure* 4.7:** Harmandir Sahib

Here we consider source node is The Beaches of Goa and destination node is Sri Harmandir Sahib. To find Shortest Path on The Beaches of Goa to Sri Harmandir Sahib. Here distance between one tourist place

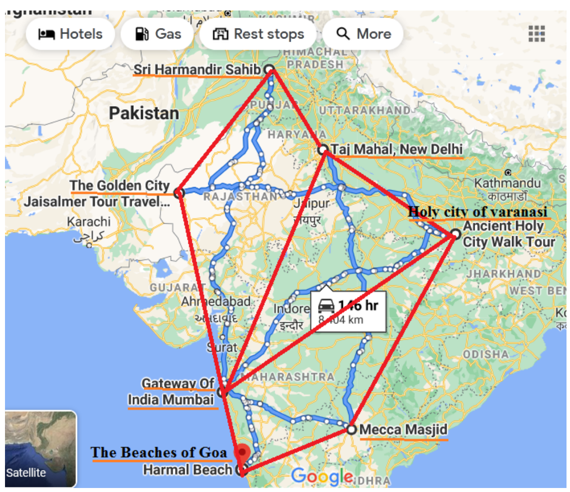


***Figure* 4.8:** A Graph Of India Famous Seven Tourist Place

to another tourist place is calculated in kilometers. The numerical value of the distance is converted to *IVITrNFG* with the help of through trapezoidal signed distance.

The given distance (kilometer) converted to neutrosophic number. We converted neutrosophic number as (a_1, a_2, a_3, a_4) are membership function & $(a_1^*, a_2^*, a_3^*, a_4^*)$ are non-membership function. These functions converted to fuzzy trapezoidal numbers using trapezoidal signed distance $\dfrac{a_1 + a_2 + a_3 + a_4}{4}$. Finally converted Interval-Valued Intuitionistic Trapezoidal Neutrosophic Fuzzy Number.

Here, Apply the *IVITrNFN* in our algorithm to find shortest path to India famous seven tourist place.

Edges	Interval-Valued, Intuitionistic Trapezoidal Fuzzy Neutrosophic Numbers
1-2	$\langle([(0.20, 0.29, 0.35, 0.56), (0.49, 0.59, 0.65, 0.87)]), ([(0.8, 0.71, 0.65, 0.44), (0.51, 0.41, 0.35, 0.13)]), ([(0.11, 0.13, 0.16, 0.2), (0.79, 0.83, 0.86, 0.92)]), ([(0.89, 0.87, 0.84, 0.8), (0.21, 0.17, 0.14, 0.08)]), ([(0.003, 0.005, 0.03, 0.08), (0.9615, 0.9699, 0.9705, 0.9801)]), ([(0.997, 0.995, 0.97, 0.92), (0.0385, 0.0301, 0.0295, 0.0199)])\rangle$
1-3	$\langle([(0.91, 0.92, 0.94, 0.99), (0.02, 0.04, 0.06, 0.12)]), ([(0.09, 0.08, 0.06, 0.01), (0.98, 0.96, 0.94, 0.88)]), ([(0.52, 0.55, 0.6, 0.69), (0.35, 0.4, 0.42, 0.47)])([(0.48, 0.45, 0.4, 0.31), (0.65, 0.6, 0.58, 0.53)]), ([(0.09, 0.12, 0.15, 0.24), (0.80, 0.83, 0.86, 0.91)]), ([(0.91, 0.88, 0.85, 0.76), (0.2, 0.17, 0.14, 0.09)])\rangle$
2-4	$\langle([(0.82, 0.86, 0.89, 0.91), (0.09, 0.11, 0.13, 0.19)]), ([(0.18, 0.14, 0.11, 0.09), (0.91, 0.89, 0.87, 0.81)]), ([(0.17, 0.2, 0.23, 0.32), (0.72, 0.76, 0.79, 0.81)])([(0.83, 0.8, 0.77, 0.68), (0.28, 0.24, 0.21, 0.19)]), ([(0.14, 0.16, 0.18, 0.24), (0.79, 0.81, 0.83, 0.85)]), ([(0.86, 0.84, 0.82, 0.76), (0.21, 0.19, 0.17, 0.15)])\rangle$
2-5	$\langle([(0.92, 0.94, 0.96, 0.98), (0.02, 0.04, 0.06, 0.08)]), ([(0.08, 0.06, 0.04, 0.02), (0.98, 0.96, 0.94, 0.92)]), ([(0.32, 0.39, 0.45, 0.64), (0.49, 0.52, 0.58, 0.61)])([(0.68, 0.61, 0.55, 0.36), (0.51, 0.48, 0.42, 0.39)]), ([(0.11, 0.059, 0.08, 0.16), (0.899, 0.919, 0.923, 0.951)]), ([(0.989, 0.941, 0.92, 0.84), (0.101, 0.081, 0.077, 0.049)])\rangle$
2-6	$\langle([(0.84, 0.86, 0.89, 0.97), (0.07, 0.09, 0.11, 0.17)]), ([(0.16, 0.14, 0.11, 0.03), (0.93, 0.91, 0.89, 0.83)]), ([(0.1, 0.2, 0.3, 0.6), (0.4, 0.7, 0.8, 0.9)])([(0.9, 0.8, 0.7, 0.4), (0.6, 0.3, 0.2, 0.1)]), ([(0.21, 0.25, 0.27, 0.35), (0.65, 0.71, 0.74, 0.82)]), ([(0.79, 0.75, 0.73, 0.65), (0.35, 0.29, 0.26, 0.18)])\rangle$
2-4	$\langle([(0.94, 0.95, 0.96, 0.99), (0.02, 0.03, 0.04, 0.07)]), ([(0.06, 0.05, 0.04, 0.01), (0.98, 0.97, 0.96, 0.93)]), ([(0.23, 0.27, 0.35, 0.51), (0.52, 0.57, 0.66, 0.89)])([(0.77, 0.73, 0.65, 0.49), (0.48, 0.43, 0.34, 0.11)]), ([(0.17, 0.21, 0.26, 0.32), (0.59, 0.68, 0.79, 0.98)]), ([(0.83, 0.79, 0.74, 0.68), (0.41, 0.32, 0.21, 0.02)])\rangle$
4-5	$\langle([(0.79, 0.85, 0.89, 0.91), (0.07, 0.09, 0.15, 0.25)]), ([(0.21, 0.15, 0.11, 0.09), (0.930.91, 0.85, 0.75)]), ([(0.25, 0.31, 0.37, 0.51), (0.47, 0.58, 0.64, 0.87)])([(0.75, 0.69, 0.63, 0.49), (0.53, 0.42, 0.36, 0.13)]), ([(0.09, 0.15, 0.26, 0.5), (0.59, 0.67, 0.75, 0.99)]), ([(0.91, 0.85, 0.74, 0.5), (0.41, 0.33, 0.25, 0.01)])\rangle$
5-7	$\langle([(0.79, 0.86, 0.89, 0.98), (0.06, 0.09, 0.12, 0.21)]), ([(0.21, 0.14, 0.11, 0.02), (0.94, 0.91, 0.88, 0.79)]), ([(0.4, 0.5, 0.6, 0.9), (0.2, 0.3, 0.4, 0.7)]), ([(0.6, 0.5, 0.4, 0.1), (0.8, 0.7, 0.6, 0.3)]), ([(0.065, 0.085, 0.127, 0.277), (0.79, 0.81, 0.86, 0.98)]), ([(0.935, 0.915, 0.873, 0.723), (0.21, 0.19, 0.14, 0.02)])\rangle$
6-7	$\langle([(0.85, 0.87, 0.89, 0.95), (0.07, 0.09, 0.11, 0.17)]), ([(0.15, 0.13, 0.11, 0.05), (0.93, 0.91, 0.89, 0.83)]), ([(0.29, 0.37, 0.41, 0.57), (0.37, 0.48, 0.59, 0.92)])([(0.71, 0.63, 0.59, 0.43), (0.63, 0.52, 0.41, 0.08)]), ([(0.09, 0.17, 0.23, 0.43), (0.59, 0.68, 0.87, 0.94)]), ([(0.91, 0.83, 0.77, 0.57), (0.41, 0.32, 0.13, 0.06)])\rangle$

Iteration 4.1 Assume the initial value

$$d_1 = \langle[(0,0,0,0),(0,0,0,0)],[(0,0,0,0),(0,0,0,0)],[(1,1,1,1),(1,1,1,1)],\\ [(1,1,1,1),(1,1,1,1)],[(1,1,1,1),(1,1,1,1)],[(1,1,1,1),(1,1,1,1)]\rangle$$

Here we assume d_1 is the beaches of Goa.

Iteration 4.2 In this iteration was calculated through proposed algorithm from the tourist place The Beaches of Goa to Gate Way of India. The labeled node is Gate Way of India and minimum provided corresponding node is The Beaches of Goa.

Minimum Node	Labeled Node	Path Node
The Beaches of Goa	Gate Way of India	$\langle([(0.20,0.29,0.35,0.56),(0.49,0.59,0.65,0.87)]),([(0.8,0.71,0.65,0.44),(0.51,0.41,0.35,0.13)]),([(0.11,0.13,0.16,0.2),(0.79,0.83,0.86,0.92)]),([(0.89,0.87,0.84,0.8),(0.21,0.17,0.14,0.08)]),([(0.003,0.005,0.03,0.08),(0.9615,0.9699,0.9705,0.9801)]),([(0.997,0.995,0.97,0.92),(0.0385,0.0301,0.0295,0.0199)])\rangle$

Iteration 4.3 The node Mecca Masjid was forerunner node of The Beaches of Goa. Here the labeled node is Mecca Masjid and the minimum provided corresponding node is The Beaches of Goa.

Minimum Node	Labeled Node	Path Node
The Beaches of Goa	Mecca Masjid	$\langle([(0.91,0.92,0.94,0.99),(0.02,0.04,0.06,0.12)]),([(0.09,0.08,0.06,0.01),(0.98,0.96,0.94,0.88)]),([(0.52,0.55,0.6,0.69),(0.35,0.4,0.42,0.47)])([(0.48,0.45,0.4,0.31),(0.65,0.6,0.58,0.53)]),([(0.09,0.12,0.15,0.24),(0.80,0.83,0.86,0.91)]),([(0.91,0.88,0.85,0.76),(0.2,0.17,0.14,0.09)])\rangle$

Iteration 4.4 The node Holy City of Varanasi has two forerunner node , they are Mecca Masjid and Gate Way of India. *IVITrNSP* is calculated to Holy City of Varanasi from Mecca Masjid and Gate Way of India. Here, the labeled node is Holy City of Varanasi and the minimum provided corresponding node is Gate Way of India.

Minimum Node	Labeled Node	Path Node
Gate Way of India	Holy City of Varanasi	$\langle([(0.856,0.901,0.928,0.96),(0.535,0.635,0.695,0.895)]),([(0.836,0.751,0.688,0.490),(0.956,0.935,0915,0.835)]),([(0.0187,0.026,0.0368,0.064),(0.5688,0.6308,0.6794,0.7452)]),([(0.7387,0.696,0.6468,0.544),(0.0588,0.0408,0.0294,0.0152)]),([(0.00042,0.0008,0.0054,0.0192),(0.7596,0.7856,0.8055,0.833)]),([(0.8574,0.8358,0.7954,0.6992),(0.008,0.0057,0.005,0.0029)])\rangle$

Iteration 4.5 The node Taj Mahal has two forerunner node, they are Gate Way of India and Holy City of Varanasi. *IVITrNSP* is calculated to Taj Mahal from Gate Way of India and Holy City of Varanasi. Here,

the labeled node is Taj Mahal and the minimum provided corresponding node is Gate Way of India.

Minimum Node	Labeled Node	Path Node
Gate Way of India	Taj Mahal	$\langle([(0.94, 0.96, 0.97, 0.99), (0.5, 0.6, 0.67, 0.88)]), ([(0.816, 0.73, 0.66, 0.45), (0.99, 0.98, 0.96, 0.93)]), ([(0.035, 0.051, 0.072, 0.128), (0.387, 0.432, 0.4988, 0.5612)]), ([(0.6052, 0.531, 0.462, 0.288), (0.107, 0.0816, 0.0588, 0.0312)]), ([(0.000033, 0.00029, 0.0024, 0.0128), (0.8644, 0.8913, 0.8958, 0.932)])([(0.986, 0.936, 0.892, 0.773), (0.00389, 0.0024, 0.0023, 0.00097)])\rangle$

Iteration 4.6 The node The Golden City was forerunner node of Gate Way of India. Here the labeled node is The Golden City and the minimum provided corresponding node is Gate Way of India.

Minimum Node	Labeled Node	Path Node
Gate Way of India	The Golden City	$\langle([(0.872, 0.9, 0.9285, 0.9868), (0.5257, 0.6269, 0.6885, 0.8921)]), ([(0.832, 0.75, 0.6885, 0.4568), (0.9657, 0.947, 0.928, 0.852)]), ([(0.011, 0.026, 0.048, 0.12), (0.316, 0.581, 0.688, 0.828)]), ([(0.801, 0.696, 0.588, 0.32), (0.126, 0.051, 0.028, 0.008)]), ([(0.00063, 0.00125, 0.0081, 0.028), (0.6249, 0.6886, 0.718, 0.8036)]), ([(0.787, 0.746, 0.708, 0.598), (0.0135, 0.0087, 0.0077, 0.0036)])\rangle$

Iteration 4.7 The node Sri Harmandir Sahib has two forerunner node , they are Taj Mahal and The Golden City. *IVITrNSP* is calculated to Sri Harmandir Sahib from Taj Mahal and The Golden City. The labeled node is Sri Harmandir Sahib and the minimum provided corresponding node is Taj Mahal.

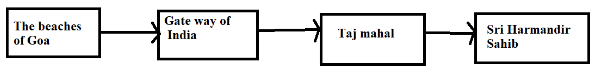
Minimum Node	Labeled Node	Path Node
Taj Mahal	Sri Harmandir Sahib	$\langle([(0.9874, 0.9944, 0.9967, 0.9998), (0.53, 0.636, 0.7096, 0.9052)]), ([(0.8546, 0.7678, 0.6974, 0.461), (0.9994, 0.9982, 0.9952, 0.9853)]), ([(0.014, 0.0255, 0.0432, 0.1152), (0.0774, 0.1296, 0.1995, 0.3928)]), ([(0.36312, 0.2655, 0.1848, 0.0288), (0.0856, 0.05712, 0.03528, 0.00936)]), ([(0.000002145, 0.000024, 0.0003, 0.0035), (0.6828, 0.72195, 0.77, 0.9134)]), ([(0.9219, 0.856, 0.779, 0.559), (0.0008, 0.00045, 0.000322, 0.0000194)])\rangle$

Since Sri Harmandir Sahib is the destination node.

We calculate SP to destination node to source node. Since

Labeled Node	**Minimum Node**
Sri Harmandir Sahib	Taj Mahal
Taj Mahal	Gate Way of India
Gate Way of India	The Beaches of Goa

Therefore the seven wonders Interval-Valued Nether Trapezoidal Neutrosophic Fuzzy Graph Shortest Path is



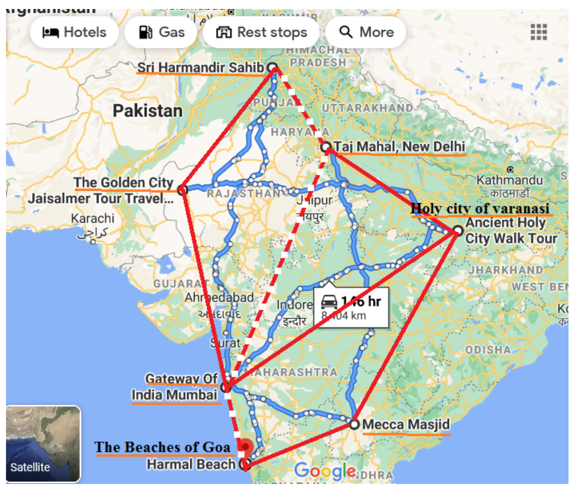


***Figure* 4.9:** *SP* from The Beaches of Goa to Sri Harmandir Sahib

5 Shortest Path On Dijkstra's Algorithm

In the above real life application, we clarify another method of SPP using Dijkstras algorithm. In this SPP, we use direct method of Dijkstras algorithm and we assume edge weight is India famous seven tourist place km.

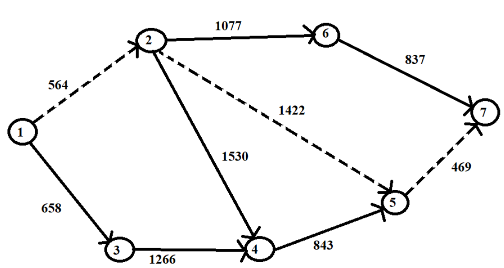


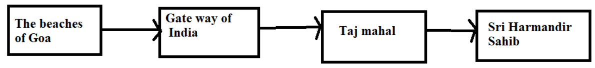
***Figure* 5.1:** *SP* for Dijkstra's Algorithm

Here, we verify India famous seven tourist place shortest path through Dijkstras Algorithm. We have the paths are

$$1 \to 2 \to 5 \to 7$$

Here these two paths Interval-Valued Intuitionistic Trapezoidal Neutrosophic Fuzzy Graphs and Dijkstra's Algorithm are same. The shortest path is

$$1 \rightarrow 2 \rightarrow 5 \rightarrow 7$$



6 Conclusion

In this article, discovering *SP* on India famous seven tourist place using Interval-Valued Intuitionistic Trapezoidal Neutrosophic Fuzzy Graph. A genuine application is given to act as an *IVITrNFG*. Finally checked most brief way *SP* on India famous seven tourist place with Dijkstra's algorithm.

Acknowledgment

The authors thank to the Editor and Referees for their valuable suggestions which led to improvement of this paper.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 253-262**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

FISHER-SHANNON ENTROPIC UNCERTAINTY RELATIONS AND THEIR POWER-PRODUCTS AS A MEASURE OF ELECTRONIC CORRELATION

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(Received: February 22, 2022; In format: March 19, 2022; Revised: May 19, 2023; Accepted: May 26, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53130

Abstract

In this paper, we have presented an analytical model of two electron systems consisting of a many particle correlated wave function with some variational parameters α, λ and μ and used it to quantify the electron-electron correlation described by the wave function containing explicitly r_{12} (inter atomic distance between two electrons) dependent term. The single particle wave functions and the charge densities have been extracted from the said correlated wave function both for the uncorrelated and correlated systems in coordinate space and its momentum analogs have been obtained by taking the Fourier transform of the coordinate analogs. We have computed and presented the results of the numerical values of the theoretic information entropies of the Shannon entropy, Fisher information entropy, Shannon power and the FisherShannon product. The numerical values are consistently found to satisfy the Beckner, Bialynicki-Birula and Mycielski (BBM) inequality relation; Stam-Cramer-Rao inequalities or Fisher based uncertainty relation and Fisher-Shannon product relation for the uncorrelated and correlated systems in both the coordinate and momentum spaces.
2020 Mathematical Sciences Classification: 62B10, 94A15, 94A17.
Keywords and Phrases: Coordinate and momentum space; uncorrelated and correlated system; Shannon information entropy; Fisher information entropy; uncertainty relations; Fisher-Shannon product.

1 Introduction

The electron correlation is a major problem in physics of atoms, molecules, and clusters as a consequence of the electronelectron repulsion. The correlation effect has a major influence on measureable quantities in atomic systems. The correlation energy (E_{corr}) [9] of a many-electron system is defined by the difference between the exact total energy (the exact non-relativistic energy) and Hartree-Fock energy, as well as by some statistical correlation coefficients [15] which assess radial and angular correlation in both the coordinate and momentum density distributions. The correlation energy had been used as a guide [16] for the amount of correlation in a given system. Recently, some information-theoretic measures of the electron correlation in atomic systems have been proposed: the so-called correlation entropy [30] which is the information entropy of the one-particle density matrix, and the sum of the Shannon information entropies of the electron density in coordinate and momentum spaces [20]. The entropic uncertainty relation has many applications both in physics and chemistry [27, 28] and because of their many applications in different areas of physics and chemistry, there have been a growing interest by many researchers in studying Shannon entropy and Fisher information in recent years. The two most important measures of the information theories are the Shannon entropy(S) [25] and Fisher information entropy(I) [10]. These two information entropies carry out a vital role in different areas of physics and chemistry. The entropic uncertainty relations in quantum information theory have been proved to be an alternative to the Heisenberg uncertainty relation in quantum mechanics [14, 17]. On one hand, the Shannon entropic uncertainty relation in coordinate and momentum spaces satisfy the Beckner, Bialynicki-Birula and Mycielski (BBM) inequality relation as [4, 6],

$$S_T = (S_\rho + S_\gamma) \geqslant D(1 + ln\pi), \tag{1.1}$$

where D represents the spatial dimension, S_ρ is the Shannon entropy in the coordinate space, S_γ is the corresponding Shannon entropy in the momentum space and S_T is the Shannon entropy sum. The entropies

S_ρ and S_γ are defined as [4,6,21,22],

$$S_\rho = -\int \rho(\vec{r}) ln[\rho(\vec{r})] d^3r, \tag{1.2}$$

$$S_\gamma = -\int \gamma(\vec{p}) ln[\gamma(\vec{p})] d^3p, \tag{1.3}$$

where $d^3r = r^2 dr d\Omega$, $d^3p = p^2 dp d\Omega$ and $d\Omega = \sin\theta d\theta d\varphi$ is the solid angle with $\psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$ being the normalized wave function in the spatial coordinate, $\rho(\vec{r}) = \int |\psi(\vec{r}, \vec{r}_2, ..., \vec{r}_N)|^2 d^3r_2...$
d^3r_N is the single particle charge density in the spatial coordinate and $\gamma(\vec{p}) = \int |\phi(\vec{p}, \vec{p}_2, ..., \vec{p}_N|^2$
$d^3p_2...d^3p_N$ is the single particle charge density in momentum space. The Shannon information entropy is usually regarded as the measure of the spatial spread of the wave function for different states [12]. One of the consequences of the *BBM* inequality is that represents the lower bound values of the Shannon entropy sum [4,6] such that if the coordinate entropy increases, then the momentum entropy will decrease in such a way that their sum bounds above (*BBM*) inequality. On the other hand, Fisher information is a local measure since it is sensitive to local rearrangement of the density. It has been reported that the higher the Fisher information, the more localized is the charge density [2,18], and conversely, the smaller the uncertainty the higher the accuracy in predicting the localization of the particles [2,18]. The Fisher information is defined as the gradient functional of the charge density of the system and is given in the coordinate and momentum spaces as [1,19]

$$I_\rho = \int \frac{1}{\rho(\vec{r})} [\vec{\nabla}\rho(\vec{r})]^2 d^3r, \tag{1.4}$$

$$I_\gamma = \int \frac{1}{\gamma(\vec{p})} [\vec{\nabla}\gamma(\vec{p})]^2 d^3p. \tag{1.5}$$

The disorder aspect of Fisher information entropy has been studied in some length by Frieden [11]. The uncertainty properties are clearly delineated by the Stam inequalities [26]. The product $I_\rho I_\gamma$ has been conjectured to exhibit a nontrivial lower bound [7] such that for three-dimensional systems it reads as:

$$I_\rho I_\gamma \geqslant 36. \tag{1.6}$$

Unlike, the Shannon entropy that satisfy the *BBM* inequality, the Fisher information fulfills the Stam inequalities [23], $I_\rho \leqslant 4 < p^2 >$, $I_\gamma \leqslant 4 < r^2 >$ and the Cramer-Rao inequalities [8] $I_\rho \geqslant \frac{9}{<r^2>}$, $I_\gamma \geqslant \frac{9}{<p^2>}$. Generally, for an arbitrary angular momentum quantum number *'l'* of any central potential model, the two products of the Fisher information must satisfied the relation [24],

$$I_\rho I_\gamma \geqslant 4 < r^2 >< p^2 > [2 - \frac{2l+1}{l(l+1)}|m|]^2, \tag{1.7}$$

where $m = 0, \pm1, \pm2$ . . . is the magnetic quantum number. With the help of the definitions of equations (1.1) to (1.5), we can define the Shannon power (J) in coordinate and momentum space as

$$J_\rho = \frac{1}{2\pi e} e^{\frac{2S_\rho}{D}}, \tag{1.8}$$

$$J_\gamma = \frac{1}{2\pi e} e^{\frac{2S_\gamma}{D}}, \tag{1.9}$$

and the Fisher-Shannon product(P) in coordinate and momentum space are defined as

$$P_\rho = \frac{I_\rho J_\rho}{D}, \tag{1.10}$$

and

$$P_\gamma = \frac{I_\gamma J_\gamma}{D}, \tag{1.11}$$

which must satisfy the following relation

$$P_{\rho\gamma} = P_\rho P_\gamma \geqslant 1, \tag{1.12}$$

where D is the spatial dimensions [29]. It is necessary to mention that throughout our all calculations, we shall use D=3 and m=$\hbar$=e=1. In this paper, we are going to study an analytical model of two electron system consisting of 'Hartree and Ingman(1933)' [13] type correlated wave function with some variational parameters α, λ and μ. The aim of our present work is to use the derived analytical model to quantify the correlation in two electron systems described by wave function containing explicitly r_{12} (inter atomic distance between two electrons) dependent term and thereafter present the results of our numerical analysis for the theoretic information entropies such as the Shannon entropy, Fisher information entropy and the Fisher-Shannon product. To begin with, we shall take an account of the effect of correlation on two electron systems using the 'Hartree and Ingman (1933)' [13] type trial wave function which can be written as

$$\psi(\vec{r}_1, \vec{r}_2, r_{12}) = ce^{-\alpha(r_1+r_2)}\chi(r_{12}), \tag{1.13}$$

where, $r_{12} = (\vec{r}_1 - \vec{r}_2)$, *'c'* is the normalization constant and the correlated function $\chi(r_{12})$ is written as

$$\chi(r_{12}) = (1 - \lambda e^{-\mu r_{12}}). \tag{1.14}$$

A few years ago, attempts were made by Bhattacharyya *et al* [5] to find out the ground-state energy of the two-electron system working with the 'Hartree and Ingman(1933)'[13] type trial wave function, $\psi(\vec{r}_1, \vec{r}_2, r_{12})$ = $e^{-\alpha(r_1+r_2)}$ (1 - $\lambda e^{-\mu r_{12}}$) with the variation parameters α, λ and μ. After minimizing the Hamiltonian with respect to the variations in the parameters of $\psi(\vec{r}_1, \vec{r}_2, r_{12})$, they obtained the values, $\alpha = 1.8395$, $\lambda = 0.586$ and $\mu = 0.379$. It is necessary to mention that we shall use these standard values for our computational purposes. It is to note that when $\lambda = 1$ and $r_{12} = 0$, the wave function takes the form as $\psi(\vec{r}_1, \vec{r}_2, r_{12}) = 0$ and the system becomes explicitly r_{12} dependent which is then referred to as the correlated system. Physically, this implies that two electrons in the atom cannot occupy the same position. And, when $\lambda = 1$ and $r_{12} = \infty$, the wave function leads to $\psi(\vec{r}_1, \vec{r}_2, r_{12}) = c\ e^{-\alpha(r_1+r_2)}$. Mechanistically, this implies when the inter-electronic separation is very large, the system becomes uncorrelated. The subscripts marked with *'uc'* and *'c'* has been used to indicate the 'uncorrelated' and 'correlated' systems respectively in all the sections of this paper. We shall use our model to compute the uncorrelated and correlated Shannon, Fisher information entropies and the Fisher-Shannon product both in the coordinate and momentum spaces for the uncorrelated and correlated systems. In applicative context it will, therefore, be quite interesting to examine how Shannon (S) and Fisher (I) information entropies along with the Fisher-Shannon product respond to important physical effects like the electron-electron correlation which plays an important role in the physics of many electron systems. To the best of our knowledge, the Shannon entropy, Shannon information and Fisher-Shannon product of the 'Hartree and Ingman (1933)' [13] type trial wave function have not been reported before in the literature.

Section 2 has been focused on obtaining the expressions for single particle wave functions [$\psi(\vec{r})$, $\phi(\vec{p})$] and single particle charge densities [$\rho(\vec{r})$,$\gamma(\vec{p})$] both in coordinate and momentum space for the uncorrelated and correlated systems.

In Section 3, we have used the expressions for single particle charge densities in both coordinate and momentum space to calculate uncorrelated [$S_{\rho_{uc}}$,$S_{\gamma_{uc}}$] and correlated [S_{ρ_c},S_{γ_c}] Shannon entropies. Similarly we have calculated uncorrelated [$I_{\rho_{uc}}$,$I_{\gamma_{uc}}$] and correlated [I_{ρ_c},I_{γ_c}] Fisher entropies. Consequently, the Fisher-Shannon products both in coordinate and momentum space for the uncorrelated and correlated systems have also been computed. We have also shown that the sum of correlated Shannon entropies is greater than that of the sum of the uncorrelated Shannon entropies in coordinate and momentum space i.e. (S_{ρ_c} + S_{γ_c}) >($S_{\rho_{uc}}$ + $S_{\gamma_{uc}}$). Each of the sums also satisfies the *BBM* inequality i.e. (S_ρ + S_γ)$\geqslant 3(1+ln\pi)$. In case of Fisher information entropies, it has been observed that the product of correlated Fisher information entropies in coordinate and momentum space $I_{\rho_c}I_{\gamma_c}$ is greater than that of the product of the uncorrelated Fisher entropies in coordinate and momentum space $I_{\rho_{uc}}I_{\gamma_{uc}}$. Both the products $I_{\rho_{uc}}I_{\gamma_{uc}}$ and $I_{\rho_c}I_{\gamma_c}$ also satisfy the Fisher based uncertainty relation $I_\rho\ I_\gamma \geqslant 36$. The inequality relation for the Fisher-Shannon products for the uncorrelated and correlated systems is $P_{\rho\gamma} = P_\rho\ P_\gamma \geqslant 1$ and the corresponding numerical results along with the verification of the relation are presented in the Table 3.5.

Finally, Section 4 has been devoted for summarizing the present work with relevant inferences.

