ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 272-276**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

IDENTITIES INVOLVING GENERALIZED BERNOULLI NUMBERS AND PARTIAL BELL POLYNOMIALS WITH THEIR APPLICATIONS

M. A. Pathan [1], Hemant Kumar[2], J. López-Bonilla[3] and Hunar Sherzad Taher [4]
[1]Centre for Mathematical and Statistical Sciences, Peechi Campus, Peechi, Kerala, India-680653
Department of Mathematics, Aligarh Muslim University, Aligarh, Uttar Pradesh, India-202002
[2]Department of Mathematics, D. A-V. Postgraduate College, Kanpur, Uttar Pradesh, India-208001
[3]ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP, CDMX, México-07738
[4]Department of Mathematics, University of Salahaddin, Erbil, Iraq-44001
Email: mapathan@gmail.com, palhemant2007@rediffmail.com, jlopezb@ipn.mx, hunar.taher@su.edu.krd
(Received: February 22, 2023; In format: February 25, 2023; Revised : May 25, 2023;Accepted : June 03, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53132

Abstract

In the present paper, we show that the partial Bell polynomials allow for obtaining identities involving the generalized Bernoulli numbers. Then, on applying these identities we derive different generating and bilateral generating functions.

2020 Mathematical Sciences Classification: 05A15, 05A19, 05A30.
Keywords and Phrases: Bernoulli polynomials, Partial Bell polynomials, Generalized Bernoulli numbers, Generating and bilateral generating functions.

1 Introduction

In [18], the partial Bell polynomials $B_{n,k}(.,.,\dots,.)$ $\forall n,k\geq 0$ are represented in the following series expansion

$$\frac{1}{k!}\left\{\sum_{m\geq 1}^{\infty} x_m\frac{t^m}{m!}\right\}^k=\sum_{n\geq k}^{\infty}B_{n,k}(x_1,x_2,\dots,x_{n-k+1})\frac{t^n}{n!}\ \forall k=0,1,2,\dots, \tag{1.1}$$

where, $B_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{n_1,n_2,\dots,n_r,\dots=0}^{\infty}\frac{n!}{n_1!n_2!..n_r!\dots}(\frac{x_1}{1!})^{n_1}(\frac{x_2}{1!})^{n_2}\dots(\frac{x_r}{r!})^{n_r}\dots$; along with $n_1+n_2+\dots+n_r+\dots=k$ and $n_1+2n_2+\dots+rn_r+\dots=n$.

Recently, Pathan et al. [11] obtained the connections between partial Bell polynomials, partition function and q-hypergeometric series.

On the other hand in [7], a generalization of (1.1) is introduced to prove the following relation

$$Q_{n+1}\left(\frac{1}{x}\right)=(n+1)\sum_{k=0}^{n}\frac{(-1)^k k!}{x^{n+k+1}}B_{n,k}\left(\frac{1}{2}\ Q_2(x),\frac{1}{3}Q_3(x),\dots,\frac{1}{n-k+2}Q_{n-k+2}(x)\right),n\geq 0, \tag{1.2}$$

in terms of the partial Bell polynomials [2,3,4,14,18] and the Bernoulli polynomials [1,13,16]

$$Q_m(y)=B_m(y)-B_m, B_m=B_m(0), m\geq 0, \tag{1.3}$$

from (1.3), we get the polynomials

$$Q_0(y)=0, Q_1(y)=y, Q_2(y)=y^2-y, Q_3(y)=y^3-\frac{3}{2}y^2+\frac{1}{2}y,$$
$$Q_4(y)=y^4-2y^3+y^2,\dots . \tag{1.4}$$

Then we employ (1.3) to find the property

$$\lim_{x\to 0}\frac{1}{x}Q_m(x)=\lim_{x\to 0}\frac{B_m(x)-B_m(0)}{x}$$
$$=\left[\frac{d}{\mathrm{dx}}B_m(x)\right](0)=mB_{m-1}. \tag{1.5}$$

Here in this research work, we make an appeal to the results (1.2)-(1.5) and then for $n,k\geq 0$ deduce various results involving identities between the generalized Bernoulli numbers $B_n^{(k)}$ (see in [6,8,9,10,16]) and the partial Bell polynomials $B_{n,k}(.,\dots,.)$ (see [11,18]). Later on applying these results we obtain many generating and bilateral generating functions.

