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RELATIONS AND IDENTITIES DUE TO DOUBLE SERIES ASSOCIATED WITH GENERAL HURWITZ-LERCH TYPE ZETA FUNCTIONS

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Abstract

In this paper, we introduce certain families of double series associated with general Hurwitz-Lerch type Zeta functions and then derive their summation formulae, series and integral identities. Again then using these identities, we obtain various known and unknown results and hypergeometric generating relations.

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1 Introduction and preliminaries

In the entire paper, in the standard notation it is provided that
$\mathbb{C}=\left\{z:z=x+iy:x,y\in\mathbb{R},i=\sqrt{(-1)}\right\},\mathbb{Z}_0^-=\{0,-1,-2,\ldots\},\ \mathbb{R}=(-\infty,\infty)$ and
$\mathbb{N}_0=\mathbb{N}\cup\{0\}=\{0,1,2,3,\ldots\}$.

The generalized Gaussian hypergeometric function has been studied and applied in computation of various problems occurring in different fields of science and technology (for example [5], [10], [11] and others) as defined by (see [13, pp. 73-74], [18, pp. 42-43])

$${}_pF_q\left(\begin{matrix}(\alpha)_{1,p};\\(\gamma)_{1,q};\end{matrix}z\right)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\gamma_i)_n}\frac{z^n}{n!}, \tag{1.1}$$

where $p,q\in\mathbb{N}_0,\alpha_i\in\mathbb{C},(i=1,2,3,\ldots,p);\gamma_i\in\mathbb{C}\backslash\mathbb{Z}_0^-,(i=1,2,3,\ldots,q);\ z\in\mathbb{C}$.

The series in (1.1) (i) converges for $|z|<\infty$, if $p\le q$; (ii) converges for $|z|<1$, if $p=q+1$; (iii) diverges for all $z,\ z\neq 0$, if $p>q+1$; (iv) converges absolutely for $|z|=1$, if $p=q+1$, and $\Re(\omega)>0,\omega=\sum_{i=1}^{q}\gamma_i-\sum_{i=1}^{p}\alpha_i$; (v) converges conditionally for $|z|=1,z\neq1$, if $p=q+1$, and $-1<\Re(\omega)\le 0$; (vi) diverges for $|z|=1$, if $p=q+1$, and $\Re(\omega)<-1$.

In reference of (1.1), when $p=2,\ q=1$, following extended Hurwitz-Lerch type hypergeometric Zeta function is studied in [2], written by

$$\phi_{\alpha,\beta;\gamma}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n\,n!}\frac{z^n}{(n+a)^s}=\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-at}t^{s-1}{}_2F_1\left(\begin{matrix}\alpha,\beta;\\\gamma;\end{matrix}ze^{-t}\right)dt,$$
$$\forall\ a,\ \alpha,\beta,s,z\in\mathbb{C},\ \Re(a)>0,\Re(s)>0 \text{ and } \gamma\in\mathbb{C}\backslash\mathbb{Z}_0^-. \tag{1.2}$$

It is remarked that the series of the extended Hurwitz-Lerch type hypergeometric Zeta function (1.2) converges if we have $\Re(s)>0$, when $|z|<1,\ (z\neq1)$.

But when $z=1$, we apply the techniques of Gaussian gamma function ([6], [7], [12]) and then Watson's theorem (see [14, p.54, Eqn. (2.3.3.13)], [18, p. 95, Problem 26]) and it is provided $\Re(\gamma)>\frac{1}{2}\Re(\alpha+\beta+1)>0$, then the series given in (1.2) converges if

$$\Re(s)>\frac{1}{2}\Re(\alpha+\beta)-\frac{1}{2}. \tag{1.3}$$

Also in Eqns. (1.1)-(1.2), for $a \neq 0$ the Pochhammer symbol [18, p. 22] is used and defined as factorial function given by

$$(a)_n = \begin{cases} a(a+1)(a+2)\dots(a+n-1); n \geq 1, \\ 1; n = 0, \end{cases}$$

and is related with the gamma function as

$$(a)_\nu = \frac{\Gamma(a+\nu)}{\Gamma(a)}, \forall \nu \in \mathbb{R}. \tag{1.4}$$

Clearly, a relation of (1.2) with the Hurwitz-Lerch Zeta function (see in [17]) is given as

$$\phi_{\alpha,1;\alpha}(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} = \phi(z,s,a), \tag{1.5}$$

converges for all $s, z \in \mathbb{C}$, and $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\Re(s) > 0$, when $|z| < 1$, $(z \neq 1)$, and when $z = 1$, the series in (1.2) is convergent for $\Re(s) > 1$.

Further by (1.5) at $z = 1$, we have a relation with shifted Hurwitz Zeta function ([3], see in also [8])

$$\phi_{\alpha,1,\alpha}(1,s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \zeta(s;a), \text{ where, } a \in \mathbb{C}\backslash\mathbb{Z}_0^- \text{ and } \Re(s) > 1. \tag{1.6}$$