2 Extraction of single particle wave function and single particle charge density from the correlated wave function

In this Section, we shall extract the expressions for the single particle wave function from the expression of the many particle correlated wave function expressed in equation (1.13) and equation (1.14) involving some adjustable parameters in coordinate and momentum spaces for both the correlated and uncorrelated systems and hence the single particle charge density. In this purpose, the many particles correlated trial wave function i.e. the 'Hartree and Ingman (1933)' [13] type wave function can be written as follows:

$$\psi(\vec{r}_1, \vec{r}_2, r_{12}) = ce^{-\alpha(r_1+r_2)}(1 - \lambda e^{-\mu r_{12}}). \tag{2.1}$$

Now integrating the wave function of equation (2.1) over $d\vec{r}_2$ we have,

$$\int \psi(\vec{r}_1, \vec{r}_2, r_{12}) d\vec{r}_2 = ce^{-\alpha r_1} \int e^{-\alpha r_2} d\vec{r}_2 - c\lambda e^{-\alpha r_1} \int e^{-\alpha r_2} e^{-\mu r_{12}} d\vec{r}_2.$$

The above integral can be written as

$$\int \psi(\vec{r}_1, \vec{r}_2, r_{12}) d\vec{r}_2 = ce^{-\alpha r_1} I_1 - c\lambda e^{-\alpha r_1} I_2 \tag{2.2}$$

where

$$I_1 = \int e^{-\alpha r_2} d\vec{r}_2 \tag{2.3}$$

and

$$I_2 = \int e^{-\alpha r_2} e^{-\mu r_{12}} d\vec{r}_2. \tag{2.4}$$

Here 'c' is the normalization constant.
Finally, the complete coordinate space wave function $\psi(\vec{r})$ can be written as follows

$$\begin{aligned} \psi(\vec{r}) &= \psi_1(\vec{r}) + \psi_2(\vec{r}) \\ &= ce^{-\alpha r_1} I_1 - c\lambda e^{-\alpha r_1} I_2 \\ &= \frac{8e^{-r\alpha} c\pi}{\alpha^3} + [\frac{4\pi c\lambda e^{-r\alpha}[\frac{8(e^{-r\alpha} - e^{-r\mu})\alpha\mu}{(-\alpha^2+\mu^2)^3} - \frac{2(e^{-r\mu})r\alpha}{(-\alpha^2+\mu^2)^2} - \frac{2(e^{-r\alpha})r\mu}{(-\alpha^2+\mu^2)^2}]}{r}], \end{aligned} \tag{2.5}$$

where

$$\psi_1(\vec{r}) = \frac{8e^{-r\alpha} c\pi}{\alpha^3} \tag{2.6}$$

and

$$\psi_2(\vec{r}) = [\frac{4\pi c\lambda e^{-r\alpha}[\frac{8(e^{-r\alpha} - e^{-r\mu})\alpha\mu}{(-\alpha^2+\mu^2)^3} - \frac{2(e^{-r\mu})r\alpha}{(-\alpha^2+\mu^2)^2} - \frac{2(e^{-r\alpha})r\mu}{(-\alpha^2+\mu^2)^2}]}{r}]. \tag{2.7}$$

We have used the standard values of the variational parameters (λ, α and μ) throughout our all calculations as $\lambda = 0.586$, $\alpha = 1.8395$ and $\mu = 0.379$.

The uncorrelated and correlated wave functions in coordinate-space are represented as

$$\psi_{uc}(\vec{r}) = \psi_1(\vec{r}) = \frac{8e^{-r\alpha} c\pi}{\alpha^3}, \tag{2.8}$$

with normalization constant $c = 0.3486$
and

$$\psi_c(\vec{r}) = \psi(\vec{r}) = \frac{8e^{-r\alpha} c\pi}{\alpha^3} + [\frac{4\pi c\lambda e^{-r\alpha}[\frac{8(e^{-r\alpha} - e^{-r\mu})\alpha\mu}{(-\alpha^2+\mu^2)^3} - \frac{2(e^{-r\mu})r\alpha}{(-\alpha^2+\mu^2)^2} - \frac{2(e^{-r\alpha})r\mu}{(-\alpha^2+\mu^2)^2}]}{r}], \tag{2.9}$$

with normalization constant $c = 0.5031$.

To study the properties of Shannon information entropy (S) and Fisher information entropy (I) in the momentum space, the Fourier transform of the coordinate space wave function is taken. For analytically calculating the required transformations the following standard integrals [3] have been used,

$$\int e^{-\gamma\xi} e^{i\vec{\mu}\cdot\vec{\xi}} = \frac{8\pi\gamma}{(\gamma^2 + \mu^2)^2}, \tag{2.10}$$

$$\int \frac{1}{\xi} e^{-\gamma\xi} e^{i\vec{\mu}\cdot\vec{\xi}} = \frac{4\pi}{(\gamma^2 + \mu^2)^2}. \tag{2.11}$$

Taking recourse of the Fourier transform of the coordinate space wave function $\psi(\vec{r})$, the momentum space wave function $\phi(\vec{p})$ can be written as $\phi(\vec{p}) = \phi_1(\vec{p}) + \phi_2(\vec{p})$.

The complete momentum space wave function $\phi(\vec{p})$ can be written as follows:

$$\phi(\vec{p}) = \phi_1(\vec{p}) + \phi_2(\vec{p})$$

$$= \frac{64\pi^2\tilde{c}}{(2\pi)^{\frac{3}{2}}\alpha^2(\alpha^2+p^2)^2} + \frac{128\pi^2\tilde{c}\lambda\alpha\mu}{(2\pi)^{\frac{3}{2}}(-\alpha^2+\mu^2)^3}[\frac{1}{(4\alpha^2+p^2)} - \frac{1}{((\alpha+\mu)^2+p^2)}] - \frac{64\pi^2\tilde{c}\lambda\alpha(\alpha+\mu)}{(2\pi)^{\frac{3}{2}}(-\alpha^2+\mu^2)^2((\alpha+\mu)^2+p^2)^2}$$

$$-\frac{128\pi^2\tilde{c}\lambda\alpha\mu}{(2\pi)^{\frac{3}{2}}(-\alpha^2+\mu^2)^2(4\alpha^2+p^2)^2} \quad (2.12)$$

The expressions for the uncorrelated and correlated wave function in momentum space are given as follows:

$$\phi_{uc}(\vec{p}) = \phi_1(\vec{p}) = \frac{64\pi^2\tilde{c}}{(2\pi)^{\frac{3}{2}}\alpha^2(\alpha^2+p^2)^2}, \quad (2.13)$$

with the normalization constant $\tilde{c} = 0.3486$
and

$$\phi_c(\vec{p}) = \phi(\vec{p}) = \frac{64\pi^2\tilde{c}}{(2\pi)^{\frac{3}{2}}\alpha^2(\alpha^2+p^2)^2} + \frac{128\pi^2\tilde{c}\lambda\alpha\mu}{(2\pi)^{\frac{3}{2}}(-\alpha^2+\mu^2)^3}[\frac{1}{(4\alpha^2+p^2)} - \frac{1}{((\alpha+\mu)^2+p^2)}]$$
$$-\frac{64\pi^2\tilde{c}\lambda\alpha(\alpha+\mu)}{(2\pi)^{\frac{3}{2}}(-\alpha^2+\mu^2)^2((\alpha+\mu)^2+p^2)^2} - \frac{128\pi^2\tilde{c}\lambda\alpha\mu}{(2\pi)^{\frac{3}{2}}(-\alpha^2+\mu^2)^2(4\alpha^2+p^2)^2}, \quad (2.14)$$

with the normalization constant $\tilde{c} = 0.5031$.

Now we shall find the expressions for the single particle charge densities in coordinate and momentum spaces for both the correlated and uncorrelated systems. The uncorrelated and correlated single-particle charge densities in coordinate space can simply be expressed as follows:

$$\rho_{uc} = |\psi_{uc}(\vec{r})|^2, \quad (2.15)$$

and

$$\rho_c = |\psi_c(\vec{r})|^2. \quad (2.16)$$

Similarly the uncorrelated and correlated single particle charge densities in momentum space are written as follows:

$$\gamma_{uc} = |\phi_{uc}(\vec{p})|^2, \quad (2.17)$$

and

$$\gamma_c = |\phi_c(\vec{p})|^2. \quad (2.18)$$

3 Computation of Shannon entropy, Fisher information entropy and the Fisher-Shannon product

In this section we present the results for the Shannon information entropy (S), Fisher information entropy (I) and Fisher-Shannon product both in coordinate and momentum space for uncorrelated and correlated systems. The expressions of uncorrelated Shannon information entropies in coordinate space [$S_{\rho_{uc}}$] and momentum space [$S_{\gamma_{uc}}$] are computed using the expressions from the equation (1.2) and equation (1.3) respectively. Similarly the uncorrelated Fisher information entropies in coordinate space [$I_{\rho_{uc}}$] and momentum space [$I_{\gamma_{uc}}$] are computed using the expressions from the equation (1.4) and equation (1.5) respectively. Now for computing the expressions for correlated Shannon information entropies in coordinate space [S_{ρ_c}] and momentum space [S_{γ_c}] the corresponding correlated wave functions have been used in the expressions of equation (1.2) and equation (1.3) respectively. Similarly we have also done for the correlated Fisher information entropies in coordinate space [I_{ρ_c}] and momentum space [I_{γ_c}] respectively using the equation (1.4) and equation (1.5).

Moreover, the expressions for uncorrelated Fisher-Shannon product in coordinate space [$P_{\rho_{uc}}$] and momentum space [$P_{\gamma_{uc}}$] and correlated Fisher-Shannon product in coordinate space [P_{ρ_c}] and momentum space [P_{γ_c}] are computed from the equation (1.10) and equation (1.11) respectively with the help of the corresponding equation (1.8) and equation (1.9) for the Shannon power in coordinate space (J_ρ) and momentum space (J_γ) for the uncorrelated and correlated systems.

The calculated values for the uncorrelated and correlated Shannon information entropies in coordinate and momentum space at different r and p values are presented in Table 3.1 respectively as follows:

***Table* 3.1:** Shannon information entropies in coordinate and momentum space for uncorrelated and correlated systems

Sl. No.	r varies from 0 to (in a.u.)	Coordinate space		p varies from 0 to (in a.u.)	Momentum space		$(S_{\rho_{uc}} + S_{\gamma_{uc}})$	$(S_{\rho_c} + S_{\gamma_c})$
		$S_{\rho_{uc}}$	S_{ρ_c}		$S_{\gamma_{uc}}$	S_{γ_c}		
1	5	2.316	2.442	5	4.134	4.024	6.450	6.466
2	10	2.316	2.442	10	4.243	4.123	6.559	6.565
3	100	2.316	2.442	100	4.250	4.130	6.566	6.572
4	1000	2.316	2.442	1000	4.250	4.130	6.566	6.572
5	5000	2.316	2.442	5000	4.250	4.130	6.566	6.572
6	10000	2.316	2.442	10000	4.250	4.130	6.566	6.572
7	100000	2.316	2.442	100000	4.250	4.130	6.566	6.572
8	1000000	2.316	2.442	1000000	4.250	4.130	6.566	6.572
9	5000000	2.316	2.442	5000000	4.250	4.130	6.566	6.572
10	Infinity	2.316	2.442	Infinity	4.250	4.130	6.566	6.572

From Table 3.1, it is observed that correlation augments the Shannon entropies in coordinate space as $S_{\rho_c} > S_{\rho_{uc}}$ and diminishes it in momentum space as $S_{\gamma_c} < S_{\gamma_u c}$. It is also evident that sum of correlated Shannon entropies i.e. $(S_{\rho_c} + S_{\gamma_c})$ is greater than the sum of uncorrelated Shannon entropies i.e. $(S_{\rho_{uc}} + S_{\gamma_{uc}})$. Thus we have verified the uncertainty relation,$(S_{\rho_c} + S_{\gamma_c}) > (S_{\rho_{uc}} + S_{\gamma_{uc}})$.

The calculated values for the uncorrelated and correlated Fisher information entropies for the coordinate and momentum space at different r and p values are presented in Table 3.2 respectively as follows:

***Table* 3.2:** Fisher information entropies in coordinate and momentum space for uncorrelated and correlated systems

Sl. No.	r varies from 0 to (in a.u.)	Coordinate space		p varies from 0 to (in a.u.)	Momentum space		$I_{\rho_{uc}}\ I_{\gamma_{uc}}$	$I_{\rho_c}\ I_{\gamma_c}$
		$I_{\rho_{uc}}$	I_{ρ_c}		$I_{\gamma_{uc}}$	I_{γ_c}		
1	5	13.535	12.588	5	3.539	3.862	47.901	48.615
2	10	13.535	12.588	10	3.546	3.868	47.995	48.690
3	100	13.535	12.588	100	3.546	3.822	47.995	48.111
4	1000	13.535	12.588	1000	3.546	3.868	47.995	48.690
5	5000	13.535	12.588	5000	3.546	3.868	47.995	48.690
6	10000	13.535	12.588	10000	3.546	3.868	47.995	48.690
7	100000	13.535	12.588	100000	3.546	3.867	47.995	48.678
8	1000000	13.535	12.588	1000000	3.546	3.861	47.995	48.602
9	5000000	13.535	12.588	5000000	3.546	3.831	47.995	48.225
10	Infinity	13.535	12.588	Infinity	3.546	3.869	47.995	48.703

From Table 3.2 it is observed that correlation diminishes the Fisher entropies in coordinate space and augments it in momentum space. We have also verified from Table 3.2 that the product of correlated $[I_{\rho_c} I_{\gamma_c}]$ and the product of uncorrelated $[I_{\rho_{uc}} I_{\gamma_{uc}}]$ Fisher entropies satisfy the inequality condition $I_{\rho_c}\ I_{\gamma_c} > I_{\rho_{uc}} I_{\gamma_{uc}}$. It is also verified in general that the products of Fisher entropies ($I_{\rho_c} I_{\gamma_c}$ and $I_{\rho_{uc}} I_{\gamma_{uc}}$) satisfy the Fisher-based uncertainty relation
$I_\rho\ I_\gamma \geqslant 36$.

To calculate the values of the uncorrelated and correlated Shannon power in coordinate space ($J_{\rho_{uc}}$,J_{ρ_c}) and momentum space ($J_{\gamma_{uc}}$,J_{γ_c}) following equation (1.8) and equation (1.9), we have used the values for the Shannon information entropies of the Table 3.1 for different r and p values and presented them in Table 3.3 as follows:

***Table* 3.3:** Shannon information entropies, Shannon power in coordinate and momentum space for uncorrelated and correlated systems

Sl. No.	r and p varies from 0 to (in a.u.)	Shannon information entropies				Shannon Power			
		Coordinate space		Momentum space		Coordinate space		Momentum space	
		$S_{\rho_{uc}}$	S_{ρ_c}	$S_{\gamma_{uc}}$	S_{γ_c}	$J_{\rho_{uc}}$	J_{ρ_c}	$J_{\gamma_{uc}}$	J_{γ_c}
1	5	2.316	2.442	4.134	4.024	0.745	0.810	2.504	2.327
2	10	2.316	2.442	4.243	4.123	0.745	0.810	2.693	2.486
3	100	2.316	2.442	4.250	4.130	0.745	0.810	2.705	2.498
4	1000	2.316	2.442	4.250	4.130	0.745	0.810	2.705	2.498
5	5000	2.316	2.442	4.250	4.130	0.745	0.810	2.705	2.498
6	10000	2.316	2.442	4.250	4.130	0.745	0.810	2.705	2.498
7	100000	2.316	2.442	4.250	4.130	0.745	0.810	2.705	2.498
8	1000000	2.316	2.442	4.250	4.130	0.745	0.810	2.705	2.498
9	5000000	2.316	2.442	4.250	4.130	0.745	0.810	2.705	2.498
10	Infinity	2.316	2.442	4.250	4.130	0.745	0.810	2.705	2.498

Moreover, to calculate the values for the uncorrelated and correlated Fisher-Shannon product in coordinate space ($P_{\rho_{uc}}$,P_{ρ_c}) and momentum space ($P_{\gamma_{uc}}$,P_{γ_c}) at different r and p values following equation (1.10) and equation (1.11), we have used the values for the Fisher information entropies and Shannon power from Table 3.2 and Table 3.3 respectively and presented them in Table 3.4 as follows:

***Table* 3.4:** Fisher information entropies, Shannon power and Fisher-Shannon product in coordinate and momentum space for uncorrelated and correlated systems

Sl. No.	r and p varies from 0 to (in a.u.)	Fisher information entropies				Shannon power				Fisher-Shannon product			
		Coordinate space		Momentum space		Coordinate space		Momentum space		Coordinate space		Momentum space	
		$I_{\rho_{uc}}$	I_{ρ_c}	$I_{\gamma_{uc}}$	I_{γ_c}	$J_{\rho_{uc}}$	J_{ρ_c}	$J_{\gamma_{uc}}$	J_{γ_c}	$P_{\rho_{uc}}$	P_{ρ_c}	$P_{\gamma_{uc}}$	P_{γ_c}
1	5	13.535	12.588	3.539	3.862	0.745	0.810	2.504	2.327	3.361	3.399	2.954	4.283
2	10	13.535	12.588	3.546	3.868	0.745	0.810	2.693	2.486	3.361	3.399	3.183	3.205
3	100	13.535	12.588	3.546	3.822	0.745	0.810	2.705	2.498	3.361	3.399	3.197	3.446
4	1000	13.535	12.588	3.546	3.868	0.745	0.810	2.705	2.498	3.361	3.399	3.197	3.488
5	5000	13.535	12.588	3.546	3.868	0.745	0.810	2.705	2.498	3.361	3.399	3.197	3.488
6	10000	13.535	12.588	3.546	3.868	0.745	0.810	2.705	2.498	3.361	3.399	3.197	3.488
7	100000	13.535	12.588	3.546	3.867	0.745	0.810	2.705	2.498	3.361	3.399	3.197	3.487
8	1000000	13.535	12.588	3.546	3.861	0.745	0.810	2.705	2.498	3.361	3.399	3.197	3.481
9	5000000	13.535	12.588	3.546	3.831	0.745	0.810	2.705	2.498	3.361	3.399	3.197	3.454
10	Infinity	13.535	12.588	3.546	3.869	0.745	0.810	2.705	2.498	3.361	3.399	3.197	3.489

Let us now verify the values obtained in the Table 3.4 for the Fisher-Shannon product, as per the requirement of the equation (1.12) the Fisher-Shannon product must satisfy the inequality relation $P_{\rho\gamma} = P_\rho\ P_\gamma \geqslant 1$. Following is the Table of verification for the Fisher-Shannon product:

***Table* 3.5:** Verification Table for Fisher-Shannon product in coordinate and momentum space for uncorrelated and correlated systems

Sl. No.	Fisher-Shannon product				Uncorrelated system	Correlated system	The inequality relation to verify $P_\rho\ P_\gamma \geqslant 1$
	Coordinate space		Momentum space				
	$P_{\rho_{uc}}$	P_{ρ_c}	$P_{\gamma_{uc}}$	$P_{\gamma_{uc}}$	$P_{\rho_{uc}}\ P_{\gamma_{uc}}$	$P_{\rho_c}\ P_{\gamma_c}$	$P_{\rho_{uc}}\ P_{\gamma_{uc}} \geqslant 1$ or $P_{\rho_c}\ P_{\gamma_c} \geqslant 1$
1	3.361	3.399	2.954	4.283	9.928	14.558	Yes
2	3.361	3.399	3.183	3.205	10.698	10.894	Yes
3	3.361	3.399	3.197	3.446	10.745	11.713	Yes
4	3.361	3.399	3.197	3.488	10.745	11.856	Yes
5	3.361	3.399	3.197	3.488	10.745	11.856	Yes
6	3.361	3.399	3.197	3.488	10.745	11.856	Yes
7	3.361	3.399	3.197	3.487	10.745	11.852	Yes
8	3.361	3.399	3.197	3.481	10.745	11.832	Yes
9	3.361	3.399	3.197	3.454	10.745	11.740	Yes
10	3.361	3.399	3.197	3.489	10.745	11.859	Yes

4 Concluding remarks

In the present work, we have used the r_{12}- dependent two electron 'Hartree and Ingman(1933)' type trial wave function to construct a single particle wave function $\psi(\vec{r})$. By taking the Fourier transform of $\psi(\vec{r})$, the wave function in momentum space i.e. $\phi(\vec{p})$ has been constructed. The wave functions $\psi(\vec{r})$ and $\phi(\vec{p})$ are used to evaluate the expressions for the single particle charge densities in coordinate and momentum spaces. These expressions have been further used to construct the analytical expressions for Shannon and Fisher entropies, Shannon power and the Fisher-Shannon product and hence to compute their values in both coordinate and momentum spaces. The expressions have been constructed by taking the correlation into account as well as without it. In Table 3.1 and Table 3.2 we have provided the values of Shannon and Fisher entropies for different values of r and p. In coordinate space, the correlation augments the values of Shannon entropies and in momentum space the correlation plays just the opposite role. In case of Fisher entropies, the correlation diminishes the values in coordinate space and augments in momentum space. Thus from the data of the two Tables we observe that correlation plays just the opposite roles in case of Shannon and Fisher information entropies. In addition to this, we have verified from Table 3.1 the uncertainty relation $(S_\rho + S_\gamma) \geqslant 3(1+ln\pi)$and the inequality condition $(S_{\rho_c} + S_{\gamma_c}) > (S_{\rho_{uc}} + S_{\gamma_{uc}})$ for Shannon entropy. Simultaneously, for Fisher entropies we have verified the relations from Table 3.2 that $I_{\rho_c} I_{\gamma_c} > I_{\rho_{uc}} I_{\gamma_{uc}}$ and $I_\rho\ I_\gamma \geqslant 36$. In Table 3.3 the numerical values relating to Shannon power for the uncorrelated and correlated systems in coordinate space ($J_{\rho_{uc}}$,J_{ρ_c}) and momentum space ($J_{\gamma_{uc}}$,J_{γ_c}) have been demonstrated. And Table 3.4 depicts altogether the numerical values of Fisher information entropies for the uncorrelated and correlated systems, Shannon power and Fisher-Shannon product in coordinate and momentum space. Moreover, the verification of Fisher-Shannon product has been checked and confirmed by the data presented in Table 3.5. Since our computed values of Shannon, Fisher information entropies and Fisher-Shannon product satisfy their respective uncertainty relationships; it validates our results obtained in a consistent way. Further the variation of information entropic measurements with coordinate (r) and momentum (p) values give us an insight into the dynamics of evolution of the system in the coordinate and momentum spaces respectively and that can easily be analyzed from the difference of numerical values computed separately for the Fisher-Shannon product in respect of the uncorrelated and correlated systems. It thus provides important evidence that the Fisher-Shannon product can be regarded as an appropriate measure of electron correlation. A more systematic and extensive analysis of this new correlation measure in many other N-electron systems is needed to get a deeper insight into it. It thus remains an interesting curiosity to investigate the efficacy of this method for studying higher electronic systems. In our further works we shall try to investigate such systems.

Conflict of Interest

The authors declare that there is no conflict of interests, financial or otherwise regarding the publication of this paper.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 263-271**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

IMPACT OF MELTING ON MHD HEAT AND MASS TRANSFER OF CASSON FLUID FLOW OVER A STRETCHING SHEET IN POROUS MEDIA IN PRESENCE OF THERMAL RADIATION AND VISCOUS DISSIPATION

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(Received: April 20, 2022; In format: May 12, 2022; Revised: May 26, 2023; Accepted: June 15, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53131

Abstract

Impact of melting on MHD heat and mass transfer of Casson fluid flow over a stretching sheet in porous media with thermal radiation and viscous dissipation have been investigated in this article. Governing PDE's are change into coupled ODE 's using a set of proper similarity transformation. Resultant equations are solved by efficient numerical scheme Runge kutta- 4^{th} order allied with shooting method. Impact of several flow parameters on flow fields are interpreted via tables and graphs. Present outcomes compared with existing results and observed excellent validation.

2020 Mathematical Sciences Classification. 76W05, 76D05 ,76Sxx, 80A19, 78A40.
Keywords and phrases. MHD, Casson fluid, Porous medium, Heat and mass transfer, Radiation, Viscous dissipation..

1 Introduction

Investigation of non-Newtonian fluids have diverse applications in geoscience, petroleum industry, atmospheric sciences, oceanography, aeromechanics etc. Principle of fluid movement, heat and mass transmission via porous media plays an important role in different fields. Various properties of non-Newtonian fluids make their fundamental equations nonlinear and non-uniform. Several models have been developed to characterize attributes of non-Newtonian fluids. One of them is the Casson model. Casson [2] introduced first Casson fluid model to characterize flow of pigment oil suspensions of printing ink type and until today many investigations regarding Casson fluid have been conducted. Casson fluid is shear thinning fluid. At zero shear rate it has infinite viscosity and zero viscosity at infinite shear rate, i.e. it performs as solid if a shear stress less than yield stress is enforced to fluid and it starts to flow when shear stress is more than yield stress. Tomato sauce, jelly, chocolate, soup, honey, human blood etc. are considered as Casson fluid. At a very high shear stress Casson fluid reduced to Newtonian fluid.

Currently heat and mass transfer through porous media is centre of comprehensive research because in laminar boundary layer flow, heat-mass and momentum transfer over stretching sheet have many applications for example in increase effectiveness of paints and lubrication, in production of fiber-glass and glass blowing, in paper making industry, plastic shaping, crystallization, extrusion rubber sheets, aerodynamics etc. Mabood et al. [4] studied melting heat transfer impact on MHD Casson fluid flow through porous media. Reddy et al. [7] investigated MHD mass and heat transfer characteristic of Casson fluid over exponentially permeable stretching surface with viscous dissipation, thermal radiation and chemical reaction. Raju et al. [5] studied melting heat transfer effect on MHD Casson fluid flow through porous media with radiation in presence of first order chemical reaction. Krishnamurthy et al. [3] analyzed impact of melting heat transfer and velocity slip boundary layer flow with thermal radiation and chemical reaction on MHD nanofluid past a nonlinear stretching sheet. Yacoob et al. [8] examine stagnation-point boundary layer flow of micropolar fluid past a linearly shrinking/stretching sheet. Bachok et al. [1] investigate time independent two-dimensional flow and heat transfer to melting shrinking/stretching sheet. Lorenzini et al. [6] investigate effect of melting heat transfer in MHD Casson fluid through moving surface in porous media with radiation.

Aim of this study is to analyze impact of melting on MHD heat and mass transfer of Casson fluid flow over a stretching sheet in porous media in presence of thermal radiation and viscous dissipation.

2 Mathematical Formulation

In present study two-dimensional time independent stagnation point flow of Casson fluid past a linear stretching sheet in porous media is considered. Permeability of porous media is K_p. Sheet is melting at constant rate into warm liquid of same material, as demonstrated in figure 2.1. Transverse magnetic field B_0 is applied uniformly to fluid. Let velocity of fluid is $u_e(x) = ax$ and stretching sheet velocity is $u_w(x) = cx$, where a and c are positive constant and x coordinate considered along the stretching sheet. Let T_m represent melting temperature and T_∞ represent free stream temperature of the fluid, where $T_\infty > T_m$.

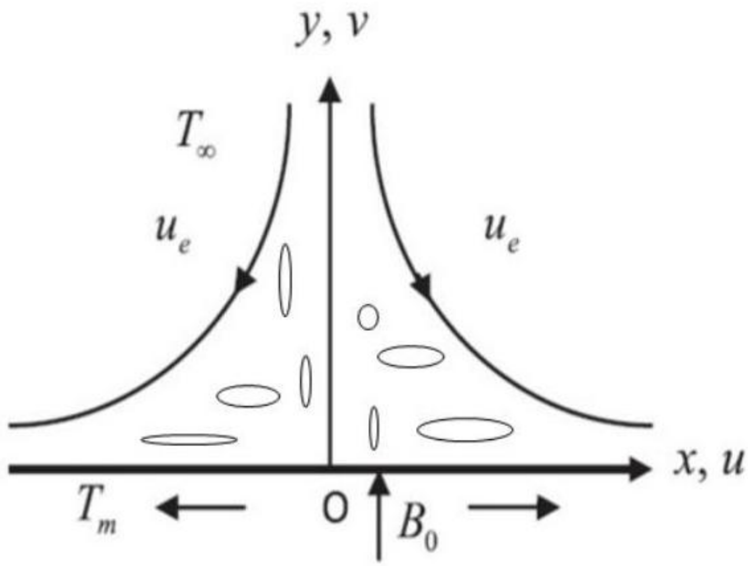

***Figure* 2.1:** Sketch of physical model

$$\tau_{ij} = \begin{cases} \left(\mu_B + (2\pi)^{\frac{1}{2}} P_y\right) 2e_{ij} & \pi > \pi_c \\ \left(\mu_B + (2\pi_c)^{\frac{1}{2}} P_y\right) 2e_{ij} & \pi < \pi_c \end{cases} \tag{2.1}$$

where μ_B represent plastic dynamic viscosity, $\pi = e_{ij}e_{ij}$ and $(i,j)^{th}$ element of deformation rate is e_{ij}, π represent rate of deformation, π_c is critical value of Casson fluid model, yield stress of fluid is P_y. Considering above postulation the governing equations of present flow are given below:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + v\left(1 + \frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho}\left(u - u_e\right) - \frac{v}{K_p}\left(u - u_e\right) \tag{2.3}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \kappa\frac{\partial^2 T}{\partial y^2} + \mu\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)^2 - \frac{\partial q_r}{\partial y}, \tag{2.4}$$

$$u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = D^*\frac{\partial^2 c}{\partial y^2}. \tag{2.5}$$

Boundary conditions are:

$$u = u_w(x) = cx, \quad T = T_m, \quad C = C_m \quad \text{at} \quad y = 0, \tag{2.6}$$

$$u \to U_e(x) = ax, \quad T \to T_\infty, \quad C \to C_\infty \quad \text{as} \quad y \to \infty. \tag{2.7}$$

and

$$k\left(\frac{\partial T}{\partial y}\right) = \left[\rho c_s\left(T_m - T_0\right) + \rho\lambda\right] v(x, 0). \tag{2.8}$$

Here β is Casson fluid parameter, v is kinematic viscosity, μ is factor of viscosity, K_p represent permeability of porous media, κ is thermal conductivity, σ is fluid electrical conductivity, at constant pressure specific heat is C_p, ρ is the density of fluid, radiative heat flux is q_r, D^* is molecular diffusivity, latent heat of fluid is λ.

Using Roseland's approximation for radiation, we obtain $q_r = -\left(\frac{4}{3}\frac{\sigma^*}{k_1}\right)\frac{\partial T^4}{\partial y}$, where σ^* represents Stefan-Boltzmann constant, k_1 represents mean absorption factor. By using Taylor series about the free stream temperature, we have

$$T^4 = 4TT_\infty^3 - 3T_\infty{}^4. \tag{2.9}$$

Now eqn. (2.4) converts to

$$\rho c_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \kappa\frac{\partial^2 T}{\partial y^2} + \mu\left(1+\frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)^2 + \frac{16\sigma^* T_\infty^3}{3k_1\rho c_p}\frac{\partial^2 T}{\partial y^2}. \tag{2.10}$$

3 Problem Solution

Introducing similarity transformation and dimensionless parameters

$$\Psi = x(av)^{\frac{1}{2}} f(\eta), \quad \eta = \left(\frac{a}{v}\right)^{\frac{1}{2}} y, \quad \theta(\eta) = \frac{T - T_m}{T_\infty - T_m}, \quad \phi(\eta) = \frac{C - C_m}{C_\infty - C_m}. \tag{3.1}$$

where Ψ is stream function interpreted as $u = \frac{\partial \Psi}{\partial v}$ and $v = -\frac{\partial \Psi}{\partial x}$.

