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(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

MATHEMATICAL STUDY OF BLOOD CIRCULATION AND BIO-CHEMICAL REACTION BASED HEAT DISTRIBUTION PROBLEM IN HUMAN DERMAL REGION

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Abstract

In this paper we study heat distribution in outer parts of human body incorporating effect of blood circulation and metabolic activities. We solve the bio heat equation of skin for steady state case using Whittaker function and Conuent Hypergeometric function. Some parameters are taken variable. A general model has been modified and solved mathematically for comparative study of heat flow in human skin. The structure of human skin is taken as heterogeneous medium and attempt has been made to solve it by analytic methods. Numerical computation has been carried out for various values of parameters.
2020 Mathematical Sciences Classification: 33C15,33C20, 92BXX
Keywords and Phrases: Physiological Heat Flow; Differential Equation; Whittaker function; Confluent Hypergeometric function.

1 Introduction

The heat of the body is produced by a slow combustion of food; and this is taking place all the time. This combustion goes on chiefly in the muscles and is much more active during exercise than when the body is at rest. Yet the internal temperature of the body during rest and moderate exercise is the same, although much more heat is produced during exercise. The loss of heat from the body takes place chiefly at the surface, through the skin. A great deal more heat is lost from the body when the surrounding air is cold, yet the body temperature remains the same. The process of heat-making which is carried on in the muscles is regulated by certain nerve centers in the brain and spinal cord, which are connected with the muscles by nerves, so that the making of heat is under constant and perfect control. When the body is exposed to cold air or water or is in any way cooled so that the temperature of the blood is lowered, nerve centers in the brain incite increased activity in the heat-making organs and more fuel is burned in the cells. In this way the heat-making process is adjusted to the needs of the body. The body temperature in health that any variation from the normal, 98.5 + degrees, gives cause for anxiety. As a result of some shock or in one who is very feeble, the temperature may fall below normal, through insufficient heat production or too great an escape of heat. More often there is a rise of temperature above the normal, and then one is said to have a fever. In fevers, heat production and loss are not so perfectly controlled as in health, because the heat centers are disturbed by the undesired substances circulating in the blood. The sweat glands are not so active as usual, and the surplus heat does not escape.

Blood circulation plays vital role in regulating the heat in a health and the flow is regulated by heart apparatus. However, certain subjects have abnormality due to age (above 40 yrs.) or under nervous stress, in such case Yoga can be useful to retain the normal rate [9, 19]. Heat regulation problems in a human body can be expressed in terms of differential equations. In this paper we generate such equations for outer body which incorporate the influence of circulation of blood and nutrient indeed bio-chemical reactions in the cells (cell metabolism). There are number of techniques developed for the solutions and listed in standard texts (Murphy [14]).However, in biological processes like physiological heat transfer these techniques have several limitations due to large number of soft parameters and the associated flexibility. Following the advent heat equation in physiological transport by Perl [16, 17]and Pennes [15]. Trezek and Cooper [4, 5] developed solutions of the boundary value problems pertaining to in-vivo tissue medium for heat flow in human dermal regions. Saxena [23] gave solution for steady state case in terms of special cases of Bessel functions.

In one dimensional boundary value problems, the differential equation can easily be transformed into an ordinary differential equations by applying a suitable transform. The required solutions can be obtained by solving this equations and inverting by any method. Other mathematical techniques can also be used for boundary value problems. In this paper Laplace transform has also been used and further solutions have been worked out in terms of special functions like Whittaker function and Confluent Hypergeometric functions.

2 The Method and the Mathematical Model

As we know that for differential equations, we can use some analytic methods and solve them with the help of Special functions. Some of the cases differential equations can reduced in Bessel equations, Whittakers equations or Kummers equations and result is expressed in terms of one of the Bessel functions, Whittaker function or in terms of Confluent Hypergeometric functions.

In this paper we use solution of Heat equation for steady state case with the help of Whittaker function which is reduced in Confluent Hypergeometric function. We should briefly discuss about Whittakers equation and Kummers equations.

