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(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

SOME RESULTS ON *d*-FRAMES

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Abstract

In this paper, we study and establish some results on properties of *d*-frames [1] and *d*-frame operators. Also, we present a result on the perturbation analysis of the *d*-frames.

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathbb{I}$ a countable index set. A sequence $\{x_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$ is said to be a Bessel sequence with Bessel bound $\lambda_2 > 0$ if $\sum_{i\in\mathbb{I}} |\langle x, x_i\rangle|^2 \leq \lambda_2||x||^2$ for all $x \in \mathcal{H}$. A Bessel sequence $\{x_i\}_{i\in\mathbb{I}}$ with Bessel bound λ_2 is said to be a frame for $\mathcal{H}$ if there exists constant $\lambda_1 > 0$ such that $\lambda_1||x||^2 \leq \sum_{i\in\mathbb{I}} |\langle x, x_i\rangle|^2 \leq \lambda_2||x||^2$, for all $x \in \mathcal{H}$. For frame $\{x_i\}_{i\in\mathbb{I}}$, the positive constants λ_1 and λ_2 are called the lower and upper frame bounds respectively. A frame $\{x_i\}_{i\in\mathbb{I}}$ is said to be a tight frame if $\lambda_1 = \lambda_2$.

By putting forward a landmark paper on frames [5], Daubechies, Grossmann and Meyer brought back the attention of the researchers towards the frame theory which was introduced by Duffin and Schaeffer [7] almost thirty years before.

In the last three decades, frames have been widely studied and applied in various fields of study viz. sampling theory, signal processing, system modeling, data analysis, etc. (For more details see [3, 4, 6, 8, 10]).

Here, it is to be noted that every Bessel sequence is not necessarily a frame always. Motivated by this fact, researchers generalised the concept of constructing the frames from the Bessel sequences in different ways. In fact they used either an operator on the Bessel sequence to make it frame or they rearranged/added/scattered the terms of the sequence to make it a frame. In the sequel, recently Mehra et al. [1] introduced *d*-frames for a Hilbert space $\mathcal{H}$ by using the concept of double sequences and studied certain properties of *d*-frame, *d*-frame operators and stability of *d*-frames. In this note, we study and establish some results on *d*-frame and their properties. Some of a results are extensions and generalizations of the results of [9] for *d*-frames in Hilbert spaces.

2 Preliminaries

Throughout this paper, $\mathcal{H}$ denotes an infinite dimensional Hilbert space and $\mathbb{N}$ denotes the set of all natural numbers. To prove our main results, we use following definitions, concept of space and results from [1].

Definition 2.1 ([1]). *The double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\mathcal{H}$ is said to be a d-frame for $\mathcal{H}$ if there exist positive constants λ_1 and λ_2 such that*

$$\lambda_1||x||^2 \leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \leq \lambda_2||x||^2, \quad for \ \ all \ \ x \in \mathcal{H}. \tag{2.1}$$

The constants λ_1 and λ_2 are called lower and upper d-frame bounds respectively.

If $\lambda_1 = \lambda_2$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called tight d-frame, and if $\lambda_1 = \lambda_2 = 1$, then it is called Parseval d-frame.

A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in Hilbert space $\mathcal{H}$ is called *d*-Bessel sequence with bound λ_2 if it satisfies upper *d*-frame inequality i.e.,

$$\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \leq \lambda_2||x||^2, \quad \forall x \in \mathcal{H}.$$

Consider the following spaces as defined in [1]:

$$\ell^2(\mathbb{N}\times\mathbb{N}) = \{\{\alpha_{\mathbf{ij}}\}_{\mathbf{i,j}\in\mathbb{N}}:\ \alpha_{\mathbf{ij}}\in\mathbb{F},\ \lim_{\mathbf{m,n}\to\infty}\sum_{\mathbf{i,j=1}}^{\mathbf{m,n}}|\alpha_{\mathbf{ij}}|^{\mathbf{2}}<\infty\}.$$

Then $\ell^2(\mathbb{N}\times\mathbb{N})$ is a Hilbert space with the norm induced by the inner product which is given by,

$$\left\langle\{\alpha_{ij}\}_{i,j\in\mathbb{N}},\{\beta_{ij}\}_{i,j\in\mathbb{N}}\right\rangle = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\alpha_{ij}\overline{\beta_{ij}},\quad \forall\{\alpha_{ij}\}_{i,j\in\mathbb{N}},\{\beta_{ij}\}_{i,j\in\mathbb{N}}\in\ell^2(\mathbb{N}\times\mathbb{N}).$$

