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(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

ON A UNIFIED OBERHETTINGER-TYPE INTEGRAL INVOLVING THE PRODUCT OF BESSEL FUNCTIONS AND SRIVASTAVA POLYNOMIALS

S. C. Pandey and K. Chaudhary
Faculty of Mathematics and Computing, Department of Mathematics and Statistics, Banasthali Vidyapith, Niwai, Rajasthan, India-304022
Email: sharedpandey@yahoo.co.in, koshi1340@gmail.com

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Abstract

The present paper is devoted to derive a generalized Oberhettinger-type integral formula. The derived form of the integral involves a finite product of the Srivastava polynomials with the first-kind Bessel functions. The outcomes are obtained in terms of the Srivastava and Daoust functions. Some of the significant particular cases are also determined.

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1 Introduction

The Bessel function frequently appears in a wide variety of problems pertaining to applied Sciences. The theory of the Bessel function is extensively used to solve several problems including radio physics, nuclear physics, atomic, acoustics, information theory and hydrodynamics. These functions can also be used to solve problems in the fields of mechanics and elasticity. In some recent investigations [7, 8, 9, 10, 13, 14, 15, 17, 18], several authors have proposed a number of interesting integral formulas associated with Bessel functions.

Srivastava [22] introduced a general class of polynomials defined by

$$S_l^m[x] = \sum_{k=0}^{\left[\frac{l}{m}\right]} \frac{(-l)_{mk}}{k!} A_{l,k} x^k, \qquad l = 0, 1, 2, ..., \tag{1.1}$$

where m is an arbitrary positive integer and the coefficients $A_{l,k}$ $(l, k \geq 0)$ are arbitrary constants may be real or complex. Also $(\varrho)_l$ represents the Pochhammer's symbol or rising factorial [23] defined by

$$(\varrho)_l = \frac{\Gamma(\varrho + l)}{\Gamma(\varrho)} = \begin{cases} 1 & \text{if } l = 0, \\ \varrho(\varrho+1)...(\varrho+l-1) & \text{if } l \in \mathbb{N}. \end{cases}$$

For applications of generalized polynomials of Srivastava [22], among others we may also refer to Chaurasia and Pandey [6], Chandel and Sengar [3, 4] and Chandel and Chauhan [5]. On suitably specializing the coefficients $A_{l,k}$ in the definition of $S_l^m[x]$ one can yields several known polynomials as its special cases including, the Jacobi polynomials, the Hermite polynomials, the Legendre polynomials, the Chebyshev polynomials of the first kind and the Chebyshev polynomials of the second kind, the Ultraspherical polynomials, the Gould-Hopper polynomials, the Laguerre polynomials and the Bessel polynomials. For more detail we refer [26].

The classical Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ can be presented in the following series form (see [20, 25])

$$P_n^{(\tau,\varsigma)}(x) = \sum_{k=0}^{n} \frac{(1+\tau)_n (1+\tau+\varsigma)_{n+k}}{(n-k)!k!(1+\tau)_k(1+\tau+\varsigma)_n} \left(\frac{x-1}{2}\right)^k, \tag{1.2}$$

which equivalently can be expressed in terms of the Gauss function as follows

$$P_n^{(\tau,\varsigma)}(x) = \frac{(1+\tau)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, & (n+\tau+\varsigma+1); \\ (1+\tau); & \end{matrix} \left(\frac{1-x}{2}\right)\right]. \tag{1.3}$$

The generalized Wright function ${}_p\Psi_q(x)$ defined by [11, 24]

$$ {}_p\Psi_q(x) = {}_p\Psi_q \left[\begin{matrix} (\alpha_i, a_i)_{1,p}; \\ (\beta_i, b_i)_{1,q}; \end{matrix} \; x \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + a_i k)}{\prod_{j=1}^{q} \Gamma(\beta_j + b_j k)} \frac{x^k}{k!}, \tag{1.4} $$

the coefficients $a_1, ..., a_p$ and $b_1, ..., b_q$, involved in (1.4), are positive real numbers such that

$$ \sum_{i=1}^{p} a_i \leq 1 + \sum_{j=1}^{q} b_j, $$

and $\Gamma(.)$ is the standard Gamma function (see, for more details, [16, 25]).

The Bessel function $J_\nu(x)$ of first kind is defined by (see [2, 20, 27])

$$ J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+\nu+k)} \frac{\left(\frac{x}{2}\right)^{\nu+2k}}{k!}, \tag{1.5} $$

where $Re(\nu) > -1$, $\nu \in \mathbb{C}$ and $x \in \mathbb{C} \setminus \{0\}$.