Using equation 2.10 into equations $2.2 - 2.5$), we get

$$\left(1+\frac{1}{\beta}\right) f''' - f'^2 + ff'' - (M + K_1)(f' - 1) + 1 = 0, \tag{3.2}$$

$$(1+R)\theta'' + \Pr Ec\left(1+\frac{1}{\beta}\right) f''^2 + \Pr f\theta' = 0, \tag{3.3}$$

$$\phi'' + Scf\phi' = 0. \tag{3.4}$$

Equation (2.6) and (2.7) reduce to

$$f(0) = -\frac{Me}{Pr}\theta'(0), \quad f'(0) = \varepsilon, \quad \theta(0) = 0, \quad \phi(0) = 0, \tag{3.5}$$

$$f'(\infty) = 1, \quad \theta(\infty) = 1, \quad \phi(\infty) = 1, \tag{3.6}$$

where $\beta = \frac{\mu_B(2\pi_c)^{1/2}}{P_y}$ is Casson fluid parameter, magnetic parameter $M = \frac{\sigma B_0^2}{\rho a}, K_1 = \frac{v}{K_p a}$ permeability parameter, radiation parameter $R = \frac{16\sigma^* T_\infty^3}{3k^*\kappa}, \varepsilon = \frac{c}{a}$ stretching parameter, $\Pr = \frac{\rho c_p v}{\kappa}$ Prandtl number, combination of stefan numbers $\frac{c_s(T_m - T_0)}{\lambda}$ and $\frac{c_p(T_\infty - Tm)}{\lambda}$ respectively for solid and liquids phases is melting parameter $Me = \frac{c_p(T_\infty - T_m)}{\lambda + c_s(T_m - T_0)}$.

The physical parameters of attention are skin friction factor C_f, Nusselt number Nu_x and Sherwood number Sh_x are described as

$$C_f = \frac{\tau_W}{\rho U_e^2}, \tag{3.7}$$

$$Nu_x = \frac{xq_W}{\kappa(T_\infty - T_m)}, \tag{3.8}$$

$$Sh_x = \frac{xL_W}{D^*(C_\infty - C_m)}, \tag{3.9}$$

where τ_W represents surface shear stress, q_W denote surface heat flux and mass flux L_W are described as

$$\tau_W = \left(1+\frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)_{y=0}, \tag{3.10}$$

$$q_W = -\kappa\left(\frac{\partial T}{\partial y}\right)_{y=0} + q_r, \tag{3.11}$$

$$L_W = -D^*\left(\frac{\partial C}{\partial y}\right)_{y=0}. \tag{3.12}$$

From equations (3.12) to 3.15 with applications of similarity transformations, we get

$$C_f = \left(1 + \frac{1}{\beta}\right) Re_x^{-\frac{1}{2}} f''(0), \tag{3.13}$$

$$Nu_x = -(1 + R) Re_x^{\frac{1}{2}} \theta'(0), \tag{3.14}$$

$$Sh_x = -\phi'(0) Re_x^{\frac{1}{2}}, \tag{3.15}$$

where $Re_x = \frac{u_e x}{v}$ represents Reynolds number.

It is remarkable observation that if we put $M = K_1 = R = Ec = Sc = 0$ and $\beta \to \infty$ in equations (3.1) to (3.3), our problem converts into model taken by Mabood et al. [4].

4 Numerical Solution

Equations (3.2) to (3.4) are solved numerically with boundary conditions (3.5) and (3.6) by applying the shooting method together with $RK4$ scheme. For calculations we utilize MATLAB computer programming. Appropriate estimates of f'', θ' and ϕ' at $\eta = 0$ are taken with shooting method to obtain boundary conditions at $\eta \to \infty$ which all are one. We assume $\Delta\eta = 0.01$ and value for $\eta_{\max} = 5$.

In Tables 4.1, 4.2 and 4.3 validation of present method is established by comparing with results of Mabood et al. [4]

***Table* 4.1:** For varying values of ε and Me comparison of numeric values of $f''(0)$ and $\theta'(0)$, when $M = K_1 = R = Ec = Sc = 0, \Pr = 1$, and $\beta \to \infty$.

Parameters	Mabood et al. [4]		Present outcomes		
ε	Me	$f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$
0.0	0	1.232588	-0.570465	1.232588	-0.570465
	1	1.037003	-0.361961	1.037003	-0.361961
0.5	0	0.713295	-0.692064	0.713295	-0.692064
	1	0.599090	-0.438971	0.599090	-0.438971
2.0	0	-1.887307	-0.979271	-1.887307	-0.979271
	1	-1.580484	-0.621187	-1.580484	-0.621187
5.0	0	-10.264749	-1.396355	-10.264749	-1.396355
	1	-8.5746752	-0.886425	-8.5746752	-0.886425
6.0	0	-13.774813	-1.511165	-13.774813	-1.511165
	1	-11.501531	-0.959514	-11.501531	-0.959514

***Table* 4.2:** For varying values of ε and Me comparison of numeric values of $f''(0)$, when $\Pr = 1$, $M = K_1 = R = Ec = Sc = 0$ and $\beta \to \infty$.

ε	Mabood et al. [4]		Present outcomes			
	$Me = 0$	$Me = 1$	$Me = 2$	$Me = 0$	$Me = 1$	$Me = 2$
0.0	1.232588	1.037003	0.946851	1.232588	1.037003	0.956851
0.1	1.146561	0.964252	0.880442	1.146561	0.964252	0.880442
0.5	0.713295	0.599089	0.547021	0.713295	0.599089	0.547021
1.0	0	0	0	0	0	0
2.0	-1.887307	-1.580484	-1.442747	-1.887307	-1.580484	-1.442747

***Table* 4.3:** For varying values of Pr and Me comparison of numeric values of $\theta'(0)$, when $M = K_1 = R = Ec = Sc = 0$ and $\beta \to \infty$.

Parameters		Mabood et al. [4]	Present outcomes
Pr	Me	$-\theta'(0)$	$-\theta'(0)$
1	0	-0.7978846	-0.7978846
	1	-0.5060545	-0.5060545
	2	-0.3826383	-0.3826383
7	0	-2.1110042	-2.1110042
	1	-1.3388943	-1.3388943
	2	-1.0123657	-1.0123657

5 Discussion of the Results

For computation default values are taken $\varepsilon = 0.5$ or $1.5, M = 0.5, K_1 = 0.2, \mathrm{Pr} = 25, R = 1, \beta = 1, Me = 1, Ec = 0.2, Sc = 1$.

Fig. 5.1 depicts impact of Magnetic parameter M on velocity, for $\varepsilon = 1.3$ velocity decreases with increasing values of M, this is because of Lorentz force which is retarded force for velocity. Effect is opposite for $\varepsilon = 0.3$. Influence of permeability parameter is illustrated in Fig. 5.2. It is concluded that with increasing values of K_1 velocity profile decrease because with increasing values of K_1 permeability decrease. Inverse effect found for $\varepsilon = 0.3$. From Fig. 5.3 fluid velocity is a decreasing function of Casson fluid parameter β because viscosity increased with increment in values of β and reverse results exist for $\varepsilon = 0.3$. Fig. 5.4 shows influence of β on temperature profile, here we conclude that fluid temperature decreases with increasing values of β due to the fact that increment in β signifies a reduction in yield stress. From Fig. 5.5 we observed that temperature increase with increment in values of Pr, according to definition of Prandtl number large values of Pr has lower thermal diffusivity. Because of the melting parameter, thickness of thermal boundary layer increases with increasing values of Pr. From Fig. 5.6 we observed that with increasing values of radiation parameter R temperature decrease. Fig. 5.7 depicts effect of melting parameter on temperature. Temperature profile decrease with increasing melting parameter because plunges of cold sheet in hot fluid, this starts to melt due to this temperature decreases. Fig. 5.8 shows the effect of Eckert number Ec on temperature profile, temperature increase due to viscous dissipation. Effect of Schmidt number shows in Fig. 5.9 which is analogous to effect of Prandtl number. Fig. 5.10 depicts impact of melting parameter Me on concentration. Concentration profile decrease with increasing values of Me.

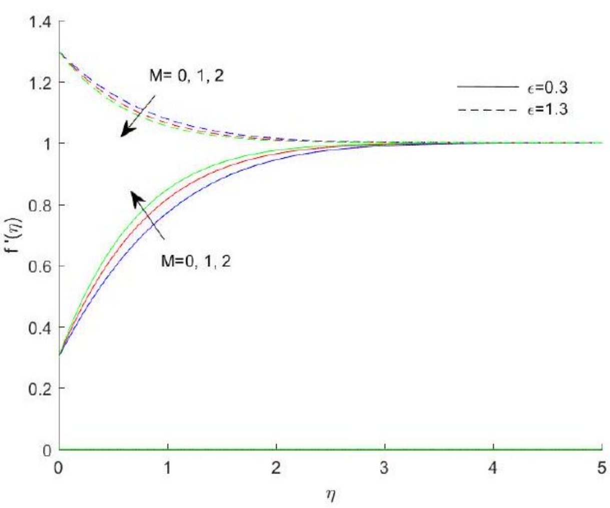

***Figure* 5.1:** Distribution of velocity for variations in M

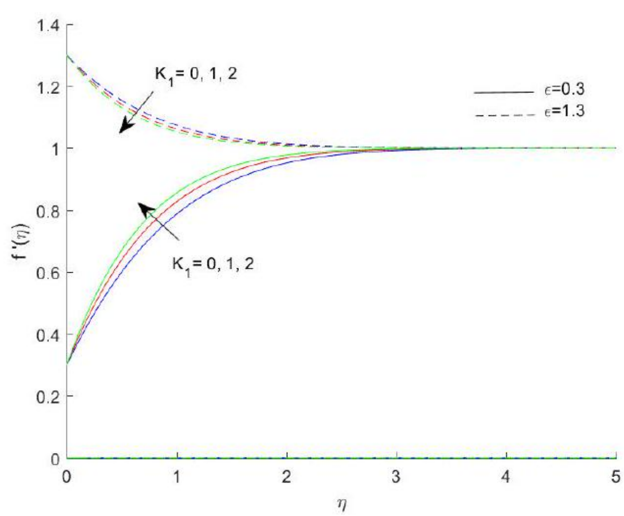

***Figure* 5.2:** Distribution of velocity for variations in K_1

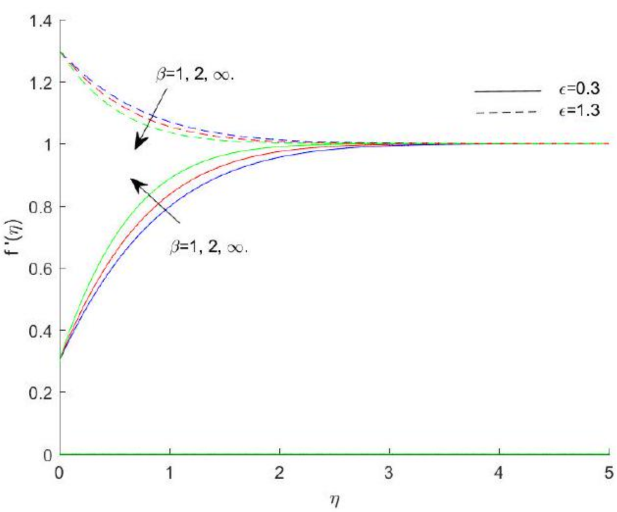

***Figure* 5.3:** Distribution of velocity for variations in β

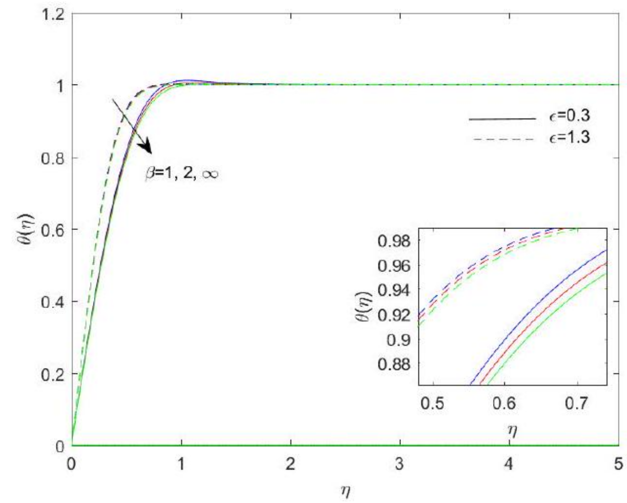

***Figure* 5.4:** Distribution of temperature for variations in β

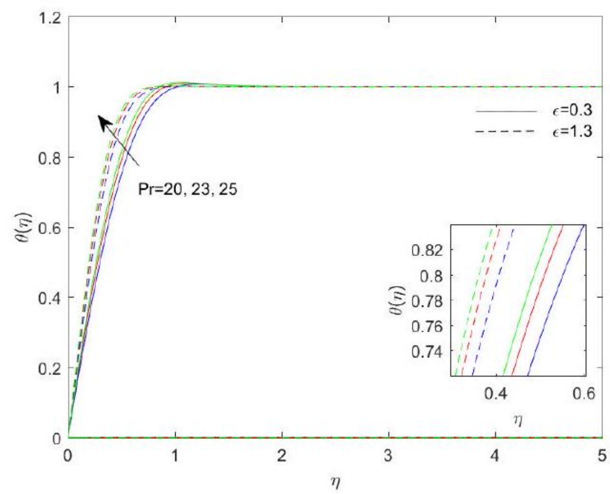

***Figure* 5.5:** Distribution of temperature for variations in Pr

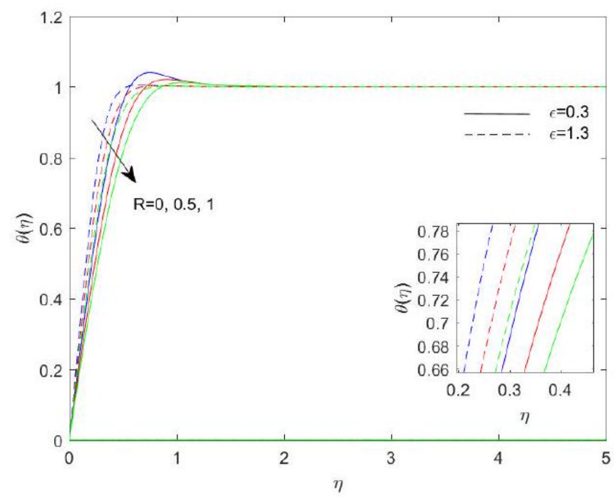

***Figure* 5.6:** Distribution of temperature for variations in R

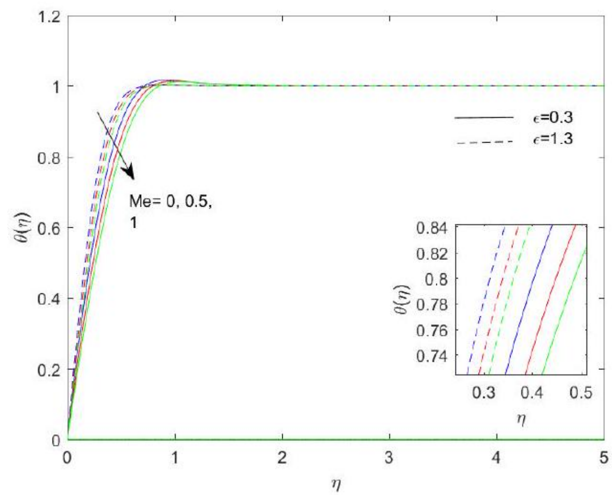

***Figure* 5.7:** Distribution of temperature for variations in Me

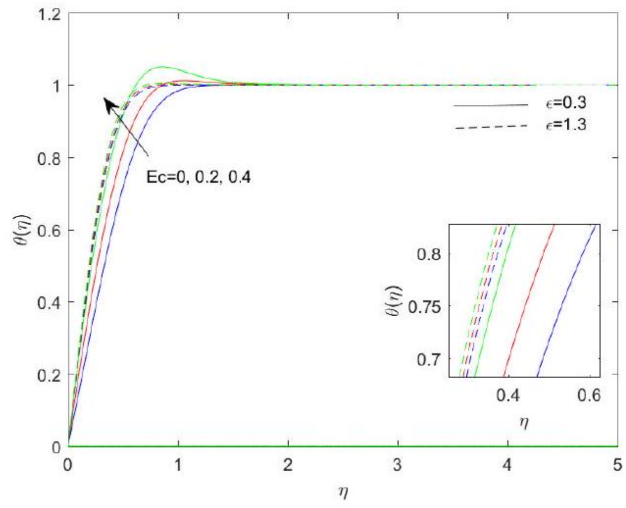

***Figure* 5.8:** Distribution of temperature for variations in Ec

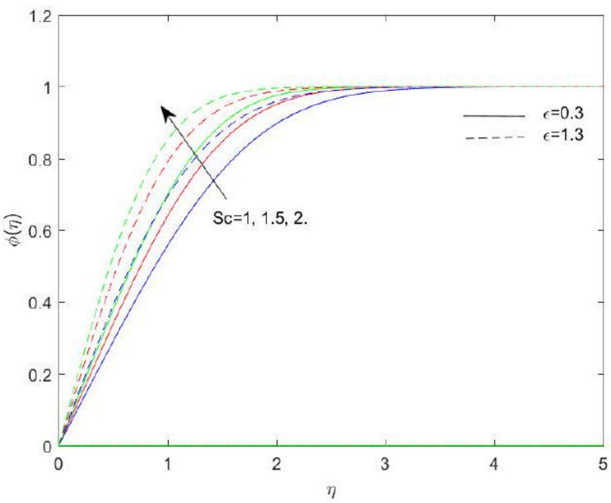

***Figure* 5.9:** Distribution of concentration for variations in Sc

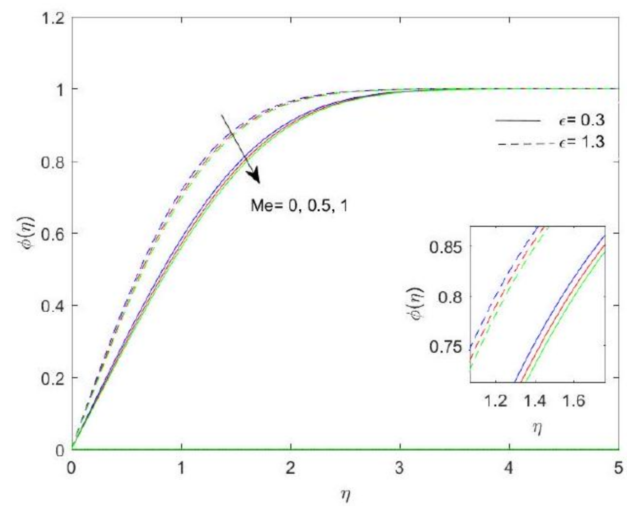

***Figure* 5.10:** Distribution of concentration for variations in Me

***Table* 5.1:** For variations in values of $\varepsilon, M, K_1, Pr, R, \beta, Me, Ec$ and Sc, Values of $f''(0), \theta'(0)$ and $\phi'(0)$.

ε	M	K_1	Pr	R	β	Me	Ec	Sc	$f''(0)$	$\theta'(0)$	$\phi'(0)$
0.3	0								0.689560	1.962200	0.571600
	1								0.844058	2.140100	0.581800
	2								0.974919	2.294100	0.588610
1.3	0								-0.362260	2.536900	0.801900
	1								-0.418174	2.553700	0.797700
	2								-0.467735	2.569500	0.794210
0.3		0							0.739190	2.018800	0.575110
		1							0.885303	2.188400	0.584100
		2							1.010939	2.337000	0.590250
1.3		0							-0.379850	2.542000	0.800530
		1							-0.433612	2.558400	0.796560
		2							-0.481646	2.574100	0.793300
0.3			20						0.768913	1.819100	0.572000
			23						0.769968	1.963200	0.575340
			25						0.770562	2.054900	0.577220
1.3			20						-0.390260	2.265200	0.792670
			23						-0.390836	2.436900	0.797180
			25						-0.391160	2.545400	0.799670
0.3				0					0.767515	2.459900	0.567530
				0.5					0.769195	2.236400	0.572900
				1					0.770562	2.054900	0.577220
1.3				0					-0.389714	3.004400	0.788400
				0.5					-0.390500	2.754800	0.794500
				1					-0.391160	2.545400	0.799670
0.3					1				0.770562	2.054900	0.577220
					2				0.887272	2.028200	0.589700
					∞				1.081560	1.997400	0.606880
1.3					1				-0.391160	2.545400	0.799670
					2				-0.450310	2.529000	0.795800
					∞				-0.548753	2.505400	0.790200
0.3						0			0.786170	2.727100	0.627570
						0.5			0.777282	2.332700	0.598750
						1			0.770562	2.054900	0.577220
1.3						0			-0.399250	3.329300	0.863780
						0.5			-0.394662	2.874700	0.827300
						1			-0.391160	2.545400	0.799670
0.3							0		0.775080	1.456900	0.591700
							0.2		0.770562	2.054900	0.577220
							0.4		0.766326	2.618400	0.563800
1.3							0		-0.391540	2.424300	0.802700
							0.2		-0.391160	2.545400	0.799670
							0.4		-0.390780	2.665400	0.796700
0.3								1	0.770562	2.054900	0.577220
								1.5	0.770562	2.054900	0.673100
								2	0.770562	2.054900	0.747800
1.3								1	-0.391160	2.545400	0.799670
								1.5	-0.391160	2.545400	0.969400
								2	-0.391160	2.545400	1.108200

6 Conclusions

In this paper a theoretical analysis of impact of melting on MHD heat and mass transfer of Casson fluid flow over a stretching sheet in porous media in the presence of thermal radiation and viscous dissipation have been done. We have acquired following results:

6.1 An increase in Magnetic parameter M, Casson fluid parameter β and Permeability parameter K_1 causes decreases in velocity profile.

6.2 Temperature profile decrease with increasing Casson fluid parameter, Melting parameter, Radiation parameter and reverse effect for Prandtl number and Eckert number.

6.3 Concentration profile increase for increasing Schmidt number and decrease for Melting parameter.

6.4 Increment in values of Magnetic parameter and Permeability parameter skin friction coefficient increase.

6.5 Local Nusselt number decrease with increasing values of radiation and melting parameter.

Acknowledgement. The authors are grateful to the Editor and Reviewer for the suggestions which led to the paper in the present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 272-276**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

IDENTITIES INVOLVING GENERALIZED BERNOULLI NUMBERS AND PARTIAL BELL POLYNOMIALS WITH THEIR APPLICATIONS

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(Received: February 22, 2023; In format: February 25, 2023; Revised : May 25, 2023;Accepted : June 03, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53132

Abstract

In the present paper, we show that the partial Bell polynomials allow for obtaining identities involving the generalized Bernoulli numbers. Then, on applying these identities we derive different generating and bilateral generating functions.

2020 Mathematical Sciences Classification: 05A15, 05A19, 05A30.
Keywords and Phrases: Bernoulli polynomials, Partial Bell polynomials, Generalized Bernoulli numbers, Generating and bilateral generating functions.

1 Introduction

In [18], the partial Bell polynomials $B_{n,k}(.,.,\ldots,.)$ $\forall n,k \geq 0$ are represented in the following series expansion

$$\frac{1}{k!}\left\{\sum_{m\geq 1}^{\infty} x_m \frac{t^m}{m!}\right\}^k = \sum_{n\geq k}^{\infty} B_{n,k}(x_1,x_2,\ldots,x_{n-k+1})\frac{t^n}{n!} \ \forall k=0,1,2,\ldots, \tag{1.1}$$

where, $B_{n,k}(x_1,x_2,\ldots,x_{n-k+1}) = \sum_{n_1,n_2,\ldots,n_r,\ldots=0}^{\infty} \frac{n!}{n_1!n_2!..n_r!\ldots}(\frac{x_1}{1!})^{n_1}(\frac{x_2}{1!})^{n_2}\ldots(\frac{x_r}{r!})^{n_r}\ldots$; along with $n_1 + n_2 + \ldots + n_r + \ldots = k$ and $n_1 + 2n_2 + \ldots + rn_r + \ldots = n$.

Recently, Pathan et al. [11] obtained the connections between partial Bell polynomials, partition function and q-hypergeometric series.

On the other hand in [7], a generalization of (1.1) is introduced to prove the following relation

$$Q_{n+1}\left(\frac{1}{x}\right) = (n+1)\sum_{k=0}^{n}\frac{(-1)^k k!}{x^{n+k+1}}B_{n,k}\left(\frac{1}{2}\ Q_2(x), \frac{1}{3}Q_3(x),\ldots,\frac{1}{n-k+2}Q_{n-k+2}(x)\right), n\geq 0, \tag{1.2}$$

in terms of the partial Bell polynomials [2,3,4,14,18] and the Bernoulli polynomials [1,13,16]

$$Q_m(y) = B_m(y) - B_m, B_m = B_m(0), m \geq 0, \tag{1.3}$$

from (1.3), we get the polynomials

$$Q_0(y) = 0, Q_1(y) = y, Q_2(y) = y^2 - y, Q_3(y) = y^3 - \frac{3}{2}y^2 + \frac{1}{2}y,$$
$$Q_4(y) = y^4 - 2y^3 + y^2, \ldots . \tag{1.4}$$

Then we employ (1.3) to find the property

$$\lim_{x\to 0}\frac{1}{x}Q_m(x) = \lim_{x\to 0}\frac{B_m(x)-B_m(0)}{x}$$
$$= \left[\frac{d}{\mathrm{dx}}B_m(x)\right](0) = mB_{m-1}. \tag{1.5}$$

Here in this research work, we make an appeal to the results (1.2)-(1.5) and then for $n,k \geq 0$ deduce various results involving identities between the generalized Bernoulli numbers $B_n^{(k)}$ (see in [6,8,9,10,16]) and the partial Bell polynomials $B_{n,k}(.,\ldots,.)$ (see [11,18]). Later on applying these results we obtain many generating and bilateral generating functions.

2 Some identities involving $B_{n,k}$ and $B_m^{(j)}$

In this section for $n, k \geq 0$, we derive certain identities between the generalized Bernoulli numbers $B_n^{(k)}$ and the partial Bell polynomials $B_{n,k}\left(B_1, B_2, \ldots, B_{n-k+1}\right)$.

Theorem 2.1. *For all $n, k \geq 0$, the partial Bell polynomials $B_{n,k}\left(B_1, B_2, \ldots, B_{n-k+1}\right)$ involving Bernoulli numbers $B_n, (n \geq 0)$ give following identities*

$$\sum_{k=0}^{n} (-1)^k k! B_{n,k}\left(B_1, B_2, \ldots, B_{n-k+1}\right) = \frac{1}{n+1}, n \geq 0, \tag{2.1}$$

and

$$\sum_{k=0}^{n} (-1)^k k! B_{n,k}\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-k+2}\right) = B_n, n \geq 0. \tag{2.2}$$

Proof. Consider the expression (1.2) and then write it in the form

$$(n+1)\sum_{k=0}^{n} (-1)^k k! B_{n,k}\left(\frac{1}{2x}\ \mathrm{Q}_2(x), \frac{1}{3x}\ \mathrm{Q}_3(x), \ldots,\ \frac{1}{(n-k+2)x} Q_{n-k+2}(x)\right) = x^{n+1} Q_{n+1}\left(\frac{1}{x}\right). \tag{2.3}$$

Then in the formula (2.3) apply the results (1.2) and $\lim_{x\to 0}$eq.(1.5), we obtain the identity (2.1) because of the limiting case $\lim_{x\to 0} x^m Q_m\left(\frac{1}{x}\right) = 1$.

Again, the inversion of (2.1) gives us the identity (2.2). □

Remark 2.1. It is remarked that Zhang-Yang [18] deduced the relation

$$B_{n,k}\left(B_1, B_2, \ldots, B_{n-k+1}\right) = \frac{1}{k!} B_n^{(k)},\ (n, k \geq 0), \tag{2.4}$$

involving the generalized Bernoulli numbers ([6, 8, 9, 10, 16]), however, (2.4) is incorrect, it must be

$$B_{n,k}\left(B_1, B_2, \ldots, B_{n-k+1}\right) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} B_n^{(j)}. \tag{2.5}$$

Theorem 2.2. *For the generalized Bernoulli numbers $B_n^{(k)}$ $(n, k \geq 0)$, there exists an identity*

$$\sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} B_n^{(j)} = \frac{1}{n+1},\ \textit{for all } n \geq 0. \tag{2.6}$$

Proof. Make an appeal to the results (2.1) and (2.5), immediately we obtain the identity (2.6). □

Theorem 2.3. *For all $n, k \geq 0$, an identity between the generalized Bernoulli numbers $B_n^{(k)}$ and the partial Bell polynomials $B_{n,k}(., \ldots, .)$ exists as*

$$\sum_{j=0}^{k} \binom{k}{j} j! B_{n,j}\left(B_1, B_2, \ldots, B_{n-j+1}\right) = B_n^{(k)}, n, k \geq 0. \tag{2.7}$$

Proof. Make an appeal to the Theorems 2.1 and 2.3 and the corrigendum in the Remark 2.2 and then in it use the property due to [13] as given by

$$\sum_{k=j}^{n} \binom{k}{j} = \binom{n+1}{j+1}. \tag{2.8}$$

Finally, the binomial inversion [5] of (2.5) gives the expression (2.7). □

Theorem 2.4. *For all $n \geq 0$, there exists following identities*

$$(-1)^k k! B_{n,k}\left(B_1, B_2, \ldots, B_{n-k+1}\right) = \frac{1}{2}\binom{n-1}{k-1}\binom{2n-1}{k}^{-1} = (-1)^k \binom{n+1}{k+1} B_n^{(k)}. \tag{2.9}$$

Proof. In the Theorems 2.1 and 2.3, make an appeal to the formula given by

$$\sum_{k=0}^{n} \frac{k}{n}\binom{n}{k}\binom{2n-1}{k}^{-1} = \frac{2}{n+1}, \tag{2.10}$$

and thus use the reduction formula for binomial coefficients we arrive the identities in (2.9). □

Remark 2.2. The relation (1.2) is equivalent to the following identity [7] (Cauchy convolution [15])

$$\sum_{k=1}^{n-1} \frac{x^k}{k!(n-k)!} Q_k\left(\frac{1}{x}\right) Q_{n-k}(x) = 0, n \geq 3. \tag{2.11}$$

3 Applications

In this section on application of the identities obtained in the Section 2 and the formula due to [12, p. 348, Problem 212] and see also in [17, p. 355, Eqn. (9)] given by

$$\sum_{n=0}^{\infty} \frac{\alpha}{\alpha+(\beta+1)n}\binom{\alpha+(\beta+1)n}{n} t^n = (1+w)^\alpha, |t|<1,$$
$$w = w(t) = t\,(1+w(t))^{\beta+1}, w(0)=0, \tag{3.1}$$

we obtain various generating and bilateral generating functions:

Example 3.1. If $\alpha > 0$ and for $n \geq 0$

$$\Psi_n = \sum_{k=0}^{n} (-1)^k k! B_{n,k}\left(B_1, B_2, \ldots, B_{n-k+1}\right). \tag{3.2}$$

Then in the disk $|t| < 1$, there exists a generating formula

$$\sum_{n=0}^{\infty} \binom{\alpha+\alpha n}{n} \Psi_n t^n = \alpha(1+\zeta)^\alpha, \tag{3.3}$$

where, $\zeta = \zeta(t) = t\,(1+\zeta(t))^\alpha, \zeta(0)=0$.

