ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 238-242**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

GROUP ANALYSIS FOR KLEIN-GORDON EQUATION VIA THEIR SYMMETRIES

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(Received: March 29, 2023*; In format : April* 04, 2023*; Revised : May* 24. 2023*; Accepted : June* 15, 2023*)*

DOI: https://doi.org/10.58250/jnanabha.2023.53128

Abstract

In the present paper we have obtained the one parameter groups and symmetry transformations associated to the classical symmetries of the Klein-Gordon (KG) equation, we have also constructed an optimal system of two dimensional sub-algebras of the KG equation which provides the preliminary classification of group invariant solutions and yield the most general group invariant solution.
2020 Mathematical Sciences Classification: 17B66, 22E70, 35Q75, 46N50, 70G65, 81R05
Keywords and Phrases: Space Invariance, Translation, Rotation, Hyperbolic Rotation.

1 Introduction

Symmetry method for differential equations, was originally developed by Lie [8] these methods without any doubt are very useful and algorithmic for analyzing and solving linear and nonlinear differential equations. Classification of differential equation as well as linearization of them are some other important applications of symmetry transformation approach. Symmetry groups of a system of partial differential equation is a group of transformations G on the space of independent and dependent variables which has the property that the elements of G transform solution of the system to other solution of the system. The general prolongation formula for computing the symmetry groups for infinitesimal generators of a group of transformations was given by Olver [11, 12, 13] for obtaining the group invariance solution of linear and non-linear differential equation. The group invariant solution of complex modified KdV equation has been studied by Hyzel [3] and for the differential equation describing the radial jet having finite fluid velocity at orifice has been studied by Naeem & Naz [9]. Pal et al. [14, 15] obtained the group invariant solution with the help of infinitesimal parameter. Hereman et al. [5], obtained the exact travelling wave solutions of KG equation with cubic nonlinearity by using direct algebraic method. Ye and Zhang [16], obtained exact travelling wave solutions of KG equation with cubic nonlinearity by using the bifurcation method and qualitative theory of dynamical systems. Dehghan and Ali [2], obtained numerical solutions of KG equation with quadratic and cubic nonlinearity by using radial basis function and analyze the accuracy of their results with the analytical solutions. Jang [6], obtained the travelling wave solutions of nonlinear KG equations. Gupta and Sharma [4] obtained the exact travelling wave solutions for the KG equation with cubic nonlinearity by using First Integral Method. Other researcher also applied the different approaches to obtain the invariant solution of KG equation [7, 10].

In this paper we extend the application of general prolongation formula to find the most general solution of the KG equation

$$\left(1/c^2\right) u_{tt} = u_{xx} + u_{yy} - \left(m^2c^2/\hbar^2\right) u$$

which use to model the two dimensional motion of free particle with mass m (Bates[1]), where u is wave function, c denotes the velocity of light and $\hbar$ is plank constant.

2 Solution of Klein-Gordon Equation

We find the group invariant solution by calculating the symmetries for two-dimensional KG equation for the motion of free particle with mass m. The equation

$$\left(1/c^2\right) u_{tt} - u_{xx} - u_{yy} + \left(m^2c^2/\hbar^2\right) u = 0 \tag{2.1}$$

which is the second order differential equation with three independent variables and one dependent variable. A vector field on X $\times$ U takes the form

$$v = \xi(x,y,t,u)\partial_x + \eta(x,y,t,u)\partial_y + \tau(x,y,t,u)\partial_t + \phi(x,y,t,u)\partial_u, \tag{2.2}$$

where ξ, η, τ and ϕ are the smooth coefficient functions. Using General Prolongation formula (Olver[13], equation (2.38), page 110) the second prolongation of v

$$pr^{(2)}v = v + \phi^x\left(\partial/\partial u_x\right) + \phi^y\left(\partial/\partial u_y\right) + \phi^t\left(\partial/\partial u_t\right) + \phi^{xx}\left(\partial/\partial u_{xx}\right) + \phi^{xy}\left(\partial/\partial u_{xy}\right)$$
$$+ \phi^{xt}\left(\partial/\partial u_{xt}\right) + \phi^{yy}\left(\partial/\partial u_{yy}\right) + \phi^{yt}\left(\partial/\partial u_{yt}\right) + \phi^{tt}\left(\partial/\partial u_{tt}\right), \tag{2.3}$$

and the coefficients present in (2.3) are calculated by using (Olver[13], equation (2.39), page 110), and by the use of infinitesimal criterion of invariance (Olver[13], equation (2.26), page 104) the two-dimensional KG equation takes the form

