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(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

AN ALGORITHMIC APPROACH TO LOCAL SOLUTION OF THE NONLINEAR SECOND ORDER ORDINARY HYBRID INTEGRODIFFERENTIAL EQUATIONS

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Abstract

In this paper, we establish a couple of approximation results for local existence and uniqueness of the solution of an *IVP* of nonlinear second order ordinary hybrid integrodifferential equations by using the Dhage monotone iteration method based on the recent hybrid fixed point theorems of Dhage (2022) and Dhage *et al.* (2022). An approximation result for Ulam-Hyers stability of the local solution of the considered hybrid differential equation is also established. Finally, our main abstract results are also illustrated with a couple of numerical examples.
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1 Introduction

Given a closed and bounded interval $J = [t_0, t_0 + a]$ in $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $a > 0$, we consider the *IVP* of nonlinear second order hybrid ordinary differential equation *(HIGDE)*,

$$\left.\begin{aligned} &x''(t) = f\Big(t, x(t), \int_{t_0}^{t} g(s, x(s))\, ds\Big), \;\; t \in J, \\ &x(t_0) = \alpha_0, \;\; x'(t_0) = \alpha_1, \end{aligned}\right\} \tag{1.1}$$

where α_0, α_1 are real numbers and the function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies some hybrid, that is, mixed hypotheses from algebra, analysis and topology to be specified later.

Definition 1.1. *A function $x \in C^1(J, \mathbb{R})$ is said to be a* solution *of the HIGDE* (1.1) *if it satisfies the equations in* (1.1) *on J, where $C^1(J, \mathbb{R})$ is the space of continuously differentiable real-valued functions defined on J. If the solution x lies in a closed ball $\overline{B_r(x_0)}$ centered at some point $x_0 \in C(J, \mathbb{R})$ of radius $r > 0$, then we say it is a* local solution *or* neighborhood solution *(in short* nbhd solution*) of the HIGDE* (1.1) *on J.*

Remark 1.1. The present idea of local or nbhd-solution is different from the usual notion of a local solution solution as mentioned in Coddington and Levinson [1]. See Dhage and Dhage [12, 13] and references given therein.

The *HIGDE* (1.1) is familiar in the subject of nonlinear analysis and can be studied for a variety of different aspects of the solution by using different methods form nonlinear functional analysis. The existence of local solution can be proved by using the Schauder fixed point principle, see for example, Coddington and Levinson [1], Lakshmikantham and Leela [17], Granas and Dugundji [15] and references therein. The approximation result for uniqueness of solution can be proved by using the Banach fixed point theorem under a Lipschitz condition which is considered to be very strong in the area of nonlinear analysis. But to the knowledge the present authors, the approximation result for local existence and uniqueness theorems without using the Lipschitz condition is not discussed so far in the theory of nonlinear differential and integral equations. In this paper, we discuss the approximation results for local existence and uniqueness of solution under weaker partial Lipschitz condition but via construction of the algorithms based on monotone iteration method and a hybrid fixed point theorem of Dhage [4]. Also see Dhage *et al.* [10, 11] and references therein.

The rest of the paper is organized as follows. Section 2 deals with the auxiliary results and main hybrid fixed point theorems involved in the Dhage iteration method. The hypotheses and main approximation results for the local existence and uniqueness of solution are given in Section 3. The approximation of the Ulam-Hyer stability is discussed in Section 4 and a couple of illustrative examples are presented in Section 5. Finally, some concluding remarks are mentioned in Section 6.

2 Auxiliary Results

We place the problem of *HIGDE* (1.1) in the function space $C(J,\mathbb{R})$ of continuous, real-valued functions defined on J. We introduce a supremum norm $\|\cdot\|$ in $C(J,\mathbb{R})$ defined by

$$\|x\| = \sup_{t\in J} |x(t)|. \tag{2.1}$$

and an order relation $\preceq$ in $C(J,\mathbb{R})$ by the cone K given by

$$K = \{x \in C(J,\mathbb{R}) \mid x(t) \geq 0 \ \forall\ t \in J\}. \tag{2.2}$$

Thus,

$$x \preceq y \iff y - x \in K, \tag{2.3}$$

or equivalently,

$$x \preceq y \iff x(t) \leq y(t) \ \forall\ t \in J.$$

It is known that the Banach space $C(J,\mathbb{R})$ together with the order relations $\preceq$ becomes an ordered Banach space which we denote for convenience, by $\big(C(J,\mathbb{R}), K\big)$. We denote the open and closed spheres centered at $x_0 \in C(J,\mathbb{R})$ of radius r, for some $r > 0$, by

$$B_r(x_0) = \{x \in C(J,\mathbb{R}) \mid \|x - x_0\| < r\} = B(x,r)$$

and

$$B_r[x_0] = \{x \in C(J,\mathbb{R}) \mid \|x - x_0\| \leq r\} = \overline{B(x,r)}$$

receptively. It is clear that $B_r[x_0] = \overline{B_r(x_0)}$. Let $M > 0$ be a real number. Denote

$$B_r^M[x_0] = \big\{x \in B_r[x_0] \,\big|\, |x(t_1) - x(t_2)| \leq M\,|t_1 - t_2| \ \text{ for } \ t_1, t_2 \in J\big\}. \tag{2.4}$$

Then, we have the following result.