Generalized Kobayashi-Stieltjes type operators [9] seem identical to extended Hurwitz-Lerch type Zeta functions, Srivastava-Daoust Double series used in initial value problems [4], Hurwitz-Lerch Zeta functions associated with double series of the Appell, Kampé de Fériet and Srivastava-Daoust functions studied in ([17], [12] and others). Srivastava [16] obtained various generating relations associated with Hurwitz-Lerch Zeta functions. In this motivation, here in our researches for exploring new ideas in the theory of Hurwitz-Lerch Zeta functions and for obtaining of generating relations, series and integral identities, we consider the parameters $x, y, s, d \in \mathbb{C}, a \in \mathbb{C}\backslash\mathbb{Z}_0^-, |x^2| < 1$ and A_n be bounded real or complex sequences $\forall n \in \mathbb{N}_0$ and $A_0 \neq 0$. Then we present following the families of double series associated with general Hurwitz-Lerch type Zeta functions defined as

$$R_1\left(A, \frac{d}{2}+\frac{1}{2}, \frac{d}{2}+1; \frac{3}{2}; x, y; s, a\right) = \sum_{m,n=0}^{\infty} \frac{A_n \left(\frac{d}{2}+\frac{1}{2}\right)_{m+n} \left(\frac{d}{2}+1\right)_{m+n}}{\left(\frac{3}{2}\right)_m} \frac{x^{2m+2n} y^n}{(n+a)^s\, m!n!}, \tag{1.7}$$

$$R_2\left(A, \frac{d}{2}, \frac{d}{2}+\frac{1}{2}; \frac{1}{2}; x, y; s, a\right) = \sum_{m,n=0}^{\infty} \frac{A_n \left(\frac{d}{2}\right)_{m+n} \left(\frac{d}{2}+\frac{1}{2}\right)_{m+n}}{\left(\frac{1}{2}\right)_m} \frac{x^{2m+2n} y^n}{(n+a)^s\, m!n!}. \tag{1.8}$$

For these double series (1.7) and (1.8), we evaluate their summation formulae and derive various interesting series and integral identities. Further applying these identities, we obtain various known and unknown results involving the Hurwitz-Lerch type Zeta functions and hypergeometric generating relations.

2 Summation Formulae

In this section, we obtain summation formulae of the families of double series associated with general Hurwitz-Lerch type Zeta functions $\forall s \in \mathbb{C}$ and $\Re(s) > 1$, defined in the Eqns. (1.7) and (1.8) in form of the generalized Dirichlet type L-functions below in Eqn. (2.2) studied in [8].

For a bounded sequence A_n, an extended Dirichlet type L-function [8] is defined by

$$L(s, A; z) = \sum_{n=1}^{\infty} \frac{A_n z^n}{n^s}, \forall s \in \mathbb{C}, |z| < 1 (z \neq 1) \tag{2.1}$$

and

$$L(s, A) = \sum_{n=1}^{\infty} \frac{A_n}{n^s} \forall s \in \mathbb{C} \text{ and } \Re(s) > 1. \tag{2.2}$$

Further we extend (2.1) and (2.2) $\forall s, z \in \mathbb{C}$, and $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$, A_n be bounded sequence, in general Hurwitz-Lerch type Zeta functions as

$$\phi(s, A, a; z) = \sum_{n=0}^{\infty} \frac{A_n z^n}{(n+a)^s}, \quad |z| < 1 (z \neq 1), a \in \mathbb{C}\backslash\mathbb{Z}_0^- \ \forall s \in \mathbb{C}, \tag{2.3}$$

and

$$\phi(s, A, a; 1) = \sum_{n=0}^{\infty} \frac{A_n}{(n+a)^s}\ , \quad a \in \mathbb{C}\backslash\mathbb{Z}_0^- \ \forall s \in \mathbb{C} \text{ and } \Re(s) > 1. \tag{2.4}$$

Lemma 2.1. *Let for all $s, z \in \mathbb{C}, a \in \mathbb{C}\backslash Z_0^-$ and A_n be bounded sequence, then summation formulas (2.3) and (2.4) for a general Hurwitz-Lerch type Zeta function exist in the form*

$$\phi(s, A, a; z) = \frac{A_0}{a^s} + \sum_{r=0}^{\infty} \binom{-s}{r} L(s+r, A; z) a^r, \tag{2.5}$$

where $|z| < 1 (z \neq 1), a \in \mathbb{C}\backslash\mathbb{Z}_0^-, \forall s \in \mathbb{C}$,
and

$$\phi(s, A, a; 1) = \frac{A_0}{a^s} + \sum_{r=0}^{\infty} \binom{-s}{r} L(s+r, A) a^r, \tag{2.6}$$

where $a \in \mathbb{C}\backslash\mathbb{Z}_0^-, \ s \in C$ and $\Re(s) > 1$.

Proof. Under the conditions of Lemma 2.1, we write (2.3) as

$$\phi(s, A, a; z) = \frac{A_0}{a^s} + \sum_{n=1}^{\infty} \frac{A_n z^n}{n^s} \left(1 + \frac{a}{n}\right)^{-s}. \tag{2.7}$$

Now applying binomial theorem and (2.1) we obtain (2.6).
Similarly making an appeal to (2.2) and (2.4) we get (2.6).
Hence Lemma 2.1 is proved. □

It is remarked that the formula (2.6) is identical to the summation formula due to Murthy and Sinha [8], when $z = 1$.

Lemma 2.2. *Under the conditions $\alpha, \beta, s, z \in \mathbb{C}, \ a, \ \gamma \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $|z| < 1 (z \neq 1)$, the function* (1.2) *follows following summation formula*

$$\phi_{\alpha,\beta;\gamma}(z, s, a) = \frac{1}{a^s} + \left(\frac{\alpha\beta}{\gamma} z\right) \sum_{r=0}^{\infty} \binom{-s}{r} \phi_{\alpha+1,\beta+1;\gamma+1}(z, s+r+1, 1) a^r, \tag{2.8}$$

and for $\alpha, \beta, s, z \in \mathbb{C}, \ a, \ \gamma \in \mathbb{C}\backslash\mathbb{Z}_0^-, \ z = 1$, there exists the formula

$$\phi_{\alpha,\beta;\gamma}(1, s, a) = \frac{1}{a^s} + \left(\frac{\alpha\beta}{\gamma}\right) \sum_{r=0}^{\infty} \binom{-s}{r} \phi_{\alpha+1,\beta+1;\gamma+1}(1, s+r+1, 1) a^r, \tag{2.9}$$

provided that

$$\Re(\gamma) > \frac{1}{2}\Re(\alpha + \beta + 1) > 0.$$

Then the inner function in right hand side of (2.9) *converges for*

$$\Re(s) + r > \frac{1}{2}\Re(\alpha + \beta) - \frac{1}{2}, r = 0, 1, 2, \ldots . \tag{2.10}$$