Remark 2.1 ([1]). Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d-Bessel sequence. Define operator $\mathcal{T}:\ell^2(\mathbb{N}\times\mathbb{N})\to\mathcal{H}$ as

$$\mathcal{T}(\{\alpha_{ij}\}_{i,j\in\mathbb{N}}) = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\alpha_{ij}x_{ij},\quad \forall\{\alpha_{ij}\}_{i,j\in\mathbb{N}}\in\ell^2(\mathbb{N}\times\mathbb{N}).$$

If $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame then operator $\mathcal{T}$ is called pre d-frame (synthesis) operator and the adjoint operator $\mathcal{T}^*$ of $\mathcal{T}$ is called analysis operator for d-frame, and defined as

$$\mathcal{T}^*(x) = \{\langle x,x_{ij}\rangle\}_{i,j\in\mathbb{N}},\quad \forall x\in\mathcal{H}.$$

Theorem 2.1 ([1])**.** *A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\mathcal{H}$ is a d-Bessel sequence with d-Bessel bound λ_2 if and only if the operator $\mathcal{T}$ is linear, well defined and bounded with $\|\mathcal{T}\|\le\sqrt{\lambda_2}$.*

Theorem 2.2 ([1])**.** *A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\mathcal{H}$ is a d-frame for $\mathcal{H}$ if and only if the operator $\mathcal{T}$ is well defined, bounded, linear and surjective.*

The d-frame operator $\mathcal{S}:\mathcal{H}\to\mathcal{H}$ for d-frame $\{x_{ij}\}_{i,j\in\mathbb{N}}$ defined as:

$$\begin{aligned}\mathcal{S}(x) &= \mathcal{T}\mathcal{T}^*(x)\\ &= \mathcal{T}(\{\langle x,x_{ij}\rangle\}_{i,j\in\mathbb{N}})\\ &= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,x_{ij}\rangle x_{ij},\quad \forall x\in\mathcal{H}.\end{aligned}$$

Since $\mathcal{T}$ and $\mathcal{T}^*$ both are linear, so $\mathcal{S}$ is also linear.

Theorem 2.3 ([1])**.** *d-frame operator $\mathcal{S}$ is bounded, self adjoint, positive and invertible.*

Definition 2.2 ([1])**.** *A d-frame $\{\tilde{x}_{ij}\}_{i,j\in\mathbb{N}}$ for a Hilbert space $\mathcal{H}$ is called alternate dual d-frame for a given d-frame $\{x_{ij}\}_{i,j\in\mathbb{N}}$, if*

$$x = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,\tilde{x}_{ij}\rangle x_{ij},\quad \forall\ x\in\mathcal{H}.$$

Remark 2.2 ([1]). $\{\mathcal{S}^{-1}(x_{ij})\}_{i,j\in\mathbb{N}}$ is a special type of dual d-frame for $\{x_{ij}\}_{i,j\in\mathbb{N}}$, called canonical dual d-frame.

3 Main Results

Proposition 3.1. *Let $\mathcal{T}$, $\mathcal{T}^*$ and $\mathcal{S}$ are operators as defined above for a d-Bessel sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$. Then*

(I) *$\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ if and only if $\mathcal{S}$ is invertible.*

(II) *$\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ if and only if the analysis operator $\mathcal{T}^*$ is invertible.*

Proof. (I) If $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ then by Theorem 2.3, $\mathcal{S}$ is invertible.