2 Some identities involving $B_{n,k}$ and $B_m^{(j)}$

In this section for $n, k \geq 0$, we derive certain identities between the generalized Bernoulli numbers $B_n^{(k)}$ and the partial Bell polynomials $B_{n,k}(B_1, B_2, \ldots, B_{n-k+1})$.

Theorem 2.1. *For all $n, k \geq 0$, the partial Bell polynomials $B_{n,k}(B_1, B_2, \ldots, B_{n-k+1})$ involving Bernoulli numbers $B_n, (n \geq 0)$ give following identities*

$$\sum_{k=0}^{n} (-1)^k k! B_{n,k}(B_1, B_2, \ldots, B_{n-k+1}) = \frac{1}{n+1}, n \geq 0, \tag{2.1}$$

and

$$\sum_{k=0}^{n} (-1)^k k! B_{n,k}\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-k+2}\right) = B_n, n \geq 0. \tag{2.2}$$

Proof. Consider the expression (1.2) and then write it in the form

$$(n+1)\sum_{k=0}^{n} (-1)^k k! B_{n,k}\left(\frac{1}{2x}\ \mathrm{Q}_2(x), \frac{1}{3x}\ \mathrm{Q}_3(x), \ldots,\ \frac{1}{(n-k+2)x}Q_{n-k+2}(x)\right) = x^{n+1}Q_{n+1}\left(\frac{1}{x}\right). \tag{2.3}$$

Then in the formula (2.3) apply the results (1.2) and $\lim_{x\to 0}$eq.(1.5), we obtain the identity (2.1) because of the limiting case $\lim_{x\to 0} x^m Q_m\left(\frac{1}{x}\right) = 1$.

Again, the inversion of (2.1) gives us the identity (2.2). □

Remark 2.1. It is remarked that Zhang-Yang [18] deduced the relation

$$B_{n,k}(B_1, B_2, \ldots, B_{n-k+1}) = \frac{1}{k!}B_n^{(k)},\ (n, k \geq 0), \tag{2.4}$$

involving the generalized Bernoulli numbers ([6, 8, 9, 10, 16]), however, (2.4) is incorrect, it must be

$$B_{n,k}(B_1, B_2, \ldots, B_{n-k+1}) = \frac{1}{k!}\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}B_n^{(j)}. \tag{2.5}$$

Theorem 2.2. *For the generalized Bernoulli numbers $B_n^{(k)}$ $(n, k \geq 0)$, there exists an identity*

$$\sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} B_n^{(j)} = \frac{1}{n+1},\ \textit{for all } n \geq 0. \tag{2.6}$$

Proof. Make an appeal to the results (2.1) and (2.5), immediately we obtain the identity (2.6). □

Theorem 2.3. *For all $n, k \geq 0$, an identity between the generalized Bernoulli numbers $B_n^{(k)}$ and the partial Bell polynomials $B_{n,k}(., \ldots, .)$ exists as*

$$\sum_{j=0}^{k} \binom{k}{j} j! B_{n,j}(B_1, B_2, \ldots, B_{n-j+1}) = B_n^{(k)}, n, k \geq 0. \tag{2.7}$$

Proof. Make an appeal to the Theorems 2.1 and 2.3 and the corrigendum in the Remark 2.2 and then in it use the property due to [13] as given by

$$\sum_{k=j}^{n} \binom{k}{j} = \binom{n+1}{j+1}. \tag{2.8}$$

Finally, the binomial inversion [5] of (2.5) gives the expression (2.7). □

Theorem 2.4. *For all $n \geq 0$, there exists following identities*

$$(-1)^k k! B_{n,k}(B_1, B_2, \ldots, B_{n-k+1}) = \frac{1}{2}\binom{n-1}{k-1}\binom{2n-1}{k}^{-1} = (-1)^k \binom{n+1}{k+1} B_n^{(k)}. \tag{2.9}$$

