ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 22-27**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

FEKETE-SZEGÖ INEQUALITY AND ZALCMAN FUNCTIONAL FOR CERTAIN SUBCLASS OF ALPHA-CONVEX FUNCTIONS

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(Received: June 28, 2022; In format: July 17, 2022; Finally revised: January 06, 2023; Accepted: January 31, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53102

Abstract

In the present investigation, we introduce a subclass of α-convex functions defined with subordination and associated with Cardioid domain in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. We establish the bounds for $|a_2|$, $|a_3|$ and $|a_4|$, Fekete-Szegö inequality and bound for the Zalcman functional for this class. The results proved earlier will follow as special cases.

2020 Mathematical Sciences Classification: 30C45, 30C50.
Keywords and Phrases: Analytic functions, Alpha-convex functions, Subordination, Cardioid domain, Coefficient bounds, Fekete-Szegö inequality, Zalcman functional.

1 Introduction

By $\mathcal{A}$, we denote the class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, defined in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. The subclass of $\mathcal{A}$, which consists of univalent functions in E, is denoted by $\mathcal{S}$.

In the theory of univalent functions, a very noted result was Bieberbach's conjecture which was established by Bieberbach [2]. It states that, for $f \in \mathcal{S}$, $|a_n| \leq n$, $n = 2, 3, ...$ and it remained as a challenge for the mathematicians for a long time. Finally, L. De-Branges [4], proved this conjecture in 1985. During the course of proving this conjecture, various results related to the coefficients were established and some new subclasses of $\mathcal{S}$ were developed.

For two analytic functions f and g in E, f is said to be subordinate to g (symbolically $f \prec g$) if there exists a function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$ such that $f(z) = g(w(z))$. Further, if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

Before defining our main classes, firstly we review some basic and relevant classes mentioned below:
$\mathcal{S}^* = \left\{ f : f \in \mathcal{A}, Re\left(\dfrac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}$, the class of starlike functions.
$\mathcal{K} = \left\{ f : f \in \mathcal{A}, Re\left(\dfrac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}$, the class of convex functions.
Mocanu [11] introduced a unifying class $\mathcal{M}(\alpha)$ as below:

$$\mathcal{M}(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}.$$

The functions in the class $\mathcal{M}(\alpha)$ are known as alpha-convex functions. In particular, $\mathcal{M}(0) \equiv \mathcal{S}^*$ and $\mathcal{M}(1) \equiv \mathcal{K}$.

For $f \in \mathcal{A}$, the relation $f \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2$ means that f lies in the region bounded by the cardioid given by

$$(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0.$$

Sharma et al. [16] introduced the classes $\mathcal{S}^*_{car}$ and $\mathcal{K}_{car}$ defined as follow:

$$\mathcal{S}^*_{car} = \left\{ f : f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, z \in E \right\}$$

and

$$\mathcal{K}_{car} = \left\{ f : f \in \mathcal{A}, \frac{(zf'(z))'}{f'(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, z \in E \right\}.$$

Obviously, $\mathcal{S}^*_{car}$ and $\mathcal{K}_{car}$ are the subclasses of starlike and convex functions associated with cardioid domain, respectively. Various subclasses of analytic functions were studied by subordinating to different kind of functions. Malik et al. [9, 10], Sharma et al. [16], Zainab et al. [18], Shi et al. [17] and Raza et al. [15] studied certain classes af analytic functions associated with cardioid domain.

Getting inspired from the above works, now we define the following subclass of α-convex functions by subordinating to $1 + \frac{4}{3}z + \frac{2}{3}z^2$.

Definition 1.1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}^{\alpha}_{car}$ if it satisfying the condition*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2.$$

The class $\mathcal{M}^{\alpha}_{car}$ is the unification of the classes $\mathcal{S}^*_{car}$ and $\mathcal{K}_{car}$ and for particular values of α, the results for the these classes can be obtained. In particular, we have the following observations:
(i) $\mathcal{M}^{0}_{car} \equiv \mathcal{S}^*_{car}$.
(ii) $\mathcal{M}^{1}_{car} \equiv \mathcal{K}_{car}$.