Conversely, If $\mathcal{S}$ is invertible $\Rightarrow\mathcal{T}$ is surjective. By Theorem 2.2, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$. □

Proof. (II) $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ $\iff$ $\mathcal{T}$ is surjective $\Rightarrow\mathcal{T}^*$ is an isomorphism $\Rightarrow\mathcal{T}^*$ is invertible. Conversely, $\mathcal{T}^*$ is invertible $\Rightarrow\mathcal{T}^*$ is surjective $\Rightarrow\mathcal{T}$ is an isomorphism $\Rightarrow\mathcal{T}$ is surjective $\iff$ $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$. □

Theorem 3.1. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d-frame for $\mathcal{H}$ with d-frame operator $\mathcal{S}$, d-frame bounds $\lambda_1\le\lambda_2$ and let $\mathcal{U}:\mathcal{H}\to\mathcal{H}$ be a bounded operator. Then $\{\mathcal{U}x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ if and only if $\mathcal{U}$ is invertible.*

Proof. Let $\mathcal{U}$ be invertible operator, then for each $x \in \mathcal{H}$,

$$\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle x,\mathcal{U}x_{ij}\rangle|^2 = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle \mathcal{U}^*x, x_{ij}\rangle|^2 \geq \lambda_1||\mathcal{U}^*x||^2 \geq ||\mathcal{U}^{-1}||^2\lambda_1||x||^2$$

and

$$\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle x,\mathcal{U}x_{ij}\rangle|^2 = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle \mathcal{U}^*x, x_{ij}\rangle|^2 \leq \lambda_2||\mathcal{U}^*x||^2 \leq ||\mathcal{U}||^2\lambda_2||x||^2.$$

Thus, $\{\mathcal{U}x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ with d-frame bounds $||\mathcal{U}^{-1}||^2\lambda_1$, $||\mathcal{U}||^2\lambda_2$.

Conversely, If $\{\mathcal{U}x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$, then its d-frame operator is invertible on $\mathcal{H}$. Now, d-frame operator of $\{\mathcal{U}x_{ij}\}_{i,j\in\mathbb{N}}$ is

$$\begin{aligned}\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,\mathcal{U}x_{ij}\rangle\mathcal{U}x_{ij} &= \mathcal{U}\big(\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,\mathcal{U}x_{ij}\rangle x_{ij}\big)\\ &= \mathcal{U}\big(\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle \mathcal{U}^*x,x_{ij}\rangle x_{ij}\big)\\ &= \mathcal{U}\mathcal{S}\mathcal{U}^*(x).\end{aligned}$$

$\mathcal{U}\mathcal{S}\mathcal{U}^*$ is invertible $\Rightarrow \mathcal{U}$ is surjective $\Rightarrow \mathcal{U}^*$ is isomorphism $\Rightarrow \mathcal{U}^*$ is surjective $\Rightarrow \mathcal{U}$ is invertible. □

Corollary 3.1. *If $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d-frame for $\mathcal{H}$ with d-frame operator $\mathcal{S}$ and $\mathcal{U}:\mathcal{H}\to\mathcal{H}$ is a bounded positive operator, then $\{x_{ij}+\mathcal{U}x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame with the d-frame operator $(I+\mathcal{U})\mathcal{S}(I+\mathcal{U}^*)$ and d-frame bounds $||I+\mathcal{U}||^{-2}\lambda_1$, $||I+\mathcal{U}||^2\lambda_2$, if and only if $I+\mathcal{U}$ is invertible.*

Corollary 3.2. *If $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ and P is an orthogonal projection on $\mathcal{H}$, then $\{x_{ij}+\alpha Px_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$, where $\alpha\neq -1$ is an scalar.*

Theorem 3.2. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be d-Bessel sequences in $\mathcal{H}$ with analysis operators $\mathcal{T}_1^*$, $\mathcal{T}_2^*$ and d-frame operators $\mathcal{S}_1$, $\mathcal{S}_2$ respectively. Then for operators $\mathcal{U}_1,\mathcal{U}_2:\mathcal{H}\to\mathcal{H}$, $\{\mathcal{U}_1x_{ij}+\mathcal{U}_2y_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ if an only if $\mathcal{T}_1^*\mathcal{U}_1^*+\mathcal{T}_2^*\mathcal{U}_2^*$ is an invertible operator. Further, d-frame operator for $\{\mathcal{U}_1x_{ij}+\mathcal{U}_2y_{ij}\}_{i,j\in\mathbb{N}}$ is $\mathcal{S}=\mathcal{U}_1\mathcal{S}_1\mathcal{U}_1^*+\mathcal{U}_2\mathcal{S}_2\mathcal{U}_2^*+\mathcal{U}_1\mathcal{T}_1\mathcal{T}_2^*\mathcal{U}_2^*+\mathcal{U}_2\mathcal{T}_2\mathcal{T}_1^*\mathcal{U}_1^*$.*