Proof. In Eqn. (2.3) setting $A_n = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!}$ and making an appeal to the formulae (1.2) and (2.5), we get the summation formula for extended Hurwitz-Lerch type hypergeometric Zeta function (1.2) as

$$\begin{aligned}
\phi_{\alpha,\beta;\gamma}(z, s, a) &= \frac{1}{a^s} + \sum_{r=0}^{\infty} \binom{-s}{r} \left\{ \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} \frac{z^n}{n^{s+r}} \right\} a^r \\
&= \frac{1}{a^s} + \left(\frac{\alpha\beta}{\gamma} z\right) \sum_{r=0}^{\infty} \binom{-s}{r} \left\{ \sum_{n=0}^{\infty} \frac{(\alpha+1)_n (\beta+1)_n}{(\gamma+1)_n n!} \frac{z^n}{(n+1)^{s+r+1}} \right\} a^r \\
&= \frac{1}{a^s} + \left(\frac{\alpha\beta}{\gamma} z\right) \sum_{r=0}^{\infty} \binom{-s}{r} \phi_{\alpha+1,\beta+1;\gamma+1}(z, s+r+1, 1) a^r.
\end{aligned} \tag{2.11}$$

For $z = 1$, by the second equality of the Eqn. (2.11) under the restrictions

$$\Re(\gamma) > \frac{1}{2}\Re(\alpha + \beta + 1) > 0,$$

and also for large values of N, we write

$$\phi_{\alpha,\beta;\gamma}(1,s,a)=\frac{1}{a^s}+\left(\frac{\alpha\beta}{\gamma}\right)\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)\sum_{n=0}^{N-1}\frac{(\alpha+1)_n(\beta+1)_n}{(\gamma+1)_n n!}\frac{1}{(n+1)^{s+r+1}}a^r$$
$$+\left(\frac{\alpha\beta}{\gamma}\right)\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)\sum_{n=N}^{\infty}\frac{(\alpha+1)_n(\beta+1)_n}{(\gamma+1)_n n!}\frac{a^r}{(n+1)^{s+r+1}}$$
$$\Rightarrow\phi_{\alpha,\beta;\gamma}(1,s,a)=\frac{1}{a^s}+\left(\frac{\alpha\beta}{\gamma}\right)\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)\sum_{n=0}^{N-1}\frac{(\alpha+1)_n(\beta+1)_n}{(\gamma+1)_n n!}\frac{1}{(n+1)^{s+r+1}}a^r$$
$$+\left(\frac{\alpha\beta}{\gamma}\right)\frac{(\alpha+1)_N(\beta+1)_N}{(\gamma+1)_N\Gamma(N+s+r+2)}\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)\sum_{n=0}^{\infty}\frac{(\alpha+N+1)_n(\beta+N+1)_n(1)_n}{(\gamma+N+1)_n(N+s+r+2)_n n!}a^r.$$

Again if we suppose that $\alpha_1,\beta_1,\gamma_1,a_1,s_1$ are the real parts of α,β,γ,a,s respectively and $\gamma_1>\frac{1}{2}(\alpha_1+\beta_1+1)$, we get an inequality

$$|\phi_{\alpha_1,\beta_1;\gamma_1}(1,s_1,a_1)|<\frac{1}{(a_1)^{s_1}}$$
$$+\left(\frac{\alpha_1\beta_1}{\gamma_1}\right)\sum_{r=0}^{\infty}\left(\begin{array}{c}-s_1\\ r\end{array}\right)\sum_{n=0}^{N-1}\frac{(\alpha_1+1)_n(\beta_1+1)_n}{(\gamma_1+1)_n n!}\frac{1}{(n+1)^{s_1+r+1}}(a_1)^r$$
$$+\left(\frac{\alpha_1\beta_1}{\gamma_1}\right)\frac{(\alpha_1+1)_N(\beta_1+1)_N}{(\gamma_1+1)_N\Gamma(N+s_1+r+2)}\sum_{r=0}^{\infty}\left(\begin{array}{c}-s_1\\ r\end{array}\right)$$
$$\times{}_3F_2\left[\begin{array}{c}\alpha_1+N+1,\beta_1+N+1,1+\frac{N}{2}+\frac{S_1}{2}+\frac{r}{2};\\ \frac{1}{2}(\alpha_1+\beta_1+2N+3),2+N+s_1+r;\end{array}1\right](a_1)^r. \quad (2.12)$$

Now applying the Watson's theorem (see [14, p.54, Eqn. (2.3.3.13)], [18, p. 95, Problem 26]) in the function ${}_3F_2[\cdot]$ of right hand side of (2.12), we find the convergence conditions as

$$\Re(s)+r>\frac{1}{2}\Re(\alpha+\beta)-\frac{1}{2}\ \forall r=0,1,2,\dots\ .$$

Hence the Lemma 2.2 is proved. □

Making an appeal to theory of the Lemmas 2.1 and 2.2, we present following theorems:

Theorem 2.1. *For all $x,y,s,\ d\in\mathbb{C},\Re(s)>1,\ a\in\mathbb{C}\backslash\mathbb{Z}_0^-,\ |x^2|<1$ and A stands for a sequence A_n be bounded real or complex sequences $\forall n\in\mathbb{N}_0$ and $A_0\neq 0$, then by the double series associated with general Hurwitz-Lerch Zeta function* (1.7)*, following summation formula exists*

$$R_1\left(A,\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\frac{3}{2};x,y;s,a\right)=\frac{A_0}{(a)^s}\,{}_2F_1\left(\begin{array}{c}\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\\ \frac{3}{2};\end{array}x^2\right)$$
$$+x^2y\left(\frac{d^2}{4}+\frac{3d}{4}+\frac{1}{2}\right)\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)R_1\left(A^+,\frac{d}{2}+\frac{3}{2},\frac{d}{2}+2;\frac{3}{2};x,y;s+r+1,1\right)a^r \quad (2.13)$$

where, A^+stands for the sequence A_{n+1} $\forall n\in\mathbb{N}_0$.