Proof. In the Theorems 2.1 and 2.3, make an appeal to the formula given by

$$\sum_{k=0}^{n} \frac{k}{n}\binom{n}{k}\binom{2n-1}{k}^{-1} = \frac{2}{n+1}, \tag{2.10}$$

and thus use the reduction formula for binomial coefficients we arrive the identities in (2.9). □

Remark 2.2. The relation (1.2) is equivalent to the following identity [7] (Cauchy convolution [15])

$$\sum_{k=1}^{n-1} \frac{x^k}{k!(n-k)!} Q_k\left(\frac{1}{x}\right) Q_{n-k}(x) = 0, n \geq 3. \tag{2.11}$$

3 Applications

In this section on application of the identities obtained in the Section 2 and the formula due to [12, p. 348, Problem 212] and see also in [17, p. 355, Eqn. (9)] given by

$$\sum_{n=0}^{\infty} \frac{\alpha}{\alpha+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} t^n = (1+w)^{\alpha}, |t|<1,$$

$$w = w(t) = t\left(1+w(t)\right)^{\beta+1}, w(0)=0, \tag{3.1}$$

we obtain various generating and bilateral generating functions:

Example 3.1. If $\alpha > 0$ and for $n \geq 0$

$$\Psi_n = \sum_{k=0}^{n} (-1)^k k! B_{n,k}\left(B_1, B_2, \ldots, B_{n-k+1}\right). \tag{3.2}$$

Then in the disk $|t|<1$, there exists a generating formula

$$\sum_{n=0}^{\infty} \binom{\alpha+\alpha n}{n} \Psi_n t^n = \alpha(1+\zeta)^{\alpha}, \tag{3.3}$$

where, $\zeta = \zeta(t) = t\left(1+\zeta(t)\right)^{\alpha}, \zeta(0)=0$.

Solution. In the Eqns. (3.2) and (3.3) make an application of the Theorem 2.1, we get

$$\sum_{n=0}^{\infty} \binom{\alpha+\alpha n}{n} \Psi_n t^n = \alpha \sum_{n=0}^{\infty} \binom{\alpha+\alpha n}{n} \frac{1}{\alpha+\alpha n} t^n. \tag{3.4}$$

Finally in the result (3.4), apply the formula (3.1) for $\beta = \alpha - 1$ we obtain the formula (3.3).

Example 3.2. If $\alpha > 0$,

$$\psi_n = \sum_{k=0}^{n} \binom{\alpha+\alpha n}{n-k} \binom{\alpha+(\beta+1)k}{k} \binom{k+1}{k} \varphi_k, \tag{3.5}$$

Then by (3.2) and (3.5), there exists a bilateral generating formula

$$\sum_{n=0}^{\infty} \psi_n \Psi_n t^n = (1+\zeta)^{\alpha} \sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} \varphi_n \zeta^n. \tag{3.6}$$

Solution. In the left hand side of (3.6) on considering (3.2) and (3.5), then on use of (2.1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \psi_n \Psi_n t^n &= \sum_{n=0}^{\infty} \frac{1}{\alpha+\alpha n} \sum_{k=0}^{n} \binom{\alpha+\alpha n}{n-k} \binom{\alpha+(\beta+1)k}{k} (\alpha k+\alpha) \varphi_k t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{\alpha+\alpha n+\alpha k}{n} \frac{\alpha k+\alpha}{\alpha+\alpha n+\alpha k} \binom{\alpha+(\beta+1)k}{k} \varphi_k t^{n+k} \\
&= \sum_{k=0}^{\infty} \binom{\alpha+(\beta+1)k}{k} \varphi_k t^k \sum_{n=0}^{\infty} \frac{\alpha+\alpha k}{\alpha+\alpha k+\alpha n} \binom{\alpha+\alpha k+\alpha n}{n} t^n \\
&= (1+\zeta)^{\alpha} \sum_{k=0}^{\infty} \binom{\alpha+(\beta+1)k}{k} \varphi_k \left\{t(1+\zeta)^{\alpha}\right\}^k \text{ (on use of (3.1)).}
\end{aligned} \tag{3.7}$$

Finally on use of (3.3) and (3.7), we find right hand side of (3.6).