Lemma 2.1. *The set $B_r^M[x_0]$ is compact in $C(J,\mathbb{R})$.*

Proof. By definition, $B_r[x_0]$ is a closed and bounded subset of the Banach space $C(J,\mathbb{R})$. Moreover, $B_r^M[x_0]$ is an equicontinuous subset of $C(J,\mathbb{R})$ in view of the condition (2.1). Now, by an application of Arzelá-Ascoli theorem, $B_r^M[x_0]$ is compact set in $C(J,\mathbb{R})$ and the proof of the lemma is complete. □

It is well-known that the hybrid fixed point theoretic technique is very much useful in the subject of nonlinear analysis for dealing with the nonlinear equations qualitatively. See Granas and Dugundji [15] and the references therein. Here, we employ the Dhage monotone iteration method or simply *Dhage iteration method* based on the following two hybrid fixed point theorems of Dhage [4] and Dhage *et al.* [10].

Theorem 2.1 (Dhage [4])**.** *Let S be a non-empty partially compact subset of a regular partially ordered Banach space $\big(E, \|\cdot\|, \preceq,\big)$ with every chain C in S is Janhavi set and let $\mathcal{T} : S \to S$ be a monotone nondecreasing, partially continuous mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.2 (Dhage [4])**.** *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in a regular partially ordered Banach space $\big(E, \|\cdot\|, \preceq,\big)$ and let $\mathcal{T} : E \to E$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying*

$$\|x_0 - \mathcal{T}x_0\| \leq (1-q)r$$

for some real number $r > 0$, then $\mathcal{T}$ has a unique comparable fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*. Furthermore, if every pair of elements in X has a lower or upper bound, then ξ^* is unique.

If a Banach X is partially ordered by an order cone K in X, then in this case we simply say X is an ordered Banach space which we denote it by (X, K). Then, we have the following useful results proved in Dhage [2, 3].

Lemma 2.2 (Dhage [2, 3]). *Every ordered Banach space (X, K) is regular.*

Lemma 2.3 (Dhage [2, 3]). *Every partially compact subset S of an ordered Banach space (X, K) is a Janhavi set in X.*

As a consequence of Lemmas 2.2 and 2.3, we obtain the following hybrid fixed point theorem which we need in what follows.

Theorem 2.3 (Dhage [4] and Dhage *et al.* [10]). *Let S be a non-empty partially compact subset of an ordered Banach space (X, K) and let $\mathcal{T} : S \to S$ be a partially continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then $\mathcal{T}$ has a fixed point $\xi^* \in S$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.4 (Dhage [4]). *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in an ordered Banach space $\big(X, K\big)$ and let $\mathcal{T} : (X, K) \to (X, K)$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying*

$$\|x_0 - \mathcal{T}x_0\| \le (1 - q)r \tag{2.5}$$

for some real number $r > 0$, then $\mathcal{T}$ has a unique comparable fixed point x^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*. Furthermore, if every pair of elements in X has a lower or upper bound, then ξ^* is unique.

The details of the notions of partial order, Janhavi set, regularity of an ordered space, monotonicity of mappings, partial continuity, partial closure, partial compactness and partial contraction etc. and related applications appear in Dhage [2, 3, 4, 5, 6], Dhage and Dhage [8], Dhage *et al.* [10, 11, 14] and references therein.

3 Local Approximation Results

We consider the following set of hypotheses in what follows.

(H_1) The function f is continuous and bounded on $J \times \mathbb{R} \times \mathbb{R}$ with bound M_f.
(H_2) $f(t, x, y)$ is nondecreasing in x and y for each $t \in J$.
(H_3) $g(t, x)$ is nondecreasing in x for each $t \in J$.
(H_4) $f(t, \alpha_0, y) \ge 0$ and $\alpha_1 \ge 0$ for all $t \in J$ and $y \ge 0$.
(H_5) $g(t, \alpha_0) \ge 0$ for all $t \in J$.

Then we have the following useful lemma.