$$\left(\phi^{tt}/c^2\right) - \phi^{xx} - \phi^{yy} + \left(m^2c^2/\hbar^2\right)\phi = Q\left(\left(u_{tt}/c^2\right) - u_{xx} - u_{yy} + \left(m^2c^2/\hbar^2\right)u\right), \tag{2.4}$$

in which $Q\left(x,y,t,u^{(2)}\right)$ depend up-to second order derivatives of u. By substituting the values of $\phi^{tt}, \phi^{xx}, \phi^{yy}$ and ϕ in equation (2.4) and equating the coefficients of the terms in the first and second order partial derivatives of u, the determining equations for the symmetry group of the two-dimensional KG equation for a free particle are found as follows

***Table* 2.1**

Monomial	Coefficient	Equation Number	Monomial	Coefficient	Equation Number
1	$\frac{\phi_{tt}}{c^2} - \phi_{xx} - \phi_{yy} + \frac{m^2c^2\phi}{\hbar^2} - \frac{m^2c^2Q\,u}{\hbar^2} = 0$	(40)	u_{yt}	$-\frac{2\eta_t}{c^2} + 2\tau_y = 0$	(20)
u_x	$-\frac{\xi_{tt}}{c^2} - \left(2\phi_{xu} - \xi_{xx} - \xi_{yy}\right) = 0$	(39)	u_{xy}	$2\eta_x + 2\xi_y = 0$	(19)
u_y	$-\frac{\eta_{tt}}{c^2} - \left(2\phi_{yu} - \eta_{xx} - \eta_{yy}\right) = 0$	(38)	u_{tt}	$\frac{\left(\phi_u - 2\tau_t\right)}{c^2} = \frac{Q}{c^2}$	(18)
u_t	$\frac{\left(2\phi_{tu} - \tau_{tt}\right)}{c^2} + \tau_{xx} + \tau_{yy} = 0$	(37)	u_{xx}	$-\left(\phi_u - 2\xi_x\right) = -Q$	(17)
u_x^2	$-\left(\phi_{uu} - 2\xi_{xu}\right) = 0$	(36)	u_{yy}	$-\left(\phi_u - 2\eta_y\right) = -Q$	(16)
u_y^2	$-\left(\phi_{uu} - 2\eta_{yu}\right) = 0$	(35)	u_xu_{tt}	$-\frac{\xi_u}{c^2} = 0$	(15)
u_t^2	$\frac{\left(\phi_{uu} - 2\tau_{tu}\right)}{c^2} = 0$	(34)	u_yu_{tt}	$-\frac{\eta_u}{c^2} = 0$	(14)
u_x^3	$\xi_{uu} = 0$	(33)	u_tu_{tt}	$-\frac{3\tau_u}{c^2} = 0$	(13)
u_y^3	$\eta_{uu} = 0$	(32)	u_xu_{xx}	$3\xi_u = 0$	(12)
u_t^3	$-\frac{\tau_{uu}}{c^2} = 0$	(31)	u_yu_{xx}	$\eta_u = 0$	(11)
u_xu_t	$-\frac{2\xi_{tu}}{c^2} + 2\tau_{xu} = 0$	(30)	u_tu_{xx}	$\tau_u = 0$	(10)
u_yu_t	$-\frac{2\eta_{tu}}{c^2} + 2\tau_{yu} = 0$	(29)	u_xu_{yy}	$\xi_u = 0$	(9)
u_xu_y	$2\xi_{yu} + 2\eta_{xu} = 0$	(28)	u_yu_{yy}	$3\eta_u = 0$	(8)
$u_xu_t^2$	$-\frac{\xi_{uu}}{c^2} = 0$	(27)	u_tu_{yy}	$\tau_u = 0$	(7)
$u_yu_t^2$	$-\frac{\eta_{uu}}{c^2} = 0$	(26)	u_tu_{xt}	$-\frac{2\xi_u}{c^2} = 0$	(6)
$u_x^2u_t$	$\tau_{uu} = 0$	(25)	u_tu_{yt}	$-\frac{2\eta_u}{c^2} = 0$	(5)
$u_y^2u_t$	$\tau_{uu} = 0$	(24)	u_xu_{xy}	$2\eta_u = 0$	(4)
$u_x^2u_y$	$\eta_{uu} = 0$	(23)	u_xu_{xt}	$2\tau_u = 0$	(3)
$u_y^2u_x$	$\xi_{uu} = 0$	(22)	u_yu_{xy}	$2\xi_u = 0$	(2)
u_{xt}	$-\frac{2\xi_t}{c^2} + 2\tau_x = 0$	(21)	u_yu_{yt}	$2\eta_u = 0$	(1)