Proof. $\{\mathcal{U}_1x_{ij}+\mathcal{U}_2y_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame for $\mathcal{H}$ if and only if its analysis operator say $\mathcal{L}^*$ is invertible, where

$$\begin{aligned}\mathcal{L}^*(x) &= \{\langle x,\mathcal{U}_1x_{ij}+\mathcal{U}_2y_{ij}\rangle\}_{i,j\in\mathbb{N}}\\ &= \{\langle x,\mathcal{U}_1x_{ij}\rangle+\langle x,\mathcal{U}_2y_{ij}\rangle\}_{i,j\in\mathbb{N}}\\ &= \{\langle \mathcal{U}_1^*x,x_{ij}\rangle\}_{i,j\in\mathbb{N}}+\{\langle \mathcal{U}_2^*x,y_{ij}\rangle\}_{i,j\in\mathbb{N}}\\ &= \mathcal{T}_1^*\mathcal{U}_1^*x+\mathcal{T}_2^*\mathcal{U}_2^*x.\end{aligned}$$

Thus, $\mathcal{L}^*=\mathcal{T}_1^*\mathcal{U}_1^*+\mathcal{T}_2^*\mathcal{U}_2^*$ is invertible.

And the d-frame operator for sequence $\{\mathcal{U}_1x_{ij}+\mathcal{U}_2y_{ij}\}_{i,j\in\mathbb{N}}$ is

$$\begin{aligned}\mathcal{S}=\mathcal{L}\mathcal{L}^* &= (\mathcal{T}_1^*\mathcal{U}_1^*+\mathcal{T}_2^*\mathcal{U}_2^*)^*(\mathcal{T}_1^*\mathcal{U}_1^*+\mathcal{T}_2^*\mathcal{U}_2^*)\\ &= \mathcal{U}_1\mathcal{S}_1\mathcal{U}_1^*+\mathcal{U}_2\mathcal{S}_2\mathcal{U}_2^*+\mathcal{U}_1\mathcal{T}_1\mathcal{T}_2^*\mathcal{U}_2^*+\mathcal{U}_2\mathcal{T}_2\mathcal{T}_1^*\mathcal{U}_1^*.\end{aligned}$$

□

Remark 3.1. In the above propositions, theorems and corollaries, if we consider classical frames in place of d-frames, we get the results of [9].

To construct a sequence of alternate dual d-frames from a given d-Bessel sequence, we prove following results.

Theorem 3.3. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d-frame for $\mathcal{H}$ with d-frame operator $\mathcal{S}$. Then for a given d-Bessel sequence $\{u_{ij}\}_{i,j\in\mathbb{N}}$, the double sequence $\{y_{ij}: y_{ij}=\mathcal{S}^{-1}x_{ij}+u_{ij}\}_{i,j\in\mathbb{N}}$ is a dual d-frame for $\{x_{ij}\}_{i,j\in\mathbb{N}}$ if and only if $\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,u_{ij}\rangle x_{ij}=0$, for all $x\in\mathcal{H}$.*

Proof. For the given d-Bessel sequence $\{u_{ij}\}_{i,j\in\mathbb{N}}$, we have $\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,u_{ij}\rangle x_{ij}=0, \forall\ x\in\mathcal{H}$. Then

$$\begin{aligned}\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,y_{ij}\rangle x_{ij} &= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,\mathcal{S}^{-1}x_{ij}\rangle x_{ij}+\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,u_{ij}\rangle x_{ij}\\ &= x\end{aligned}$$

$\Rightarrow \{y_{ij}\}_{i,j\in\mathbb{N}}$ is a dual d-frame for $\{x_{ij}\}_{i,j\in\mathbb{N}}$.