Lemma 3.1. *If $h \in L^1(J, \mathbb{R})$, then the IVP of ordinary second order linear differential equation*

$$x''(t) = h(t), \;\; t \in J, \quad x(t_0) = \alpha_0, \quad x'(t_0) = \alpha_1, \tag{3.1}$$

is equivalent to the integral equation

$$x(t) = \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t - s)\, h(s)\, ds, \;\; , t \in J. \tag{3.2}$$

Theorem 3.1. *Sppose that the hypotheses (H_1), (H_3) and (H_4) hold. Furthermore, if the inequalities $|\alpha_1|\, a + M_f\, a^2 \le r$ and $|\alpha_1| + 2\, M_f\, a \le M$ hold, then the HIGDE* (1.1) *has a solution x^* in $B_r^M[\alpha_0]$, where $x_0 \equiv \alpha_0$, and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} &x_0(t) = \alpha_0, \;\; t \in J, \\ &x_{n+1}(t) = \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t - s)\, f\Big(s, x_n(s), \int_{t_0}^{s} g(\tau, x_n(\tau))\, d\tau\Big)\, ds, \;\; t \in J, \end{aligned}\right\} \tag{3.3}$$

where $n = 0, 1, \ldots$; converges monotone nondecreasingly to x^*.

Proof. Set $X = C(J,\mathbb{R})$. Clearly, (X,K) is a partially ordered Banach space. Let x_0 be a constant function on J such that $x_0(t) = \alpha_0$ for all $t \in J$ and define a closed ball $B_r^M[x_0]$ in X defined by (2.3). By Lemma 2.1, $B_r^M[x_0]$ is a compact subset of X. By Lemma 3.1, the *HIGDE* (1.1) is equivalent to the nonlinear hybrid integral equation *(HIE)*

$$x(t) = \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^{t}(t-s)\, f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds,\ \ t \in J. \tag{3.4}$$

Now, define an operator $\mathcal{T}$ on $B_r^M[x_0]$ into X by

$$\mathcal{T}x(t) = \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^{t}(t-s)\, f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds,\ \ t \in J. \tag{3.5}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.3 on $B_r^M[x_0]$ in the following series of steps.

Step I: *The operator $\mathcal{T}$ maps $B_r^M[x_0]$ into itself.*

Firstly, we show that $\mathcal{T}$ maps $B_r^M[x_0]$ into itself. Let $x \in B_r^M[x_0]$ be arbitrary element. Then,

$$\begin{aligned}
|\mathcal{T}x(t) - x_0(t)| &\le |\alpha_1(t-t_0)| + \left|\int_{t_0}^{t}(t-s)\, f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds\right| \\
&\le |\alpha_1|\,a + \int_{t_0}^{t}|t-s|\,\left|f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\right|ds \\
&= |\alpha_1|\,a + M_f\,a\int_{t_0}^{t_0+a} ds \\
&= |\alpha_1|\,a + M_f\,a^2 \\
&\le r.
\end{aligned}$$

for all $t \in J$. Taking the supremum over t in the above inequality yields

$$\|\mathcal{T}x - x_0\| \le |\alpha_1|\,a + M_f\,a^2 \le r$$

which implies that $\mathcal{T}x \in B_r[x_0]$ for all $x \in B_r^M[x_0]$. Next, let $t_1, t_2 \in J$ be arbitrary. Then, we have

$$\begin{aligned}
&\big|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\big| \\
&\le |\alpha_1|\,|t_1-t_2| + \bigg|\int_{t_0}^{t_1}(t_1-s)\ f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds \\
&\qquad\qquad - \int_{t_0}^{t_2}(t_2-s)\ f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds\bigg| \\
&\le |\alpha_1|\,|t_1-t_2| + + \bigg|\int_{t_0}^{t_1}(t_1-s)\ f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds \\
&\qquad\qquad - \int_{t_0}^{t_2}(t_2-s)\ f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds\bigg| \\
&\qquad + \bigg|\int_{t_0}^{t_1}(t_1-s)\, f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds \\
&\qquad\qquad - \int_{t_0}^{t_2}(t_2-s)\ f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds\bigg| \\
&\le |\alpha_1|\,|t_1-t_2| + \int_{t_0}^{t_1}|t_1-t_2|\,\left|f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\right|ds \\
&\qquad + \left|\int_{t_1}^{t_2}|t_2-s|\,\left|f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\right|ds\right| \\
&\le |\alpha_1|\,|t_1-t_2| + \int_{t_0}^{t_0+a}|t_1-t_2|\,M_f\,ds + \left|\int_{t_1}^{t_2} a\,M_f\,ds\right| \\
&\le |\alpha_1|\,|t_1-t_2| + 2M_f\,a\,|t_1-t_2| \\
&= \big(|\alpha_1| + 2M_f\,a\big)\,|t_1-t_2|
\end{aligned}$$

$$\leq M\,|t_1 - t_2|$$

where, $|\alpha_1| + 2M_f\, a \leq M$. Therefore, $\mathcal{T}x \in B_r^M[x_0]$ for all $x \in B_r^M[x_0]$ As a result, we have $\mathcal{T}(B_r^M[x_0]) \subset B_r^M[x_0]$.