Solution. In the Eqns. (3.2) and (3.3) make an application of the Theorem 2.1, we get

$$\sum_{n=0}^{\infty} \binom{\alpha+\alpha n}{n} \Psi_n t^n = \alpha \sum_{n=0}^{\infty} \binom{\alpha+\alpha n}{n} \frac{1}{\alpha+\alpha n} t^n. \tag{3.4}$$

Finally in the result (3.4), apply the formula (3.1) for $\beta = \alpha - 1$ we obtain the formula (3.3).

Example 3.2. If $\alpha > 0$,

$$\psi_n = \sum_{k=0}^{n} \binom{\alpha+\alpha n}{n-k}\binom{\alpha+(\beta+1)k}{k}\binom{k+1}{k}\varphi_k, \tag{3.5}$$

Then by (3.2) and (3.5), there exists a bilateral generating formula

$$\sum_{n=0}^{\infty} \psi_n \Psi_n t^n = (1+\zeta)^\alpha \sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} \varphi_n \zeta^n. \tag{3.6}$$

Solution. In the left hand side of (3.6) on considering (3.2) and (3.5), then on use of (2.1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \psi_n \Psi_n t^n &= \sum_{n=0}^{\infty} \frac{1}{\alpha+\alpha n} \sum_{k=0}^{n} \binom{\alpha+\alpha n}{n-k}\binom{\alpha+(\beta+1)k}{k}(\alpha k+\alpha)\varphi_k t^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \binom{\alpha+\alpha n+\alpha k}{n} \frac{\alpha k+\alpha}{\alpha+\alpha n+\alpha k}\binom{\alpha+(\beta+1)k}{k}\varphi_k t^{n+k} \\
&= \sum_{k=0}^{\infty} \binom{\alpha+(\beta+1)k}{k}\varphi_k t^k \sum_{n=0}^{\infty} \frac{\alpha+\alpha k}{\alpha+\alpha k+\alpha n}\binom{\alpha+\alpha k+\alpha n}{n} t^n \\
&= (1+\zeta)^\alpha \sum_{k=0}^{\infty} \binom{\alpha+(\beta+1)k}{k}\varphi_k \left\{t(1+\zeta)^\alpha\right\}^k \text{ (on use of (3.1)).}
\end{aligned} \tag{3.7}$$

Finally on use of (3.3) and (3.7), we find right hand side of (3.6).

Example 3.3. In (3.5) if

$$\varphi_k = {}_{p+1}F_{q+1}\left[\begin{matrix} -k, \alpha_1, \ldots, \alpha_p; \\ \alpha+\beta k+1, \beta_1, \ldots, \beta_q; \end{matrix} x\right], \tag{3.8}$$

where, the generalized hypergeometric function ${}_pF_q(.)$ is defined by [17, p. 42]

$$ {}_pF_q\left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\gamma_i)_n} \frac{z^n}{n!}, \tag{3.9}$$

where $p, q \in \mathbb{N} \cup \{0\}$, $\alpha_i \in \mathbb{C}(i = 1, 2, 3, \ldots, p); \gamma_i \in \mathbb{C}(i = 1, 2, 3, \ldots, q); z \in \mathbb{C}$; also all

$$\gamma_i \neq 0, -1, -2, \ldots, \ (i = 1, 2, 3, \ldots, q).$$

The series in (3.9) (i) converges for $|z| < \infty$, if $p \leq q$; (ii) converges for $|z| < 1$, if $p = q+1$; (iii) diverges for all z, $z \neq 0$, if $p > q+1$; (iv) converges absolutely for $|z| = 1$, if $p = q+1$, and $\Re(\omega) > 0, \omega = \sum_{i=1}^{q} \gamma_i - \sum_{i=1}^{p} \alpha_i$; (v) converges conditionally for $|z| = 1, z \neq 1$, if $p = q + 1$, and $-1 < \Re(\omega) \leq 0$; (vi) diverges for $|z| = 1$, if $p = q + 1$, and $\Re(\omega) < -1$.

Then on application of the example 3.2, there exists a bilateral generating function

$$\sum_{n=0}^{\infty} \psi_n \Psi_n t^n = \frac{(1+\zeta)^{\alpha} (1+W(\zeta))^{\alpha+1}}{1-\beta W(\zeta)} {}_pF_q \left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} - xW(\zeta) \right], \tag{3.10}$$

where $W(\zeta)$ is given in following (3.12).

Solution. Make an appeal to the functions (3.8) and (3.9) in Example (3.2) we find

$$\begin{aligned} \sum_{n=0}^{\infty} \psi_n \Psi_n t^n &= (1+\zeta)^{\alpha} \sum_{n=0}^{\infty} \binom{\alpha + (\beta+1)n}{n} {}_{p+1}F_{q+1} \left[\begin{matrix} -n, \alpha_1, \ldots, \alpha_p; \\ \alpha + \beta n + 1, \beta_1, \ldots, \beta_q; \end{matrix} x \right] \zeta^n \\ &= (1+\zeta)^{\alpha} \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_p)_k}{(\beta_1)_k \ldots (\beta_q)_k} \frac{(-x\zeta)^k}{k!} \sum_{n=0}^{\infty} \binom{\alpha + (\beta+1)k + (\beta+1)n}{n} \zeta^n \\ &= (1+\zeta)^{\alpha} \frac{(1+W(\zeta))^{\alpha+1}}{1-\beta W(\zeta)} \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_p)_k}{(\beta_1)_k \ldots (\beta_q)_k} \frac{\left(-x\zeta (1+W(\zeta))^{\beta+1}\right)^k}{k!}. \end{aligned} \tag{3.11}$$

Now in (3.11) define

$$W(\zeta) = \zeta (1+W(\zeta))^{\beta+1}, \ W(0) = 0, \tag{3.12}$$

we derive the bilateral generating function (3.10).

4 Concluding remarks

The identities obtained in the Section 2 are very powerful tool to derive different results involving generalized Bernoulli numbers $B_n^{(k)}$ and the partial Bell polynomials $B_{n,k}(B_1, B_2, \ldots, B_{n-k+1})$.

We derive various generating and bilateral generating functions which are applicable in computing of many problems occurring in the science and technology. The polynomials in form of generalized hypergeometric functions are specialized in Legendre, Bessel, Hermite, Laggurre and Jacobi polynomials etc. found in the literature of generating functions. Hence Section 3 has very important and applicable techniques.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 277-286**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

AN ALGORITHMIC APPROACH TO LOCAL SOLUTION OF THE NONLINEAR SECOND ORDER ORDINARY HYBRID INTEGRODIFFERENTIAL EQUATIONS

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(Received: February 14, 20232; In format: February 22, 2023; Revised: April 29, 2023; Accepted: May 26, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53133

Abstract

In this paper, we establish a couple of approximation results for local existence and uniqueness of the solution of an *IVP* of nonlinear second order ordinary hybrid integrodifferential equations by using the Dhage monotone iteration method based on the recent hybrid fixed point theorems of Dhage (2022) and Dhage *et al.* (2022). An approximation result for Ulam-Hyers stability of the local solution of the considered hybrid differential equation is also established. Finally, our main abstract results are also illustrated with a couple of numerical examples.

2020 Mathematical Sciences Classification: 34A12, 34A34, 34A45, 47H10
Keywords and Phrases: Ordinary differential equation; Dhage iteration method; Approximation theorems; Local existence and uniqueness; Ulam-Hyers stability.

1 Introduction

Given a closed and bounded interval $J = [t_0, t_0 + a]$ in $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $a > 0$, we consider the *IVP* of nonlinear second order hybrid ordinary differential equation *(HIGDE)*,

$$\left.\begin{aligned} &x''(t) = f\Big(t, x(t), \int_{t_0}^{t} g(s, x(s))\, ds\Big), \;\; t \in J, \\ &x(t_0) = \alpha_0, \;\; x'(t_0) = \alpha_1, \end{aligned}\right\} \tag{1.1}$$

where α_0, α_1 are real numbers and the function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies some hybrid, that is, mixed hypotheses from algebra, analysis and topology to be specified later.

Definition 1.1. *A function $x \in C^1(J, \mathbb{R})$ is said to be a* solution *of the HIGDE* (1.1) *if it satisfies the equations in* (1.1) *on J, where $C^1(J, \mathbb{R})$ is the space of continuously differentiable real-valued functions defined on J. If the solution x lies in a closed ball $\overline{B_r(x_0)}$ centered at some point $x_0 \in C(J, \mathbb{R})$ of radius $r > 0$, then we say it is a* local solution *or* neighborhood solution *(in short* nbhd solution*) of the HIGDE* (1.1) *on J.*

Remark 1.1. The present idea of local or nbhd-solution is different from the usual notion of a local solution solution as mentioned in Coddington and Levinson [1]. See Dhage and Dhage [12, 13] and references given therein.

The *HIGDE* (1.1) is familiar in the subject of nonlinear analysis and can be studied for a variety of different aspects of the solution by using different methods form nonlinear functional analysis. The existence of local solution can be proved by using the Schauder fixed point principle, see for example, Coddington and Levinson [1], Lakshmikantham and Leela [17], Granas and Dugundji [15] and references therein. The approximation result for uniqueness of solution can be proved by using the Banach fixed point theorem under a Lipschitz condition which is considered to be very strong in the area of nonlinear analysis. But to the knowledge the present authors, the approximation result for local existence and uniqueness theorems without using the Lipschitz condition is not discussed so far in the theory of nonlinear differential and integral equations. In this paper, we discuss the approximation results for local existence and uniqueness of solution under weaker partial Lipschitz condition but via construction of the algorithms based on monotone iteration method and a hybrid fixed point theorem of Dhage [4]. Also see Dhage *et al.* [10, 11] and references therein.

The rest of the paper is organized as follows. Section 2 deals with the auxiliary results and main hybrid fixed point theorems involved in the Dhage iteration method. The hypotheses and main approximation results for the local existence and uniqueness of solution are given in Section 3. The approximation of the Ulam-Hyer stability is discussed in Section 4 and a couple of illustrative examples are presented in Section 5. Finally, some concluding remarks are mentioned in Section 6.

2 Auxiliary Results

We place the problem of *HIGDE* (1.1) in the function space $C(J,\mathbb{R})$ of continuous, real-valued functions defined on J. We introduce a supremum norm $\|\cdot\|$ in $C(J,\mathbb{R})$ defined by

$$\|x\| = \sup_{t\in J} |x(t)|. \tag{2.1}$$

and an order relation $\preceq$ in $C(J,\mathbb{R})$ by the cone K given by

$$K = \{x \in C(J,\mathbb{R}) \mid x(t) \geq 0 \ \forall\ t \in J\}. \tag{2.2}$$

Thus,

$$x \preceq y \iff y - x \in K, \tag{2.3}$$

or equivalently,

$$x \preceq y \iff x(t) \leq y(t) \ \forall\ t \in J.$$

It is known that the Banach space $C(J,\mathbb{R})$ together with the order relations $\preceq$ becomes an ordered Banach space which we denote for convenience, by $\big(C(J,\mathbb{R}), K\big)$. We denote the open and closed spheres centered at $x_0 \in C(J,\mathbb{R})$ of radius r, for some $r > 0$, by

$$B_r(x_0) = \{x \in C(J,\mathbb{R}) \mid \|x - x_0\| < r\} = B(x,r)$$

and

$$B_r[x_0] = \{x \in C(J,\mathbb{R}) \mid \|x - x_0\| \leq r\} = \overline{B(x,r)}$$

receptively. It is clear that $B_r[x_0] = \overline{B_r(x_0)}$. Let $M > 0$ be a real number. Denote

$$B_r^M[x_0] = \big\{x \in B_r[x_0] \,\big|\, |x(t_1) - x(t_2)| \leq M\,|t_1 - t_2| \ \text{ for } \ t_1, t_2 \in J\big\}. \tag{2.4}$$

Then, we have the following result.

Lemma 2.1. *The set $B_r^M[x_0]$ is compact in $C(J,\mathbb{R})$.*

Proof. By definition, $B_r[x_0]$ is a closed and bounded subset of the Banach space $C(J,\mathbb{R})$. Moreover, $B_r^M[x_0]$ is an equicontinuous subset of $C(J,\mathbb{R})$ in view of the condition (2.1). Now, by an application of Arzelá-Ascoli theorem, $B_r^M[x_0]$ is compact set in $C(J,\mathbb{R})$ and the proof of the lemma is complete. $\square$

It is well-known that the hybrid fixed point theoretic technique is very much useful in the subject of nonlinear analysis for dealing with the nonlinear equations qualitatively. See Granas and Dugundji [15] and the references therein. Here, we employ the Dhage monotone iteration method or simply *Dhage iteration method* based on the following two hybrid fixed point theorems of Dhage [4] and Dhage *et al.* [10].

Theorem 2.1 (Dhage [4])**.** *Let S be a non-empty partially compact subset of a regular partially ordered Banach space $\big(E, \|\cdot\|, \preceq,\big)$ with every chain C in S is Janhavi set and let $\mathcal{T} : S \to S$ be a monotone nondecreasing, partially continuous mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.2 (Dhage [4])**.** *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in a regular partially ordered Banach space $\big(E, \|\cdot\|, \preceq,\big)$ and let $\mathcal{T} : E \to E$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying*

$$\|x_0 - \mathcal{T}x_0\| \leq (1-q)r$$

for some real number $r > 0$, then $\mathcal{T}$ has a unique comparable fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*. Furthermore, if every pair of elements in X has a lower or upper bound, then ξ^* is unique.

If a Banach X is partially ordered by an order cone K in X, then in this case we simply say X is an ordered Banach space which we denote it by (X, K). Then, we have the following useful results proved in Dhage [2, 3].

Lemma 2.2 (Dhage [2, 3]). *Every ordered Banach space (X, K) is regular.*

Lemma 2.3 (Dhage [2, 3]). *Every partially compact subset S of an ordered Banach space (X, K) is a Janhavi set in X.*

As a consequence of Lemmas 2.2 and 2.3, we obtain the following hybrid fixed point theorem which we need in what follows.

Theorem 2.3 (Dhage [4] and Dhage *et al.* [10]). *Let S be a non-empty partially compact subset of an ordered Banach space (X, K) and let $\mathcal{T} : S \to S$ be a partially continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then $\mathcal{T}$ has a fixed point $\xi^* \in S$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.4 (Dhage [4]). *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in an ordered Banach space $\big(X, K\big)$ and let $\mathcal{T} : (X, K) \to (X, K)$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying*

$$\|x_0 - \mathcal{T}x_0\| \leq (1 - q)r \tag{2.5}$$

for some real number $r > 0$, then $\mathcal{T}$ has a unique comparable fixed point x^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*. Furthermore, if every pair of elements in X has a lower or upper bound, then ξ^* is unique.

The details of the notions of partial order, Janhavi set, regularity of an ordered space, monotonicity of mappings, partial continuity, partial closure, partial compactness and partial contraction etc. and related applications appear in Dhage [2, 3, 4, 5, 6], Dhage and Dhage [8], Dhage *et al.* [10, 11, 14] and references therein.

3 Local Approximation Results

We consider the following set of hypotheses in what follows.

(H_1) The function f is continuous and bounded on $J \times \mathbb{R} \times \mathbb{R}$ with bound M_f.

(H_2) $f(t, x, y)$ is nondecreasing in x and y for each $t \in J$.

(H_3) $g(t, x)$ is nondecreasing in x for each $t \in J$.

(H_4) $f(t, \alpha_0, y) \geq 0$ and $\alpha_1 \geq 0$ for all $t \in J$ and $y \geq 0$.

(H_5) $g(t, \alpha_0) \geq 0$ for all $t \in J$.

Then we have the following useful lemma.

Lemma 3.1. *If $h \in L^1(J, \mathbb{R})$, then the IVP of ordinary second order linear differential equation*

$$x''(t) = h(t), \ \ t \in J, \quad x(t_0) = \alpha_0, \quad x'(t_0) = \alpha_1, \tag{3.1}$$

is equivalent to the integral equation

$$x(t) = \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t - s)\, h(s)\, ds, \ \ , t \in J. \tag{3.2}$$

Theorem 3.1. *Sppose that the hypotheses (H_1), (H_3) and (H_4) hold. Furthermore, if the inequalities $|\alpha_1|\, a + M_f\, a^2 \leq r$ and $|\alpha_1| + 2\, M_f\, a \leq M$ hold, then the HIGDE* (1.1) *has a solution x^* in $B_r^M[\alpha_0]$, where $x_0 \equiv \alpha_0$, and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} &x_0(t) = \alpha_0, \ \ t \in J, \\ &x_{n+1}(t) = \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t - s)\, f\Big(s, x_n(s), \int_{t_0}^{s} g(\tau, x_n(\tau))\, d\tau\Big)\, ds, \ \ t \in J, \end{aligned}\right\} \tag{3.3}$$

where $n = 0, 1, \ldots$; converges monotone nondecreasingly to x^*.

Proof. Set $X = C(J,\mathbb{R})$. Clearly, (X,K) is a partially ordered Banach space. Let x_0 be a constant function on J such that $x_0(t) = \alpha_0$ for all $t \in J$ and define a closed ball $B_r^M[x_0]$ in X defined by (2.3). By Lemma 2.1, $B_r^M[x_0]$ is a compact subset of X. By Lemma 3.1, the *HIGDE* (1.1) is equivalent to the nonlinear hybrid integral equation *(HIE)*

$$x(t) = \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^{t}(t-s)\,f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds,\ \ t \in J. \tag{3.4}$$

Now, define an operator $\mathcal{T}$ on $B_r^M[x_0]$ into X by

$$\mathcal{T}x(t) = \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^{t}(t-s)\,f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds,\ \ t \in J. \tag{3.5}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.3 on $B_r^M[x_0]$ in the following series of steps.

Step I: *The operator $\mathcal{T}$ maps $B_r^M[x_0]$ into itself.*

Firstly, we show that $\mathcal{T}$ maps $B_r^M[x_0]$ into itself. Let $x \in B_r^M[x_0]$ be arbitrary element. Then,

$$\begin{aligned}
|\mathcal{T}x(t) - x_0(t)| &\le |\alpha_1(t-t_0)| + \left|\int_{t_0}^{t}(t-s)\,f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds\right| \\
&\le |\alpha_1|\,a + \int_{t_0}^{t}|t-s|\,\left|f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\right|ds \\
&= |\alpha_1|\,a + M_f\,a\int_{t_0}^{t_0+a} ds \\
&= |\alpha_1|\,a + M_f\,a^2 \\
&\le r.
\end{aligned}$$

for all $t \in J$. Taking the supremum over t in the above inequality yields

$$\|\mathcal{T}x - x_0\| \le |\alpha_1|\,a + M_f\,a^2 \le r$$

which implies that $\mathcal{T}x \in B_r[x_0]$ for all $x \in B_r^M[x_0]$. Next, let $t_1, t_2 \in J$ be arbitrary. Then, we have

$$\begin{aligned}
&\big|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\big| \\
&\le |\alpha_1|\,|t_1-t_2| + \bigg|\int_{t_0}^{t_1}(t_1-s)\ f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds \\
&\qquad\qquad - \int_{t_0}^{t_2}(t_2-s)\ f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds\bigg| \\
&\le |\alpha_1|\,|t_1-t_2| + + \bigg|\int_{t_0}^{t_1}(t_1-s)\ f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds \\
&\qquad\qquad - \int_{t_0}^{t_2}(t_2-s)\ f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds\bigg| \\
&\qquad + \bigg|\int_{t_0}^{t_1}(t_1-s)\,f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds \\
&\qquad\qquad - \int_{t_0}^{t_2}(t_2-s)\ f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds\bigg| \\
&\le |\alpha_1|\,|t_1-t_2| + \int_{t_0}^{t_1}|t_1-t_2|\,\left|f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\right|ds \\
&\qquad + \left|\int_{t_1}^{t_2}|t_2-s|\,\left|f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\right|ds\right| \\
&\le |\alpha_1|\,|t_1-t_2| + \int_{t_0}^{t_0+a}|t_1-t_2|\,M_f\,ds + \left|\int_{t_1}^{t_2} a\,M_f\,ds\right| \\
&\le |\alpha_1|\,|t_1-t_2| + 2M_f\,a\,|t_1-t_2| \\
&= \big(|\alpha_1| + 2M_f\,a\big)\,|t_1-t_2|
\end{aligned}$$

$$\leq M\,|t_1 - t_2|$$

where, $|\alpha_1| + 2M_f\, a \leq M$. Therefore, $\mathcal{T}x \in B_r^M[x_0]$ for all $x \in B_r^M[x_0]$ As a result, we have $\mathcal{T}(B_r^M[x_0]) \subset B_r^M[x_0]$.

Step II: *$\mathcal{T}$ is a monotone nondecreasing operator.*

Let $x, y \in B_r^M[x_0]$ be any two elements such that $x \succeq y$. Then, by hypotheses (H_2) and (H_3),

$$\begin{aligned}
\mathcal{T}x(t) &= \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s)\, f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big)\, ds \\
&\geq \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s)\, f\Big(s, y(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big)\, ds \\
&= \mathcal{T}y(t)
\end{aligned}$$

for all $t \in J$. So, $\mathcal{T}x \succeq \mathcal{T}y$, that is, $\mathcal{T}$ is monotone nondecreasing on $B_r^M[x_0]$.

Step III: *$\mathcal{T}$ is partially continuous operator.*

Let C be a chain in $B_r^M[x_0]$ and let $\{x_n\}$ be a sequence in C converging to a point $x \in C$. Then, by dominated cnonvergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{T}x_n &= \lim_{n\to\infty} \left[\alpha_0 + \int_{t_0}^{t} (t-s)\, f\Big(s, x_n(s), \int_{t_0}^{s} g(\tau, x_n(\tau))\, d\tau\Big)\, ds\right] \\
&= \alpha_0 + \alpha_1(t - t_0) + \lim_{n\to\infty} \int_{t_0}^{t} (t-s)\, f\Big(s, x_n(s), \int_{t_0}^{s} g(\tau, x_n(\tau))\, d\tau\Big)\, ds \\
&= \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s)\, \left[\lim_{n\to\infty} f\Big(s, x_n(s), \int_{t_0}^{s} g(\tau, x_n(\tau))\, d\tau\Big)\right]\, ds \\
&= \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s)\, f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big)\, ds \\
&= \mathcal{T}x(t)
\end{aligned}$$

for all $t \in J$. Therefore, $\mathcal{T}x_n \to \mathcal{T}x$ pointwise on J. As $\{\mathcal{T}x_n\} \subset B_r^M[x_0]$, $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of points in X. As a result, we have that $\mathcal{T}x_n \to \mathcal{T}x$ uniformly on J. Hence $\mathcal{T}$ is partially continuous operator on $B_r^M[x_0]$.

Step IV: *The element $x_0 \in B_r^M[x_0]$ satisfies the relation $x_0 \preceq \mathcal{T}x_0$.*

Since the hypotheses (H_4) and (H_5) hold, one has

$$\begin{aligned}
x_0(t) &= \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s) f\Big(s, x_0(s), \int_{t_0}^{s} g(\tau, x_0(\tau))\, d\tau\Big)\, ds \\
&\leq x_0(t) + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s) f\Big(s, \alpha_0(s), \int_{t_0}^{s} g(\tau, \alpha_0)\, d\tau\Big)\, ds \\
&= \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s) f f\Big(s, x_0(s), \int_{t_0}^{s} g(\tau, x_0(\tau))\, d\tau\Big)\, ds \\
&= \mathcal{T}x_0(t)
\end{aligned}$$

for all $t \in J$. This shows that the constant function x_0 in $B_r^M[x_0]$ serves as to satisfy the operator inequality $x_0 \preceq \mathcal{T}x_0$.

Thus, the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.3, and so $\mathcal{T}$ has a fixed point x^* in $B_r^M[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the *HIE* (3.4) and consequently the *HIGDE* (1.1) has a local solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nondecreasing and converges to x^*. This completes the proof. □

Next, we prove an approximation result for existence and uniqueness of the solution simultaneously under weaker form of Lipschitz condition. We need the following hypotheses in what follows.

(H_6) There exists a constant $k > 0$ such that

$$0 \leq f(t, x_1, x_2) - f(t, y_1, y_2) \leq \ell_1(x_1 - y_1) + \ell_2(x_2 - y_2)$$

for all $t \in J$ and $x_1, y_1, x_2, y_2 \in \mathbb{R}$ with $x_1 \geq y_1$, $x_2 \geq y_2$, where $\big(\ell_1 a + \ell_2 k\, a^2\big) < 1$.

(H$_7$) There exists a constant $k > 0$ such that

$$0 \le g(t,x) - g(t,y) \le k\,(x-y)$$

for all $t \in J$ and $x, y \in \mathbb{R}$ with $x \ge y$.

Theorem 3.2. *Suppose that the hypotheses (H$_1$), (H$_6$) and (H$_7$) hold. Furthermore, if*

$$|\alpha_1|\,a + M_f\,a^2 \le \left[1 - \left(\ell_1 a^2 + \ell_2 k\,a^3\right)\right] r, \quad \left(\ell_1 a^2 + \ell_2 k\,a^3\right) < 1, \tag{3.6}$$

for some real number $r > 0$, then the HIGDE (1.1) has a unique solution x^* in $B_r[x_0]$ defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nondecreasing and converges to x^*.

Proof. Set $(X,K) = \big(C(J,\mathbb{R}), \preceq\big)$ which is a lattice w.r.t. the lattice *join* and *meet* operations defined by $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$, and so every pair of elements of X has a lower and an upper bound. Let $r > 0$ be a fixed real number and consider closed sphere $B_r[x_0]$ centred at x_0 of radius r in the partially ordered Banach space (X,K).

Define an operator $\mathcal{T}$ on X into X by (3.5). Clearly, $\mathcal{T}$ is monotone nondecreasing on X. To see this, let $x, y \in X$ be two elements such that $x \succeq y$. Then, by hypotheses (H$_6$) and (H$_7$), we obtain

$$\begin{aligned}
&\mathcal{T}x(t) - \mathcal{T}y(t) \\
&= \int_{t_0}^{t} (t-s)\, \left[f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds - f\Big(s, y(s), \int_{t_0}^{s} g(\tau, y(\tau))\,d\tau\Big)\right] ds \\
&\ge 0,
\end{aligned}$$

for all $t \in J$. Therefore, $\mathcal{T}x \succeq \mathcal{T}y$ and consequently $\mathcal{T}$ is monotone nondecresing on X.

Next, we show that $\mathcal{T}$ is a partial contraction on X. Let $x, y \in X$ be such that $x \succeq y$. Then, by hypotheses (H$_6$) and (H$_7$), we obtain

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}y(t)| &= \Big| \int_{t_0}^{t} (t-s) f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds \\
&\qquad - \int_{t_0}^{t} (t-s) f\Big(s, y(s), \int_{t_0}^{s} g(\tau, y(\tau))\,d\tau\Big)\,ds \Big| \\
&\le \Big| \int_{t_0}^{t} (t-s) \Big[f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds \\
&\qquad - f\Big(s, y(s), \int_{t_0}^{s} g(\tau, y(\tau))\,d\tau\Big)\Big]\,ds \Big| \\
&\le \Big| \int_{t_0}^{t} (t-s) \Big[\ell_1 \big(x(s) - y(s)\big) + \ell_2 \int_{t_0}^{s} k\,(t-s)\big(x(\tau) - y(\tau)\big)\,d\tau \Big]\,ds \Big| \\
&= \ell_1 \int_{t_0}^{t} a\,|x(s) - y(s)|\,ds + \ell_2 k \int_{t_0}^{t} a\,\big(x(s) - y(s)\big)\big) \\
&\le \ell_1 a \int_{t_0}^{t_0+a} \|x - y\|\,ds + \ell_2\,k\,a^2 \int_{t_0}^{t_0+a} \|x - y\|\,ds \\
&= \big[\ell_1 a^2 + \ell_2 k\,a^3\big]\,\|x - y\| \\
&= \lambda \|x - y\|
\end{aligned}$$

for all $t \in J$, where $\lambda = \ell_1 a^2 + \ell_2 k\,a^3 < 1$. Taking the supremum over t in the above inequality yields

$$\|\mathcal{T}x - \mathcal{T}y\| \le \lambda\,\|x - y\|$$

for all comparable elements $x, y \in X$. This shows that $\mathcal{T}$ is a partial contraction on X with contraction constant $k\,a$. Furthermore, it can be shown as in the proof of Theorem 3.1 that the element $x_0 \in B_r^M[x_0]$ satisfies the relation $x_0 \preceq \mathcal{T}x_0$ in view of hypothesis (H_4). Finally, by hypotheses $(H_4) - (H_5)$ and condition (3.6), one has

$$\|x_0 - \mathcal{T}x_0\| \le |\alpha_1|\,a + \sup_{t \in J} \Big| \int_{t_0}^{t} (t-s)\, f\Big(s, \alpha_0, \int_{t_0}^{s} g(\tau, \alpha_0)\,d\tau\Big)\,ds \Big|$$

$$
\begin{aligned}
&\le |\alpha_1|\, a + \sup_{t\in J}\int_{t_0}^{t} |t-s|\, \Big| f\Big(s, \alpha_0, \int_{t_0}^{s} g(\tau, \alpha_0)\, d\tau\Big)\Big|\, ds \\
&\le |\alpha_1|\, a + M_f\, a^2 \\
&\le \big[1 - \big(\ell_1 a^2 + \ell_2 k\, a^3\big)\big]\, r
\end{aligned}
$$

which shows that the condition (2.5) of Theorem 2.4 is satisfied. Hence $\mathcal{T}$ has a unique fixed point x^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the *HIE* (3.4) and consequently the *HIGDE* (1.1) has a unique local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nondecreasing and converges to x^*. This completes the proof. □

Remark 3.1. The conclusion of Theorems 3.1 and 3.2 also remains true if we replace the hypothesis (H_4) with the following one.

(H_4') $f(t, \alpha_0, y) \le 0$ and $\alpha_1 \le 0$ for all $t \in J$ and $y \ge 0$.

In this case, the *HIGDE* (1.1) has a local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nonincreasing and converges to x^*.

Remark 3.2. If the initial condition in the equation (1.1) is such that $\alpha_0 > 0$, then under the conditions of Theorem 3.1, the *HIGDE* (1.1) has a local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to the positive solution x^*. Similarly, under the conditions of Theorem 3.2, the *HIGDE* (1.1) has a unique local positive solution x^* defined on J and the sequence of successive approximations defined by (3.3) $\{x_n\}_{n=0}^{\infty}$ converges monotone nondecreasingly to the unique positive solution x^*.