Conversely, If $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a dual d-frame for $\{x_{ij}\}_{i,j\in\mathbb{N}}$ where $y_{ij}=\mathcal{S}^{-1}x_{ij}+u_{ij}$, then

$$\begin{aligned}x &= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,y_{ij}\rangle x_{ij},\quad \forall\ x\in\mathcal{H}\\ &= x+\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,u_{ij}\rangle x_{ij}\end{aligned}$$

$\Rightarrow \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,u_{ij}\rangle x_{ij}=0$. □

Theorem 3.4. *If $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d-frame for $\mathcal{H}$ with d-frame operator $\mathcal{S}$ and dual $\{y_{ij}\}_{i,j\in\mathbb{N}}$. Then the sequence $\{g_{ij}\}_{i,j\in\mathbb{N}}$ define by $g_{ij}=\mathcal{S}^{-1}x_{ij}-x_{ij}+\mathcal{S}y_{ij}$ is also a dual for $\{x_{ij}\}_{i,j\in\mathbb{N}}$.*

Proof.

$$\begin{aligned}\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,g_{ij}\rangle x_{ij} &= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,\mathcal{S}^{-1}x_{ij}-x_{ij}+\mathcal{S}y_{ij}\rangle x_{ij}\\ &= x-\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,\mathcal{S}x_{ij}\rangle \mathcal{S}^{-1}x_{ij}+\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x,\mathcal{S}y_{ij}\rangle x_{ij}\\ &= x-\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle \mathcal{S}^*x,x_{ij}\rangle \mathcal{S}^{-1}x_{ij}+\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle \mathcal{S}^*x,y_{ij}\rangle x_{ij}\\ &= x-\mathcal{S}^*x+\mathcal{S}^*x\\ &= x.\end{aligned}$$

□

To prove our next theorem, we use following results from Casazza et al. [2].

Lemma 3.1 ([2])**.** *Let $\mathcal{U}:\mathcal{H}\to\mathcal{H}$ be a linear operator and assume that there exist constant $\alpha,\beta\in[0,1[$ such that*

$$||\mathcal{U}x-x||\le\alpha||x||+\beta||\mathcal{U}x||,\ \forall\ x\in\mathcal{H}.$$

Then $\mathcal{U}$ is a bounded linear invertible operator on $\mathcal{H}$, and

$$\frac{1-\alpha}{1+\beta}\ ||x||\le||\mathcal{U}x||\le\frac{1+\alpha}{1-\beta}\ ||x||,$$

$$\frac{1-\beta}{1+\alpha}\ ||x||\le||\mathcal{U}^{-1}x||\le\frac{1+\beta}{1-\alpha}\ ||x||,\ \forall\ x\in\mathcal{H}.$$

Lemma 3.2 ([2])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ are two Hilbert spaces, $\mathcal{U}:\mathcal{X}\to\mathcal{Y}$ be a bounded operator, $\mathcal{X}_0$ a dense subspace of $\mathcal{X}$ and $\mathcal{V}:\mathcal{X}\to\mathcal{Y}$ a linear mapping. If*

$$||\mathcal{U}x-\mathcal{V}x||\le\alpha||\mathcal{U}x||+\beta||\mathcal{V}x||+\gamma||x||,\ \forall\ x\in\mathcal{X}_0,$$

where $\beta\in[0,1[$, then $\mathcal{V}$ is a bounded linear operator on a dense subspace of $\mathcal{X}$ and hence has a unique extension to a bounded linear operator (of the same norm) on $\mathcal{X}$.

Theorem 3.5. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d-frame with bounds λ_1,λ_2, and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be a double sequence in $\mathcal{H}$ and assume that $\exists$ constant $\alpha,\beta,\gamma\ge0$ such that max$\left(\alpha+\frac{\gamma}{\sqrt{\lambda_1}},\beta\right)<1$ and*

$$\begin{aligned}\lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}c_{ij}(x_{ij}-y_{ij})\right\| &\le \alpha\lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}c_{ij}x_{ij}\right\|+\beta\lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n}c_{ij}y_{ij}\right\|\\ &+ \gamma\|\{c_{ij}\}_{i,j\in\mathbb{N}}\|,\qquad \forall\{c_{ij}\}_{i,j\in\mathbb{N}}\in\ell^2(\mathbb{N}\times\mathbb{N}).\end{aligned}\tag{3.1}$$

Then, $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is also a d-frame for $\mathcal{H}$ with bounds $\lambda_1\left(1-\frac{\alpha+\beta+\frac{\gamma}{\sqrt{\lambda_1}}}{1+\beta}\right)^2$ and $\lambda_2\left(1+\frac{\alpha+\beta+\frac{\gamma}{\sqrt{\lambda_2}}}{1-\beta}\right)^2$.