Step II: *$\mathcal{T}$ is a monotone nondecreasing operator.*

Let $x, y \in B_r^M[x_0]$ be any two elements such that $x \succeq y$. Then, by hypotheses (H_2) and (H_3),

$$\begin{aligned}
\mathcal{T}x(t) &= \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s)\, f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big)\, ds \\
&\geq \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s)\, f\Big(s, y(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big)\, ds \\
&= \mathcal{T}y(t)
\end{aligned}$$

for all $t \in J$. So, $\mathcal{T}x \succeq \mathcal{T}y$, that is, $\mathcal{T}$ is monotone nondecreasing on $B_r^M[x_0]$.

Step III: *$\mathcal{T}$ is partially continuous operator.*

Let C be a chain in $B_r^M[x_0]$ and let $\{x_n\}$ be a sequence in C converging to a point $x \in C$. Then, by dominated cnonvergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{T}x_n &= \lim_{n\to\infty} \left[\alpha_0 + \int_{t_0}^{t} (t-s)\, f\Big(s, x_n(s), \int_{t_0}^{s} g(\tau, x_n(\tau))\, d\tau\Big)\, ds\right] \\
&= \alpha_0 + \alpha_1(t - t_0) + \lim_{n\to\infty} \int_{t_0}^{t} (t-s)\, f\Big(s, x_n(s), \int_{t_0}^{s} g(\tau, x_n(\tau))\, d\tau\Big)\, ds \\
&= \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s)\, \left[\lim_{n\to\infty} f\Big(s, x_n(s), \int_{t_0}^{s} g(\tau, x_n(\tau))\, d\tau\Big)\right]\, ds \\
&= \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s)\, f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\, d\tau\Big)\, ds \\
&= \mathcal{T}x(t)
\end{aligned}$$

for all $t \in J$. Therefore, $\mathcal{T}x_n \to \mathcal{T}x$ pointwise on J. As $\{\mathcal{T}x_n\} \subset B_r^M[x_0]$, $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of points in X. As a result, we have that $\mathcal{T}x_n \to \mathcal{T}x$ uniformly on J. Hence $\mathcal{T}$ is partially continuous operator on $B_r^M[x_0]$.

Step IV: *The element $x_0 \in B_r^M[x_0]$ satisfies the relation $x_0 \preceq \mathcal{T}x_0$.*

Since the hypotheses (H_4) and (H_5) hold, one has

$$\begin{aligned}
x_0(t) &= \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s) f\Big(s, x_0(s), \int_{t_0}^{s} g(\tau, x_0(\tau))\, d\tau\Big)\, ds \\
&\leq x_0(t) + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s) f\Big(s, \alpha_0(s), \int_{t_0}^{s} g(\tau, \alpha_0)\, d\tau\Big)\, ds \\
&= \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s) f f\Big(s, x_0(s), \int_{t_0}^{s} g(\tau, x_0(\tau))\, d\tau\Big)\, ds \\
&= \mathcal{T}x_0(t)
\end{aligned}$$

for all $t \in J$. This shows that the constant function x_0 in $B_r^M[x_0]$ serves as to satisfy the operator inequality $x_0 \preceq \mathcal{T}x_0$.

Thus, the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.3, and so $\mathcal{T}$ has a fixed point x^* in $B_r^M[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the *HIE* (3.4) and consequently the *HIGDE* (1.1) has a local solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nondecreasing and converges to x^*. This completes the proof. □

Next, we prove an approximation result for existence and uniqueness of the solution simultaneously under weaker form of Lipschitz condition. We need the following hypotheses in what follows.

(H_6) There exists a constant $k > 0$ such that

$$0 \leq f(t, x_1, x_2) - f(t, y_1, y_2) \leq \ell_1(x_1 - y_1) + \ell_2(x_2 - y_2)$$

for all $t \in J$ and $x_1, y_1, x_2, y_2 \in \mathbb{R}$ with $x_1 \geq y_1$, $x_2 \geq y_2$, where $\big(\ell_1 a + \ell_2 k\, a^2\big) < 1$.

(H_7) There exists a constant $k > 0$ such that
$$0 \le g(t,x) - g(t,y) \le k\,(x-y)$$
for all $t \in J$ and $x, y \in \mathbb{R}$ with $x \ge y$.

Theorem 3.2. *Suppose that the hypotheses (H_1), (H_6) and (H_7) hold. Furthermore, if*
$$|\alpha_1|\,a + M_f\,a^2 \le \left[1 - \left(\ell_1 a^2 + \ell_2 k\,a^3\right)\right] r, \quad \left(\ell_1 a^2 + \ell_2 k\,a^3\right) < 1, \tag{3.6}$$
for some real number $r > 0$, then the HIGDE (1.1) has a unique solution x^* in $B_r[x_0]$ defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nondecreasing and converges to x^*.

Proof. Set $(X, K) = \big(C(J,\mathbb{R}), \preceq\big)$ which is a lattice w.r.t. the lattice *join* and *meet* operations defined by $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$, and so every pair of elements of X has a lower and an upper bound. Let $r > 0$ be a fixed real number and consider closed sphere $B_r[x_0]$ centred at x_0 of radius r in the partially ordered Banach space (X, K).