4 Approximation of Local Ulam-Hyers Stability

The Ulam-Hyers stability for various dynamic systems has already been discussed by several authors under the conditions of classical Schauder fixed point theorem (see Tripathy [18], Huang *et al.* [16] and references therein). Here, in the present paper, we discuss the approximation of the Ulam-Hyers stability of local solution of the *HIGDE* (1.1) under the conditions of hybrid fixed point principle stated in Theorem 2.4. We need the following definition in what follows.

Definition 4.1. *The HIGDE* (1.1) *is said to be locally Ulam-Hyers stable if for $\epsilon > 0$ and for each solution $y \in B_r[x_0]$ of the inequality*

$$
\left.\begin{aligned}
&\Big| y''(t) - f\Big(t, y(t), \int_{t_0}^{t} g(s, y(s))\, ds\Big)\Big| \le \epsilon, \quad t \in J, \\
&y(t_0) = \alpha_0, \quad y'(t_0) = \alpha_1,
\end{aligned}\right\} \tag{$*$}
$$

there exists a constant $K_f > 0$ such that

$$
\big| y(t) - \xi(t)\big| \le K_f \epsilon \tag{$**$}
$$

for all $t \in J$, where $\xi \in B_r[x_0]$ is a local solution of the HIGDE (1.1) *defined on J. The solution ξ of the HIGDE* (1.1) *is called Ulam-Hyers stable local solution on J.*

Theorem 4.1. *Assume that all the hypotheses of Theorem* 3.2 *hold. Then the HIGDE* (1.1) *has a unique Ulam-Hyers stable local solution $x^* \in B_r[x_0]$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations given by* (3.3) *converges monotone nondecreasingly to x^*.*

Proof. Let $\epsilon > 0$ be given and let $y \in B_r[x_0]$ be a solution of the functional inequality (4.1) on J, that is, we have

$$
\left.\begin{aligned}
&\Big| y''(t) - f\Big(t, y(t), \int_{t_0}^{t} g(s, y(s))\, ds\Big)\Big| \le \epsilon, \quad t \in J, \\
&y(t_0) = \alpha_0, \quad y'(t_0) = \alpha_1,
\end{aligned}\right\} \tag{4.1}
$$

By Theorem 3.2, the *HIGDE* (1.1) has a unique local solution $\xi \in B_r[x_0]$. Then by Lemma 2.1, one has

$$
\xi(t) = x_o + \alpha_1 (t - t_0) + \int_{t_0}^{t} (t-s)\, f\Big(s, \xi(s), \int_{t_0}^{s} g(\tau, \xi(\tau))\, d\tau\Big)\, ds, \quad t \in J. \tag{4.2}
$$

Now, by integration of (4.1) yields the estimate:

$$\left| y(t) - \alpha_0 - \alpha_1(t-t_0) - \int_{t_0}^{t} (t-s)\, f\Big(s, y(s), \int_{t_0}^{s} g(\tau, y(\tau))\, d\tau\Big)\, ds \right| \le \frac{a^2}{2}\,\epsilon, \tag{4.3}$$

for all $t \in J$.

Next, from (4.2) and (4.3) we obtain

$$\begin{aligned}
&\big|y(t) - \xi(t)\big| \\
&= \left| y(t) - \alpha_0 - \alpha_1(t-t_0) - \int_{t_0}^{t} (t-s) f\Big(s, \xi(s), \int_{t_0}^{s} g(\tau, \xi(\tau))\, d\tau\Big)\, ds \right| \\
&\le \left| y(t) - \alpha_0 - \alpha_1(t-t_0) - \int_{t_0}^{t} (t-s) f\Big(s, y(s), \int_{t_0}^{s} g(\tau, y(\tau))\, d\tau\Big)\, ds \right| \\
&\quad + \left| \int_{t_0}^{t} (t-s)\Big[f\Big(s, y(s), \int_{t_0}^{s} g(\tau, y(\tau))\, d\tau\Big) - f\Big(s, \xi(s), \int_{t_0}^{s} g(\tau, \xi(\tau))\, d\tau\Big)\Big]\, ds \right| \\
&\le \frac{a^2}{2}\,\epsilon + \left| \int_{t_0}^{t} a\Big[\ell_1\big(y(s)-\xi(s)\big) + \ell_2 \int_{t_0}^{t} k\,(t-s)\big(y(\tau)-\xi(\tau)\big)\, d\tau\Big]\, ds \right| \\
&= \frac{a^2}{2}\,\epsilon + \ell_1 a \int_{t_0}^{t} |y(s)-\xi(s)|\, ds + \ell_2 k\, a^2 \int_{t_0}^{t} |y(s)-\xi(s)|\, ds \\
&\le \frac{a^2}{2}\,\epsilon + +\ell_1 a \int_{t_0}^{t_0+a} \|y-\xi\|\, ds + \ell_2\, k\, a^2 \int_{t_0}^{t_0+a} \|y-\xi\|\, ds \\
&= \frac{a^2}{2}\,\epsilon + a^2\big(\ell_1 + \ell_2 k\, a\big)\|y-\xi\| \\
&= \frac{a^2}{2}\,\epsilon + \lambda\|y-\xi\|
\end{aligned}$$

for all $t \in J$, where $\lambda = a^2\big(\ell_1 + \ell_2 k\, a\big) < 1$. Taking the supremum over t, we obtain

$$\|y-\xi\| \le \frac{a^2}{2}\,\epsilon + a^2\big(\ell_1 + \ell_2 k\, a\big)\|y-\xi\|$$

or

$$\|y-\xi\| \le \left[\frac{a^2}{2[1 - a^2\big(\ell_1 + \ell_2 k\, a\big)]}\right]\epsilon$$

where, $a^2\big(\ell_1 + \ell_2 k\, a\big) < 1$. Letting $K_f = \left[\frac{a^2}{2[1 - a^2\big(\ell_1 + \ell_2 k\, a\big)]}\right] > 0$, we obtain

$$\big|y(t) - \xi(t)\big| \le K_f \epsilon$$

for all $t \in J$. As a result, ξ is a Ulam-Hyers stable local solution of the *HIGDE* (1.1) on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nondecreasing and converges to ξ. Consequently the *HIGDE* (1.1) is a locally Ulam-Hyers stable on J. This completes the proof. □

Remark 4.1. If the given initial condition in the equation (1.1) is such that $x_0 > 0$, then under the conditions of Theorem 4.1, the *HIGDE* (1.1) has a unique Ulam-Hyers stable local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to x^*.

5 The Examples

In this section, we indicate a couple of examples illustrating the abstract ideas involved in the main approximation results, Theorems 3.1, 3.2 and 4.1 of this paper.

Example 5.1. Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the *IVP* of nonlinear first order HIGDE,

$$x''(t) = \tan h x(t) + \int_0^t \tan h x(s)\, ds, \ \ t \in [0,1]; \quad x(0) = \frac{1}{4}, \ \ x'(0) = 1. \tag{5.1}$$

Here, $\alpha_0 = \frac{1}{4}$, $\alpha_1 = 1$, $g(t,x) = \tan hx$, $(t,x) \in [0,1] \times \mathbb{R}$ and $f(t,x,y) = \tan hx + y$ for $(t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$. We show that the functions g and f satisfy all the conditions of Theorem 3.1. Clearly, f is bounded on $[0,1] \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = 2$ and so the hypothesis (H_1) is satisfied. Also the function $f(t,x,y)$ is nondecreasing in x and y for each $t \in [0,1]$. Therefore, hypothesis (H_2) is satisfied. Next, $g(t,x)$ is nondecreasing in x for each $t \in [0,1]$, so the hypothesis (H_3) is satisfied. Moreover, $f(t,\alpha_0,y) = f(t,\frac{1}{4},y) = \tan h(\frac{1}{4}) + y \geq 0$ and $\alpha_1 \geq 0$ for each $t \in [0,1]$ and $y \geq 0$, so the hypothesis (H_4) holds. Finally, $g(t,\alpha_0) = \tan h(\frac{1}{4}) \geq 0$ for all $t \in [0,1]$ and hypothesis (H_5) is satisfied. If we take $r = 2$ and $M = 1$, all the conditions of Theorem 3.1 are satisfied. Hence, the *HIGDE* (5.1) has a local solution x^* in the closed ball $B_2^1[\frac{1}{4}]$ of $C(J,\mathbb{R})$ which is positive in view of Remark 3.2. Moreover, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$x_0(t) = \frac{1}{4}, \;\; t \in [0,1],$$

$$x_{n+1}(t) = \frac{1}{4} + \alpha_1(t - t_0) + \int_0^t \tan hx_n(s)\,ds + \int_0^t (t-s)\,\tan hx_n(s)\,ds, \;\; t \in [0,1],$$

is monotone nondecreasing and converges to the positive solution x^* defined on $[0,1]$.

Example 5.2. Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the *IVP* of nonlinear first order *HIGDE*,

$$x''(t) = \frac{1}{4}\,\tan^{-1} x(t) + \frac{1}{4}\int_0^t \tan^{-1} x(s)\,, \;\; t \in [0,1]; \quad x(0) = \frac{1}{4}, \;\; x'(0) = 1. \tag{5.2}$$

Here, $\alpha_0 = \dfrac{1}{4}$, $\alpha_1 = 1$, and $g(t,x) = \tan^{-1} x$ for $(t,x) \in [0,1] \times \mathbb{R}$. Again, $f(t,x,y) = \dfrac{1}{4}\,\tan^{-1} x + \dfrac{1}{4} y$ for each $t \in [0,1]$. We show that f satisfies all the conditions of Theorem 3.2. Clearly, f is bounded on $[0,1] \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = \frac{11}{14}$ and so, the hypothesis (H_1) is satisfied. Next, let $x,y \in \mathbb{R}$ be such that $x \geq y$. Then there exists a constant ξ with $x < \xi < y$ satisfying

$$0 \leq g(t,x) - g(t,y) \leq \frac{1}{1+\xi^2}(x-y) \leq (x-y)$$

for all $t \in [0,1]$. So the hypothesis (H_7) holds with $k = 1$. Moreover, $g(t,\alpha_0) = g\left(t,\frac{1}{4}\right) = \tan^{-1}\left(\frac{1}{4}\right) \geq 0$ for each $t \in [0,1]$, and so the hypothesis (H_4) holds. Similarly,

$$f(t,\alpha_0,y) = \frac{1}{4}\,\tan^{-1}\alpha_0 + \frac{1}{4}\,y = \tan^{-1}\left(\frac{1}{4}\right) + \frac{1}{4}\,y \geq 0$$

and $\alpha_1(t - t_0) = t \geq 0$ for all $t \in [0,1]$ and for all positive number y, so the hypothesis (H_4) is satisfied. Next, let $x_1, y_1, x_2, y_2 \in \mathbb{R}$ with $x_1 \geq y_1$, $x_2 \geq y_2$. Then,

$$0 \leq f(t,x_1,x_2) - f(t,y_1,y_2) \leq \frac{1}{4}\cdot(x_1 - y_2) + \frac{1}{4}\,(x_2 - y_2)$$

for each $t \in [0,1]$. Therefore, hypothesis (H_6) holds with $\ell_1 = \frac{1}{4} = \ell_2$. If we take $r = 2$, then we have

$$M_f a = \frac{11}{14} \leq \left(1 - \frac{1}{2}\right)\cdot 2 = \left[1 - \left(\ell_1 a + \ell_2 k\,a^2\right)\right] r$$

and so, the condition (3.6) is satisfied. Thus, all the conditions of Theorem 3.2 are satisfied. Hence, the *HIGDE* (5.2) has a unique local solution x^* in the closed ball $B_2[\frac{1}{4}]$ of $C(J,\mathbb{R})$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$x_0(t) = \frac{1}{4}, \;\; t \in [0,1],$$

$$x_{n+1}(t) = \frac{1}{4} + \frac{1}{4}\int_0^t \tan^{-1} x_n(s)\,ds + \int_0^t (t-s)\,\tan^{-1} x_n(s)\,ds, \;\; t \in [0,1],$$

monotone nondecreasing converges to x^*. Moreover, the unique local solution x^* is Ulam-Hyers stable on $[0,1]$ in view of Definition 4.1. Consequently the *HIGDE* (5.2) is a locally Ulam-Hyers stable on the interval $[0,1]$.

Remark 5.1. The local approximation results of this paper includes similar results for the nonlinear *IVP*s of second order ordinary differential equations

$$\left.\begin{aligned} x''(t) &= f\big(t,x(t)\big), \;\; t \in J, \\ x(t_0) &= \alpha_0, \;\; x'(t_0) = \alpha_1, \end{aligned}\right\} \tag{5.3}$$

proved in Dhage *et al.* [11] as the special cases.

Remark 5.2. The approximation results of this paper may be extended to nonlinear *IVP*s of higher order ordinary differential equations

$$\left.\begin{aligned} x^{(n)}(t) &= f\Big(t, x(t), \int_{t_0}^{t} g(s, x(s))\,ds\Big),\ \ t \in J,\\ x^{(i)}(t_0) &= \alpha_{(i)},\ \ i = 0, 1, 2, \dots, n-1, \end{aligned}\right\} \tag{5.4}$$

by using the arguments similar to Theorems 3.1 and 3.2 with appropriate modifications.

6 Concluding Remark

Finally, while concluding this paper, we remark that unlike the Schauder fixed point theorem we do not require any convexity argument in the proof of main existence theorem, Theorem 3.1. Similarly, we do not require the usual Lipschitz condition in the proof of uniqueness theorem, Theorem 3.2, but a weaker form of one sided or partial Lipschitz condition is enough to serve the purpose. However, in both the cases we are able to ac*HIE*ve the existence of local solution by convergence of the successive approximations. Moreover, the differential equation (1.1) considered in this paper is of very simple form, however other complex nonlinear *IVPs* of *HIGDEs* may be considered and the present study can also be extended to such sophisticated nonlinear differential equations with appropriate modifications. These and other such problems form the further research scope in the subject of nonlinear differential and integral equations with applications. Some of the results in this direction will be reported elsewhere.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 287-292**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

SOME RESULTS ON d-FRAMES

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(Received: July 17, 2022; In format: July 24, 2022; Revised: June 03, 2023; Accepted: June 07, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53134

Abstract

In this paper, we study and establish some results on properties of d-frames [1] and d-frame operators. Also, we present a result on the perturbation analysis of the d-frames.
2020 Mathematical Sciences Classification: 42C15, 42C40, 46C50.
Keywords and Phrases: Double sequence, Frame, d-Frame, Perturbation.

1 Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathbb{I}$ a countable index set. A sequence $\{x_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$ is said to be a Bessel sequence with Bessel bound $\lambda_2 > 0$ if $\sum\limits_{i\in\mathbb{I}} |\langle x, x_i\rangle|^2 \le \lambda_2 ||x||^2$ for all $x \in \mathcal{H}$. A Bessel sequence $\{x_i\}_{i\in\mathbb{I}}$ with Bessel bound λ_2 is said to be a frame for $\mathcal{H}$ if there exists constant $\lambda_1 > 0$ such that $\lambda_1||x||^2 \le \sum_{i\in\mathbb{I}} |\langle x, x_i\rangle|^2 \le \lambda_2||x||^2$, for all $x \in \mathcal{H}$. For frame $\{x_i\}_{i\in\mathbb{I}}$, the positive constants λ_1 and λ_2 are called the lower and upper frame bounds respectively. A frame $\{x_i\}_{i\in\mathbb{I}}$ is said to be a tight frame if $\lambda_1 = \lambda_2$.

By putting forward a landmark paper on frames [5], Daubechies, Grossmann and Meyer brought back the attention of the researchers towards the frame theory which was introduced by Duffin and Schaeffer [7] almost thirty years before.

In the last three decades, frames have been widely studied and applied in various fields of study viz. sampling theory, signal processing, system modeling, data analysis, etc. (For more details see [3, 4, 6, 8, 10]).

Here, it is to be noted that every Bessel sequence is not necessarily a frame always. Motivated by this fact, researchers generalised the concept of constructing the frames from the Bessel sequences in different ways. In fact they used either an operator on the Bessel sequence to make it frame or they rearranged/added/scattered the terms of the sequence to make it a frame. In the sequel, recently Mehra et al. [1] introduced d-frames for a Hilbert space $\mathcal{H}$ by using the concept of double sequences and studied certain properties of d-frame, d-frame operators and stability of d-frames. In this note, we study and establish some results on d-frame and their properties. Some of a results are extensions and generalizations of the results of [9] for d-frames in Hilbert spaces.

2 Preliminaries

Throughout this paper, $\mathcal{H}$ denotes an infinite dimensional Hilbert space and $\mathbb{N}$ denotes the set of all natural numbers. To prove our main results, we use following definitions, concept of space and results from [1].

Definition 2.1 ([1]). *The double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\mathcal{H}$ is said to be a d-frame for $\mathcal{H}$ if there exist positive constants λ_1 and λ_2 such that*

$$\lambda_1||x||^2 \le \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \le \lambda_2||x||^2, \quad for \ \ all \ \ x \in \mathcal{H}. \tag{2.1}$$

The constants λ_1 and λ_2 are called lower and upper d-frame bounds respectively.

If $\lambda_1 = \lambda_2$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called tight d-frame, and if $\lambda_1 = \lambda_2 = 1$, then it is called Parseval d-frame.

A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in Hilbert space $\mathcal{H}$ is called d-Bessel sequence with bound λ_2 if it satisfies upper d-frame inequality i.e.,

$$\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \le \lambda_2||x||^2, \quad \forall x \in \mathcal{H}.$$

Consider the following spaces as defined in [1]:

$$\ell^2(\mathbb{N}\times\mathbb{N}) = \{\{\alpha_{\mathbf{ij}}\}_{\mathbf{i,j}\in\mathbb{N}}:\ \alpha_{\mathbf{ij}}\in\mathbb{F},\ \lim_{\mathbf{m,n}\to\infty}\sum_{\mathbf{i,j=1}}^{\mathbf{m,n}}|\alpha_{\mathbf{ij}}|^{\mathbf{2}}<\infty\}.$$

Then $\ell^2(\mathbb{N}\times\mathbb{N})$ is a Hilbert space with the norm induced by the inner product which is given by,

$$\left\langle\{\alpha_{ij}\}_{i,j\in\mathbb{N}},\{\beta_{ij}\}_{i,j\in\mathbb{N}}\right\rangle = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\alpha_{ij}\overline{\beta_{ij}},\quad \forall\{\alpha_{ij}\}_{i,j\in\mathbb{N}},\{\beta_{ij}\}_{i,j\in\mathbb{N}}\in\ell^2(\mathbb{N}\times\mathbb{N}).$$

Remark 2.1 ([1]). Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d-Bessel sequence. Define operator $\mathcal{T}:\ell^2(\mathbb{N}\times\mathbb{N})\to\mathcal{H}$ as

$$\mathcal{T}(\{\alpha_{ij}\}_{i,j\in\mathbb{N}}) = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\alpha_{ij}x_{ij},\quad \forall\{\alpha_{ij}\}_{i,j\in\mathbb{N}}\in\ell^2(\mathbb{N}\times\mathbb{N}).$$

If $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame then operator $\mathcal{T}$ is called pre d-frame (synthesis) operator and the adjoint operator $\mathcal{T}^*$ of $\mathcal{T}$ is called analysis operator for d-frame, and defined as

$$\mathcal{T}^*(x) = \{\langle x,x_{ij}\rangle\}_{i,j\in\mathbb{N}},\quad \forall x\in\mathcal{H}.$$

Theorem 2.1 ([1])**.** *A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\mathcal{H}$ is a d-Bessel sequence with d-Bessel bound λ_2 if and only if the operator $\mathcal{T}$ is linear, well defined and bounded with $\|\mathcal{T}\|\le\sqrt{\lambda_2}$.*

Theorem 2.2 ([1])**.** *A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\mathcal{H}$ is a d-frame for $\mathcal{H}$ if and only if the operator $\mathcal{T}$ is well defined, bounded, linear and surjective.*

The d-frame operator $\mathcal{S}:\mathcal{H}\to\mathcal{H}$ for d-frame $\{x_{ij}\}_{i,j\in\mathbb{N}}$ defined as:

$$\begin{aligned}\mathcal{S}(x) &= \mathcal{T}\mathcal{T}^*(x)\\ &= \mathcal{T}(\{\langle x,x_{ij}\rangle\}_{i,j\in\mathbb{N}})\\ &= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,x_{ij}\rangle x_{ij},\quad \forall x\in\mathcal{H}.\end{aligned}$$

Since $\mathcal{T}$ and $\mathcal{T}^*$ both are linear, so $\mathcal{S}$ is also linear.

Theorem 2.3 ([1])**.** *d-frame operator $\mathcal{S}$ is bounded, self adjoint, positive and invertible.*

Definition 2.2 ([1])**.** *A d-frame $\{\tilde{x}_{ij}\}_{i,j\in\mathbb{N}}$ for a Hilbert space $\mathcal{H}$ is called alternate dual d-frame for a given d-frame $\{x_{ij}\}_{i,j\in\mathbb{N}}$, if*

$$x = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,\tilde{x}_{ij}\rangle x_{ij},\quad \forall\ x\in\mathcal{H}.$$

Remark 2.2 ([1]). $\{\mathcal{S}^{-1}(x_{ij})\}_{i,j\in\mathbb{N}}$ is a special type of dual d-frame for $\{x_{ij}\}_{i,j\in\mathbb{N}}$, called canonical dual d-frame.

3 Main Results

Proposition 3.1. *Let $\mathcal{T}$, $\mathcal{T}^*$ and $\mathcal{S}$ are operators as defined above for a d-Bessel sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$. Then*

(I) *$\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ if and only if $\mathcal{S}$ is invertible.*

(II) *$\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ if and only if the analysis operator $\mathcal{T}^*$ is invertible.*

Proof. (I) If $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ then by Theorem 2.3, $\mathcal{S}$ is invertible.

Conversely, If $\mathcal{S}$ is invertible $\Rightarrow\mathcal{T}$ is surjective. By Theorem 2.2, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$. □

Proof. (II) $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}\iff\mathcal{T}$ is surjective $\Rightarrow\mathcal{T}^*$ is an isomorphism $\Rightarrow\mathcal{T}^*$ is invertible. Conversely, $\mathcal{T}^*$ is invertible $\Rightarrow\mathcal{T}^*$ is surjective $\Rightarrow\mathcal{T}$ is an isomorphism $\Rightarrow\mathcal{T}$ is surjective $\iff\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$. □

Theorem 3.1. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d-frame for $\mathcal{H}$ with d-frame operator $\mathcal{S}$, d-frame bounds $\lambda_1\le\lambda_2$ and let $\mathcal{U}:\mathcal{H}\to\mathcal{H}$ be a bounded operator. Then $\{\mathcal{U}x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ if and only if $\mathcal{U}$ is invertible.*

Proof. Let $\mathcal{U}$ be invertible operator, then for each $x \in \mathcal{H}$,

$$\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \mathcal{U}x_{ij}\rangle|^2 = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle \mathcal{U}^* x, x_{ij}\rangle|^2 \geq \lambda_1 ||\mathcal{U}^* x||^2 \geq ||\mathcal{U}^{-1}||^2 \lambda_1 ||x||^2$$

and

$$\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \mathcal{U}x_{ij}\rangle|^2 = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle \mathcal{U}^* x, x_{ij}\rangle|^2 \leq \lambda_2 ||\mathcal{U}^* x||^2 \leq ||\mathcal{U}||^2 \lambda_2 ||x||^2.$$

Thus, $\{\mathcal{U}x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ with d-frame bounds $||\mathcal{U}^{-1}||^2\lambda_1$, $||\mathcal{U}||^2\lambda_2$.

Conversely, If $\{\mathcal{U}x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$, then its d-frame operator is invertible on $\mathcal{H}$. Now, d-frame operator of $\{\mathcal{U}x_{ij}\}_{i,j\in\mathbb{N}}$ is

$$\begin{aligned}
\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, \mathcal{U}x_{ij}\rangle \mathcal{U}x_{ij} &= \mathcal{U}\big(\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, \mathcal{U}x_{ij}\rangle x_{ij}\big) \\
&= \mathcal{U}\big(\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle \mathcal{U}^* x, x_{ij}\rangle x_{ij}\big) \\
&= \mathcal{U}\mathcal{S}\mathcal{U}^*(x).
\end{aligned}$$

$\mathcal{U}\mathcal{S}\mathcal{U}^*$ is invertible $\Rightarrow \mathcal{U}$ is surjective $\Rightarrow \mathcal{U}^*$ is isomorphism $\Rightarrow \mathcal{U}^*$ is surjective $\Rightarrow \mathcal{U}$ is invertible. □

Corollary 3.1. *If $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d-frame for $\mathcal{H}$ with d-frame operator $\mathcal{S}$ and $\mathcal{U} : \mathcal{H} \to \mathcal{H}$ is a bounded positive operator, then $\{x_{ij} + \mathcal{U}x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame with the d-frame operator $(I+\mathcal{U})\mathcal{S}(I+\mathcal{U}^*)$ and d-frame bounds $||I+\mathcal{U}||^{-2}\lambda_1$, $||I+\mathcal{U}||^2\lambda_2$, if and only if $I+\mathcal{U}$ is invertible.*

Corollary 3.2. *If $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ and P is an orthogonal projection on $\mathcal{H}$, then $\{x_{ij} + \alpha P x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$, where $\alpha \neq -1$ is an scalar.*

Theorem 3.2. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be d-Bessel sequences in $\mathcal{H}$ with analysis operators $\mathcal{T}_1^*$, $\mathcal{T}_2^*$ and d-frame operators $\mathcal{S}_1$, $\mathcal{S}_2$ respectively. Then for operators $\mathcal{U}_1, \mathcal{U}_2 : \mathcal{H} \to \mathcal{H}$, $\{\mathcal{U}_1 x_{ij} + \mathcal{U}_2 y_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ if an only if $\mathcal{T}_1^*\mathcal{U}_1^* + \mathcal{T}_2^*\mathcal{U}_2^*$ is an invertible operator. Further, d-frame operator for $\{\mathcal{U}_1 x_{ij} + \mathcal{U}_2 y_{ij}\}_{i,j\in\mathbb{N}}$ is $\mathcal{S} = \mathcal{U}_1\mathcal{S}_1\mathcal{U}_1^* + \mathcal{U}_2\mathcal{S}_2\mathcal{U}_2^* + \mathcal{U}_1\mathcal{T}_1\mathcal{T}_2^*\mathcal{U}_2^* + \mathcal{U}_2\mathcal{T}_2\mathcal{T}_1^*\mathcal{U}_1^*$.*

Proof. $\{\mathcal{U}_1 x_{ij} + \mathcal{U}_2 y_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ if and only if its analysis operator say $\mathcal{L}^*$ is invertible, where

$$\begin{aligned}
\mathcal{L}^*(x) &= \{\langle x, \mathcal{U}_1 x_{ij} + \mathcal{U}_2 y_{ij}\rangle\}_{i,j\in\mathbb{N}} \\
&= \{\langle x, \mathcal{U}_1 x_{ij}\rangle + \langle x, \mathcal{U}_2 y_{ij}\rangle\}_{i,j\in\mathbb{N}} \\
&= \{\langle \mathcal{U}_1^* x, x_{ij}\rangle\}_{i,j\in\mathbb{N}} + \{\langle \mathcal{U}_2^* x, y_{ij}\rangle\}_{i,j\in\mathbb{N}} \\
&= \mathcal{T}_1^*\mathcal{U}_1^* x + \mathcal{T}_2^*\mathcal{U}_2^* x.
\end{aligned}$$

Thus, $\mathcal{L}^* = \mathcal{T}_1^*\mathcal{U}_1^* + \mathcal{T}_2^*\mathcal{U}_2^*$ is invertible.

And the d-frame operator for sequence $\{\mathcal{U}_1 x_{ij} + \mathcal{U}_2 y_{ij}\}_{i,j\in\mathbb{N}}$ is

$$\begin{aligned}
\mathcal{S} = \mathcal{L}\mathcal{L}^* &= (\mathcal{T}_1^*\mathcal{U}_1^* + \mathcal{T}_2^*\mathcal{U}_2^*)^*(\mathcal{T}_1^*\mathcal{U}_1^* + \mathcal{T}_2^*\mathcal{U}_2^*) \\
&= \mathcal{U}_1\mathcal{S}_1\mathcal{U}_1^* + \mathcal{U}_2\mathcal{S}_2\mathcal{U}_2^* + \mathcal{U}_1\mathcal{T}_1\mathcal{T}_2^*\mathcal{U}_2^* + \mathcal{U}_2\mathcal{T}_2\mathcal{T}_1^*\mathcal{U}_1^*.
\end{aligned}$$

□

Remark 3.1. In the above propositions, theorems and corollaries, if we consider classical frames in place of d-frames, we get the results of [9].

To construct a sequence of alternate dual d-frames from a given d-Bessel sequence, we prove following results.

Theorem 3.3. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d-frame for $\mathcal{H}$ with d-frame operator $\mathcal{S}$. Then for a given d-Bessel sequence $\{u_{ij}\}_{i,j\in\mathbb{N}}$, the double sequence $\{y_{ij} : y_{ij} = \mathcal{S}^{-1}x_{ij} + u_{ij}\}_{i,j\in\mathbb{N}}$ is a dual d-frame for $\{x_{ij}\}_{i,j\in\mathbb{N}}$ if and only if $\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, u_{ij}\rangle x_{ij} = 0$, for all $x \in \mathcal{H}$.*

Proof. For the given d-Bessel sequence $\{u_{ij}\}_{i,j\in\mathbb{N}}$, we have $\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,u_{ij}\rangle x_{ij}=0, \forall\ x\in\mathcal{H}$. Then

$$\begin{aligned}\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,y_{ij}\rangle x_{ij} &= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,\mathcal{S}^{-1}x_{ij}\rangle x_{ij}+\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,u_{ij}\rangle x_{ij}\\ &= x\end{aligned}$$

$\Rightarrow \{y_{ij}\}_{i,j\in\mathbb{N}}$ is a dual d-frame for $\{x_{ij}\}_{i,j\in\mathbb{N}}$.

Conversely, If $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a dual d-frame for $\{x_{ij}\}_{i,j\in\mathbb{N}}$ where $y_{ij}=\mathcal{S}^{-1}x_{ij}+u_{ij}$, then

$$\begin{aligned}x &= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,y_{ij}\rangle x_{ij},\quad \forall\ x\in\mathcal{H}\\ &= x+\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,u_{ij}\rangle x_{ij}\end{aligned}$$

$\Rightarrow \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,u_{ij}\rangle x_{ij}=0.$ □

Theorem 3.4. *If $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d-frame for $\mathcal{H}$ with d-frame operator $\mathcal{S}$ and dual $\{y_{ij}\}_{i,j\in\mathbb{N}}$. Then the sequence $\{g_{ij}\}_{i,j\in\mathbb{N}}$ define by $g_{ij}=\mathcal{S}^{-1}x_{ij}-x_{ij}+\mathcal{S}y_{ij}$ is also a dual for $\{x_{ij}\}_{i,j\in\mathbb{N}}$.*

Proof.

$$\begin{aligned}\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,g_{ij}\rangle x_{ij} &= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,\mathcal{S}^{-1}x_{ij}-x_{ij}+\mathcal{S}y_{ij}\rangle x_{ij}\\ &= x-\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,\mathcal{S}x_{ij}\rangle\mathcal{S}^{-1}x_{ij}+\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,\mathcal{S}y_{ij}\rangle x_{ij}\\ &= x-\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle \mathcal{S}^{*}x,x_{ij}\rangle\mathcal{S}^{-1}x_{ij}+\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle \mathcal{S}^{*}x,y_{ij}\rangle x_{ij}\\ &= x-\mathcal{S}^{*}x+\mathcal{S}^{*}x\\ &= x.\end{aligned}$$

□

To prove our next theorem, we use following results from Casazza et al. [2].