The requirement for equations (1) to (15) is that ξ, η and τ are independent of u, equations (16), (17) and (18) give $\xi_x = \tau_t = \eta_y$, equations (19), (20) and (21) give $\left(\xi_t/c^2\right) = \tau_x, \left(\eta_t/c^2\right) = \tau_y$ and $\eta_x = -\xi_y$, equations (34), (35) and (36) give $\phi = \beta u + \alpha$ where $\alpha = \alpha(x, y, t)$ and $\beta = \beta(x, y, t)$ are functions. From the equation (37), (38) and (39) we get $\beta_x = 0, \beta_y = 0$ and $\beta_t = 0$, from (40) we find $\beta = Q = c_4\left(\hbar^2/m^2c^2\right)$. The most general infinitesimal symmetry of the two-dimensional KG equation in ideal fluid has coefficient function of the form $\xi = c_5 y + (c_6 t + c_2)/c^2, \eta = -c_5\, x + (c_7 t + c_2)/c^2, \tau = c_6 x + c_7 y + c_1 c^2$ and $\phi = c_4\left(\hbar^2/m^2c^2\right)u + \alpha$ where $c_1, \ldots, c_7$ are arbitrary constant and α is an arbitrary solution of the KG equation. The Lie algebras of infinitesimal symmetries of two-dimensional KG equation for a free particle with mass m is spanned by the seven vector fields $v_1 = c^2\partial_t, v_2 = c^2\partial_x, v_3 = c^2\partial_y, v_4 = \left(\hbar^2/m^2c^2\right)u\partial_u, v_5 = y\partial_x - x\partial_y, v_6 = c^2 t\partial_x + x\partial t, v_7 = c^2 t\partial_x + x\partial_t$, and the infinite-dimensional sub-algebra $v_\alpha = \alpha\partial_u$ where α is an arbitrary solution of two-dimensional KG equation. The commutation relation between these vector fields are given by the following

***Table* 2.2:** Commutation-Relation

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_α
v_1	0	0	0	0	0	c^2v_2	c^2v_3	$c^2v_{\alpha_t}$
v_2	0	0	0	0	$-v_3$	v_1	0	$c^2v_{\alpha_x}$
v_3	0	0	0	0	v_2	0	v_1	$c^2v_{\alpha_y}$
v_4	0	0	0	0	0	0	0	$-\left(\frac{\hbar^2}{m^2c^2}\right)v_\alpha$
v_5	0	v_3	$-v_2$	0	0	v_7	$-v_6$	v_{α_1}
v_6	$-c^2v_2$	$-v_1$	0	0	$-v_7$	0	$-c^2v_5$	v_{α_2}
v_7	$-c^2v_3$	0	$-v_1$	0	v_6	c^2v_5	0	v_{α_3}
v_α	$-c^2v_{\alpha_t}$	$-c^2v_{\alpha_x}$	$-c^2v_{\alpha_y}$	$\left(\frac{\hbar^2}{m^2c^2}\right)v_\alpha$	$-v_{\alpha_1}$	$-v_{\alpha_2}$	$-v_{\alpha_3}$	0

where $\alpha_1 = y\alpha_x - x\alpha_y, \alpha_2 = tc^2\alpha_x + x\alpha_t, \alpha_3 = tc^2\alpha_y + y\alpha_t$.

The one-parameter groups G_i generated by the v_i are given as follows

$G_1 : \left(x, y, t + c^2\varepsilon, u\right), G_2 : \left(x + c^2\varepsilon, y, t, u\right), G_3 : \left(x, y + c^2\varepsilon, t, u\right)$,

$G_4 : \left(x, y, t, e^{\left(\frac{\hbar^2\varepsilon}{m^2c^2}\right)}u\right), G_5 : (x\cos\varepsilon + y\sin\varepsilon, y\cos\varepsilon - x\sin\varepsilon, t, u)$,

$G_6 : (x\cosh c\varepsilon + tc\sinh c\varepsilon, y, t\cosh c\varepsilon + (x/c)\sinh c\varepsilon, u)$,

$G_7 : (x, y\cosh c\varepsilon + tc\sinh\varepsilon, t\cosh c\varepsilon + (y/c)\sinh c, \varepsilon)$,

$G_\alpha : (x, y, t, u + \varepsilon\alpha)$ where each G_i is a symmetry group.