Define an operator $\mathcal{T}$ on X into X by (3.5). Clearly, $\mathcal{T}$ is monotone nondecreasing on X. To see this, let $x, y \in X$ be two elements such that $x \succeq y$. Then, by hypotheses (H_6) and (H_7), we obtain
$$\begin{aligned}
&\mathcal{T}x(t) - \mathcal{T}y(t)\\
&= \int_{t_0}^{t} (t-s)\left[f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds - f\Big(s, y(s), \int_{t_0}^{s} g(\tau, y(\tau))\,d\tau\Big)\right] ds\\
&\ge 0,
\end{aligned}$$
for all $t \in J$. Therefore, $\mathcal{T}x \succeq \mathcal{T}y$ and consequently $\mathcal{T}$ is monotone nondecresing on X.

Next, we show that $\mathcal{T}$ is a partial contraction on X. Let $x, y \in X$ be such that $x \succeq y$. Then, by hypotheses (H_6) and (H_7), we obtain
$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}y(t)| &= \bigg|\int_{t_0}^{t} (t-s) f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds\\
&\qquad - \int_{t_0}^{t} (t-s) f\Big(s, y(s), \int_{t_0}^{s} g(\tau, y(\tau))\,d\tau\Big)\,ds\bigg|\\
&\le \bigg|\int_{t_0}^{t} (t-s)\Big[f\Big(s, x(s), \int_{t_0}^{s} g(\tau, x(\tau))\,d\tau\Big)\,ds\\
&\qquad - f\Big(s, y(s), \int_{t_0}^{s} g(\tau, y(\tau))\,d\tau\Big)\Big]\,ds\bigg|\\
&\le \bigg|\int_{t_0}^{t} (t-s)\Big[\ell_1\big(x(s) - y(s)\big) + \ell_2 \int_{t_0}^{s} k\,(t-s)\big(x(\tau) - y(\tau)\big)\,d\tau\Big]\,ds\bigg|\\
&= \ell_1 \int_{t_0}^{t} a\,|x(s) - y(s)|\,ds + \ell_2 k \int_{t_0}^{t} a\,\big(x(s) - y(s)\big)\big)\\
&\le \ell_1 a \int_{t_0}^{t_0+a} \|x-y\|\,ds + \ell_2\,k\,a^2 \int_{t_0}^{t_0+a} \|x-y\|\,ds\\
&= \left[\ell_1 a^2 + \ell_2 k\,a^3\right] \|x-y\|\\
&= \lambda \|x-y\|
\end{aligned}$$
for all $t \in J$, where $\lambda = \ell_1 a^2 + \ell_2 k\,a^3 < 1$. Taking the supremum over t in the above inequality yields
$$\|\mathcal{T}x - \mathcal{T}y\| \le \lambda\,\|x-y\|$$
for all comparable elements $x, y \in X$. This shows that $\mathcal{T}$ is a partial contraction on X with contraction constant $k\,a$. Furthermore, it can be shown as in the proof of Theorem 3.1 that the element $x_0 \in B_r^M[x_0]$ satisfies the relation $x_0 \preceq \mathcal{T}x_0$ in view of hypothesis (H_4). Finally, by hypotheses (H_4) − (H_5) and condition (3.6), one has
$$\|x_0 - \mathcal{T}x_0\| \le |\alpha_1|\,a + \sup_{t\in J}\bigg|\int_{t_0}^{t} (t-s)\,f\Big(s, \alpha_0, \int_{t_0}^{s} g(\tau, \alpha_0)\,d\tau\Big)\,ds\bigg|$$

$$\leq |\alpha_1|\, a + \sup_{t\in J} \int_{t_0}^{t} |t-s| \Big| f\Big(s, \alpha_0, \int_{t_0}^{s} g(\tau, \alpha_0)\, d\tau\Big)\Big|\, ds$$
$$\leq |\alpha_1|\, a + M_f\, a^2$$
$$\leq \big[1 - \big(\ell_1 a^2 + \ell_2 k\, a^3\big)\big]\, r$$

which shows that the condition (2.5) of Theorem 2.4 is satisfied. Hence $\mathcal{T}$ has a unique fixed point x^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the *HIE* (3.4) and consequently the *HIGDE* (1.1) has a unique local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nondecreasing and converges to x^*. This completes the proof. □

Remark 3.1. The conclusion of Theorems 3.1 and 3.2 also remains true if we replace the hypothesis (H_4) with the following one.

(H_4') $f(t, \alpha_0, y) \leq 0$ and $\alpha_1 \leq 0$ for all $t \in J$ and $y \geq 0$.

In this case, the *HIGDE* (1.1) has a local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nonincreasing and converges to x^*.