Lemma 3.1 ([2])**.** *Let $\mathcal{U}:\mathcal{H}\to\mathcal{H}$ be a linear operator and assume that there exist constant $\alpha,\beta\in[0,1[$ such that*

$$||\mathcal{U}x-x||\le\alpha||x||+\beta||\mathcal{U}x||,\ \ \forall\ x\in\mathcal{H}.$$

Then $\mathcal{U}$ is a bounded linear invertible operator on $\mathcal{H}$, and

$$\frac{1-\alpha}{1+\beta}\ ||x||\le||\mathcal{U}x||\le\frac{1+\alpha}{1-\beta}\ ||x||,$$
$$\frac{1-\beta}{1+\alpha}\ ||x||\le||\mathcal{U}^{-1}x||\le\frac{1+\beta}{1-\alpha}\ ||x||,\ \ \forall\ x\in\mathcal{H}.$$

Lemma 3.2 ([2])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ are two Hilbert spaces, $\mathcal{U}:\mathcal{X}\to\mathcal{Y}$ be a bounded operator, $\mathcal{X}_0$ a dense subspace of $\mathcal{X}$ and $\mathcal{V}:\mathcal{X}\to\mathcal{Y}$ a linear mapping. If*

$$||\mathcal{U}x-\mathcal{V}x||\le\alpha||\mathcal{U}x||+\beta||\mathcal{V}x||+\gamma||x||,\ \ \forall\ x\in\mathcal{X}_0,$$

where $\beta\in[0,1[$, then $\mathcal{V}$ is a bounded linear operator on a dense subspace of $\mathcal{X}$ and hence has a unique extension to a bounded linear operator (of the same norm) on $\mathcal{X}$.

Theorem 3.5. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d-frame with bounds λ_1,λ_2,and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be a double sequence in $\mathcal{H}$ and assume that $\exists$ constant $\alpha,\beta,\gamma\ge0$ such that max$\left(\alpha+\frac{\gamma}{\sqrt{\lambda_1}},\beta\right)<1$ and*

$$\begin{aligned}\lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}c_{ij}(x_{ij}-y_{ij})\right\| &\le \alpha\lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}c_{ij}x_{ij}\right\|+\beta\lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}c_{ij}y_{ij}\right\|\\ &+ \gamma\|\{c_{ij}\}_{i,j\in\mathbb{N}}\|,\qquad\forall\{c_{ij}\}_{i,j\in\mathbb{N}}\in\ell^2(\mathbb{N}\times\mathbb{N}).\end{aligned}\tag{3.1}$$

Then, $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is also a d-frame for $\mathcal{H}$ with bounds $\lambda_1\left(1-\frac{\alpha+\beta+\frac{\gamma}{\sqrt{\lambda_1}}}{1+\beta}\right)^2$ and $\lambda_2\left(1+\frac{\alpha+\beta+\frac{\gamma}{\sqrt{\lambda_2}}}{1-\beta}\right)^2$.

Proof. If $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame, then for pre d-frame operator $\mathcal{T}$ of $\{x_{ij}\}_{i,j\in\mathbb{N}}$, we have

$$\|\mathcal{T}(\{c_{ij}\}_{i,j\in\mathbb{N}})\| = \lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n} c_{ij}x_{ij}\right\| \le \sqrt{\lambda_2}\,\|\{c_{ij}\}_{i,j\in\mathbb{N}}\|, \quad \{c_{ij}\}_{i,j\in\mathbb{N}}\in\ell^2(\mathbb{N}\times\mathbb{N}). \tag{3.2}$$

Define an operator $\mathcal{U}:\ell^2(\mathbb{N}\times\mathbb{N})\to\mathcal{H}$ such that

$$\mathcal{U}(\{c_{ij}\}_{i,j\in\mathbb{N}}) = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n} c_{ij}y_{ij}. \tag{3.3}$$

For equations (3.2) and (3.3), the equation (3.1) gives

$$||\mathcal{T}(\{c_{ij}\}_{i,j\in\mathbb{N}})-\mathcal{U}(\{c_{ij}\}_{i,j\in\mathbb{N}})|| \le \alpha||\mathcal{T}(\{c_{ij}\}_{i,j\in\mathbb{N}})||+\beta||\mathcal{U}(\{c_{ij}\}_{i,j\in\mathbb{N}})||+\gamma||(\{c_{ij}\}_{i,j\in\mathbb{N}})||,$$
$$\forall\ (\{c_{ij}\}_{i,j\in\mathbb{N}})\in\ell^2(\mathbb{N}\times\mathbb{N}).$$

Therefore, from Lemma 3.2, $\mathcal{U}$ is bounded linear operator on $\ell^2(\mathbb{N}\times\mathbb{N})$.
Using triangle inequality, we have

$$\begin{aligned}
&\|\mathcal{U}(\{c_{ij}\}_{i,j\in\mathbb{N}})\| \le \|\mathcal{T}(\{c_{ij}\}_{i,j\in\mathbb{N}})-\mathcal{U}(\{c_{ij}\}_{i,j\in\mathbb{N}})\|+\|\mathcal{T}(\{c_{ij}\}_{i,j\in\mathbb{N}})\|\\
&\Rightarrow \|\mathcal{U}(\{c_{ij}\}_{i,j\in\mathbb{N}})\| \le \frac{1+\alpha}{1-\beta}\,||\mathcal{T}(\{c_{ij}\}_{i,j\in\mathbb{N}})||+\frac{\gamma}{1-\beta}\,||(\{c_{ij}\}_{i,j\in\mathbb{N}})||
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n} c_{ij}y_{ij}\right\| &\le \frac{1+\alpha}{1-\beta}\lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n} c_{ij}x_{ij}\right\|+\frac{\gamma}{1-\beta}\,||(\{c_{ij}\}_{i,j\in\mathbb{N}})||\\
&\le \left(\frac{(1+\alpha)\sqrt{\lambda_2}+\gamma}{1-\beta}\right)||(\{c_{ij}\}_{i,j\in\mathbb{N}})||.
\end{aligned}$$

Thus, $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a d-Bessel sequence with bound $\left(\frac{(1+\alpha)\sqrt{\lambda_2}+\gamma}{1-\beta}\right)^2 = \lambda_2\left(1+\frac{\alpha+\beta+\frac{\gamma}{\sqrt{\lambda_2}}}{1-\beta}\right)^2$.

Since $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame, so for d-frame operator $\mathcal{S}$, the double sequence $\{\mathcal{S}^{-1}x_{ij}\}_{i,j\in\mathbb{N}}$ is the dual d-frame of $\{x_{ij}\}_{i,j\in\mathbb{N}}$ with upper bound λ_1^{-1}.
define an operator $\mathcal{T}^\dagger:\mathcal{H}\to\ell^2(\mathbb{N}\times\mathbb{N})$ such that

$$\begin{aligned}
\mathcal{T}^\dagger(x) &= \mathcal{T}^*\mathcal{S}^{-1}(x)\\
&= \left\{\left\langle x,\mathcal{S}^{-1}(x_{ij})\right\rangle\right\}_{i,j\in\mathbb{N}}, \quad \forall x\in\mathcal{H}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|\mathcal{T}^\dagger(x)\right\|^2 &= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\left\langle x,\mathcal{S}^{-1}(x_{ij})\right\rangle\right|^2\\
&\le {\lambda_1}^{-1}\left\|x\right\|^2, \quad \forall x\in\mathcal{H}.
\end{aligned}$$

Replacing $\{c_{ij}\}_{i,j\in\mathbb{N}}$ by $\mathcal{T}^\dagger(x)$, in (3.1) and using (3.3), we get

$$\|x-\mathcal{U}\mathcal{T}^\dagger(x)\| \le \left(\alpha+\frac{\gamma}{\sqrt{\lambda_1}}\right)\|x\|+\beta||\mathcal{U}\mathcal{T}^\dagger(x)||, \quad \forall x\in\mathcal{H}. \tag{3.4}$$

Applying Lemma 3.1, equation (3.4) implies that the operator $\mathcal{U}\mathcal{T}^\dagger$ is invertible and

$$||\mathcal{U}\mathcal{T}^\dagger|| \le \frac{1+\alpha+\frac{\gamma}{\sqrt{\lambda_1}}}{1-\beta}, \qquad \|(\mathcal{U}\mathcal{T}^\dagger)^{-1}\| \le \frac{1+\beta}{1-(\alpha+\frac{\gamma}{\sqrt{\lambda_1}})}, \quad \forall\ x\in\mathcal{H}.$$

For $x\in\mathcal{H}$,

$$\begin{aligned}
x &= (\mathcal{U}\mathcal{T}^\dagger)(\mathcal{U}\mathcal{T}^\dagger)^{-1}(x)\\
&= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left\langle(\mathcal{U}\mathcal{T}^\dagger)^{-1}(x),\mathcal{S}^{-1}(x_{ij})\right\rangle y_{ij}.
\end{aligned}$$

$$\Rightarrow \|x\|^4 = \langle x,x\rangle^2 = \left|\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left\langle(\mathcal{U}\mathcal{T}^\dagger)^{-1}(x),\mathcal{S}^{-1}(x_{ij})\right\rangle\left\langle y_{ij},x\right\rangle\right|^2$$

$$
\begin{aligned}
&\leq \frac{1}{\lambda_1}\left\|(\mathcal{U}\mathcal{T}^{\dagger})^{-1}(x)\right\|^2 \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle y_{ij},x\rangle\right|^2 \\
&\leq \frac{1}{\lambda_1}\left(\frac{1+\beta}{1-(\alpha+\frac{\gamma}{\sqrt{\lambda_1}})}\right)^2\|x\|^2 \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle y_{ij},x\rangle\right|^2.
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle y_{ij},x\rangle\right|^2 &\geq \lambda_1\left(\frac{1-(\alpha+\frac{\gamma}{\sqrt{\lambda_1}})}{1+\beta}\right)^2\|x\|^2 \\
&= \lambda_1\left(1-\frac{\alpha+\beta+\frac{\gamma}{\sqrt{\lambda_1}}}{1+\beta}\right)^2\|x\|^2, \quad \forall \ x\in\mathcal{H}.
\end{aligned}
$$

Thus, d-Bessel sequence $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame with bounds $\lambda_1\left(1-\frac{\alpha+\beta+\frac{\gamma}{\sqrt{\lambda_1}}}{1+\beta}\right)^2$ and $\lambda_2\left(1+\frac{\alpha+\beta+\frac{\gamma}{\sqrt{\lambda_2}}}{1-\beta}\right)^2$. □

Remark 3.2. Taking $\beta = 0$, Theorem 4.1 of [1] becomes a particular case of Theorem 3.5.

4 Conclusion

In this paper, we studied cases in which new d-frames can be constructed from the existing ones and established the results on stability of d-frame under small perturbation. Also, we proved the result to construct an alternate dual d-frame from a given specific d-Bessel sequence. The frame theory has many exciting applications in different areas of study. So the concept of d-frame can have many applications specially in signal processing. This can be taken as a future scope of interdisciplinary research.

Acknowledgement. The authors are thankful to the Editors and Reviewers for their valuable suggestions to improve the paper.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 293-299**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

STUDY OF RICCI SOLITONS IN f-KENMOTSU MANIFOLDS WITH THE QUARTER-SYMMETRIC METRIC CONNECTION

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(Received: September 28, 2022; In format: October 31, 2023; Revised: May 16, 2023; Accepted : May 17, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53135

Abstract

In this paper, the non-existence of ξ-projectively flat 3-dimensional f-Kenmotsu manifold with quarter-symmetric metric connection has been established. Moreover, we prove that 3-dimensional f-Kenmotsu manifold with the quarter-symmetric metric connection is an η- Einstein manifold and the Ricci soliton is given as expanding or shrinking under certain restrictions on f.
2020 Mathematical Sciences Classification: 53C15, 53C20
Keywords and Phrases: Kenmotsu manifold, Ricci solitons, Quarter-symmetric metric connection.

1 Introduction

In 1924, the notion of semi-symmetric connections on a manifold was introduced by Friedman and Schouten[7] and the notion of quarter-symmetric connections which are generalization of the semi-symmetric connections was defined and studied by Golab[14]. Kenmotsu, in 1972, studied a class of contact Riemannian manifold together with some special conditions and given it a name as Kenmotsu manifold.

A manifold M, with the structure (ϕ, ξ, η, g) is called normal if $[\phi, \phi] + 2d\eta \bigotimes \xi = 0$ and it is almost cosymplectic if $d\eta = 0$ and $d\phi = 0$. M is cosymplectic if it is normal and almost cosymplectic. Olszak and Rosca [12] studied f-Kenmotsu Manifolds in a geometrical aspect, and gave some curvature conditions. The other mathematicians studied that a Ricci-symmetric f-Kenmotsu Manifold is an Einstein Manifold. Later on, authors, in 2010, also proved that Ricci semi- symmetric α- Kenmotsu manifolds are Einstein manifolds. By f-Kenmotsu Manifolds we mean an almost contact metric manifold which is locally conformal almost cosymplectic and normal.

In 1983, the concept of Ricci solitons in contact geometry was studied by Sharma and Sinha [15]. Later, in contact metric manifold Crasmareanu [4], Bejan [2] and others deeply studied Ricci solitons.

In 2012, Ricci solitons on Kenmotsu manifolds were studied exclusively by Nagraja and Premlatha [11] and a study on quarter-symmetric metric connection were done by Sular, Özgur and De [13] and De and De [6] in different ways.

Section 1 is introductory and in section 2, we have some fundamental notions used in this study. Section 3 deals with the introduction of f-Kenmotsu Manifold. In the next section 4, we study f-Kenmotsu manifold with quarter-symmetric metric connection and proved that this manifold is not always ξ - projective flat. In the last section we prove that f-Kenmotsu manifold with the quarter-symmetric metric connection is η-Einstein manifold and the Ricci soliton defined on this manifold is classified with respect to the values of f and λ.

2 Preliminaries

Let us consider a 3-dimensional differentiable manifold M with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a (1,1) tensor field, ξ is a vecter field, η is a 1-form and g is Riemannian metric, satisfying

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi, \\ \eta \circ \phi &= 0, \\ \phi\xi &= 0, \\ \eta(\xi) &= 1, \\ g(X, \xi) &= \eta(X), \end{aligned}$$

$$g(X,\phi Y) = -g(\phi X, Y),$$
$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.1}$$

for any vector fields $X, Y \in \chi(M)$. Then M is called an almost contact manifold. For an almost contact manifold M, we have

$$(\nabla_X \phi)Y = \nabla_X \phi Y - \phi(\nabla_X Y), \tag{2.2}$$
$$(\nabla_X \eta)Y = \nabla_X \eta Y - \eta(\nabla_X Y). \tag{2.3}$$

Let $\{e_1, e_2, e_3, ..., e_n\}$ be orthonormal basis of $T_p(M)$. R be Riemannian curvature tensor, S be Ricci curvature tensor, Q be Ricci operator, then $\forall\, X, Y \in \chi(M)$ it follows that [5]

$$S(X,Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i), \tag{2.4}$$

$$QX = -\sum_{i=1}^{n} R(e_i, X)e_i, \tag{2.5}$$

$$S(X,Y) = g(QX, Y). \tag{2.6}$$

In f-Kenmotsu manifold, if the Ricci tensor S satisfy the condition

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y) \tag{2.7}$$

α, β be certain scalars, then the manifold M is said to be η-Einstein manifold. If β= 0, the manifold is Einstein manifold.

In a 3-dimensional Riemannian manifold, the curvature tensor R is defind as

$$\begin{aligned} R(X,Y)Z &= S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \\ &\quad -\frac{\tau}{2}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.8}$$

where S is the Ricci tensor, Q is Ricci operator and τ is the scalar curvature.

Now, let M be an n-dimensional Riemannian manifold with the Riemannian connection ∇. A linear connection $\bar{\nabla}$ is said to be a quarter-symmetric connection on M if the torsion tensor $\bar{T}$ of the connection $\bar{\nabla}$ satisfies

$$\bar{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{2.9}$$

where $\bar{T} \neq 0$ and η is a 1-form.

If moreover $\bar{\nabla} g = 0$, then the connection is called quarter-symmetric metric connection.

If $\bar{\nabla} g \neq 0$, the connection is called quarter-symmetric non-metric connection[17].

For $n \geq 1$, the manifold M is locally projectively flat iff the projective curvature tensor P vanishes. We define the projective curvature tensor P as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{2.10}$$

for any $X, Y, Z \in \chi(M)$ where S is the Ricci tensor and R is the curvature tensor of M.

If $P(X,Y)\xi = 0$ for any $X, Y \in \chi(M)$, the manifold M is called ξ-projective flat[16].

A Ricci Soliton is defined on a Riemannian manifold (M, g) as a natural generalization of an Einstein metric. We define Ricci Soliton as a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ be a real scalar such that

$$L_V g + 2S + 2\lambda g = 0 \tag{2.11}$$

where L_V denotes the Lie derivative operator along the vector field V and S is a Ricci tensor of M. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively.

3 f-Kenmotsu manifolds

A 3-dimensional almost contact manifold M with the structure (ϕ, ξ, η, g) is an f-Kenmotsu manifold if the covariant derivative of ϕ satisfies [17],

$$(\nabla_X \phi)Y = f[g(\phi X, Y)\xi - \eta(Y)\phi X] \tag{3.1}$$

where $f \in C^\infty(M, R)$ such that $df \wedge \eta = 0$.

If $f^2 + f' \neq 0$, where f'=ξf, then M is called Regular [4]. If $f = \alpha = constant \neq 0$, M is called α-Kenmotsu Manifold. If $f = 1$, the manifold is called Kenmotsu manifold.

By (2.1) and (2.3), we have

$$(\nabla_X \eta)Y = fg(\phi X, \phi Y). \tag{3.2}$$

From (3.1), we have [15]

$$\nabla_X \xi = f[X - \eta(X)\xi]. \tag{3.3}$$

Also from (2.6), in 3-dimensional f-Kenmotsu manifold

$$\begin{aligned} R(X,Y)Z &= (\frac{\tau}{2} + 2f^2 + 2f')(X \wedge Y)Z \\ &\quad -(\frac{\tau}{2} + 3f^2 + 3f')[\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z] \end{aligned} \tag{3.4}$$

and

$$S(X,Y) = (\frac{\tau}{2} + f^2 + f')g(X,Y) - (\frac{\tau}{2} + 3f^2 + 3f')\eta(X)\eta(Y). \tag{3.5}$$

Thus from (3.5), we get

$$S(X,\xi) = -2(f^2 + f')\eta(X) \tag{3.6}$$

by (3.4) and (3.5),we get

$$R(X,Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y] \tag{3.7}$$

$$R(\xi,X)\xi = -(f^2 + f')(\eta(X)\xi - X), \tag{3.8}$$

$$QX = (\frac{\tau}{2} + f^2 + f')X - (\frac{\tau}{2} + 3f^2 + 3f')\eta(X)\xi. \tag{3.9}$$

From (2.10) and using (3.6) and (3.7), we have

Theorem 3.1. *A 3-dimensional f-Kenmotsu manifold is always ξ-projectively flat.*

4 f-Kenmotsu Manifolds with the quarter-symmetric metric connection

Let ∇ be a Riemannian connection of f-Kenmotsu manifold and $\bar{\nabla}$ be a linear connection then this linear connection $\bar{\nabla}$ defined as

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y, \tag{4.1}$$

where $X, Y \in \chi(M)$ be any vector field and η be 1-form, is called the quarter-symmetric metric connection. Now, using (2.2),(3.1) and (4.1) we have

$$(\bar{\nabla}_X \phi)Y = f[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{4.2}$$

for any vector field $X, Y \in \chi(M)$, where ϕ be (1,1) tensor field, ξ is a vector field, η is 1-form and $f \in C^\infty(M, \mathbf{R})$ so that $df \wedge \eta = 0$. As a result of $df \wedge \eta = 0$, we have

$$df = f', \quad X(f) = f'\eta(X), \tag{4.3}$$

where f'=ξf [11].

If $f = 0$, manifold is cosymplectic. If $f = \alpha \neq 0$, then the manifold is α-Kenmotsu. An f-Kenmotsu manifold with quarter -symmetric metric connection is called regular if $f^2 + f' - 2f\phi \neq 0$.

By (2.2),(4.2) we have

$$\bar{\nabla}_X \xi = f[X - \eta(X)\xi]. \tag{4.4}$$

Using (2.2),(2.1) and (3.2), we get

$$(\bar{\nabla}_X \eta)Y = fg(\phi X, \phi Y). \tag{4.5}$$

We define the curvature tensor $\bar{R}$ of f-Kenmotsu manifold M with respect to quarter-symmetric metric connection $\bar{\nabla}$ as

$$\bar{R}(X,Y)\xi = \bar{\nabla}_X\bar{\nabla}_Y\xi - \bar{\nabla}_Y\bar{\nabla}_X\xi - \bar{\nabla}_{[X,Y]}\xi. \tag{4.6}$$

Using (4.1), (4.4) and (3.3), we obtain

$$\begin{aligned} \bar{\nabla}_X\bar{\nabla}_Y\xi &= X(f)Y - \eta(Y)X(f)\xi + f\nabla_XY - f\eta(X)\phi Y - f^2\eta(Y)X \\ &\quad + f^2\eta(X)\eta(Y)\xi - fX\eta(Y)\xi, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \bar{\nabla}_Y\bar{\nabla}_X\xi &= Y(f)X - \eta(X)Y(f)\xi + f\nabla_YX - f\eta(Y)\phi X - f^2\eta(X)Y \\ &\quad + f^2\eta(Y)\eta(X)\xi - fY\eta(X)\xi \end{aligned} \tag{4.8}$$

and

$$\bar{\nabla}_{[X,Y]}\xi = f\nabla_XY - f\nabla_YX - fX\eta(Y)\xi + fY\eta(X)\xi. \tag{4.9}$$

Using (4.9) and (4.8) in (4.6), we have

$$\begin{aligned} \bar{R}(X,Y)\xi &= X(f)Y - Y(f)X - \eta(Y)X(f)\xi + \eta(X)Y(f)\xi - f\eta(X)\phi Y \\ &\quad + f\eta(Y)\phi X - f^2\eta(Y)X + f^2\eta(X)Y. \end{aligned} \tag{4.10}$$

Now using (4.3) in (4.10), we have

$$\bar{R}(X,Y)\xi = -(f^2+f')(\eta(Y)X - \eta(X)Y) + f(\eta(Y)\phi X - \eta(X)\phi Y). \tag{4.11}$$

From (4.11), we get

$$\bar{R}(\xi,Y)\xi = -(f^2+f')(\eta(Y)\xi - Y) - f\phi Y, \tag{4.12}$$

and

$$\bar{R}(X,\xi)\xi = -(f^2+f')(X - \eta(X)\xi) + f\phi X. \tag{4.13}$$

In (4.11) taking inner product with Z, we get

$$\begin{aligned} g(\bar{R}(X,Y)\xi,Z) &= -(f^2+f')(\eta(Y)g(X,Z) - \eta(X)g(Y,Z)) \\ &\quad + f(\eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z)). \end{aligned} \tag{4.14}$$

Now we have,

Lemma 4.1. *Let M be 3-dimensional f-Kenmotsu manifold with the quarter-symmetric metric connection.If $\bar{S}$ be Ricci curvature and $\bar{Q}$ be Ricci operator. Then*

$$\bar{S}(X,\xi) = -2(f^2+f')\eta(X), \tag{4.15}$$

and

$$\bar{Q}\xi = -2(f^2+f')\xi. \tag{4.16}$$

Proof. Contracting (4.14) with Y and Z, taking summation over $i = 1,2,3,...,n$ and using (2.4) the proof of (4.15) is completed. Also by (2.6) and (2.1) in (4.15), we get (4.16).

Lemma 4.2. *Let M be 3-dimensional f-Kenmotsu manifold with quarter-symmetric metric connection.If $\bar{S}$ be Ricci tensor, τ be scaler curvature tensor and $\bar{Q}$ Ricci operator. Then it follows that*

$$\begin{aligned} \bar{S}(X,Y) &= (\frac{\tau}{2}+f^2+f')g(X,Y) - (\frac{\tau}{2}+3f^2+3f')\eta(X)\eta(Y) \\ &\quad + fg(\phi X,Y), \end{aligned} \tag{4.17}$$

and

$$\bar{Q}X = (\frac{\tau}{2}+f^2+f')X - (\frac{\tau}{2}+3f^2+3f')\eta(X)\xi + f\phi X. \tag{4.18}$$

Proof. Contracting with Y in (4.13), we get

$$g(\bar{R}(X,\xi)\xi,Y) = -(f^2+f')(g(X,Y) - \eta(X)\eta(Y)) + fg(\phi X,Y). \tag{4.19}$$

Putting X=ξ, Y=X, Z=Y in (2.8), using (4.15) and taking contraction with ξ, we obtain

$$\begin{aligned} g(\bar{R}(\xi,X)Y,\xi) &= \bar{S}(X,Y) - (2f^2+2f'+\frac{\tau}{2})g(X,Y) + (4f^2+4f'+\frac{\tau}{2})\eta(X)\eta(Y) \\ &\quad -\frac{\tau}{2}[g(X,Y) - \eta(X)\eta(Y)]. \end{aligned} \tag{4.20}$$

With the help of (4.19) and (4.20), we have (4.17) . Now using (4.17)and (2.6), we get

$$g(\bar{Q}X - [(\frac{\tau}{2} + f^2 + f')X - (\frac{\tau}{2} + 3f^2 + 3f')\eta(X)\xi + f\phi X], Y) = 0. \tag{4.21}$$

Since $Y \neq 0$ in (4.21), which leads the proof of (4.18).

Example 4.1 (A 3-dimensional f-Kenmotsu manifold with quarter-symmetric metric connection). Let us consider a 3-dimensional manifold $M = (x, y, z) \in R^3$, z≠0, where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = z^2 \frac{\partial}{\partial x}, e_2 = z^2 \frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined as

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Considering a (1,1) tensor field ϕ defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0,$$

then using linearity of g and ϕ, for any $Z,W \in \chi\ (M)$, we get

$$\eta(e_3) = 1,$$
$$\phi^2(Z) = -Z + \eta(Z)e_3,$$
$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W).$$

Now by computation directly, we get

$$[e_1, e_2] = 0, [e_2, e_3] = -\frac{2}{z}e_2, [e_1, e_3] = -\frac{2}{z}e_1.$$

By the use of these above equations we have

$$\nabla_{e_1} e_1 = \frac{2}{z}e_3,\ \nabla_{e_2} e_2 = \frac{2}{z}e_3,\ \nabla_{e_3} e_3 = 0,\ \nabla_{e_2} e_1 = \nabla_{e_1} e_2 = \nabla_{e_3} e_1 = \nabla_{e_3} e_3 = 0. \tag{4.22}$$

Now in this example we consider for quarter-symmetric metric connection. By using (4.1) and (4.21), we have

$$\bar{\nabla}_{e_i} e_i = \frac{2}{z}e_3,\ \bar{\nabla}_{e_3} e_3 = 0,\ \bar{\nabla}_{e_i} e_3 = -\frac{2}{z}e_i,\ \bar{\nabla}_{e_i} e_j = 0,\ \bar{\nabla}_{e_3} e_1 = e_2,\ \bar{\nabla}_{e_3} e_2 = -e_1 \tag{4.23}$$

where $i \neq j = 1,\ 2$. We know that

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{4.24}$$

Using (4.23) and (4.24), we get

$$\begin{aligned}
\bar{R}(e_1, e_3)e_3 &= -\frac{2}{z}(\frac{3e_1}{z} + e_2), \\
\bar{R}(e_2, e_3)e_3 &= -\frac{2}{z}(\frac{3e_2}{z} - e_1), \\
\bar{R}(e_i, e_j)e_3 &= 0,\ i, j = 1, 2 \\
\bar{R}(e_i, e_j)e_j &= \frac{4}{z^2}e_i,\ i, j = 1, 2 \\
\bar{R}(e_1, e_3)e_2 &= -\frac{2}{z}e_3, \\
\bar{R}(e_2, e_3)e_1 &= \frac{2}{z}e_3, \\
\bar{R}(e_3, e_i)e_i &= -\frac{6}{z^2}e_3.
\end{aligned} \tag{4.25}$$

where $i \neq j$=1,2.

Using (2.4) and (4.25), we verify that

$$\bar{S}(e_i, e_i) = -\frac{2}{z^2},\ i = 1, 2, \quad \bar{S}(e_3, e_3) = -\frac{12}{z^2}. \tag{4.26}$$

Now using (2.10),(4.25) and (4.26), we find that

$$\bar{P}(e_1, e_2)e_3 = 0, \quad \bar{P}(e_i, e_3)e_3 = -\frac{2}{z}(\frac{2e_1}{z} + e_2).$$

This leads the following Proposition:

Proposition 4.1 *A 3-dimensional f-Kenmotsu manifold with the quarter-symmetric metric connection is not necessarily ξ-projectively flat.*

5 Ricci Solitons in f-Kenmotsu Manifold with the quarter-symmetric metric connection

Consider a 3-dimensional f-Kenmotsu manifold with the quarter-symmetric metric connection. Let V be pointwise collinear with ξ (i.e.$V = b\xi$, where b is a function). Then

$$(L_V g + 2S + 2\lambda g)(X, Y) = 0,$$

implies

$$\begin{aligned} 0 &= (Xb)\eta(Y) + bg(\bar{\nabla}_X \xi, Y) + (Yb)\eta(X) + bg(X, \bar{\nabla}_Y \xi) + 2\bar{S}(X, Y) \\ &\quad + 2\lambda g(X, Y). \end{aligned} \tag{5.1}$$

Using (4.4) in (5.1), we get

$$2bfg(X, Y) - 2bf\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2\bar{S}(X, Y) + 2\lambda g(X, Y) = 0. \tag{5.2}$$

Substitute Y with ξ in (5.2), we have

$$Xb + (\xi b)\eta(X) - 4(f^2 + f')\eta(X) + 2\lambda\eta(X) = 0. \tag{5.3}$$

Again substituting X with ξ in (5.3)

$$\xi b = 2(f^2 + f') - \lambda. \tag{5.4}$$

Putting (5.4) in (4.3), we get

$$b = [2(f^2 + f') - \lambda]\eta. \tag{5.5}$$

Applying d on (5.5)

$$0 = db = [2(f^2 + f') - \lambda]d\eta. \tag{5.6}$$

Since $d\eta \neq 0$,we have

$$[2(f^2 + f') - \lambda] = 0. \tag{5.7}$$

Now using (5.5) and (5.7) it is obtained that b is constant. Hence from (5.2), we can verify

$$\bar{S}(X, Y) = -(bf + \lambda)g(X, Y) - bf\eta(X)\eta(Y). \tag{5.8}$$

which results that M is η-Einstein manifold. Thus we have:

Theorem 5.1. *If in a 3-dimensional f-Kenmotsu manifold M with quarter-symmetric metric connection, the metric g is a Ricci soliton and V is a pointwise collinear with ξ, then V is a constant multiple of ξ and M is η-Einstein manifold of the form (5.8) and Ricci Soliton is expanding or shrinking according as $\lambda = 2(f^2 + f')$ is positive or negative.*

6 Conclusion

In this study, we have some curvature conditions for 3-dimensional f-Kenmotsu manifolds with quarter-symmetric metric connection. We have also shown that these manifolds are not always ξ-projective flat. Finally, we have that 3-dimensional f-Kenmotsu manifold with the quarter-symmetric metric connection is also an η-Einstein manifold and the Ricci soliton defined expanding or shrinking on this manifold is named with respect to the values of f and λ.