Example 3.3. In (3.5) if

$$\varphi_k = {}_{p+1}F_{q+1}\left[\begin{matrix} -k, \alpha_1, \ldots, \alpha_p; \\ \alpha+\beta k+1, \beta_1, \ldots, \beta_q; \end{matrix} x\right], \tag{3.8}$$

where, the generalized hypergeometric function ${}_pF_q(.)$ is defined by [17, p. 42]

$${}_pF_q\left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_n}{\prod_{i=1}^{q} (\gamma_i)_n} \frac{z^n}{n!}, \tag{3.9}$$

where $p, q \in \mathbb{N} \cup \{0\}$, $\alpha_i \in \mathbb{C}(i = 1, 2, 3, \ldots, p)$; $\gamma_i \in \mathbb{C}(i = 1, 2, 3, \ldots, q)$; $z \in \mathbb{C}$; also all

$$\gamma_i \neq 0, -1, -2, \ldots, \ (i = 1, 2, 3, \ldots, q).$$

The series in (3.9) (i) converges for $|z| < \infty$, if $p \leq q$; (ii) converges for $|z| < 1$, if $p = q+1$; (iii) diverges for all z, $z \neq 0$, if $p > q+1$; (iv) converges absolutely for $|z| = 1$, if $p = q+1$, and $\Re(\omega) > 0, \omega = \sum_{i=1}^{q} \gamma_i - \sum_{i=1}^{p} \alpha_i$; (v) converges conditionally for $|z| = 1, z \neq 1$, if $p = q + 1$, and $-1 < \Re(\omega) \leq 0$; (vi) diverges for $|z| = 1$, if $p = q + 1$, and $\Re(\omega) < -1$.

Then on application of the example 3.2, there exists a bilateral generating function

$$\sum_{n=0}^{\infty} \psi_n \Psi_n t^n = \frac{(1+\zeta)^{\alpha} (1 + W(\zeta))^{\alpha+1}}{1 - \beta W(\zeta)} {}_pF_q \left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} - xW(\zeta)\right], \tag{3.10}$$

where $W(\zeta)$ is given in following (3.12).

Solution. Make an appeal to the functions (3.8) and (3.9) in Example (3.2) we find

$$\begin{aligned} \sum_{n=0}^{\infty} \psi_n \Psi_n t^n &= (1+\zeta)^{\alpha} \sum_{n=0}^{\infty} \binom{\alpha + (\beta+1)n}{n} {}_{p+1}F_{q+1} \left[\begin{matrix} -n, \alpha_1, \ldots, \alpha_p; \\ \alpha + \beta n + 1, \beta_1, \ldots, \beta_q; \end{matrix} x\right] \zeta^n \\ &= (1+\zeta)^{\alpha} \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_p)_k}{(\beta_1)_k \ldots (\beta_q)_k} \frac{(-x\zeta)^k}{k!} \sum_{n=0}^{\infty} \binom{\alpha + (\beta+1)k + (\beta+1)n}{n} \zeta^n \\ &= (1+\zeta)^{\alpha} \frac{(1 + W(\zeta))^{\alpha+1}}{1 - \beta W(\zeta)} \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_p)_k}{(\beta_1)_k \ldots (\beta_q)_k} \frac{\left(-x\zeta (1 + W(\zeta))^{\beta+1}\right)^k}{k!}. \end{aligned} \tag{3.11}$$

Now in (3.11) define

$$W(\zeta) = \zeta (1 + W(\zeta))^{\beta+1}, \ W(0) = 0, \tag{3.12}$$

we derive the bilateral generating function (3.10).