We know that the Whittaker functions are solutions of the differential equation [24].

$$\frac{d^2w}{dz^2} - \left(-\frac{1}{4} + \frac{k}{z} + \frac{1/4 - \mu^2}{z^2}\right) w = 0. \tag{2.1}$$

It has a regular singular point at $z = 0$ and an irregular singular point at $z = \infty$. Two solution are given by the Whittaker functions $M_{k,z}(z)$ and $W_{k,z}(z)$ defined in terms of Kummer's Confluent Hypergeometric functions M and U by

$$M_{k,z}(z) = e^{-z/2} z^{\left(\mu+\frac{1}{2}\right)} M\left(\mu - k + \frac{1}{2}, 1 + 2\mu; z\right), \tag{2.2}$$

$$W_{k,z}(z) = e^{-z/2} z^{\left(\mu+\frac{1}{2}\right)} U\left(\mu - k + \frac{1}{2}, 1 + 2\mu; z\right), \tag{2.3}$$

The Kummer's equation may be written as

$$z\frac{d^2w}{dz^2} - (b - z)\frac{dw}{dz} - aw = 0, \tag{2.4}$$

with a regular singular point at $z = 0$ and irregular singular point at $z = \infty$. It has two linearly independent solutions $M(a, b; z)$ and $U(a, b; z)$. Kummers function (of first kind) M is a generalized hypergeometric series introduced is given by (Kummer[10]):

$$M(z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!} = {}_1F_1(a; b; z), \tag{2.5}$$

where $a^{(0)} = 1$, and $a^{(n)} = a(a + 1)(a + 2)...(a + n - 1)$ is the rising factorial. This function ${}_1F_1(a; b; z)$ is known as Confluent Hypergeometric function.

Now we are using these functions in our problem discussed below.

In epidermis and dermis regions of human body, the temperature distribution depends on various physical and biological quantities. These quantities are related to the local tissue temperature T through the following Bio-Heat equation for in vivo tissue is given by Perl [16].

$$\rho c\frac{\partial T}{\partial t} = div\,(KgradT) + m_b c_b\,(T_b - T) + S, \tag{2.6}$$

The one dimensional equation for constant thermal conductivity is written as

$$\rho c\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2} + m_b c_b\,(T_b - T) + S, \tag{2.7}$$

where $\rho, c, K, t, m_b, c_b, T_b$ and S are respectively tissue density, heat capacity, thermal conductivity, time, blood mass flow rate, heat capacity of blood and blood temperature and rate of metabolic heat generation at a point. Here, a one dimensional form of variation of temperature is taken in x-direction, perpendicular to the outer skin surface. The equation (2.7) is solved separately for epidermis and dermis under following conditions:

(i) At outer skin $(x = 0)$:

$$K_1\frac{\partial T}{\partial x} = h\,(T - T_a) + LE, \tag{2.8}$$

where
K_1 = value of K (Thermal conductivity) in epidermis,
h = heat transfer coefficient of convection and radiation,
L = latent heat of of tissue,
E = rate of sweat evaporation,
T_a = atmospheric temperature.

(ii) For epidermis $(0 < x < a)$:
Due to lack of blood flow and no metabolic activity in epidermis we take,

$$m_b c_b = S = 0. \tag{2.9}$$

(iii) Interface $(x = a)$:

$$K_1 \frac{\partial T}{\partial x} = K_2 \frac{\partial T}{\partial x}, \tag{2.10}$$

K_2 = value of K (thermal conductivity)in dermis.

(iv) For dermis $(a < x < b)$:

$$m_b c_b = \bar{M}\bar{X}, \qquad S = s(T_b - T),$$

where $\bar{X} = \left[\frac{(x-a)}{(b-a)}\right]^2$, $\bar{M}$ and s are values of $m_b c_b$ and S in subdermal region.

(v) At Subdermal boundary $(x = b)$:
$T = T_b$ where T_b is blood temperature which is almost same as body core temperature.

We simplified and solve equations (2.7) with conditions mentioned in $(i), (ii), (iii), (iv)$ and (v) and solve with the help of Laplace transform. The solutions for the both regions are obtained in the following form:

For Epidermis:

$$\bar{T} = A_1 \exp(y\sqrt{p}) + A_2 \exp(-y\sqrt{p}), \tag{2.11}$$

For Dermis:

$$\bar{T} = z^{-1/2}\left[A_3 M_{-p/4,1/4}(z^2) + A_4 M_{-p/4,-1/4}(z^2)\right], \tag{2.12}$$

where
$\bar{T}$ is Laplace transform of $(T_b - T)/T_b$, p is parameter of the transform,
$M_{k,m}$ denotes Whittaker's function of first kind. Also A_1, A_2, A_3 and A_4 are determined with the help of the above conditions. Thus the same are obtained as:

$$A_1 = \frac{n_1 l_3}{D}, \qquad A_2 = \frac{n_2 l_3}{D},$$
$$A_3 = \frac{n_3}{D}, \qquad A_4 = \frac{n_3 l_6}{D},$$
$$D = n_1 l_1 - n_2 l_2, \qquad \bar{a} = \frac{a}{(b-a)},$$
$$l_1 = \sqrt{p} - h, \qquad l_2 = l_1 + 2h,$$
$$l_3 = \frac{-(hT_a + \alpha)}{p}, \qquad l_4 = \frac{K_2}{K_1\sqrt{p}},$$
$$l_5 = \frac{l_4(1+p)}{2}, \qquad l_6 = -\frac{1}{\beta}\frac{{}_1F_1(a_2, C_2, ; \beta^2)}{{}_1F_1(a_1, C_1; \beta^2)},$$
$$n_1 = (1 + m_1), \qquad n_2 = (1 + m_1)e^{2\bar{a}\sqrt{p}},$$
$$n_3 = 2l_3 e^{(-\bar{a}\sqrt{p})}, \qquad m_1 = l_4 l_6 + l_5,$$

where

$$y = \frac{x}{(b-a)}, \qquad z = (y - \bar{a}),$$
$$\alpha = \frac{LE(b-a)}{K_1 T_b}, \qquad \beta = (m_b c_b + s)^{1/4}.$$

${}_1F_1$ denotes confluent hypergeometric funtion.
For steady state problem the solution (2.11) and (2.12) take the following form:
For Epidermis:

$$T = T_b(\mathbb{A}_1 - \mathbb{A}_2 y). \tag{2.13}$$

For Dermis:

$$T = T_b\left[1 - e^{-\frac{z^2}{2}}\left\{\mathbb{A}_3 z \;\, {}_1F_1(a_1, c_1; z^2) + \mathbb{A}_4 z \;\, {}_1F_1(a_2, c_2; z^2)\right\}\right]. \tag{2.14}$$

Values of all the notations used in equations (2.13) and (2.14) are defined in Appendix-A.

3 Numerical Results

This model has been solved with some numerical assumptions. Taking two layers the solution for T is obtained for the following values of physical and physiological constants have been taken as prescribed by Cooper and Trezek [4, 5] and Saxena[23].

$$\begin{aligned} L &= 579\,\text{cal/gm} \\ T_b &= 37^\circ\text{C} \quad \text{(The Core Temperature)} \\ \rho &= 1.05\,\text{gm/cm}^3 \\ c &= 0.83\,\text{cal/gm} \\ h &= 0.02\,\text{cal/cm}^2\text{-min}^\circ\text{C} \quad \text{(The Heat Transfer Coefficient).} \end{aligned}$$

For Epidermis

$$\begin{aligned} K_1 &= 0.040\,\text{cal/cm-min}^\circ\text{C} \\ \bar{M}_1 &= 0.000\,\text{cal/cm}^3\text{-min}^\circ\text{C} \\ s &= 0.00\,\text{cal/cm}^3\text{-min.} \end{aligned}$$

For Dermis

$$\begin{aligned} K_2 &= 0.060\,\text{cal/cm-min}^\circ\text{C} \\ \bar{M}_2 &= 0.030\,\text{cal/cm}^3\text{-min}^\circ\text{C} \\ s &= 0.0357\,\text{cal/cm}^3\text{-min.} \end{aligned}$$

The numerical calculations have been made for the following four cases of atmospheric temperature T_a together with the respective values of rate of evaporation E.

***Table* 3.1:** Cases for distinguish temperatures and evaporation rates.

S.No.	Temperature (T_a) (°C)	Evaporation (E) (gm/cm^2-min)
(i)	15	0.00
(ii)	20	0.45×10^{-3}
(iii)	25	0.79×10^{-3}
(iv)	30	0.9×10^{-3}

We can assign different values of the constants a and b depending on the sample of thickness of the skin under study for different different parts of body and persons. The set of values of a and b we considered here are as follows:

Set-I	$a = 0.2$ cm	$b = 0.7$ cm
Set-II	$a = 0.4$ cm	$b = 1.2$ cm
Set-III	$a = 0.6$ cm	$b = 1.3$ cm

The graphs have been plotted between temperature T and position x for different sets of the values of T_a and E.

These assumptions are generally based on the values taken by several researchers including Trezek and Cooper[4, 5] who have not only carried out numerical computation of the models in simplified form but also conducted laboratory investigations on certain mammals in in-vitro stage. Experiments have also been conducted by Hodgson [8] extensively on human being (in-vivo) sitting in a sophisticated climatic chamber designed by himself. He measured several parameters including sweat evaporation under different conditions. Some values are also available in classical monographs like Ruch and Patton [18] and have been used widely by Saxena and his subsequent workers ([1, 2, 3, 6, 7, 11, 12, 13, 20, 21, 22, 23]). Based on the above our numerical calculations are exhibited in the graphs.