Proof. If $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame, then for pre d-frame operator $\mathcal{T}$ of $\{x_{ij}\}_{i,j\in\mathbb{N}}$, we have

$$\|\mathcal{T}(\{c_{ij}\}_{i,j\in\mathbb{N}})\| = \lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n} c_{ij}x_{ij}\right\| \leq \sqrt{\lambda_2}\,\|\{c_{ij}\}_{i,j\in\mathbb{N}}\|, \quad \{c_{ij}\}_{i,j\in\mathbb{N}} \in \ell^2(\mathbb{N}\times\mathbb{N}). \tag{3.2}$$

Define an operator $\mathcal{U} : \ell^2(\mathbb{N}\times\mathbb{N}) \to \mathcal{H}$ such that

$$\mathcal{U}(\{c_{ij}\}_{i,j\in\mathbb{N}}) = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n} c_{ij}y_{ij}. \tag{3.3}$$

For equations (3.2) and (3.3), the equation (3.1) gives

$$||\mathcal{T}(\{c_{ij}\}_{i,j\in\mathbb{N}}) - \mathcal{U}(\{c_{ij}\}_{i,j\in\mathbb{N}})|| \leq \alpha||\mathcal{T}(\{c_{ij}\}_{i,j\in\mathbb{N}})|| + \beta||\mathcal{U}(\{c_{ij}\}_{i,j\in\mathbb{N}})|| + \gamma||(\{c_{ij}\}_{i,j\in\mathbb{N}})||,$$
$$\forall\ (\{c_{ij}\}_{i,j\in\mathbb{N}}) \in \ell^2(\mathbb{N}\times\mathbb{N}).$$

Therefore, from Lemma 3.2, $\mathcal{U}$ is bounded linear operator on $\ell^2(\mathbb{N}\times\mathbb{N})$.
Using triangle inequality, we have

$$\begin{aligned}
&\|\mathcal{U}(\{c_{ij}\}_{i,j\in\mathbb{N}})\| \leq \|\mathcal{T}(\{c_{ij}\}_{i,j\in\mathbb{N}}) - \mathcal{U}(\{c_{ij}\}_{i,j\in\mathbb{N}})\| + \|\mathcal{T}(\{c_{ij}\}_{i,j\in\mathbb{N}})\| \\
&\Rightarrow \|\mathcal{U}(\{c_{ij}\}_{i,j\in\mathbb{N}})\| \leq \frac{1+\alpha}{1-\beta}\ ||\mathcal{T}(\{c_{ij}\}_{i,j\in\mathbb{N}})|| + \frac{\gamma}{1-\beta}\ ||(\{c_{ij}\}_{i,j\in\mathbb{N}})||
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n} c_{ij}y_{ij}\right\| &\leq \frac{1+\alpha}{1-\beta}\lim_{m,n\to\infty}\left\|\sum_{i,j=1}^{m,n} c_{ij}x_{ij}\right\| + \frac{\gamma}{1-\beta}\ ||(\{c_{ij}\}_{i,j\in\mathbb{N}})|| \\
&\leq \left(\frac{(1+\alpha)\sqrt{\lambda_2}+\gamma}{1-\beta}\right)||(\{c_{ij}\}_{i,j\in\mathbb{N}})||.
\end{aligned}$$

Thus, $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a d-Bessel sequence with bound $\left(\frac{(1+\alpha)\sqrt{\lambda_2}+\gamma}{1-\beta}\right)^2 = \lambda_2\left(1+\frac{\alpha+\beta+\frac{\gamma}{\sqrt{\lambda_2}}}{1-\beta}\right)^2$.

Since $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame, so for d-frame operator $\mathcal{S}$, the double sequence $\{\mathcal{S}^{-1}x_{ij}\}_{i,j\in\mathbb{N}}$ is the dual d-frame of $\{x_{ij}\}_{i,j\in\mathbb{N}}$ with upper bound λ_1^{-1}.
define an operator $\mathcal{T}^\dagger : \mathcal{H} \to \ell^2(\mathbb{N}\times\mathbb{N})$ such that

$$\begin{aligned}
\mathcal{T}^\dagger(x) &= \mathcal{T}^*\mathcal{S}^{-1}(x) \\
&= \left\{\left\langle x, \mathcal{S}^{-1}(x_{ij})\right\rangle\right\}_{i,j\in\mathbb{N}}, \quad \forall x \in \mathcal{H}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|\mathcal{T}^\dagger(x)\right\|^2 &= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\left\langle x, \mathcal{S}^{-1}(x_{ij})\right\rangle\right|^2 \\
&\leq {\lambda_1}^{-1}\left\|x\right\|^2, \quad \forall x \in \mathcal{H}.
\end{aligned}$$