Proof. We write the formula (1.7) as

$$R_1\left(A,\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\frac{3}{2};x,y;s,a\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_{m+n}\left(\frac{d}{2}+1\right)_{m+n}}{\left(\frac{3}{2}\right)_m}\frac{x^{2m}(yx^2)^n}{(n+a)^s m!n!}.$$

Then for $a\in\mathbb{C}\backslash\mathbb{Z}_0^-$, we derive

$$R_1\left(A,\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\frac{3}{2};x,y;s,a\right)$$
$$=\frac{A_0}{(a)^s}\sum_{m=0}^{\infty}\frac{\left(\frac{d}{2}+\frac{1}{2}\right)_m\left(\frac{d}{2}+1\right)_m}{\left(\frac{3}{2}\right)_m}\frac{x^{2m}}{m!}+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_{m+n}\left(\frac{d}{2}+1\right)_{m+n}}{\left(\frac{3}{2}\right)_m}\frac{x^{2m}(yx^2)^n}{(n+a)^s m!n!}$$
$$=\frac{A_0}{(a)^s}\sum_{m=0}^{\infty}\frac{\left(\frac{d}{2}+\frac{1}{2}\right)_m\left(\frac{d}{2}+1\right)_m}{\left(\frac{3}{2}\right)_m}\frac{(x^2)^m}{m!}$$

$$+x^2y\left(\frac{d^2}{4}+\frac{3d}{4}+\frac{1}{2}\right)\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)a^r\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{A_{n+1}\left(\frac{d}{2}+\frac{3}{2}\right)_{m+n}\left(\frac{d}{2}+2\right)_{m+n}}{\left(\frac{3}{2}\right)_m(n+1)^{s+r+1}}\frac{x^{2m}(yx^2)^n}{m!n!}. \quad (2.14)$$

□

Theorem 2.2. *For all $x,y,s,d\in\mathbb{C},\Re(s)>1,\ a\in\mathbb{C}\backslash\mathbb{Z}_0^-,\ |x^2|<1$ and A stands for the sequence A_n be bounded real or complex sequences $\forall n\in\mathbb{N}_0$ and $A_0\neq 0$, then by the double series associated with general Hurwitz-Lerch Zeta function* (1.8), *following summation formula exists*

$$R_2\left(A,\frac{d}{2},\frac{d}{2}+\frac{1}{2};\frac{1}{2};x,y;s,a\right)=\frac{A_0}{a^s}\,{}_2F_1\left(\begin{array}{c}\frac{d}{2},\frac{d}{2}+\frac{1}{2};\\ \frac{1}{2};\end{array}x^2\right)$$
$$+\left(\frac{d^2}{4}+\frac{d}{4}\right)x^2y\sum_{r=0}^{\infty}\left(\begin{array}{c}-s\\ r\end{array}\right)R_2\left(A^+,\frac{d}{2}+1,\frac{d}{2}+\frac{3}{2};\frac{1}{2};x,y;s+r+1,1\right)a^r. \quad (2.15)$$

Proof. Considering the double series associated with general Hurwitz-Lerch Zeta function (1.8) and applying the same techniques as in the proof of the Theorem 2.1, we establish the required result (2.15). □

3 Series and integral identities

In this section, we derive series and integral identities associated with general Hurwitz-Lerch Zeta functions due to double series defined in the Eqns. (1.7) and (1.8).

Theorem 3.1. *If $|x^2|<1$, then the double series associated with general Hurwitz-Lerch Zeta function* (1.7) *generates following series identity*

$$\sum_{n=0}^{\infty}\frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!}\sum_{m=0}^{n}\frac{A_m(-n)_m\left(-\frac{1}{2}-n\right)_m}{m!}\frac{y^m}{(m+a)^s}$$
$$=\frac{1}{2xd(1-x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2y}{(1-x)^2}\right)^n}{n!(n+a)^s}$$
$$-\frac{1}{2xd(1+x)^d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2y}{(1+x)^2}\right)^n}{n!(n+a)^s}, \quad (3.1)$$

provided that all conditions of the Theorem 2.1 *are satisfied.*

Proof. Consider the double series associated with general Hurwitz-Lerch Zeta function (1.7) in the form

$$R_1\left(A,\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\frac{3}{2};x,y;s,a\right)$$
$$=\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}+1\right)_n x^{2n}y^n}{(n+a)^s n!}\sum_{m=0}^{\infty}\frac{\left(\frac{d}{2}+\frac{1}{2}+n\right)_m\left(\frac{d}{2}+1+n\right)_m}{\left(\frac{3}{2}\right)_m}\frac{x^{2m}}{m!}. \quad (3.2)$$