Remark 3.2. If the initial condition in the equation (1.1) is such that $\alpha_0 > 0$, then under the conditions of Theorem 3.1, the *HIGDE* (1.1) has a local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to the positive solution x^*. Similarly, under the conditions of Theorem 3.2, the *HIGDE* (1.1) has a unique local positive solution x^* defined on J and the sequence of successive approximations defined by (3.3) $\{x_n\}_{n=0}^{\infty}$ converges monotone nondecreasingly to the unique positive solution x^*.

4 Approximation of Local Ulam-Hyers Stability

The Ulam-Hyers stability for various dynamic systems has already been discussed by several authors under the conditions of classical Schauder fixed point theorem (see Tripathy [18], Huang *et al.* [16] and references therein). Here, in the present paper, we discuss the approximation of the Ulam-Hyers stability of local solution of the *HIGDE* (1.1) under the conditions of hybrid fixed point principle stated in Theorem 2.4. We need the following definition in what follows.

Definition 4.1. *The HIGDE* (1.1) *is said to be locally Ulam-Hyers stable if for $\epsilon > 0$ and for each solution $y \in B_r[x_0]$ of the inequality*

$$\left.\begin{aligned} &\Big| y''(t) - f\Big(t, y(t), \int_{t_0}^{t} g(s, y(s))\, ds\Big)\Big| \leq \epsilon, \quad t \in J,\\ &y(t_0) = \alpha_0, \quad y'(t_0) = \alpha_1, \end{aligned}\right\} \tag{$*$}$$

there exists a constant $K_f > 0$ such that

$$\big|y(t) - \xi(t)\big| \leq K_f \epsilon \tag{$**$}$$

for all $t \in J$, where $\xi \in B_r[x_0]$ is a local solution of the HIGDE (1.1) *defined on J. The solution ξ of the HIGDE* (1.1) *is called Ulam-Hyers stable local solution on J.*

Theorem 4.1. *Assume that all the hypotheses of Theorem* 3.2 *hold. Then the HIGDE* (1.1) *has a unique Ulam-Hyers stable local solution $x^* \in B_r[x_0]$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations given by* (3.3) *converges monotone nondecreasingly to x^*.*

Proof. Let $\epsilon > 0$ be given and let $y \in B_r[x_0]$ be a solution of the functional inequality (4.1) on J, that is, we have

$$\left.\begin{aligned} &\Big| y''(t) - f\Big(t, y(t), \int_{t_0}^{t} g(s, y(s))\, ds\Big)\Big| \leq \epsilon, \quad t \in J,\\ &y(t_0) = \alpha_0, \quad y'(t_0) = \alpha_1, \end{aligned}\right\} \tag{4.1}$$

By Theorem 3.2, the *HIGDE* (1.1) has a unique local solution $\xi \in B_r[x_0]$. Then by Lemma 2.1, one has

$$\xi(t) = x_o + \alpha_1(t-t_0) + \int_{t_0}^{t} (t-s)\, f\Big(s, \xi(s), \int_{t_0}^{s} g(\tau, \xi(\tau))\, d\tau\Big)\, ds, \quad t \in J. \tag{4.2}$$

Now, by integration of (4.1) yields the estimate:

$$\left|y(t)-\alpha_0-\alpha_1(t-t_0)-\int_{t_0}^{t}(t-s)\,f\Big(s,y(s),\int_{t_0}^{s}g(\tau,y(\tau))\,d\tau\Big)\,ds\right|\le\frac{a^2}{2}\,\epsilon, \tag{4.3}$$

for all $t\in J$.

Next, from (4.2) and (4.3) we obtain

$$\begin{aligned}
&\big|y(t)-\xi(t)\big|\\
&=\left|y(t)-\alpha_0-\alpha_1(t-t_0)-\int_{t_0}^{t}(t-s)f\Big(s,\xi(s),\int_{t_0}^{s}g(\tau,\xi(\tau))\,d\tau\Big)\,ds\right|\\
&\le\left|y(t)-\alpha_0-\alpha_1(t-t_0)-\int_{t_0}^{t}(t-s)f\Big(s,y(s),\int_{t_0}^{s}g(\tau,y(\tau))\,d\tau\Big)\,ds\right|\\
&\quad+\left|\int_{t_0}^{t}(t-s)\Big[f\Big(s,y(s),\int_{t_0}^{s}g(\tau,y(\tau))\,d\tau\Big)-f\Big(s,\xi(s),\int_{t_0}^{s}g(\tau,\xi(\tau))\,d\tau\Big)\Big]\,ds\right|\\
&\le\frac{a^2}{2}\,\epsilon+\left|\int_{t_0}^{t}a\Big[\ell_1\big(y(s)-\xi(s)\big)+\ell_2\int_{t_0}^{t}k\,(t-s)\big(y(\tau)-\xi(\tau)\big)\,d\tau\Big]\,ds\right|\\
&=\frac{a^2}{2}\,\epsilon+\ell_1 a\int_{t_0}^{t}|y(s)-\xi(s)|\,ds+\ell_2 k\,a^2\int_{t_0}^{t}|y(s)-\xi(s)|\,ds\\
&\le\frac{a^2}{2}\,\epsilon++\ell_1 a\int_{t_0}^{t_0+a}\|y-\xi\|\,ds+\ell_2\,k\,a^2\int_{t_0}^{t_0+a}\|y-\xi\|\,ds\\
&=\frac{a^2}{2}\,\epsilon+a^2\big(\ell_1+\ell_2 k\,a\big)\|y-\xi\|\\
&=\frac{a^2}{2}\,\epsilon+\lambda\|y-\xi\|
\end{aligned}$$