Replacing $\{c_{ij}\}_{i,j\in\mathbb{N}}$ by $\mathcal{T}^\dagger(x)$, in (3.1) and using (3.3), we get

$$\|x - \mathcal{U}\mathcal{T}^\dagger(x)\| \leq \left(\alpha + \frac{\gamma}{\sqrt{\lambda_1}}\right)\|x\| + \beta||\mathcal{U}\mathcal{T}^\dagger(x)||, \quad \forall x \in \mathcal{H}. \tag{3.4}$$

Applying Lemma 3.1, equation (3.4) implies that the operator $\mathcal{U}\mathcal{T}^\dagger$ is invertible and

$$||\mathcal{U}\mathcal{T}^\dagger|| \leq \frac{1+\alpha+\frac{\gamma}{\sqrt{\lambda_1}}}{1-\beta}, \qquad \|(\mathcal{U}\mathcal{T}^\dagger)^{-1}\| \leq \frac{1+\beta}{1-(\alpha+\frac{\gamma}{\sqrt{\lambda_1}})}, \quad \forall\ x \in \mathcal{H}.$$

For $x \in \mathcal{H}$,

$$\begin{aligned}
x &= (\mathcal{U}\mathcal{T}^\dagger)(\mathcal{U}\mathcal{T}^\dagger)^{-1}(x) \\
&= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left\langle(\mathcal{U}\mathcal{T}^\dagger)^{-1}(x), \mathcal{S}^{-1}(x_{ij})\right\rangle y_{ij}.
\end{aligned}$$

$$\Rightarrow \|x\|^4 = \langle x, x\rangle^2 = \left|\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left\langle(\mathcal{U}\mathcal{T}^\dagger)^{-1}(x), \mathcal{S}^{-1}(x_{ij})\right\rangle\left\langle y_{ij}, x\right\rangle\right|^2$$

$$
\begin{aligned}
&\leq \frac{1}{\lambda_1}\big\|(\mathcal{U}\mathcal{T}^{\dagger})^{-1}(x)\big\|^2 \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\big|\langle y_{ij},x\rangle\big|^2 \\
&\leq \frac{1}{\lambda_1}\bigg(\frac{1+\beta}{1-(\alpha+\frac{\gamma}{\sqrt{\lambda_1}})}\bigg)^2\|x\|^2 \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\big|\langle y_{ij},x\rangle\big|^2.
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\big|\langle y_{ij},x\rangle\big|^2 &\geq \lambda_1\bigg(\frac{1-(\alpha+\frac{\gamma}{\sqrt{\lambda_1}})}{1+\beta}\bigg)^2\|x\|^2 \\
&= \lambda_1\bigg(1-\frac{\alpha+\beta+\frac{\gamma}{\sqrt{\lambda_1}}}{1+\beta}\bigg)^2\|x\|^2, \quad \forall \;\; x\in\mathcal{H}.
\end{aligned}
$$

Thus, d-Bessel sequence $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame with bounds $\lambda_1\bigg(1-\frac{\alpha+\beta+\frac{\gamma}{\sqrt{\lambda_1}}}{1+\beta}\bigg)^2$ and $\lambda_2\bigg(1+\frac{\alpha+\beta+\frac{\gamma}{\sqrt{\lambda_2}}}{1-\beta}\bigg)^2$. $\square$

Remark 3.2. Taking $\beta = 0$, Theorem 4.1 of [1] becomes a particular case of Theorem 3.5.

4 Conclusion

In this paper, we studied cases in which new d-frames can be constructed from the existing ones and established the results on stability of d-frame under small perturbation. Also, we proved the result to construct an alternate dual d-frame from a given specific d-Bessel sequence. The frame theory has many exciting applications in different areas of study. So the concept of d-frame can have many applications specially in signal processing. This can be taken as a future scope of interdisciplinary research.

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