4 Concluding remarks

The identities obtained in the Section 2 are very powerful tool to derive different results involving generalized Bernoulli numbers $B_n^{(k)}$ and the partial Bell polynomials $B_{n,k}(B_1, B_2, \ldots, B_{n-k+1})$.

We derive various generating and bilateral generating functions which are applicable in computing of many problems occurring in the science and technology. The polynomials in form of generalized hypergeometric functions are specialized in Legendre, Bessel, Hermite, Laggurre and Jacobi polynomials etc. found in the literature of generating functions. Hence Section 3 has very important and applicable techniques.

References

[1] T. Arakawa, T. Ibukiyama and M. Kaneko, *Bernoulli Numbers and Zeta Functions,* Springer, Japan, 2014.

[2] H. Belbachir, E. V. P. Spreafico and M. Rachidi, Linear recursive relations for Bernoulli numbers and applications, *J. Ramanujan Soc. Maths. and Math. Sci.,* **8**(1) (2020), 07-30.

[3] D. Birmajer, J. B. Gil and M. D. Weiner, Some convolution identities and an inverse relation involving partial Bell polynomials, *The Electronic J. Combinatorics,* **19**(4) (2012) # P34, 1-14.

[4] D. Birmajer, J. B. Gil and M. D. Weiner, Linear recurrence sequences and their convolutions via Bell polynomials, arXiv: 1405.7727v2 [math.CO] 26 Nov. 2014.

[5] K. N. Boyadzhiev, *Notes on the Binomial Transform,* World Scientific, Singapore, 2018.

[6] K. N. Boyadzhiev, New series identities with Cauchy, Stirling, and harmonic numbers, and Laguerre polynomials, arXiv: 1911.00186 [math.NT] 1 Nov. 2019.

[7] Jian Cao, Feng Qi and J. López-Bonilla, Some properties of differences between the Bernoulli polynomials and numbers, submitted to *Electronic Research Archive*, Special Issue: *Classical problems in Number Theory and Combination*, 2023.

[8] M. N. Coffey, Series representation of the Riemann and Hurwitz zeta functions and series and integral representations of the first Stieltjes constant, arXiv:1106.5146v1 [math-ph] 25 Jun 2011.

[9] H. Luo and G. Liu, An identity involving Nrlund numbers and Stirling numbers of the first kind, *Scientia Magna,* **4**(2) (2008), 45-48.

[10] A. M. Mathai and H. J. Haubold, *Special Functions for Applied Scientists,* Springer, New York, 2008.

[11] M. A. Pathan, J. D. Bulnes, J. López-Bonilla and H. Kumar, Connection between partial Bell polynomials and $(q;q)$; partition function, and certain q-hypergeometric series, *J. Ramanujan Soc. Maths. Math. Sci.,* **10** (1) (2022), 1-12.

[12] G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Vol. I. (Translated from the German by D. Aeppli,) Springer-Verlag, New York, Heidelberg and Berlin, 1972.

[13] J. Quaintance and H. W. Gould, *Combinatorial Identities for Stirling Numbers,* World Scientific, Singapore, 2016.

[14] M. Shattuck, Some combinatorial formulas for the partial r-Bell polynomials, *Notes on Number Theory and Discrete Maths.,* **23**(1) (2017), 63-76.

[15] R. Sivaramakrishnan, *Classical Theory of Arithmetic Functions,* Marcel Dekker Inc, New York, 1989.

[16] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals,* Elsevier, London, 2012.

[17] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood)/Wiley, Chichester/New York, 1984.

[18] Z. Zhang and J. Yang, Notes on some identities related to the partial Bell polynomials*, Tamsui Oxford J. Information and Math. Sci.,* **28**(1) (2012), 39-48.