for all $t\in J$, where $\lambda=a^2\big(\ell_1+\ell_2 k\,a\big)<1$. Taking the supremum over t, we obtain

$$\|y-\xi\|\le\frac{a^2}{2}\,\epsilon+a^2\big(\ell_1+\ell_2 k\,a\big)\|y-\xi\|$$

or

$$\|y-\xi\|\le\left[\frac{a^2}{2[1-a^2\big(\ell_1+\ell_2 k\,a\big)]}\right]\,\epsilon$$

where, $a^2\big(\ell_1+\ell_2 k\,a\big)<1$. Letting $K_f=\left[\dfrac{a^2}{2[1-a^2\big(\ell_1+\ell_2 k\,a\big)]}\right]>0$, we obtain

$$\big|y(t)-\xi(t)\big|\le K_f\epsilon$$

for all $t\in J$. As a result, ξ is a Ulam-Hyers stable local solution of the *HIGDE* (1.1) on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nondecreasing and converges to ξ. Consequently the *HIGDE* (1.1) is a locally Ulam-Hyers stable on J. This completes the proof. □

Remark 4.1. If the given initial condition in the equation (1.1) is such that $x_0>0$, then under the conditions of Theorem 4.1, the *HIGDE* (1.1) has a unique Ulam-Hyers stable local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to x^*.

5 The Examples

In this section, we indicate a couple of examples illustrating the abstract ideas involved in the main approximation results, Theorems 3.1, 3.2 and 4.1 of this paper.

Example 5.1. Given a closed and bounded interval $J=[0,1]$ in $\mathbb{R}$, consider the *IVP* of nonlinear first order HIGDE,

$$x''(t)=\tan hx(t)+\int_0^t\tan hx(s)\,ds,\;\; t\in[0,1];\quad x(0)=\frac{1}{4},\;\; x'(0)=1. \tag{5.1}$$

Here, $\alpha_0 = \frac{1}{4}$, $\alpha_1 = 1$, $g(t,x) = \tan hx$, $(t,x) \in [0,1] \times \mathbb{R}$ and $f(t,x,y) = \tan hx + y$ for $(t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$. We show that the functions g and f satisfy all the conditions of Theorem 3.1. Clearly, f is bounded on $[0,1] \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = 2$ and so the hypothesis (H_1) is satisfied. Also the function $f(t,x,y)$ is nondecreasing in x and y for each $t \in [0,1]$. Therefore, hypothesis (H_2) is satisfied. Next, $g(t,x)$ is nondecreasing in x for each $t \in [0,1]$, so the hypothesis (H_3) is satisfied. Moreover, $f(t,\alpha_0,y) = f(t,\frac{1}{4},y) = \tan h(\frac{1}{4}) + y \geq 0$ and $\alpha_1 \geq 0$ for each $t \in [0,1]$ and $y \geq 0$, so the hypothesis (H_4) holds. Finally, $g(t,\alpha_0) = \tan h(\frac{1}{4}) \geq 0$ for all $t \in [0,1]$ and hypothesis (H_5) is satisfied. If we take $r = 2$ and $M = 1$, all the conditions of Theorem 3.1 are satisfied. Hence, the *HIGDE* (5.1) has a local solution x^* in the closed ball $B_2^1[\frac{1}{4}]$ of $C(J,\mathbb{R})$ which is positive in view of Remark 3.2. Moreover, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$x_0(t) = \frac{1}{4}, \;\; t \in [0,1],$$

$$x_{n+1}(t) = \frac{1}{4} + \alpha_1(t-t_0) + \int_0^t \tan hx_n(s)\,ds + \int_0^t (t-s)\,\tan hx_n(s)\,ds, \;\; t \in [0,1],$$

is monotone nondecreasing and converges to the positive solution x^* defined on $[0,1]$.