Now making an appeal to the result due to Sneddon [15, p. 42, Example II (1 (iii))] given by

$$\sum_{m=0}^{\infty}\frac{\left(\frac{d}{2}+\frac{1}{2}\right)_m\left(\frac{d}{2}+1\right)_m}{\left(\frac{3}{2}\right)_m}\frac{x^{2m}}{m!}=\frac{1}{2dx}(1-x)^{-d}-\frac{1}{2dx}(1+x)^{-d},x\neq\pm 1, \quad (3.3)$$

in right hand side of (3.2), we derive

$$R_1\left(A,\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\frac{3}{2};x,y;s,a\right)$$

$$=\frac{\Gamma\left(\frac{d}{2}\right)}{4x(1-x)^d\Gamma\left(\frac{d}{2}+1\right)}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2y}{(1-x)^2}\right)^n}{n!(n+a)^s}$$
$$-\frac{\Gamma\left(\frac{d}{2}\right)}{4x(1+x)^d\Gamma\left(\frac{d}{2}+1\right)}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2y}{(1+x)^2}\right)^n}{n!(n+a)^s}. \quad (3.4)$$

Further in the double series (1.7), making and appeal to the series rearrangement techniques [18, p. 100], we obtain

$$
\begin{aligned}
&R_1\left(A, \frac{d}{2}+\frac{1}{2}, \frac{d}{2}+1; \frac{3}{2}; x, y; s, a\right)\\
&= \sum_{m=0}^{\infty} \frac{\left(\frac{d}{2}+\frac{1}{2}\right)_m \left(\frac{d}{2}+1\right)_m x^{2m}}{m!} \sum_{n=0}^{m} \frac{A_n}{\left(\frac{3}{2}\right)_{m-n}} \frac{y^n}{(n+a)^s (m-n)! n!}\\
&= \sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!} \sum_{m=0}^{n} \frac{A_m (-n)_m \left(-\frac{1}{2}-n\right)_m}{m!} \frac{y^m}{(m+a)^s}.
\end{aligned} \tag{3.5}
$$

Finally, employing (3.4) and (3.5) we establish the identity (3.1). □

Srivastava [16] obtained various generating relations associated with some families of the extended Hurwitz-Lerch Zeta functions, then to make extension in this area we derive following generating relations for our defined families of the extended Hurwitz-Lerch Zeta functions (1.7) and (1.8), given by

Corollary 3.1. *In the Theorem* 3.1 *set* $A_n = \frac{\prod_{i=1}^{p} (\alpha_i)_n}{\prod_{i=1}^{q} (\gamma_i)_n}, \forall n = 0, 1, 2, 3, \ldots,$ *and define an extended semi-hypergeometric Hurwitz-Lerch Zeta function*

$$
{}_{p+2}H_q\left(\begin{matrix} (\alpha)_{1,p}, -n, -\frac{1}{2}-n; \\ (\gamma)_{1,q}; \end{matrix}\; y, s, a\right) = \sum_{m=0}^{n} \frac{\prod_{i=1}^{p} (\alpha_i)_n (-n)_m \left(-\frac{1}{2}-n\right)_m}{\prod_{i=1}^{q} (\gamma_i)_n m!} \frac{y^m}{(m+a)^s}, \tag{3.6}
$$

and then make an appeal to equality (3.1) *for* $|x^2| < 1$, *there exists a generating relation of extended generalized hypergeometric Hurwitz-Lerch Zeta function due to the formula* (1.7) *given by*

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!}\; {}_{p+2}H_q\left(\begin{matrix} (\alpha)_{1,p}, -n, -\frac{1}{2}\;\; n; \\ (\gamma)_{1,q}; \end{matrix}\; y, s, a\right)\\
&= \frac{1}{2xd\,(1-x)^d}\; {}_{p+2}H_q\left(\begin{matrix} (\alpha)_{1,p}, \frac{d}{2}+\frac{1}{2}, \;\frac{d}{2}; \\ (\gamma)_{1,q}; \end{matrix}\; \frac{x^2 y}{(1-x)^2}, s, a\right)\\
&\quad - \frac{1}{2xd\,(1+x)^d}\; {}_{p+2}H_q\left(\begin{matrix} (\alpha)_{1,p}, \frac{d}{2}+\frac{1}{2}, \;\frac{d}{2}; \\ (\gamma)_{1,q}; \end{matrix}\; \frac{x^2 y}{(1+x)^2}, s, a\right).
\end{aligned} \tag{3.7}
$$

Theorem 3.2. *The double series associated with general Hurwitz-Lerch Zeta function* (1.7), *generates the following integral identity*

$$
\begin{aligned}
&\int_0^{\infty} e^{-at} t^{s-1} \left\{\sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!} \sum_{m=0}^{n} \frac{A_m (-n)_m \left(-\frac{1}{2}-n\right)_m \left(ye^{-t}\right)^m}{m!}\right\} dt\\
&= \frac{1}{2xd} \int_0^{\infty} e^{-at} t^{s-1} \left\{\frac{1}{(1-x)^d} \sum_{n=0}^{\infty} \frac{A_n \left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{d}{2}\right)_n \left(\frac{x^2 y e^{-t}}{(1-x)^2}\right)^n}{n!}\right.\\
&\qquad \left. - \frac{1}{(1+x)^d} \sum_{n=0}^{\infty} \frac{A_n \left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{d}{2}\right)_n \left(\frac{x^2 y e^{-t}}{(1+x)^2}\right)^n}{n!}\right\} dt,
\end{aligned} \tag{3.8}
$$

where, $\Re(a) > 0, \Re(s) > 0, |x^2| < 1.$

Proof. Making an appeal to the equality (3.5) and to the Euler integral formula ([6], [7], [17]), we find an integral representation for $\Re(a) > 0, \Re(s) > 0$ as

$$
\begin{aligned}
&R_1\left(A, \frac{d}{2}+\frac{1}{2}, \frac{d}{2}+1; \frac{3}{2}; x, y; s, a\right)\\
&= \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-at} t^{s-1} \left\{\sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!} \sum_{m=0}^{n} \frac{A_m (-n)_m \left(-\frac{1}{2}-n\right)_m \left(ye^{-t}\right)^m}{m!}\right\} dt.
\end{aligned} \tag{3.9}
$$