Fekete and Szegö [5] established the estimate $|a_3 - \mu a_2^2|$, where μ is real and $f \in \mathcal{S}$. Further, the upper bound of $|a_3 - \mu a_2^2|$ for various classes of analytic functions were extensively studied by several authors. There is another very useful functional $J_{n,m}(f) = a_n a_m - a_{m+n-1}$, $n, m \in \mathbb{N} - \{1\}$, which was investigated by Ma [8] and it is known as generalized Zalcman functional. The functional $J_{2,3}(f) = a_2 a_3 - a_4$ is a specific case of the generalized Zalcman functional. Various authors including Khan et al. [7], Mohamad and Wahid [12] and Cho et al. [3], computed the upper bound for the functional $J_{2,3}(f)$ over different subclasses of analytic functions as it plays very important role in finding the bounds for the third Hankel determinant.

In the present paper, we establish the upper bounds for the initial coefficients, Fekete-Szegö inequality and bound for the Zalcman functional for the class $\mathcal{M}^{\alpha}_{car}$. Also various known results follow as particular cases.

2 Preliminary Results

By $\mathcal{P}$, we denote the class of analytic functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

whose real parts are positive in E.

To prove our main results, we shall make use of the following lemmas:

Lemma 2.1. [2] ([14, 6]) *If $p \in \mathcal{P}$, then*

$$|p_k| \leq 2, k \in \mathbb{N},$$

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2},$$

$$|p_{i+j} - \mu p_i p_j| \leq 2, 0 \leq \mu \leq 1,$$

and for complex number ρ, we have

$$|p_2 - \rho p_1^2| \leq 2 \max\{1, |2\rho - 1|\}.$$

Lemma 2.2. ([1]). *Let $p \in \mathcal{P}$, then*

$$|Jp_1^3 - Kp_1p_2 + Lp_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|.$$

In particular, it is proved in [14] *that*

$$|p_1^3 - 2p_1p_2 + p_3| \leq 2.$$

3 Main Results

Theorem 3.1. *If $f \in \mathcal{M}^{\alpha}_{car}$, then*

$$|a_2| \leq \frac{4}{3(1+\alpha)}, \tag{3.1}$$

$$|a_3| \leq \frac{3\alpha^2+30\alpha+11}{9(1+2\alpha)(1+\alpha)^2}, \tag{3.2}$$

and

$$|a_4| \leq \frac{180\alpha^3+940\alpha^2+444\alpha+68}{81(1+2\alpha)(1+3\alpha)(1+\alpha)^3}. \tag{3.3}$$

The bounds are sharp.

Proof. As $f \in \mathcal{M}^{\alpha}_{car}$, by the principle of subordination, we have

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = 1 + \frac{4}{3}w(z) + \frac{2}{3}(w(z))^2. \tag{3.4}$$

Define $p(z) = \dfrac{1+w(z)}{1-w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$, which implies $w(z) = \dfrac{p(z)-1}{p(z)+1}$.

On expanding, we have

$$(1-\alpha)\tfrac{zf'(z)}{f(z)} + \alpha\tfrac{(zf'(z))'}{f'(z)} = 1 + (1+\alpha)a_2 z + \left[2(1+2\alpha)a_3 - (1+3\alpha)a_2^2\right] z^2$$

$$+ \left[3(1+3\alpha)a_4 - 3(1+5\alpha)a_2a_3 + (1+7\alpha)a_2^3\right] z^3 + ... \tag{3.5}$$

Also

$$1 + \frac{4}{3}w(z) + \frac{2}{3}(w(z))^2 = 1 + \frac{2}{3}p_1 z + \left(\frac{2}{3}p_2 - \frac{p_1^2}{6}\right) z^2 + \left(\frac{2}{3}p_3 - \frac{1}{3}p_1p_2\right) z^3 + ... \tag{3.6}$$

Using (3.5) and (3.6), (3.4) yields

$$1 + (1+\alpha)a_2 z + \left[2(1+2\alpha)a_3 - (1+3\alpha)a_2^2\right] z^2$$
$$+ \left[3(1+3\alpha)a_4 - 3(1+5\alpha)a_2a_3 + (1+7\alpha)a_2^3\right] z^3 + ...$$

$$= 1 + \frac{2}{3}p_1 z + \left(\frac{2}{3}p_2 - \frac{p_1^2}{6}\right) z^2 + \left(\frac{2}{3}p_3 - \frac{1}{3}p_1p_2\right) z^3 + ... \tag{3.7}$$