Example 5.2. Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the *IVP* of nonlinear first order *HIGDE*,

$$x''(t) = \frac{1}{4}\,\tan^{-1} x(t) + \frac{1}{4}\int_0^t \tan^{-1} x(s)\,, \;\; t \in [0,1]; \quad x(0) = \frac{1}{4}, \;\; x'(0) = 1. \tag{5.2}$$

Here, $\alpha_0 = \dfrac{1}{4}$, $\alpha_1 = 1$, and $g(t,x) = \tan^{-1} x$ for $(t,x) \in [0,1] \times \mathbb{R}$. Again, $f(t,x,y) = \dfrac{1}{4}\,\tan^{-1} x + \dfrac{1}{4} y$ for each $t \in [0,1]$. We show that f satisfies all the conditions of Theorem 3.2. Clearly, f is bounded on $[0,1] \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = \frac{11}{14}$ and so, the hypothesis (H_1) is satisfied. Next, let $x,y \in \mathbb{R}$ be such that $x \geq y$. Then there exists a constant ξ with $x < \xi < y$ satisfying

$$0 \leq g(t,x) - g(t,y) \leq \frac{1}{1+\xi^2}(x-y) \leq (x-y)$$

for all $t \in [0,1]$. So the hypothesis (H_7) holds with $k = 1$. Moreover, $g(t,\alpha_0) = g\left(t,\frac{1}{4}\right) = \tan^{-1}\left(\frac{1}{4}\right) \geq 0$ for each $t \in [0,1]$, and so the hypothesis (H_4) holds. Similarly,

$$f(t,\alpha_0,y) = \frac{1}{4}\,\tan^{-1}\alpha_0 + \frac{1}{4}\,y = \tan^{-1}\left(\frac{1}{4}\right) + \frac{1}{4}\,y \geq 0$$

and $\alpha_1(t-t_0) = t \geq 0$ for all $t \in [0,1]$ and for all positive number y, so the hypothesis (H_4) is satisfied. Next, let $x_1,y_1,x_2,y_2 \in \mathbb{R}$ with $x_1 \geq y_1$, $x_2 \geq y_2$. Then,

$$0 \leq f(t,x_1,x_2) - f(t,y_1,y_2) \leq \frac{1}{4}\cdot(x_1-y_2) + \frac{1}{4}\,(x_2-y_2)$$

for each $t \in [0,1]$. Therefore, hypothesis (H_6) holds with $\ell_1 = \frac{1}{4} = \ell_2$. If we take $r = 2$, then we have

$$M_f a = \frac{11}{14} \leq \left(1 - \frac{1}{2}\right)\cdot 2 = \left[1 - \left(\ell_1 a + \ell_2 k\,a^2\right)\right] r$$

and so, the condition (3.6) is satisfied. Thus, all the conditions of Theorem 3.2 are satisfied. Hence, the *HIGDE* (5.2) has a unique local solution x^* in the closed ball $B_2[\frac{1}{4}]$ of $C(J,\mathbb{R})$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$x_0(t) = \frac{1}{4}, \;\; t \in [0,1],$$

$$x_{n+1}(t) = \frac{1}{4} + \frac{1}{4}\int_0^t \tan^{-1} x_n(s)\,ds + \int_0^t (t-s)\,\tan^{-1} x_n(s)\,ds, \;\; t \in [0,1],$$

monotone nondecreasing converges to x^*. Moreover, the unique local solution x^* is Ulam-Hyers stable on $[0,1]$ in view of Definition 4.1. Consequently the *HIGDE* (5.2) is a locally Ulam-Hyers stable on the interval $[0,1]$.

Remark 5.1. The local approximation results of this paper includes similar results for the nonlinear *IVP*s of second order ordinary differential equations

$$\left.\begin{aligned} &x''(t) = f\big(t,x(t)\big), \;\; t \in J, \\ &x(t_0) = \alpha_0, \;\; x'(t_0) = \alpha_1, \end{aligned}\right\} \tag{5.3}$$

proved in Dhage *et al.* [11] as the special cases.

Remark 5.2. The approximation results of this paper may be extended to nonlinear *IVP*s of higher order ordinary differential equations

$$\left.\begin{aligned} x^{(n)}(t) &= f\Big(t, x(t), \int_{t_0}^{t} g(s, x(s))\,ds\Big), \ \ t \in J, \\ x^{(i)}(t_0) &= \alpha_{(i)}, \ \ i = 0, 1, 2, \dots, n-1, \end{aligned}\right\} \tag{5.4}$$

by using the arguments similar to Theorems 3.1 and 3.2 with appropriate modifications.

6 Concluding Remark

Finally, while concluding this paper, we remark that unlike the Schauder fixed point theorem we do not require any convexity argument in the proof of main existence theorem, Theorem 3.1. Similarly, we do not require the usual Lipschitz condition in the proof of uniqueness theorem, Theorem 3.2, but a weaker form of one sided or partial Lipschitz condition is enough to serve the purpose. However, in both the cases we are able to ac*HIE*ve the existence of local solution by convergence of the successive approximations. Moreover, the differential equation (1.1) considered in this paper is of very simple form, however other complex nonlinear *IVPs* of *HIGDEs* may be considered and the present study can also be extended to such sophisticated nonlinear differential equations with appropriate modifications. These and other such problems form the further research scope in the subject of nonlinear differential and integral equations with applications. Some of the results in this direction will be reported elsewhere.

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