Again starting with the equality (3.4) and applying the same techniques as in (3.9), we obtain the result

$$R_1\left(A, \frac{d}{2}+\frac{1}{2}, \frac{d}{2}+1; \frac{3}{2}; x, y; s, a\right) = \frac{1}{\Gamma(s)}\int_0^\infty e^{-at}t^{s-1}\left\{\frac{1}{2xd(1-x)^d}\sum_{n=0}^\infty \frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2ye^{-t}}{(1-x)^2}\right)^n}{n!} - \frac{1}{2xd(1+x)^d}\sum_{n=0}^\infty \frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2ye^{-t}}{(1+x)^2}\right)^n}{n!}\right\} dt. \quad (3.10)$$

The relations (3.9) and (3.10) immediately give the integral equality (3.8). □

Theorem 3.3. *The double series associated with general Hurwitz-Lerch Zeta function* (1.7) *generates the following general generating relation for* $|x^2| < 1$,

$$\sum_{n=0}^\infty \frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!}\sum_{m=0}^n \frac{A_m(-n)_m\left(-\frac{1}{2}-n\right)_m(ye^{-t})^m}{m!} = \frac{1}{2xd(1-x)^d}\sum_{n=0}^\infty \frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2ye^{-t}}{(1-x)^2}\right)^n}{n!} - \frac{1}{2xd(1+x)^d}\sum_{n=0}^\infty \frac{A_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}\right)_n\left(\frac{x^2ye^{-t}}{(1+x)^2}\right)^n}{n!}. \quad (3.11)$$

Proof. Making an appeal to the result (3.8) of the Theorem 3.2 we get an identity. This identity gives us the general generating relation (3.11). □

In the similar manner by the double series associated with general Hurwitz-Lerch Zeta function (1.8), we derive:

Theorem 3.4. *Double series associated with general Hurwitz-Lerch Zeta function* (1.8) *generates following series identity for* $|x^2| < 1$, *as*

$$\sum_{n=0}^\infty \frac{\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n(x^2)^n}{\left(\frac{1}{2}\right)_n n!}\sum_{m=0}^n \frac{A_m(-n)_m\left(\frac{1}{2}-n\right)_m y^m}{(m+a)^s m!} = \frac{1}{2(1-x)^d}\sum_{n=0}^\infty \frac{A_n\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{x^2y}{(1-x)^2}\right)^n}{(n+a)^s n!} + \frac{1}{2(1+x)^d}\sum_{n=0}^\infty \frac{A_n\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{x^2y}{(1+x)^2}\right)^n}{(n+a)^s n!}. \quad (3.12)$$

provided that all conditions of the Theorem 2.2 are satisfied.

Proof. Considering the formula (1.8) and making an appeal to the revised result due to Sneddon [15, p. 42, Example II (1 (ii))]

$$\sum_{m=0}^\infty \frac{\left(\frac{d}{2}\right)_m\left(\frac{d}{2}+\frac{1}{2}\right)_m}{\left(\frac{1}{2}\right)_m}\frac{(x^2)^m}{m!} = \frac{1}{2}\left\{(1-x)^{-d}+(1+x)^{-d}\right\},$$

we arrive at

$$R_2\left(A, \frac{d}{2}, \frac{d}{2}+\frac{1}{2}; \frac{1}{2}; x, y; s, a\right) = \frac{1}{2(1-x)^d}\sum_{n=0}^\infty \frac{A_n\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{x^2y}{(1-x)^2}\right)^n}{(n+a)^s n!} + \frac{1}{2(1+x)^d}\sum_{n=0}^\infty \frac{A_n\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{x^2y}{(1+x)^2}\right)^n}{(n+a)^s n!}. \quad (3.13)$$

provided that all conditions of the Theorem 2.2 are satisfied.

Further for the same conditions of (3.13). making an appeal to formula (1.8) and series rearrangement techniques, we obtain

$$R_2\left(A, \frac{d}{2}, \frac{d}{2}+\frac{1}{2}; \frac{1}{2}; x, y; s, a\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{A_n \left(\frac{d}{2}\right)_m \left(\frac{d}{2}+\frac{1}{2}\right)_m}{\left(\frac{1}{2}\right)_{m-n}} \frac{\left(x^2\right)^m y^n}{(n+a)^s (m-n)!n!}. \tag{3.14}$$

But for all n such that $0 \le n \le m$, we have

$$\left(\frac{1}{2}\right)_{m-n} = \frac{(-1)^n \left(\frac{1}{2}\right)_m}{\left(\frac{1}{2}-m\right)_n} \text{ and } \frac{1}{(m-n)!} = \frac{(-1)^n (-m)_n}{m!}.$$

Therefore,

$$R_2\left(A, \frac{d}{2}, \frac{d}{2}+\frac{1}{2}; \frac{1}{2}; x, y; s, a\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}\right)_n \left(\frac{d}{2}+\frac{1}{2}\right)_n \left(x^2\right)^n}{\left(\frac{1}{2}\right)_n n!} \sum_{m=0}^{n} \frac{A_m (-n)_m \left(\frac{1}{2}-n\right)_m y^m}{(m+a)^s m!}. \tag{3.15}$$

Finally, making an appeal to the results (3.13) and (3.15), we establish the formula (3.12). ☐