On equating the coefficients of z, z^2 and z^3 in (3.7) and on simplification, we obtain

$$a_2 = \frac{2}{3(1+\alpha)}p_1, \tag{3.8}$$

$$a_3 = \frac{1}{2(1+2\alpha)}\left[\frac{2}{3}p_2 + \left(\frac{5+18\alpha-3\alpha^2}{18(1+\alpha)^2}\right)p_1^2\right], \tag{3.9}$$

and

$$a_4 = \frac{1}{9(1+3\alpha)}\left[2p_3 + \frac{1+7\alpha-2\alpha^2}{(1+\alpha)(1+2\alpha)}p_1p_2 + \frac{-45\alpha^3+37\alpha^2-15\alpha-1}{18(1+2\alpha)(1+\alpha)^3}p_1^3\right]. \tag{3.10}$$

Using first inequality of Lemma 2.1 in (3.8), the result (3.1) is obvious.

From (3.9), we have

$$|a_3| = \frac{1}{3(1+2\alpha)}\left|p_2 - \frac{3}{2}\left(\frac{3\alpha^2-18\alpha-5}{18(1+\alpha)^2}\right)p_1^2\right|. \tag{3.11}$$

Using fourth inequality of Lemma 2.1 in (3.11), the result (3.2) can be easily obtained.

Furthermore, on applying Lemma 2.2 in (3.10), the result (3.3) is obvious. □

Remark 3.1. The results of Theorem 3.1 are sharp and the equality is attained in (3.1), (3.2) and (3.3) for the function f given by

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = 1 + \frac{4}{3}z + \frac{2}{3}z^2.$$

Proof. The expansion of $(1-\alpha)\dfrac{zf'(z)}{f(z)}+\alpha\dfrac{(zf'(z))'}{f'(z)}=1+\frac{4}{3}z+\frac{2}{3}z^2$, yields

$1+(1+\alpha)a_2z+\left[2(1+2\alpha)a_3-(1+3\alpha)a_2^2\right]z^2+\left[3(1+3\alpha)a_4-3(1+5\alpha)a_2a_3+(1+7\alpha)a_2^3\right]z^3+...=1+\frac{4}{3}z+\frac{2}{3}z^2$.

On equating the coefficients of z, it gives

$(1+\alpha)a_2=\frac{4}{3}$, which implies $|a_2|=\dfrac{4}{3(1+\alpha)}$ and it gives equality in (3.1).

Equating the coefficients of z^2, we obtain

$$2(1+2\alpha)a_3-(1+3\alpha)a_2^2=\frac{2}{3}.$$

On sustituting the value of a_2, we can easily obtain

$$|a_3|=\frac{3\alpha^2+30\alpha+11}{9(1+2\alpha)(1+\alpha)^2},$$

which shows equality in (3.2).

Further equating the coefficients of z^3, we get

$$3(1+3\alpha)a_4-3(1+5\alpha)a_2a_3+(1+7\alpha)a_2^3=0.$$

On substituting the values of a_2 and a_3 and after simplification, it is obvious to get

$$|a_4|=\frac{180\alpha^3+940\alpha^2+444\alpha+68}{81(1+2\alpha)(1+3\alpha)(1+\alpha)^3},$$

which shows equality in (3.3).

For $\alpha=0$, Theorem 3.1 yields the following result proved by Shi et al. [17]: □

Corollary 3.1. *If $f\in\mathcal{S}^*_{car}$, then*

$$|a_2|\le\frac{4}{3},|a_3|\le\frac{11}{9},|a_4|\le\frac{68}{81}.$$

On putting $\alpha=1$ in Theorem 3.1, the following result due to Shi et al. [17] can be easily obtained:

Corollary 3.2. *If $f\in\mathcal{K}_{car}$, then*

$$|a_2|\le\frac{2}{3},|a_3|\le\frac{11}{27},|a_4|\le\frac{17}{81}.$$

Theorem 3.2. *If $f\in\mathcal{M}^{\alpha}_{car}$, then*

$$|a_3-\mu a_2^2|\le\frac{2}{3(1+2\alpha)}max\left\{1,\frac{16\mu(1+2\alpha)-3\alpha^2-30\alpha-11}{6(1+\alpha)^2}\right\}. \tag{3.12}$$

Proof. From (3.8) and (3.9), we have

$$|a_3-\mu a_2^2|=\frac{1}{3(1+2\alpha)}\left|p_2-\frac{16\mu(1+2\alpha)+3\alpha^2-18\alpha-5}{12(1+\alpha)^2}p_1^2\right|. \tag{3.13}$$

Using fourth inequality of Lemma 2.1, (3.13) yields

$$|a_3-\mu a_2^2|\le\frac{2}{3(1+2\alpha)}max\left\{1,\frac{16\mu(1+2\alpha)-3\alpha^2-30\alpha-11}{6(1+\alpha)^2}\right\}. \tag{3.14}$$

Hence, the result (3.12) is obvious from (3.14).