Acknowledgement. The authors are thankful to the Department of Mathematics and Astronomy, University of Lucknow, Lucknow, for giving full support for this study.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 300-307**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

COMMON FIXED-POINT THEOREM USING ψ -WEAK CONTRACTION FOR EIGHT SELF-MAPPINGS IN FUZZY METRIC SPACE

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(*Received : February* 28, 2023 ; *In format: March* 31, 2023; Revised: June 08, 2023; Accepted; June 09, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53136

Abstract

Applying weakly compatible for eight self-mappings in fuzzy metric space, we demonstrate common fixed-point theorems in this analysis after already formulating the generalised ψ- weak contraction condition, which involves third and fourth components of $\mathcal{M}(x, y, t)$.
2020 Mathematical Sciences Classification: 47H10, 54H25.
Keywords and Phrases: ψ-Weak Contraction, weakly Compatible mappings, fuzzy metric space.

1 Introduction

The idea of fuzzy sets was developed by Zadeh [22] in 1965 as a novel approach to depict the ambiguity in daily life. The development of fuzzy mathematics started at this point. In 1975, Kramosil and Michalek [10] defined the fuzzy metric space with the help of continuous t-norm by using the concepts of fuzziness. Fuzzy set theory is used in a wide range of real-world applications, including neural networks, fixed theory, health care, image processing, and control theory. When Zadeh [22] introduced the idea of a fuzzy set, which served as the basis for fuzzy mathematics, it marked a turning point in the history of mathematics.

Fuzzy mathematics has developed rapidly over the past three decades as a result, and recent studies have revealed that practically all fields of mathematics, including arithmetic, topology, graph theory, probability theory, logic, etc., have been fuzzyfied [1, 2, 4, 8, 9, 12, 13]. Communication, image processing, control theory, mathematical programming, neural network theory, stability theory, engineering, and medical sciences are among the applied areas where fuzzy set theory is used (medical genetics, nervous system). It makes sense that fuzzy fixed point theory has become more popular among experts in the discipline and that fuzzy mathematics has opened up new opportunities for fixed point theorists. For more details on this topic, one can see [5, 11, 14, 15, 16, 17, 18, 20]

2 Preliminaries

Definition 2.1 ([19]). *Let $(\mathfrak{B}, M, *)$ be a fuzzy metric space and $\mathcal{G}$ and $\mathcal{H}$ be two self-mappings of this space. When $\{x_n\}$ is a sequence in $\mathfrak{B}$ such that $\lim_{n\to\infty} \mathcal{G}x_n = \lim_{n\to\infty} \mathcal{H}x_n = u$ for some $u \in \mathfrak{B}$, the mappings $\mathcal{G}$ and $\mathcal{H}$ are known as compatible if $\lim_{n\to\infty} \mathcal{M}(\mathcal{G}\mathcal{H}x_n, \mathcal{H}\mathcal{G}x_n, t) = 1$, for all $t > 0$.*

Jungck [6, 7] presented the idea of weakly compatible mappings in 1986 and demonstrated that weakly compatible maps are compatible maps, despite the possibility that the opposite is also true. Later Subrahmanyam [19] extended the definition as follows:

Definition 2.2 ([19]). *If G and H commute at their coincidence sites, they are considered to be weakly compatible.*

Definition 2.3 ([3]). *If $\mathfrak{B}$ is arbitrary set, $*$ is a continuous t-norm, $\mathcal{M}$ is a fuzzy set in $\mathfrak{B}^2 \times [0, \infty)$, the triplet $(\mathfrak{B}, \mathcal{M}, *)$ meets the following requirements for being a fuzzy metric space:*
(i) $\mathcal{M}(x, y, t) > 0$,
(ii) $\mathcal{M}(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,

(iii) $\mathcal{M}(x,y,t) = \mathcal{M}(y,x,t)$,
(iv) $(\mathcal{M}(x,y,t) * \mathcal{M}(y,z,s)) \leq \mathcal{M}(x,z,t+s)$,
(v) $\mathcal{M}(x,y,.) : [0,\infty) \to [0,1]$ *is left continuous for all* $x,y,z \in \mathfrak{B}$ *and* $s,t > 0$,
(vi) $\lim_{t\to\infty} \mathcal{M}(x,y,t) = 1$, *for all* $x,y, \in \mathfrak{B}$.
$\mathcal{M}(x,y,t)$ *is a measure of how close together* x *and* y *are with regard to* t.

Definition 2.4 ([14]). *Let* $(\mathfrak{B},\mathcal{M},*)$ *be a fuzzy metric space. A sequence* $\{x_n\}$ *in* $\mathfrak{B}$ *is defined as:*
(i) Converge to $x \in \mathfrak{B}$ *if* $\lim_{n\to\infty} \mathcal{M}(x_n,x,t) = 1$ *fo each* $t > 0$.
(ii) Cauchy sequence if $\lim_{n\to\infty} \mathcal{M}(x_n,x_p,t) = 1$ *for all* $t > 0$ *and* $p > 0$.
(iii) Complete if every Cauchy sequence in $\mathfrak{B}$ *is convergent in* $\mathfrak{B}$.

Proposition 2.1 ([6]). *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be compatible mappings of a fuzzy metric space* $(\mathfrak{B},\mathcal{M},*)$ *into itself. If* $\mathcal{A}t = \mathcal{B}t$ *for some* t *in* $\mathfrak{B}$, *then* $\mathcal{AB}t = \mathcal{AA}t = \mathcal{BB}t = \mathcal{BA}t$.

Proposition 2.2 ([6]). *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be compatible mappings of a fuzzy metric space* $(\mathfrak{B},\mathcal{M},*)$ *into itself.*
Suppose that $lim_n\mathcal{A}x_n = lim_n\mathcal{B}x_n = t$ *for some* t *in* $\mathfrak{B}$. *Then the following holds:*
(i) $\mathcal{BA}x_n = \mathcal{A}t$ *if* $\mathcal{A}$ *is continuous at* t;
(ii) $\mathcal{AB}x_n = \mathcal{B}t$ *if* $\mathcal{B}$ *is continuous at* t;
(iii) $\mathcal{AB}t = \mathcal{BA}t$ *and* $\mathcal{A}t = \mathcal{B}t$ *if* $\mathcal{A}$ *and* $\mathcal{B}$ *are continuous at* t.

Lemma 2.1 ([19]). *Let* $(\mathfrak{B},\mathcal{M},*)$ *be a fuzzy metric space. If there exists* $q \in (0,1)$ *such that* $\mathcal{M}(x,y,qt) \geq \mathcal{M}(x,y,t)$ *for all* $x,y \in \mathcal{B}$, *and* $t > 0$, *then* $x = y$.

Lemma 2.2 ([19]). *Let* $\{y_n\}$ *be a sequence in a fuzzy metric space* $(\mathfrak{B},\mathcal{M},*)$. *If there exists* $q \in (0,1)$ *such that* $\mathcal{M}(y_{(n+2)},y_{(n+1)},qt) \geq \mathcal{M}(y_{(n+1)},y_n,t), t > 0, n \in N$, *then* y_n *is a Cauchy sequence in* $\mathfrak{B}$.

Lemma 2.3 ([20]). *Let* $(\mathfrak{B},\mathcal{M},*)$ *be a fuzzy metric space. If there is a sequence* $\{x_n\} \in X$, *such that for every* $n \in N$,

$$\mathcal{M}(x_n,x_{(n+1)},t) \geq \mathcal{M}(x_0,x_1,k^n t)$$

for every $k > 1$, *then the sequence is a Cauchy sequence.*

3 Main Results

Let Ψ be set of all continuous functions $\psi : [0,1]^4 \to [0,1]$ increasing in any coordinate and $\psi(t,t,t,t) > t$.

Theorem 3.1. *Let* $(\mathfrak{B},\mathcal{M},*)$ *be a complete fuzzy metric space. Let* $\mathcal{N},\mathcal{P},\mathcal{Q},\mathcal{S},\mathcal{T},\mathcal{K},\mathcal{L}$ *and* $\mathcal{W}$ *are eight self-mappings of a complete fuzzy metric space* $(\mathfrak{B},\mathcal{M},*)$ *into itself satisfying*
$(C_1)\mathcal{TK}(\mathfrak{B}) \subseteq \mathcal{NP}(\mathfrak{B}), \mathcal{WL}(\mathfrak{B}) \subseteq \mathcal{QS}(\mathfrak{B})$,
$(C_2)\mathcal{QS} = \mathcal{SQ}, \mathcal{NP} = \mathcal{PN}, \mathcal{TK} = \mathcal{KT}, \mathcal{WL} = \mathcal{LW}, (\mathcal{TK})\mathcal{S} = \mathcal{S}(\mathcal{TK}), (\mathcal{WL})\mathcal{P} = \mathcal{P}(\mathcal{WL}), (\mathcal{NP})\mathcal{L} = \mathcal{L}(\mathcal{NP}), (\mathcal{QS})\mathcal{K} = \mathcal{K}(\mathcal{QS})$
(C_3) *One of* $\mathcal{NP}(\mathfrak{B}), \mathcal{WL}(\mathfrak{B}), \mathcal{QS}(\mathfrak{B})$ *or* $\mathcal{TK}(\mathfrak{B})$ *is complete ,*
(C_4) *The pair* $(\mathcal{TK},\mathcal{QS})$ *and* $(\mathcal{WL},\mathcal{NP})$ *are weakly compatible,*
(C_5) $M^3(\mathcal{TK}u,\mathcal{WL}v,t)$

$$\geq \psi \left\{ \begin{array}{c} M^2(\mathcal{QS}u,\mathcal{TK}u,ht)M(\mathcal{WL}v,\mathcal{NP}v,ht), \\ M(\mathcal{QS}u,\mathcal{TK}u,ht)M^2(\mathcal{WL}v,\mathcal{NP}v,ht), \\ M(\mathcal{QS}u,\mathcal{TK}u,ht)M(\mathcal{TK}u,\mathcal{WL}v,ht)M(\mathcal{WL}v,\mathcal{NP}v,ht), \\ M(\mathcal{WL}v,\mathcal{NP}v,ht)M(\mathcal{QS}u,\mathcal{NP}v,ht)M(\mathcal{QS}u,\mathcal{TK}u,ht) \end{array} \right\}$$

for all $u,v \in \mathfrak{B}, h > 1$ *and* $\psi \in \Psi$.
Then $\mathcal{N},\mathcal{P},\mathcal{Q},\mathcal{S},\mathcal{T},\mathcal{K},\mathcal{L}$ *and* $\mathcal{W}$ *have a unique common fixed point in* $\mathfrak{B}$.

Proof. Let $x_0 \in \mathfrak{B}$ be an arbitrary point. By (C_1) we can search a point x_1 such that $\mathcal{TK}(x_0) = \mathcal{NP}(x_1) = y_0$. For this point x_1 one can search a point $x_2 \in \mathfrak{B}$ such that $\mathcal{WL}(x_1) = \mathcal{QS}(x_2) = y_1$. By continuing in this manner, a sequence $\{x_n\}$ can be created, such that $y_{2n} = \mathcal{JK}(x_{2n}) = \mathcal{NP}(x_{2n+1})$,

$$y_{2n+1} = \mathcal{WL}(x_{2n+1}) = \mathcal{QS}(x_{2n+2}), \text{for each } \ n \geq 0. \tag{3.1}$$

□

For simplicity, we take $\alpha_m(t) = \mathcal{M}(y_m, y_{m+1}, t)$.

Initially, we establish that $\{y_n\}$ is Cauchy sequence.

Case I. If n is even, considering $u = x_{2n}$ and $v = x_{2n+1}$ in (C_5), we get

$$M^3(\mathcal{TK}x_{2n}, \mathcal{WL}x_{2n+1}, t)$$
$$\geq \psi \left\{ \begin{array}{c} M^2(\mathcal{QS}x_{2n}, \mathcal{TK}x_{2n}, ht)M(\mathcal{WL}x_{2n+1}, \mathcal{NP}x_{2n+1}, ht), \\ M(\mathcal{QS}x_{2n}, \mathcal{TK}x_{2n}, ht)M^2(\mathcal{WL}x_{2n+1}, \mathcal{NP}x_{2n+1}, ht), \\ M(\mathcal{QS}x_{2n}, \mathcal{TK}x_{2n}, ht)M(\mathcal{TK}x_{2n}, \mathcal{WL}x_{2n+1}, ht)M(\mathcal{WL}x_{2n+1}, \mathcal{NP}x_{2n+1}, ht), \\ M(\mathcal{WL}x_{2n+1}, \mathcal{NP}x_{2n+1}, ht)M(\mathcal{QS}x_{2n}, \mathcal{NP}x_{2n+1}, ht)M(\mathcal{QS}x_{2n}, \mathcal{TK}x_{2n}, ht) \end{array} \right\}.$$

Using (3.1), we have

$$M^3(y_{2n}, y_{2n+1}, t)$$
$$\geq \psi \left\{ \begin{array}{c} M^2(y_{2n-1}, y_{2n}, ht)M(y_{2n+1}, y_{2n}, ht), \\ M(y_{2n-1}, y_{2n}, ht)M^2(y_{2n+1}, y_{2n}, ht), \\ M(y_{2n-1}, y_{2n}, ht)M(y_{2n}, y_{2n+1}, ht)M(y_{2n+1}, y_{2n}, ht), \\ M(y_{2n+1}, y_{2n}, ht)M(y_{2n-1}, y_{2n}, ht)M(y_{2n-1}, y_{2n}, ht) \end{array} \right\}.$$

On using $\alpha_{2n}(t) = \mathcal{M}(y_{2n}, y_{2n+1}, t)$ in the above inequality, we have

$$\alpha^3_{2n}(t) \geq \psi \left\{ \begin{array}{c} \alpha^2_{2n-1}(ht)\alpha_{2n}(ht), (\alpha_{2n-1}(ht)\alpha^2_{2n}(ht), \\ \alpha_{2n-1}(ht)\alpha^2_{2n}(ht), (\alpha_{2n}(ht)\alpha^2_{2n-1}(ht) \end{array} \right\}. \tag{3.2}$$

We claim that $\alpha_{2n}(ht) \geq \alpha_{2n-1}(ht)$

If $\alpha_{2n}(ht) < \alpha_{2n-1}(ht)$, then (3.2) reduces to

$$\alpha^3_{2n}(t) \geq \psi \left\{ \alpha^3_{2n}(ht), \alpha^3_{2n}(ht), \alpha^3_{2n}(ht), \alpha^3_{2n}(ht)) \right\}.$$

Using property of ψ we get

$$\alpha^3_{2n}(t) > \alpha^3_{2n}(ht) \implies \alpha_{2n}(t) > \alpha_{2n}(ht),$$

a contradiction.

Therefore $\alpha_{2n}(ht) \geq \alpha_{2n-1}(ht)$.

Like in similar manner, if n is odd, then we can achieve $\alpha_{2n+1}(ht) \geq \alpha_{2n}(ht)$.

It follows that the sequence $\alpha_n(t)$ is increasing in $[0, 1]$, thus (3.2) reduces to

$$\alpha^3_{2n}(t) \geq \psi \left\{ \alpha^3_{2n-1}(ht), \alpha^3_{2n-1}(ht), \alpha^3_{2n-1}(ht)\alpha^3_{2n-1}(ht) \right\}.$$

Using property of ψ we get

$$\alpha^3_{2n}(t) > \alpha^3_{2n-1}(ht) \Rightarrow \alpha_{2n}(t) \geq \alpha_{2n-1}(t).$$

Similarly for an odd integer $2n + 1$, we have $\alpha_{2n+1}(t) \geq \alpha_{2n}(ht)$,

Hence $\alpha_n(t) \geq \alpha_{n-1}(ht)$, that is,

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, ht) \geq ... \geq M(y_0, y_1, h^n t).$$

Hence by Lemma 2.3 $\{y_n\}$ is a Cauchy sequence in $\mathfrak{B}$.

Case II. $\mathrm{N}\mathcal{P}(\mathfrak{B})$ is complete. In this case $\{y_{2n}\} = \{\mathcal{NP}x_{2n+1}\}$ is a Cauchy sequence in $N\mathcal{P}(\mathfrak{B})$, which is complete then the sequence $\{y_{2n}\}$ converges to some point $z \in N\mathcal{P}(\mathfrak{B})$. Consequently, the subsequences $\{\mathcal{TK}x_{2n}\}$, $\{\mathcal{QS}x_{2n}\}$, $\{\mathcal{NP}x_{2n+1}\}$, and $\{\mathcal{WL}x_{2n+1}\}$ also converges to the same point z. As $z \in N\mathcal{P}(\mathfrak{B})$, there exists $r \in \mathfrak{B}$ such that $z = \mathcal{NP}r$.

Now we claim that $z = \mathcal{WL}r$. For this putting $u = x_{2n}$ and $v = r$ in (C_5), we get

$$M^3(\mathcal{TK}x_{2n}, W\mathcal{L}r, t)$$
$$\geq \psi \left\{ \begin{array}{c} M^2(\mathcal{QS}x_{2n}, \mathcal{TK}x_{2n}, ht)M(W\mathcal{L}r, N\mathcal{P}r, ht), \\ M(\mathcal{QS}x_{2n}, \mathcal{TK}x_{2n}, ht)M^2(W\mathcal{L}r, N\mathcal{P}r, ht), \\ M(\mathcal{QS}x_{2n}, \mathcal{TK}x_{2n}, ht)M(\mathcal{TK}x_{2n}, W\mathcal{L}r, ht)M(W\mathcal{L}r, N\mathcal{P}r, ht) \\ M(W\mathcal{L}r, N\mathcal{P}r, ht)M(\mathcal{QS}x_{2n}, N\mathcal{P}r, ht)M(\mathcal{QS}x_{2n}, \mathcal{TK}x_{2n}, ht) \end{array} \right\}.$$

Taking limit $n \to \infty$ and using $z = \mathcal{NP}r$ in above inequality we have,

$$M^3(z, \mathcal{WL}r, t) \geq \psi \left\{ \begin{array}{c} M^2(z, z, ht)M(\mathcal{WL}r, z, ht), \\ M(z, z, ht)M^2(\mathcal{WL}r, z, ht), \\ M(z, z, ht)M(z, \mathcal{WL}r, ht)M(\mathcal{WL}r, z, ht) \\ M(\mathcal{WL}r, z, ht)M(z, z, ht)M(z, z, ht) \end{array} \right\},$$

$$M^3(z,\mathcal{WL}r,t) \geq \psi \left\{ \begin{array}{c} 1.1.M(\mathcal{WL}r,z,ht), \\ 1.M^2(\mathcal{WL}r,z,ht), \\ 1.M(z,\mathcal{WL}r,ht)M(\mathcal{WL}r,z,ht), \\ M(\mathcal{WL}r,z,ht).1.1 \end{array} \right\}.$$

Suppose $\mathcal{WL}r \neq z$, then $M(z,\mathcal{WL}r,ht) < 1$, using this in above inequality we get

$$M^3(z,\mathcal{WL}r,t) \geq \psi \left\{ M^3(z,\mathcal{WL}r,ht), M^3(z,\mathcal{WL}r,ht), M^3(z,\mathcal{WL}r,ht), M^3(z,\mathcal{WL}r,ht) \right\}.$$

Using property of ψ we get

$$M^3(z,\mathcal{WL}r,t) > M^3(z,\mathcal{WL}r,ht)$$

$$\implies M(z,\mathcal{WL}r,t) > M(z,\mathcal{WL}r,ht), \text{ a contradiction.}$$

Hence $\mathcal{WL}r = z$

Thus $\mathcal{WL}r = z = \mathcal{NP}r$. Since $(\mathcal{WL},\mathcal{NP})$ are weakly compatible, so we have $\mathcal{WL}z = \mathcal{NP}z$.

Next, we will show that $\mathcal{P}z = z$, for this putting $u = x_{2n}$ and $v = \mathcal{P}r$ in (C_5), we get

$$M^3(\mathcal{TK}x_{2n},\mathcal{WLP}r,t) \geq \psi \left\{ \begin{array}{c} M^2(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht)M(\mathcal{WLP}r,\mathcal{NPP}r,ht), \\ M(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht)M^2(\mathcal{WLP}r,\mathcal{NPP}r,ht), \\ M(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht)M(\mathcal{TK}x_{2n},\mathcal{WLP}r,ht)M(\mathcal{WLP}r,\mathcal{NPP}r,ht) \\ M(\mathcal{WLP}r,\mathcal{NPP}r,ht)M(\mathcal{QS}x_{2n},\mathcal{NPP}r,ht)M(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht) \end{array} \right\}.$$

From (C_2) $\mathcal{WLP} = \mathcal{PWL}$ and $\mathcal{NP} = \mathcal{PN}$ using in above inequality we get,

$$M^3(\mathcal{TK}x_{2n},\mathcal{PWL}r,t) \geq \psi \left\{ \begin{array}{c} M^2(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht)M(\mathcal{PWL}r,\mathcal{PNP}r,ht), \\ M(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht)M^2(\mathcal{PWL}r,\mathcal{PNP}r,ht), \\ M(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht)M(\mathcal{TK}x_{2n},\mathcal{PWL}r,ht)M(\mathcal{PWL}r,\mathcal{PNP}r,ht) \\ M(\mathcal{PWL}r,\mathcal{PNP}r,ht)M(\mathcal{QS}x_{2n},\mathcal{PNP}r,ht)M(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht) \end{array} \right\}.$$

Taking limit $n \longrightarrow \infty$ and using $\mathcal{WL}r = z = \mathcal{NP}r$ in above inequality we have,

$$M^3(z,\mathcal{P}z,t) \geq \psi \left\{ \begin{array}{c} M^2(z,z,ht)M(\mathcal{P}z,\mathcal{P}z,ht), \\ M(z,z,ht)M^2(\mathcal{P}z,\mathcal{P}z,ht), \\ M(z,z,ht)M(z,\mathcal{P}z,ht)M(\mathcal{P}z,\mathcal{P}z,ht) \\ M(\mathcal{P}z,\mathcal{P}z,ht)M(z,\mathcal{P}z,ht)M(z,z,ht) \end{array} \right\}.$$

Suppose $\mathcal{P}z \neq z$, then $M(z,\mathcal{P}z,ht) < 1$, using this in above inequality we get

$$M^3(z,\mathcal{P}z,t) \geq \psi \left\{ M^3(z,\mathcal{P}z,ht), M^3(z,\mathcal{P}z,ht), M^3(z,\mathcal{P}z,ht), M^3(z,\mathcal{P}z,ht) \right\}.$$

Using property of ψ we get

$$M^3(z,\mathcal{P}z,t) > M^3(z,\mathcal{P}z,ht).$$

$$\implies M(z,\mathcal{P}z,t) > M(z,\mathcal{P}z,ht), \text{ a contradiction.}$$

Hence $z = \mathcal{P}z$.

Thus $\mathcal{P}z = \mathcal{NP}z = z \implies \mathcal{N}z = z$.

Thus $\mathcal{N}z = \mathcal{P}z = \mathcal{WL}z = z$.

Next, we will show that $\mathcal{L}z = z$, for this putting $u = x_{2n}$ and $v = \mathcal{L}r$ in (C_5), we get

$$M^3(\mathcal{TK}x_{2n},\mathcal{WLL}r,t) \geq \psi \left\{ \begin{array}{c} M^2(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht)M(\mathcal{WLL}r,\mathcal{NPL}r,ht), \\ M(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht)M^2(\mathcal{WLL}r,\mathcal{NPL}r,ht), \\ M(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht)M(\mathcal{TK}x_{2n},\mathcal{WLL}r,ht)M(\mathcal{WLL}r,\mathcal{NPL}r,ht), \\ M(\mathcal{WLL}r,\mathcal{NPL}r,ht)M(\mathcal{QS}x_{2n},\mathcal{NPL}r,ht)M(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht). \end{array} \right\}.$$

From (C_2) $\mathcal{WL} = \mathcal{WL}$ and $(\mathcal{NP})\mathcal{L} = \mathcal{L}(\mathcal{PN})$ using in above inequality we get,

$$M^3(\mathcal{TK}x_{2n},\mathcal{LWL}r,t) \geq \psi \left\{ \begin{array}{c} M^2(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht)M(\mathcal{LWL}r,\mathcal{LNP}r,ht), \\ M(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht)M^2(\mathcal{LWL}r,\mathcal{LNP}r,ht), \\ M(\mathcal{QS}x_{2n},\mathcal{TK}x_{2n},ht)M(\mathcal{TK}x_{2n},\mathcal{LWL}r,ht)M(\mathcal{LWL}r,\mathcal{LNP}r,ht), \\ M(\mathcal{LWL}r,\mathcal{LNP}r,ht)M(\mathcal{QS}x_{2n},\mathcal{LNP}r,ht)M(\mathcal{Q}\delta x_{2n},\mathcal{TK}x_{2n},ht) \end{array} \right\}.$$

Taking limit $n \longrightarrow \infty$ and using $\mathcal{WL}r = z = \mathcal{NP}r$ in above inequality we have,

$$M^3(z, \mathcal{L}z, t) \geq \psi \begin{Bmatrix} M^2(z, z, ht)M(\mathcal{L}z, \mathcal{L}z, ht), \\ M(z, z, ht)M^2(\mathcal{L}z, \mathcal{L}z, ht), \\ M(z, z, ht)M(z, \mathcal{L}z, ht)M(\mathcal{L}z, \mathcal{L}z, ht), \\ M(\mathcal{L}z, \mathcal{L}z, ht)M(z, \mathcal{L}z, ht)M(z, z, ht) \end{Bmatrix}.$$

Suppose $\mathcal{L}z \neq z$, then $M(z, \mathcal{L}z, ht) < 1$, using this in above inequality we get

$$M^3(z, \mathcal{L}z, t) \geq \psi \left\{ M^3(z, \mathcal{L}z, ht), M^3(z, \mathcal{L}z, ht), M^3(z, \mathcal{L}z, ht), M^3(z, \mathcal{L}z, ht) \right\}.$$

Using property of ψ we get

$$M^3(z, \mathcal{L}z, t) > M^3(z, \mathcal{L}z, ht).$$

$$\implies M(z, \mathcal{L}z, t) > M(z, \mathcal{L}z, ht), \text{ a contradiction.}$$

Hence $z = \mathcal{L}z$.
Thus, $\mathcal{L}z = W\mathcal{L}z = z \implies Wz = z$.
Thus $Nz = Pz = Wz = \mathcal{L}z = z$.
As $W\mathcal{L}(\mathfrak{B}) \subseteq \mathcal{QS}(\mathfrak{B})$, there exists $m \in \mathfrak{B}$ such that $z = W\mathcal{L}z = \mathcal{QS}m$.

Next, we will show that $\mathcal{TK}m = z$, for this putting $u = m$ and $v = x_{2n+1}$ in (C_5), we have

$$M^3(\mathcal{TK}m, W\mathcal{L}x_{2n+1}, t)$$
$$\geq \psi \begin{Bmatrix} M^2(\mathcal{QS}m, \mathcal{TK}m, ht)M(W\mathcal{L}x_{2n+1}, \mathcal{NP}x_{2n+1}, ht), \\ M(\mathcal{QS}m, \mathcal{TK}m, ht)M^2(W\mathcal{L}x_{2n+1}, \mathcal{NP}x_{2n+1}, ht), \\ M(\mathcal{QS}m, \mathcal{TK}m, ht)M(\mathcal{TK}m, W\mathcal{L}x_{2n+1}, ht)M(W\mathcal{L}x_{2n+1}, \mathcal{NP}x_{2n+1}, ht) \\ M(W\mathcal{L}x_{2n+1}, \mathcal{NP}x_{2n+1}, ht)M(\mathcal{QS}m, \mathcal{NP}x_{2n+1}, ht)M(\mathcal{QS}m, \mathcal{TK}m, ht). \end{Bmatrix}.$$

Taking limit $n \longrightarrow \infty$ and using $z = W\mathcal{L}z = \mathcal{QS}m$ in above inequality we have,

$$M^3(\mathcal{TK}m, z, t) \geq \psi \begin{Bmatrix} M^2(z, \mathcal{TK}m, ht)M(z, z, ht), \\ M(z, \mathcal{TK}m, ht)M^2(z, z, ht), \\ M(z, \mathcal{TK}m, ht)M(\mathcal{TK}m, z, ht)M(z, z, ht) \\ M(z, z, ht)M(z, z, ht)M(z, \mathcal{TK}m, ht). \end{Bmatrix}.$$

Suppose $\mathcal{TK}m \neq z$, then $M(\mathcal{TK}m, z, ht) < 1$, using this in above inequality we get

$$M^3(\mathcal{TK}m, z, t) \geq \psi \begin{Bmatrix} M^3(\mathcal{TK}m, z, ht)M^3(\mathcal{TK}m, z, ht), \\ M^3(\mathcal{TK}m, z, ht)M^3(\mathcal{TK}m, z, ht) \end{Bmatrix}.$$

Using property of ψ we get

$$M^3(\mathcal{TK}m, z, t) > M^3(\mathcal{TK}m, z, ht).$$

$$\implies M(\mathcal{TK}m, z, t) > M(\mathcal{TK}m, z, ht), \text{ a contradiction.}$$

Hence $\mathcal{TK}m = z$.

Since $(\mathcal{TK}, \mathcal{QS})$ are weakly compatible, so $\mathcal{TK}$ and $\mathcal{QS}$ commute their coincidence point m, then we have $\mathcal{TK}z = \mathcal{QS}z$.