Corollary 3.2. *In the Theorem* 3.4 *setting* $A_n = \frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\gamma_i)_n}, \forall n = 0, 1, 2, 3, \ldots,$ *and defining an extended semi-hypergeometric Hurwitz-Lerch Zeta function*

$$_{p+2}H_q\left(\begin{array}{c}(\alpha)_{1,p}, -n, \frac{1}{2}-n; \\ (\gamma)_{1,q};\end{array} y, s, a\right) = \sum_{m=0}^{n} \frac{\prod_{i=1}^{p}(\alpha_i)_n (-n)_m \left(\frac{1}{2}-n\right)_m}{\prod_{i=1}^{q}(\gamma_i)_n m!} \frac{y^m}{(m+a)^s}, \tag{3.16}$$

and then making an appeal to equality (3.12), *we obtain the generating relation of extended generalized hypergeometric Hurwitz-Lerch Zeta function defined by* (1.8)

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}\right)_n \left(\frac{d}{2}+\frac{1}{2}\right)_n x^{2n}}{\left(\frac{1}{2}\right)_n n!}\ {}_{p+2}H_q\left(\begin{array}{c}(\alpha)_{1,p}, -n, \frac{1}{2}-n;) \\ (\gamma)_{1,q};\end{array} y, s, a\right) \\
&= \frac{1}{2(1-x)^d}\ {}_{p+2}H_q\left(\begin{array}{c}(\alpha)_{1,p}, \frac{d}{2}, \frac{d}{2}+\frac{1}{2}; \\ (\gamma)_{1,q};\end{array} \frac{x^2 y}{(1-x)^2}, s, a\right) \\
&+ \frac{1}{2(1-x)^d}\ {}_{p+2}H_q\left(\begin{array}{c}(\alpha)_{1,p}, \frac{d}{2}, \frac{d}{2}+\frac{1}{2}; \\ (\gamma)_{1,q};\end{array} \frac{x^2 y}{(1+x)^2}, s, a\right).
\end{aligned} \tag{3.17}$$

Theorem 3.5. *The double series associated with general Hurwitz-Lerch Zeta function* (1.8) *generates following integral identity*

$$\begin{aligned}
&\int_0^{\infty} e^{-at} t^{s-1} \sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}\right)_n \left(\frac{d}{2}+\frac{1}{2}\right)_n \left(x^2\right)^n}{\left(\frac{1}{2}\right)_n n!} \sum_{m=0}^{n} \frac{A_m (-n)_m \left(\frac{1}{2}-n\right)_m \left(ye^{-t}\right)^m}{m!} dt \\
&= \frac{1}{2}\int_0^{\infty} e^{-at} t^{s-1} \Bigg\{ \frac{1}{(1-x)^d} \sum_{n=0}^{\infty} \frac{A_n \left(\frac{d}{2}\right)_n \left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{x^2 y e^{-t}}{(1-x)^2}\right)^n}{n!} \\
&+ \frac{1}{(1+x)^d} \sum_{n=0}^{\infty} \frac{A_n \left(\frac{d}{2}\right)_n \left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{x^2 y e^{-t}}{(1+x)^2}\right)^n}{n!} \Bigg\} dt,
\end{aligned} \tag{3.18}$$

where $\Re(a) > 0,\ \Re(s) > 0.$

Proof. By equation (3.15) we immediately obtain the result (3.18) on applying Euler integral formule. ☐

Theorem 3.6. *The double series associated with general Hurwitz-Lerch Zeta function* (1.8) *generates following general generating relation*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\left(\frac{d}{2}\right)_n \left(\frac{d}{2}+\frac{1}{2}\right)_n \left(x^2\right)^n}{\left(\frac{1}{2}\right)_n n!} \sum_{m=0}^{n} \frac{A_m (-n)_m \left(\frac{1}{2}-n\right)_m \left(ye^{-t}\right)^m}{m!} \\
&= \frac{1}{2(1-x)^d} \sum_{n=0}^{\infty} \frac{A_n \left(\frac{d}{2}\right)_n \left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{x^2 y e^{-t}}{(1-x)^2}\right)^n}{n!} + \frac{1}{2(1+x)^d} \sum_{n=0}^{\infty} \frac{A_n \left(\frac{d}{2}\right)_n \left(\frac{d}{2}+\frac{1}{2}\right)_n \left(\frac{x^2 y e^{-t}}{(1+x)^2}\right)^n}{n!}.
\end{aligned} \tag{3.19}$$

Proof. Make an appeal to the Theorem 3.5 and by the identity of Eqn. (3.18) we establish the result (3.19). ☐

This result (3.19) is identical to the generating relation due to H. Exton [1, (1999)].

4 Applications

In this section, we present some known and unknown generating relations and summation formulae. Making an appeal to the Corollary 3.1 and the identity (3.8) of the Theorem 3.2 we derive

$$\sum_{n=0}^{\infty}\frac{\left(\frac{d}{2}+\frac{1}{2}\right)_n\left(\frac{d}{2}+1\right)_n x^{2n}}{\left(\frac{3}{2}\right)_n n!}\ {}_{p+2}F_q\left(\begin{matrix}(\alpha)_{1,p},-n,-\frac{1}{2}-n;\\(\gamma)_{1,q};\end{matrix}\ ye^{-t}\right)$$
$$=\frac{1}{2xd(1-x)^d}\ {}_{p+2}F_q\left(\begin{matrix}(\alpha)_{1,p},\frac{d}{2}+\frac{1}{2},\frac{d}{2};\\(\gamma)_{1,q};\end{matrix}\ \frac{x^2ye^{-t}}{(1-x)^2}\right)$$
$$-\frac{1}{2xd(1+x)^d}\ {}_{p+2}F_q\left(\begin{matrix}(\alpha)_{1,p},\frac{d}{2}+\frac{1}{2},\frac{d}{2};\\(\gamma)_{1,q};\end{matrix}\ \frac{x^2ye^{-t}}{(1+x)^2}\right). \quad (4.1)$$