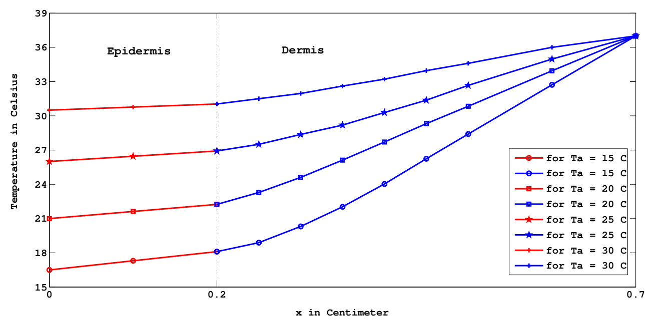

***Figure* 3.1:** Graph between in depth x and temperature T for Set-I

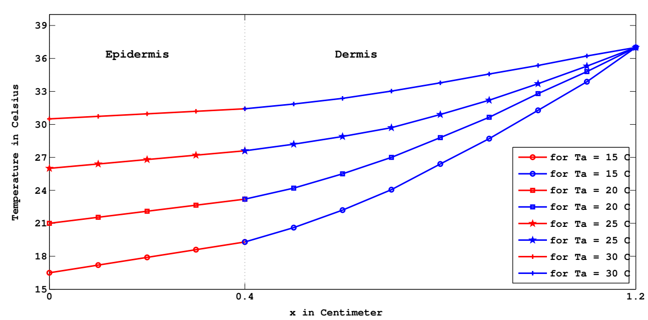

***Figure* 3.2:** Graph between in depth x and temperature T for Set-II

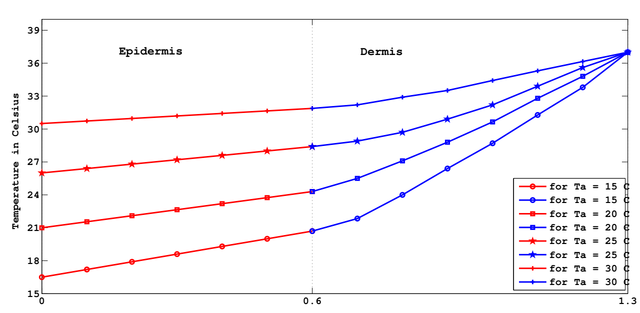

***Figure* 3.3:** Graph between in depth x and temperature T for Set-III

4 Discussion

Temperature profiles in epidermis and dermis for different environment conditions and anatomy. The curvilinear variation is clearly visible in dermis due to additional terms of blood flow and biochemical reactions.

These graphs are only illustrations for certain sample cases and can be extended for subjects placed under different atmospheric conditions.

As indicated above three sets of skin layers with different thicknesses have been considered for computation and for different values of atmospheric temperature. The patterns of graph reflect both these assumptions. In figure-3.1 and figure-3.2, the rise of temperature is more in comparison to figure-3.3 which has rapid rise in dermis in comparison to the earlier two cases.

The mathematical solution derived in this paper provide sufficient freedom for the assumption and occurrence of biophysical parameters namely, thermal conductivity, metabolic cell reactions and micro circulation of blood. This aspect is vital for the theoretical study as in-vivo situation demands it. The temperature profile thus obtained can further be use for more advanced studies pertaining to extreme climates and thermoregulation related diseases like malignant tumor.

This study is confined to human subjects at rest. Same can be extended to persons undergoing some physical activity or exercise like Yoga as indicated earlier [9, 19, 21]. In such cases the assumptions regarding blood circulation rate $\bar{M}$ and metabolic rate S have to be connected with practical results. Accordingly these two parameters have to be flexible, either in steps or continuously time dependent. This will open an opportunity for new investigations.

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Appendix-A

$$
\begin{aligned}
a_1 &= \frac{3}{4}, \\
a_2 &= \frac{1}{4}, \\
c_1 &= \frac{3}{2}, \\
c_2 &= \frac{1}{2}, \\
\mathbb{A}_1 &= \frac{l_1(1-m_1\bar{a})}{R}, \\
\mathbb{A}_2 &= \frac{m_1 l_1}{R}, \\
\mathbb{A}_3 &= \frac{l_1 l_3}{R}, \\
\mathbb{A}_4 &= \frac{l_1}{R}, \\
D &= n_1 l_1 - n_2 l_2, \\
\bar{a} &= \frac{a}{(b-a)}, \\
l_1 &= hT_a + \alpha, \\
l_2 &= \frac{K_2}{2K_1}, \\
R &= h - m_1(h\bar{a}+1).
\end{aligned}
$$