For $\mu=1$, the result (3.12) yields

$$|a_3-a_2^2|\le\frac{2}{3(1+2\alpha)}max\left\{1,\frac{5+2\alpha-3\alpha^2}{6(1+\alpha)^2}\right\}.$$

But $\dfrac{5+2\alpha-3\alpha^2}{6(1+\alpha)^2}\le1$, for $0\le\alpha\le1$.

Hence, we have

$$|a_3-a_2^2|\le\frac{2}{3(1+2\alpha)}. \tag{3.15}$$

For $\alpha=0$, the following result is obvious from Theorem 3.2: □

Corollary 3.3. *If $f \in \mathcal{S}^*_{car}$, then*

$$|a_3 - \mu a_2^2| \leq \frac{2}{3} max\left\{1, \frac{16\mu - 11}{6}\right\}.$$

For $\alpha = 1$, Theorem 3.2 agrees with the following result:

Corollary 3.4. *If $f \in \mathcal{K}_{car}$, then*

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} max\left\{1, \frac{12\mu - 11}{6}\right\}.$$

For $\mu = 1, \alpha = 0$, Theorem 3.2 yields the following result:

Corollary 3.5. *If $f \in \mathcal{S}^*_{car}$, then*

$$|a_3 - a_2^2| \leq \frac{2}{3}.$$

For $\mu = 1, \alpha = 1$, Theorem 3.2 gives the following result:

Corollary 3.6. *If $f \in \mathcal{K}_{car}$, then*

$$|a_3 - a_2^2| \leq \frac{2}{9}.$$

Theorem 3.3. *If $f \in \mathcal{M}^{\alpha}_{car}$, then*

$$|a_2 a_3 - a_4| \leq \frac{72\alpha^3 + 216\alpha^2 + 340\alpha + 68}{81(1+\alpha)^2(1+2\alpha)(1+3\alpha)}. \tag{3.16}$$

Proof. Using (3.8), (3.9) and (3.10), we have $a_2 a_3 - a_4$

$$= \frac{1}{81(1+\alpha)^2(1+2\alpha)(1+3\alpha)}\left[(9\alpha^2 + 49\alpha + 8)p_1^3 - 9(1+\alpha)(-1+\alpha-2\alpha^2)p_1 p_2 + 18(1+\alpha)^2(-1-2\alpha)p_3\right]. \tag{3.17}$$

Taking modulus and on applying Lemma 2.2, the result (3.16) is obvious from (3.17). □

For $\alpha = 0$, Theorem 3.3 yields the following result:

Corollary 3.7. *If $f \in \mathcal{S}^*_{car}$, then*

$$|a_2 a_3 - a_4| \leq \frac{68}{81}.$$

For $\alpha = 1$, Theorem 3.3 yields the following result:

Corollary 3.8. *If $f \in \mathcal{K}_{car}$, then*

$$|a_2 a_3 - a_4| \leq \frac{29}{162}.$$

4 Conclusion and Open Problems

Till now, many researchers have studied the coefficient problems for various fundamental subclasses of analytic functions, but not much work has been done on the coefficients of subclasses of alpha-convex functions as it involves some lengthy and complicated calculations. In the present investigation, a new subclass of alpha-convex functions is introduced by subordinating to the cardioid domain. We establish the bounds for the first three coefficients, Fekete-Szegö inequality and Zalcman functional for the class $\mathcal{M}^{\alpha}_{car}$. The results obtained here, generalize the results of various authors. The results of this paper can be extended towards the estimation of third and fourth Hankel determinants and also this work will motivate the other researchers to study some more generalized classes of functions.

Acknowledgement. The authors are very much grateful to the Editor and referees for their valuable suggestions to revise the paper.

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