Further making an appeal to the Corollary 3.2 and the identity (3.18) of the Theorem 3.5, we obtain another generating relation

$$\sum_{n=0}^{\infty}\frac{\left(\frac{d}{2}\right)_n\left(\frac{d}{2}+\frac{1}{2}\right)_n x^{2n}}{\left(\frac{1}{2}\right)_n n!}\ {}_{p+2}F_q\left(\begin{matrix}(\alpha)_{1,p},-n,\frac{1}{2}-n;\\(\gamma)_{1,q};\end{matrix}\ ye^{-t}\right)$$
$$=\frac{1}{2(1-x)^d}\ {}_{p+2}F_q\left(\begin{matrix}(\alpha)_{1,p},\frac{d}{2},\frac{d}{2}+\frac{1}{2};\\(\gamma)_{1,q};\end{matrix}\ \frac{x^2ye^{-t}}{(1-x)^2}\right)$$
$$+\frac{1}{2(1+x)^d}\ {}_{p+2}F_q\left(\begin{matrix}(\alpha)_{1,p},\frac{d}{2},\frac{d}{2}+\frac{1}{2};\\(\gamma)_{1,q};\end{matrix}\ \frac{x^2ye^{-t}}{(1+x)^2}\right). \quad (4.2)$$

Now in the results (4.1) and (4.2), setting $p=0, q=1, \gamma_1=\frac{3}{2}$ and $d=1$ so that $A_m=\frac{1}{\left(\frac{3}{2}\right)_m}$ and supposing that for all $n\in\mathbb{N}_0, x,y\in\mathbb{C}$ and $t\in[0^+,\infty)$, then for following sequences of functions defined by

$$H_n^{(1)}(y,t)={}_2F_1\left(\begin{matrix}-n,-n-\frac{1}{2};\\\frac{3}{2};\end{matrix}ye^{-t}\right)\ \text{and}\ H_n^{(2)}(y,t)={}_2F_1\left(\begin{matrix}-n,n-\frac{1}{2};\\\frac{1}{2};\end{matrix}ye^{-t}\right), \quad (4.3)$$

there exist following summation formulae

$$\sum_{n=0}^{\infty}H_n^{(1)}(y,t)x^{2n}=\frac{y^{-1/2}e^{t/2}}{4x^2}\left[\log\left\{\frac{1-x+xy^{1/2}e^{-t/2}}{1-x-xy^{1/2}e^{-t/2}}\right\}-\log\left\{\frac{1+x+xy^{1/2}e^{-t/2}}{1+x-xy^{1/2}e^{-t/2}}\right\}\right], \quad (4.4)$$

and

$$\sum_{n=0}^{\infty}H_n^{(2)}(y,t)x^{2n}=\frac{y^{-1/2}e^{t/2}}{4x}\left[\log\left\{\frac{1-x+xy^{1/2}e^{-t/2}}{1-x-xy^{1/2}e^{-t/2}}\right\}+\log\left\{\frac{1+x+xy^{1/2}e^{-t/2}}{1+x-xy^{1/2}e^{-t/2}}\right\}\right], \quad (4.5)$$

respectively.

Further for all $n\in\mathbb{N}_0, \Re(a)>\frac{1}{2}, x,y\in\mathbb{C}$ and $\Re(s)>0$, considering sequence of functions

$$H_n^{(3)}(a,y,s)=\int_0^{\infty}e^{-at}t^{s-1}\ {}_2F_1\left(\begin{matrix}-n,-n-\frac{1}{2};\\\frac{3}{2};\end{matrix}ye^{-t}\right)dt \quad (4.6)$$

and

$$H_n^{(4)}(a,y,s)=\int_0^{\infty}e^{-at}t^{s-1}\ {}_2F_1\left(\begin{matrix}-n,n-\frac{1}{2};\\\frac{1}{2};\end{matrix}ye^{-t}\right)dt \quad (4.7)$$

and making an appeal to the Theorems 3.2 and 3.5 in Eqns. (4.4) and (4.5), the following summation formulae are computed as

$$\sum_{n=0}^{\infty}H_n^{(3)}(a,y,s)x^{2n}$$
$$=\frac{y^{-1/2}}{4x^2}\int_0^{\infty}e^{-\left(a-\frac{1}{2}\right)t}t^{s-1}\left[\log\left\{\frac{1-x+xy^{1/2}e^{-t/2}}{1-x-xy^{1/2}e^{-t/2}}\right\}-\log\left\{\frac{1+x+xy^{1/2}e^{-t/2}}{1+x-xy^{1/2}e^{-t/2}}\right\}\right]dt. \quad (4.8)$$

and

$$\sum_{n=0}^{\infty}H_n^{(4)}(a,y,s)x^{2n}$$
$$=\frac{y^{-1/2}}{4x}\int_0^{\infty}e^{-\left(a-\frac{1}{2}\right)t}t^{s-1}\left[\log\left\{\frac{1-x+xy^{1/2}e^{-t/2}}{1-x-xy^{1/2}e^{-t/2}}\right\}+\log\left\{\frac{1+x+xy^{1/2}e^{-t/2}}{1+x-xy^{1/2}e^{-t/2}}\right\}\right]dt, \quad (4.9)$$

respectively.

Several other results, integral identities and generating relations may be derived on making an application of our formulae evaluated in previous sections, due to lack of space we omit them.

5 Conclusion

The summation formulae of the families of double series associated with general Hurwitz-Lerch type Zeta functions presented in the Section 2 may be useful in computational work. The identities found in the Section 3 applicable in evaluation of various generating relations of hypergeometric functions and the Zeta functions found in the literature. The sequence of functions given in the Section 4 may be useful in various problems of science and technology.

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