If we take $u = f(x, y, t)$ be a solution of the KG equation then the functions

$u^{(1)} = f\left(x, y, t - c^2\varepsilon\right), u^{(2)} = f\left(x - c^2\varepsilon, y, t\right), u^{(3)} = f\left(x, y - c^2\varepsilon, t\right)$,

$u^{(4)} = e^{\left(\frac{\hbar^2\varepsilon}{m^2c^2}\right)}f(x, y, t), u^{(5)} = f(x\cos\varepsilon - y\sin\varepsilon, y\cos\varepsilon + x\sin\varepsilon, t)$,

$u^{(6)} = f(x\cosh c\varepsilon - tc\sinh c\varepsilon, y, t\cosh c\varepsilon - (x/c)\sinh c\varepsilon)$,

$u^{(7)} = f(x, y\cosh c\varepsilon - tc\sinh c\varepsilon, t\cosh c\varepsilon - (y/c)\sinh\varepsilon)$,

$u^{(\alpha)} = f(x, y, t) + \varepsilon\alpha(x, y, t)$ where ε is any real number and $\alpha(x, y, t)$ is any other solution to two dimensional KG equation for a free particle with mass m. At the end the most general solution that we can obtain from a given solution $u = f(x, y, t)$, by group transformations is in the form given below

$$\begin{aligned} u = e^{\left(\frac{\hbar^2\varepsilon_4}{m^2c^2}\right)} f\big(&x\cosh c\varepsilon_6 \cdot \cos\varepsilon_5 - tc\sinh c\varepsilon_6 - y\sin\varepsilon_5 - c^2\varepsilon_2, \\ &y\cosh c\varepsilon_7 \cdot \cos\varepsilon_5 - tc\sinh c\varepsilon_7 + x\sin\varepsilon_5 - c^2\varepsilon_3, \\ &t\cosh c\varepsilon_7 \cdot \cosh c\varepsilon_6 - (y/c)\sinh c\varepsilon_7 - (x/c)\sinh c\varepsilon_6 - c^2\varepsilon_1\big) + \alpha(x, y, t), \end{aligned} \quad (2.5)$$

where $\varepsilon_1, \ldots, \varepsilon_7$ are real constant and α be an arbitrary solution to two-dimensional KG equation for free particle with mass m.

3 Special Case

If we take $\hbar = c = 1$ then equation (2.5) reduces to

$$
\begin{aligned}
u = e^{\left(\frac{\varepsilon_4}{m^2}\right)} f\,(& x\cosh\varepsilon_6 \cdot \cos\varepsilon_5 - t\sinh\varepsilon_6 - y\sin\varepsilon_5 - \varepsilon_2, \\
& y\cosh\varepsilon_7 \cdot \cos\varepsilon_5 - t\sinh\varepsilon_7 + x\sin\varepsilon_5 - \varepsilon_3, \\
& t\cosh\varepsilon_7 \cdot \cosh\varepsilon_6 - y\sinh\varepsilon_7 - x\sinh\varepsilon_6 - \varepsilon_1) \\
& + \alpha(x, y, t),
\end{aligned}
$$

where $\varepsilon_1, \ldots \varepsilon_7$ are real constant and α be an arbitrary solution to two-dimensional KG equation for free particle with $\hbar = c = 1$ and mass m.

If we take y = 0 and $\hbar = c = 1$ then equation (2.5) reduces to

$$
u = e^{\left(\frac{\varepsilon_4}{m^2}\right)} f\,(x\cosh\varepsilon_6 - t\sinh\varepsilon_6 - \varepsilon_2, t\cosh c\varepsilon_6 - x\sinh\varepsilon_6 - \varepsilon_1) + \alpha(x, t),
$$

where $\varepsilon_1, \ldots \varepsilon_7$ are real constant and α be an arbitrary solution to one-dimensional KG equation for free particle with $\hbar = c = 1$ and mass m.

4 Conclusion

In our investigation the symmetry group G_4 and G_α reflects the linearity of two-dimensional KG equation for free particle with mass m. The group G_1 is space invariance symmetry group. The group G_2 and G_3 are time invariance group. The group G_5 represent rotational symmetry group. The group G_6 and G_7 are well-known hyperbolic rotational symmetry group.

Acknowledgement. We are very much thankful to the Editor and Reviewer for their valuable suggestions to bring the paper in its present form.

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