Next we will show that $\mathcal{TK}z = z$, for this putting $u = z$ and $v = x_{2n+1}$ in (C_5), we have

$$M^3(\mathcal{TK}z, W\mathcal{L}x_{2n+1}, t)$$
$$\geq \psi \begin{Bmatrix} M^2(\mathcal{QS}z, \mathcal{TK}z, ht)M(W\mathcal{L}x_{2n+1}, \mathcal{NP}x_{2n+1}, ht), \\ M(\mathcal{QS}z, \mathcal{TK}z, ht)M^2(W\mathcal{L}x_{2n+1}, \mathcal{NP}x_{2n+1}, ht), \\ M(\mathcal{QS}z, \mathcal{TK}z, ht)M(\mathcal{TK}z, W\mathcal{L}x_{2n+1}, ht)M(W\mathcal{L}x_{2n+1}, \mathcal{NP}x_{2n+1}, ht) \\ M(W\mathcal{L}x_{2n+1}, \mathcal{NP}x_{2n+1}, ht)M(\mathcal{QS}z, \mathcal{NP}x_{2n+1}, ht)M(\mathcal{QS}z, \mathcal{TK}z, ht) \end{Bmatrix}.$$

Taking limit $n \longrightarrow \infty$ and using $\mathcal{TK}z = \mathcal{QS}z$ in above inequality we have

$$M^3(\mathcal{TK}z, z, t) \geq \psi \begin{Bmatrix} M^2(\mathcal{TK}z, \mathcal{TK}z, ht)M(z, z, ht), \\ M(\mathcal{TK}z, \mathcal{TK}z, ht)M^2(z, z, ht), \\ M(\mathcal{TK}z, \mathcal{TK}z, ht)M(\mathcal{TK}z, z, ht)M(z, z, ht) \\ M(z, z, ht)M(\mathcal{TK}z, z, ht)M(\mathcal{TK}z, \mathcal{TK}z, ht) \end{Bmatrix}.$$

Suppose $\mathcal{TK}z \neq z$, then $M(\mathcal{TK}z, z, ht) < 1$, using this in above inequality we get

$$M^3(\mathcal{TK}z, z, t) \geq \psi \begin{Bmatrix} M^3(\mathcal{TK}z, z, ht)M^3(\mathcal{TK}z, z, ht), \\ M^3(\mathcal{TK}z, z, ht)M^3(\mathcal{TK}z, z, ht) \end{Bmatrix}.$$

Using property of ψ, we get

$$M^3(\mathcal{TK}z, z, t) > M^3(\mathcal{TK}z, z, ht).$$

$$\implies M(\mathcal{TK}z, z, t) > M(\mathcal{TK}z, z, ht), \text{ a contradiction.}$$

Hence $\mathcal{TK}z = z$.

Thus $\mathcal{TK}z = \mathcal{QS}z = z$.

Next we will show that $\mathcal{S}z = z$, for this putting $\mu = \mathcal{S}z$ and $v = x_{2n+1}$ in (C_5), we have

$$M^3(\mathcal{TKS}z, W\mathcal{L}x_{2n+1}, t) \geq \psi \left\{ \begin{array}{c} M^2(\mathcal{QSS}z, \mathcal{TKS}z, ht)M(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht), \\ M(\mathcal{QSS}z, \mathcal{TKS}z, ht)M^2(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht), \\ M(\mathcal{QSS}z, \mathcal{TKS}z, ht)M(\mathcal{TKS}z, W\mathcal{L}x_{2n+1}, ht)M(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht) \\ M(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht)M(\mathcal{QSS}z, N\mathcal{P}x_{2n+1}, ht)M(\mathcal{QSS}z, \mathcal{TKS}z, ht) \end{array} \right\}.$$

From (C_2) $\mathcal{QS} = \mathcal{SQ}$ and $(\mathcal{TK})\mathcal{S} = \mathcal{S}(\mathcal{TK})$ using in above inequality we have,

$$M^3(\mathcal{STK}z, W\mathcal{L}x_{2n+1}, t) \geq \psi \left\{ \begin{array}{c} M^2(\mathcal{SQS}z, \mathcal{STK}z, ht)M(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht), \\ M(\mathcal{SQS}z, \mathcal{STK}z, ht)M^2(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht), \\ M(\mathcal{SQS}z, \mathcal{STK}z, ht)M(\mathcal{STK}z, W\mathcal{L}x_{2n+1}, ht)M(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht), \\ M(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht)M(\mathcal{SQS}z, N\mathcal{P}x_{2n+1}, ht)M(\mathcal{SQS}z, \mathcal{STK}z, ht) \end{array} \right\}.$$

Taking limit $n \longrightarrow \infty$ and using $\mathcal{TK}z = \mathcal{QS}z = z$ in above inequality we have,

$$M^3(\mathcal{S}z, z, t) \geq \psi \left\{ \begin{array}{c} M^2(\mathcal{S}z, \mathcal{S}z, ht)M(z, z, ht), \\ M(\mathcal{S}z, \mathcal{S}z, ht)M^2(z, z, ht), \\ M(\mathcal{S}z, \mathcal{S}z, kt)M(\mathcal{S}z, z, kt)M(z, z, ht), \\ M((z, z, kt)M(\mathcal{S}z, z, kt)M(\mathcal{S}z, \mathcal{S}z, ht) \end{array} \right\}.$$

Suppose $\mathcal{S}z \neq z$, then $M(\mathcal{S}z, z, ht) < 1$, using this in above inequality we get

$$M^3(\mathcal{S}z, z, t) \geq \psi \left\{ \begin{array}{c} M^3(\mathcal{S}z, z, ht), M^3(\mathcal{S}z, z, ht), \\ M^3(\mathcal{S}z, z, ht), M^3(\mathcal{S}z, z, ht) \end{array} \right\}$$

using property of ψ we get

$$M^3(\mathcal{S}z, z, t) > M^3(\mathcal{S}z, z, ht)$$

$$\implies M(\mathcal{S}z, z, t) > M(\mathcal{S}z, z, ht), \text{ a contradiction.}$$

Hence $\mathcal{S}z = z$. Then $z = \mathcal{QS}z = \mathcal{Q}z$. Therefore $z = \mathcal{S}z = \mathcal{Q}z = \mathcal{TK}z$.

Next we will show that $\mathcal{K}z = z$, for this putting $u = \mathcal{K}z$ and $v = x_{2n+1}$ in $(\mathcal{C}_5)$, we have

$$M^3(\mathcal{TKK}z, W\mathcal{L}x_{2n+1}, t) \geq \psi \left\{ \begin{array}{c} M^2(\mathcal{QSK}z, \mathcal{TKK}z, ht)M(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht), \\ M(\mathcal{QSK}z, \mathcal{TKK}z, ht)M^2(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht), \\ M(\mathcal{QSK}z, \mathcal{TKK}z, ht)M(\mathcal{TKK}z, W\mathcal{L}x_{2n+1}, ht)M(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht), \\ M(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht)M(\mathcal{QSK}z, N\mathcal{P}x_{2n+1}, ht)M(\mathcal{QSK}z, \mathcal{TKK}z, ht) \end{array} \right\}.$$

From $\mathcal{C}_2$, using $\mathcal{TK} = \mathcal{KT}, (\mathcal{QS})\mathcal{K} = \mathcal{K}(\mathcal{QS})$ in above inequality we have

$$M^3(\mathcal{TKK}z, W\mathcal{L}x_{2n+1}, t) \geq \psi \left\{ \begin{array}{c} M^2(\mathcal{KQS}z, \mathcal{KTK}z, ht)M(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht), \\ M(\mathcal{KQS}z, \mathcal{KTK}z, ht)M^2(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht), \\ M(\mathcal{KQS}z, \mathcal{KTK}z, ht)M(\mathcal{KTK}z, W\mathcal{L}x_{2n+1}, ht)M(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht), \\ M(W\mathcal{L}x_{2n+1}, N\mathcal{P}x_{2n+1}, ht)M(\mathcal{KQS}zN\mathcal{P}x_{2n+1}, ht)M(\mathcal{KQS}z, \mathcal{KTK}z, ht) \end{array} \right\}.$$

Taking limit $n \longrightarrow \infty$ and using $\mathcal{TK}z = \mathcal{QS}z = z$ in above inequality we have,

$$M^3(\mathcal{K}z, z, t) \geq \psi \left\{ \begin{array}{c} M^2(\mathcal{K}z, \mathcal{K}z, ht)M(z, z, ht), \\ M(\mathcal{K}z, \mathcal{K}z, ht)M^2(z, z, ht), \\ M(\mathcal{K}z, \mathcal{K}z, ht)M(\mathcal{K}z, z, ht)M(z, z, ht), \\ M(z, z, ht)M(\mathcal{K}z, z, ht)M(\mathcal{K}z, \mathcal{K}z, ht) \end{array} \right\}.$$

Suppose $\mathcal{K}z \neq z$, then $M(\mathcal{K}z, z, ht) < 1$, using this in above imnequality we get

$$M^3(\mathcal{K}z, z, t) \geq \psi \left\{ \begin{array}{c} M^3(\mathcal{K}z, z, ht), M^3(\mathcal{K}z, z, ht), \\ M^3(\mathcal{K}z, z, ht), M^3(\mathcal{K}z, z, ht) \end{array} \right\}.$$

Using property of ψ we get

$$M^3(\mathcal{K}z,z,t) > M^3(\mathcal{K}z,z,ht)$$
$$\implies M(\mathcal{K}z,z,t) > M(\mathcal{K}z,z,ht), \text{ a contradiction,}$$

Hence $\mathcal{K}z = z$.
Thus $\mathcal{TK}z = \mathcal{T}z = z$.
Thus $\mathcal{Q}z = \mathcal{S}z = \mathcal{K}z = z$.
Hence $Nz = \mathcal{P}z = \mathcal{Q}z = \mathcal{S}z = \mathcal{T}z = \mathcal{K}z = \mathcal{L}z = Wz = z$.
Hence z be a unique fixed point of $N, \mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T}, \mathcal{K}, \mathcal{L}$, and W.

4 Application

A fixed point theorem for a single mapping satisfies an analogue of a Banach contraction principle for an integral type inequality was discovered by Branciari in 2002.
As an application of Theorem 3.1, we now show the following theorem.

Theorem 4.1. *Let $N, \mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T}, \mathcal{K}, \mathcal{L}$ and W be eight self mappings of a complete fuzzy metric space $(\mathbb{B}, M, *)$ satisfying the conditions $(\mathcal{C}_1), (\mathcal{C}_2), (\mathcal{C}_3), (\mathcal{C}_4)$ and the following condition.*

$$\int_0^{M^3(x,y,t)} \psi(w)dw \geq \int_0^{\sigma(u,v)} \psi(w)dw$$
$$\sigma(u,v) = \psi \left\{ \begin{array}{c} M^2(\mathcal{QS}u, \mathcal{TK}u, ht)M(W\mathcal{L}v, N\mathcal{P}v, ht), \\ M(\mathcal{QS}u, \mathcal{TK}u, ht)M^2(W\mathcal{L}v, N\mathcal{P}v, ht), \\ M(\mathcal{QS}u, \mathcal{TKK}z, ht)M(\mathcal{TK}u, W\mathcal{L}v, ht)M(W\mathcal{L}v, N\mathcal{P}v, ht) \\ M(W\mathcal{L}v, N\mathcal{P}v, ht)M(\mathcal{QS}u, N\mathcal{P}v, ht)M(\mathcal{QS}u, \mathcal{TK}u, ht) \end{array} \right\}$$

for all $u, v \in \mathbb{B}$, where $\psi : [0,1]^4 \to [0,1]$ is increasing in any cooridanate and $\psi(t,t,t,t) > t$ for every $t \in [0,1)$, where $\psi : [0,1]^4 \to [0,1]$ is a "Lebesgue-integrable function" which is summable, nonnegative, and such that, for each $\epsilon > 0$, $\int_0^{\epsilon} \psi(\omega)d\omega > 0$. Then $\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T}, \mathcal{K}, \mathcal{L}$ and $\mathcal{W}$ have a unique common fixed point in $\mathcal{W}$.

Proof. The theorem's proof proceeds in a manner similar to that of Theorem 3.1. □

5 Conclusion

For eight self-mappings in fuzzy metric space that contain third and fourth power of the distance measure $\mathcal{M}(x,y,t)$, we demonstrate the common fixed-point theorem.

Authors Contributions.
Each author contributed in the writing of this work. All authors read and approved the final draft.
Acknowledgement. The authors are very much thankful to the referee for his valuable suggestions to bring the paper in its present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 308-314**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

MATHEMATICAL STUDY OF BLOOD CIRCULATION AND BIO-CHEMICAL REACTION BASED HEAT DISTRIBUTION PROBLEM IN HUMAN DERMAL REGION

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(Received: January 07, 2023*; In format: March* 30, 2023*; Revised: June* 10, 2023*; Accepted: June* 12, 2023*)*

DOI: https://doi.org/10.58250/jnanabha.2023.53137

Abstract

In this paper we study heat distribution in outer parts of human body incorporating effect of blood circulation and metabolic activities. We solve the bio heat equation of skin for steady state case using Whittaker function and Conuent Hypergeometric function. Some parameters are taken variable. A general model has been modified and solved mathematically for comparative study of heat flow in human skin. The structure of human skin is taken as heterogeneous medium and attempt has been made to solve it by analytic methods. Numerical computation has been carried out for various values of parameters.

2020 Mathematical Sciences Classification: 33C15,33C20, 92BXX
Keywords and Phrases: Physiological Heat Flow; Differential Equation; Whittaker function; Confluent Hypergeometric function.

1 Introduction

The heat of the body is produced by a slow combustion of food; and this is taking place all the time. This combustion goes on chiefly in the muscles and is much more active during exercise than when the body is at rest. Yet the internal temperature of the body during rest and moderate exercise is the same, although much more heat is produced during exercise. The loss of heat from the body takes place chiefly at the surface, through the skin. A great deal more heat is lost from the body when the surrounding air is cold, yet the body temperature remains the same. The process of heat-making which is carried on in the muscles is regulated by certain nerve centers in the brain and spinal cord, which are connected with the muscles by nerves, so that the making of heat is under constant and perfect control. When the body is exposed to cold air or water or is in any way cooled so that the temperature of the blood is lowered, nerve centers in the brain incite increased activity in the heat-making organs and more fuel is burned in the cells. In this way the heat-making process is adjusted to the needs of the body. The body temperature in health that any variation from the normal, 98.5 + degrees, gives cause for anxiety. As a result of some shock or in one who is very feeble, the temperature may fall below normal, through insufficient heat production or too great an escape of heat. More often there is a rise of temperature above the normal, and then one is said to have a fever. In fevers, heat production and loss are not so perfectly controlled as in health, because the heat centers are disturbed by the undesired substances circulating in the blood. The sweat glands are not so active as usual, and the surplus heat does not escape.

Blood circulation plays vital role in regulating the heat in a health and the flow is regulated by heart apparatus. However, certain subjects have abnormality due to age (above 40 yrs.) or under nervous stress, in such case Yoga can be useful to retain the normal rate [9, 19]. Heat regulation problems in a human body can be expressed in terms of differential equations. In this paper we generate such equations for outer body which incorporate the influence of circulation of blood and nutrient indeed bio-chemical reactions in the cells (cell metabolism). There are number of techniques developed for the solutions and listed in standard texts (Murphy [14]).However, in biological processes like physiological heat transfer these techniques have several limitations due to large number of soft parameters and the associated flexibility. Following the advent heat equation in physiological transport by Perl [16, 17]and Pennes [15]. Trezek and Cooper [4, 5] developed solutions of the boundary value problems pertaining to in-vivo tissue medium for heat flow in human dermal regions. Saxena [23] gave solution for steady state case in terms of special cases of Bessel functions.

In one dimensional boundary value problems, the differential equation can easily be transformed into an ordinary differential equations by applying a suitable transform. The required solutions can be obtained by solving this equations and inverting by any method. Other mathematical techniques can also be used for boundary value problems. In this paper Laplace transform has also been used and further solutions have been worked out in terms of special functions like Whittaker function and Confluent Hypergeometric functions.

2 The Method and the Mathematical Model

As we know that for differential equations, we can use some analytic methods and solve them with the help of Special functions. Some of the cases differential equations can reduced in Bessel equations, Whittakers equations or Kummers equations and result is expressed in terms of one of the Bessel functions, Whittaker function or in terms of Confluent Hypergeometric functions.

In this paper we use solution of Heat equation for steady state case with the help of Whittaker function which is reduced in Confluent Hypergeometric function. We should briefly discuss about Whittakers equation and Kummers equations.

We know that the Whittaker functions are solutions of the differential equation [24].

$$\frac{d^2w}{dz^2} - \left(-\frac{1}{4} + \frac{k}{z} + \frac{1/4 - \mu^2}{z^2}\right) w = 0. \tag{2.1}$$

It has a regular singular point at $z = 0$ and an irregular singular point at $z = \infty$. Two solution are given by the Whittaker functions $M_{k,z}(z)$ and $W_{k,z}(z)$ defined in terms of Kummer's Confluent Hypergeometric functions M and U by

$$M_{k,z}(z) = e^{-z/2} z^{\left(\mu+\frac{1}{2}\right)} M\left(\mu - k + \frac{1}{2}, 1 + 2\mu; z\right), \tag{2.2}$$

$$W_{k,z}(z) = e^{-z/2} z^{\left(\mu+\frac{1}{2}\right)} U\left(\mu - k + \frac{1}{2}, 1 + 2\mu; z\right), \tag{2.3}$$

The Kummer's equation may be written as

$$z\frac{d^2w}{dz^2} - (b - z)\frac{dw}{dz} - aw = 0, \tag{2.4}$$

with a regular singular point at $z = 0$ and irregular singular point at $z = \infty$. It has two linearly independent solutions $M(a, b; z)$ and $U(a, b; z)$. Kummers function (of first kind) M is a generalized hypergeometric series introduced is given by (Kummer[10]):

$$M(z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!} = {}_1F_1(a; b; z), \tag{2.5}$$

where $a^{(0)} = 1$, and $a^{(n)} = a(a + 1)(a + 2)...(a + n - 1)$ is the rising factorial. This function ${}_1F_1(a; b; z)$ is known as Confluent Hypergeometric function.

Now we are using these functions in our problem discussed below.

In epidermis and dermis regions of human body, the temperature distribution depends on various physical and biological quantities. These quantities are related to the local tissue temperature T through the following Bio-Heat equation for in vivo tissue is given by Perl [16].

$$\rho c\frac{\partial T}{\partial t} = div\left(KgradT\right) + m_b c_b \left(T_b - T\right) + S, \tag{2.6}$$

The one dimensional equation for constant thermal conductivity is written as

$$\rho c\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2} + m_b c_b \left(T_b - T\right) + S, \tag{2.7}$$

where $\rho, c, K, t, m_b, c_b, T_b$ and S are respectively tissue density, heat capacity, thermal conductivity, time, blood mass flow rate, heat capacity of blood and blood temperature and rate of metabolic heat generation at a point. Here, a one dimensional form of variation of temperature is taken in x-direction, perpendicular to the outer skin surface. The equation (2.7) is solved separately for epidermis and dermis under following conditions:

(i) At outer skin $(x = 0)$:

$$K_1\frac{\partial T}{\partial x} = h\left(T - T_a\right) + LE, \tag{2.8}$$

where
K_1 = value of K (Thermal conductivity) in epidermis,
h = heat transfer coefficient of convection and radiation,
L = latent heat of of tissue,
E = rate of sweat evaporation,
T_a = atmospheric temperature.

(ii) For epidermis $(0 < x < a)$:
Due to lack of blood flow and no metabolic activity in epidermis we take,

$$m_b c_b = S = 0. \tag{2.9}$$

(iii) Interface $(x = a)$:

$$K_1 \frac{\partial T}{\partial x} = K_2 \frac{\partial T}{\partial x}, \tag{2.10}$$

K_2 = value of K (thermal conductivity)in dermis.

(iv) For dermis $(a < x < b)$:

$$m_b c_b = \bar{M}\bar{X}, \qquad S = s(T_b - T),$$

where $\bar{X} = \left[\frac{(x-a)}{(b-a)}\right]^2$, $\bar{M}$ and s are values of $m_b c_b$ and S in subdermal region.

(v) At Subdermal boundary $(x = b)$:
$T = T_b$ where T_b is blood temperature which is almost same as body core temperature.

We simplified and solve equations (2.7) with conditions mentioned in $(i), (ii), (iii), (iv)$ and (v) and solve with the help of Laplace transform. The solutions for the both regions are obtained in the following form:

For Epidermis:

$$\bar{T} = A_1 \exp(y\sqrt{p}) + A_2 \exp(-y\sqrt{p}), \tag{2.11}$$

For Dermis:

$$\bar{T} = z^{-1/2}\left[A_3 M_{-p/4,1/4}(z^2) + A_4 M_{-p/4,-1/4}(z^2)\right], \tag{2.12}$$

where
$\bar{T}$ is Laplace transform of $(T_b - T)/T_b$, p is parameter of the transform,
$M_{k,m}$ denotes Whittaker's function of first kind. Also A_1, A_2, A_3 and A_4 are determined with the help of the above conditions. Thus the same are obtained as:

$$\begin{aligned}
A_1 &= \frac{n_1 l_3}{D}, & A_2 &= \frac{n_2 l_3}{D},\\
A_3 &= \frac{n_3}{D}, & A_4 &= \frac{n_3 l_6}{D},\\
D &= n_1 l_1 - n_2 l_2, & \bar{a} &= \frac{a}{(b-a)},\\
l_1 &= \sqrt{p} - h, & l_2 &= l_1 + 2h,\\
l_3 &= \frac{-(hT_a + \alpha)}{p}, & l_4 &= \frac{K_2}{K_1\sqrt{p}},\\
l_5 &= \frac{l_4(1+p)}{2}, & l_6 &= -\frac{1}{\beta}\frac{{}_1F_1(a_2, C_2, ; \beta^2)}{{}_1F_1(a_1, C_1; \beta^2)},\\
n_1 &= (1 + m_1), & n_2 &= (1 + m_1)e^{2\bar{a}\sqrt{p}},\\
n_3 &= 2l_3 e^{(-\bar{a}\sqrt{p})}, & m_1 &= l_4 l_6 + l_5,
\end{aligned}$$

where

$$\begin{aligned}
y &= \frac{x}{(b-a)}, & z &= (y - \bar{a}),\\
\alpha &= \frac{LE(b-a)}{K_1 T_b}, & \beta &= (m_b c_b + s)^{1/4}.
\end{aligned}$$

$_1F_1$ denotes confluent hypergeometric funtion.
For steady state problem the solution (2.11) and (2.12) take the following form:
For Epidermis:

$$T = T_b(\mathbb{A}_1 - \mathbb{A}_2 y). \tag{2.13}$$

For Dermis:

$$T = T_b\left[1 - e^{-\frac{z^2}{2}}\left\{\mathbb{A}_3 z \;\, _1F_1(a_1, c_1; z^2) + \mathbb{A}_4 z \;\, _1F_1(a_2, c_2; z^2)\right\}\right]. \tag{2.14}$$

Values of all the notations used in equations (2.13) and (2.14) are defined in Appendix-A.

3 Numerical Results

This model has been solved with some numerical assumptions. Taking two layers the solution for T is obtained for the following values of physical and physiological constants have been taken as prescribed by Cooper and Trezek [4, 5] and Saxena[23].

$$\begin{aligned} L &= 579\,\text{cal/gm} \\ T_b &= 37^\circ\text{C} \quad \text{(The Core Temperature)} \\ \rho &= 1.05\,\text{gm/cm}^3 \\ c &= 0.83\,\text{cal/gm} \\ h &= 0.02\,\text{cal/cm}^2\text{-min}^\circ\text{C} \quad \text{(The Heat Transfer Coefficient).} \end{aligned}$$

For Epidermis

$$\begin{aligned} K_1 &= 0.040\,\text{cal/cm-min}^\circ\text{C} \\ \bar{M}_1 &= 0.000\,\text{cal/cm}^3\text{-min}^\circ\text{C} \\ s &= 0.00\,\text{cal/cm}^3\text{-min.} \end{aligned}$$

For Dermis

$$\begin{aligned} K_2 &= 0.060\,\text{cal/cm-min}^\circ\text{C} \\ \bar{M}_2 &= 0.030\,\text{cal/cm}^3\text{-min}^\circ\text{C} \\ s &= 0.0357\,\text{cal/cm}^3\text{-min.} \end{aligned}$$

The numerical calculations have been made for the following four cases of atmospheric temperature T_a together with the respective values of rate of evaporation E.

***Table* 3.1:** Cases for distinguish temperatures and evaporation rates.

S.No.	Temperature (T_a) (°C)	Evaporation (E) (gm/cm^2-min)
(i)	15	0.00
(ii)	20	0.45 $\times 10^{-3}$
(iii)	25	0.79 $\times 10^{-3}$
(iv)	30	0.9 $\times 10^{-3}$

We can assign different values of the constants a and b depending on the sample of thickness of the skin under study for different different parts of body and persons. The set of values of a and b we considered here are as follows:

Set-I	$a = 0.2$ cm	$b = 0.7$ cm
Set-II	$a = 0.4$ cm	$b = 1.2$ cm
Set-III	$a = 0.6$ cm	$b = 1.3$ cm

The graphs have been plotted between temperature T and position x for different sets of the values of T_a and E.

These assumptions are generally based on the values taken by several researchers including Trezek and Cooper[4, 5] who have not only carried out numerical computation of the models in simplified form but also conducted laboratory investigations on certain mammals in in-vitro stage. Experiments have also been conducted by Hodgson [8] extensively on human being (in-vivo) sitting in a sophisticated climatic chamber designed by himself. He measured several parameters including sweat evaporation under different conditions. Some values are also available in classical monographs like Ruch and Patton [18] and have been used widely by Saxena and his subsequent workers ([1, 2, 3, 6, 7, 11, 12, 13, 20, 21, 22, 23]). Based on the above our numerical calculations are exhibited in the graphs.

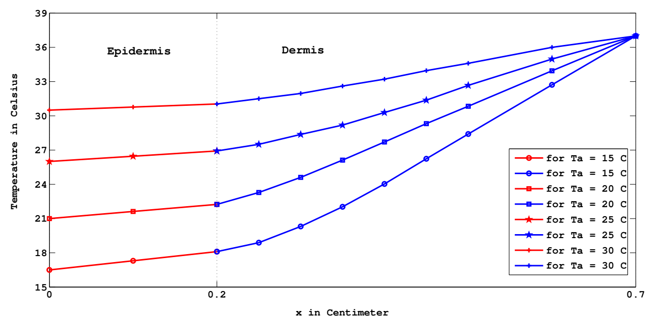

***Figure* 3.1:** Graph between in depth x and temperature T for Set-I

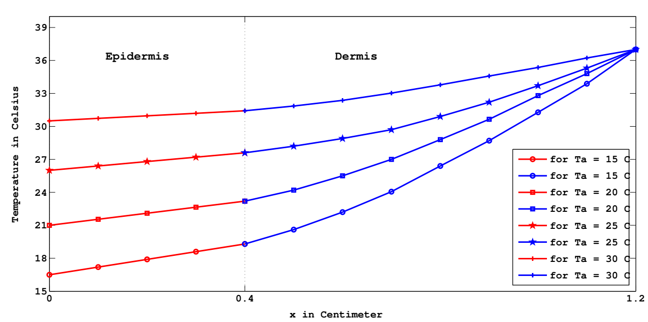

***Figure* 3.2:** Graph between in depth x and temperature T for Set-II

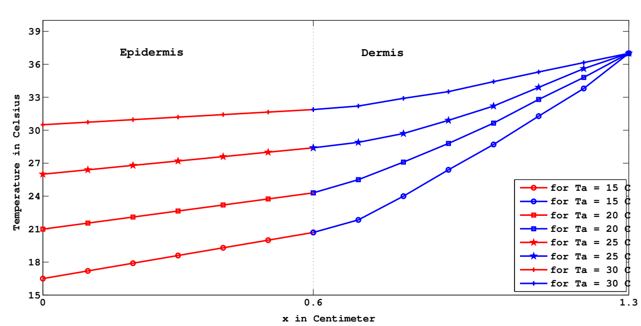

***Figure* 3.3:** Graph between in depth x and temperature T for Set-III

4 Discussion

Temperature profiles in epidermis and dermis for different environment conditions and anatomy. The curvilinear variation is clearly visible in dermis due to additional terms of blood flow and biochemical reactions.

These graphs are only illustrations for certain sample cases and can be extended for subjects placed under different atmospheric conditions.

As indicated above three sets of skin layers with different thicknesses have been considered for computation and for different values of atmospheric temperature. The patterns of graph reflect both these assumptions. In figure-3.1 and figure-3.2, the rise of temperature is more in comparison to figure-3.3 which has rapid rise in dermis in comparison to the earlier two cases.

The mathematical solution derived in this paper provide sufficient freedom for the assumption and occurrence of biophysical parameters namely, thermal conductivity, metabolic cell reactions and micro circulation of blood. This aspect is vital for the theoretical study as in-vivo situation demands it. The temperature profile thus obtained can further be use for more advanced studies pertaining to extreme climates and thermoregulation related diseases like malignant tumor.

This study is confined to human subjects at rest. Same can be extended to persons undergoing some physical activity or exercise like Yoga as indicated earlier [9, 19, 21]. In such cases the assumptions regarding blood circulation rate $\bar{M}$ and metabolic rate S have to be connected with practical results. Accordingly these two parameters have to be flexible, either in steps or continuously time dependent. This will open an opportunity for new investigations.

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Appendix-A

$$
\begin{aligned}
a_1 &= \frac{3}{4}, \\
a_2 &= \frac{1}{4}, \\
c_1 &= \frac{3}{2}, \\
c_2 &= \frac{1}{2}, \\
\mathbb{A}_1 &= \frac{l_1(1-m_1\bar{a})}{R}, \\
\mathbb{A}_2 &= \frac{m_1 l_1}{R}, \\
\mathbb{A}_3 &= \frac{l_1 l_3}{R}, \\
\mathbb{A}_4 &= \frac{l_1}{R}, \\
D &= n_1 l_1 - n_2 l_2, \\
\bar{a} &= \frac{a}{(b-a)}, \\
l_1 &= hT_a + \alpha, \\
l_2 &= \frac{K_2}{2K_1}, \\
R &= h - m_1(h\bar{a}+1).
\end{aligned}
$$

ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)

Jñānābha

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1.	Place of Publication	D.V. Postgraduate College Orai-285001, U.P., India
2.	Periodicity of Publication	Bi-annual
3.	Printer's Name Nationality Address	Creative Laser Graphics (Iqbal Ahmad) Indian IIT Gate, Kanpur
4.	Publisher's Name	Dr. R.C. Singh Chandel For Vijñāna Parishad of India
	Nationality Address	Indian D.V. Postgraduate College Orai-285001, U.P. India
5.	Editor's Name Nationality Address	Dr. R.C. Singh Chandel Indian D.V. Postgraduate College Orai-285001, U.P. India
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