Srivastava and Daoust [24] proposed multivariable generalized hypergeometric function, given as

$$ F^{A:B';...;B^{(n)}}_{C:D';...;D^{(n)}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = F^{A:B';...;B^{(n)}}_{C:D';...;D^{(n)}} \left[\begin{matrix} [(a):\theta', ..., \theta^{(n)}] : [(b'):\phi']; ...; \\ [(c):\psi', ..., \psi^{(n)}] : [(d'):\delta']; ...; \end{matrix} \right. $$

$$ \left. \begin{matrix} [(b^{(n)}):\phi^{(n)}]; \\ [(d^{(n)}):\delta^{(n)}]; \end{matrix} x_1, ..., x_n \right] = \sum_{m_1,\,...,m_n=0}^{\infty} \Omega(m_1, ..., m_n) \frac{x^{m_1}}{m_1!} \cdots \frac{x^{m_n}}{m_n!}, \tag{1.6} $$

where, for convenience

$$ \Omega(m_1, ..., m_n) = \frac{\prod_{j=1}^{A} (a_j)_{m_1\theta'_j + ... + m_n\theta_j^{(n)}} \prod_{j=1}^{B'} (b'_j)_{m_1\phi'_j} \cdots \prod_{j=1}^{B^{(n)}} (b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C} (c_j)_{m_1\psi'_j + ... + m_n\psi_j^{(n)}} \prod_{j=1}^{D'} (d'_j)_{m_1\delta'_j} \cdots \prod_{j=1}^{D^{(n)}} (d_j^{(n)})_{m_n\delta_j^{(n)}}}, $$

the coefficients $\theta_j^{(k)}$, $j = 1, ..., A$; $\phi_j^{(k)}$, $j = 1, ..., B^{(k)}$; $\psi_j^{(k)}$, $j = 1, ..., C$; $\delta_j^{(k)}$, $j = 1, ..., D^{(k)}$ are real and positive, and (a) abbreviates the array of A parameters $a_1, ..., a_A$, $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}$, $j = 1, ..., B^{(k)}$; $\forall k \in \{1, ..., n\}$, with similar interpretations for (c) and $(d^{(k)})$, $\forall k \in \{1, ..., n\}$; etcetera.

In the present work, we recall the following integral mentioned in the classical monograph by Oberhettinger (see [19], p. 22)

$$ \int_0^{\infty} x^{\delta-1} \left(x + h + \sqrt{x^2 + 2hx}\right)^{-\eta} dx = 2\eta h^{-\eta} \left(\frac{h}{2}\right)^{\delta} \frac{\Gamma(2\delta)\Gamma(\eta-\delta)}{\Gamma(1+\delta+\eta)}, \tag{1.7} $$

provided $0 < Re(\delta) < Re(\eta)$.

The intention of this paper is to propose a unified integral involving the Oberhettinger-type that includes a finite product of the Bessel functions and Srivastava polynomials. The main result in the current investigation is presented in terms of a Theorem. Further, two Corollaries of the main result are derived. Some other interesting well-known special cases of the main result are also determined.

2. Main results

In this section, we derive an integral formula involving a finite product of general class of polynomials and Bessel functions. The outcome is expressed in terms of an elegant form Srivastava and Daoust function, defined above in (1.6).

Theorem 2.1. *For* $\mu, \lambda_i, \nu_q \in \mathbb{C}$, $l_p, s_p \geq 0$, $Re(\nu_q) > -1$, $Re(\sigma_j^{(p)}) > Re(\eta_p) > 0$ *and* $Re\left(\lambda_i + \sum_{q=1}^{w} \lambda_k^{(q)} \nu_q\right) > Re\left(\mu + \sum_{q=1}^{w} \delta_q \nu_q\right) > 0$ $(i = 1, 2, ..., n_1, j = 1, 2, ..., n_2,\ k = 1, 2, ..., n_3,\ p = 1, 2, ..., v,\ q = 1, 2, ..., w)$ *the following integral formula holds true:*

$$ \int_0^{\infty} x^{\mu-1} \prod_{i=1}^{n_1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda_i} \prod_{p=1}^{v} S_{l_p}^{m_p} \left[\xi_p x^{\eta_p} \prod_{j=1}^{n_2} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\sigma_j^{(p)}} \right] $$

$$\times \prod_{q=1}^{w} J_{\nu_q}\left[\zeta_q x^{\delta_q}\prod_{k=1}^{n_3}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_k^{(q)}}\right]dx$$

$$= \frac{\zeta_1^{\nu_1}\ldots\zeta_w^{\nu_w}}{\Gamma(1+\nu_1)\ldots\Gamma(1+\nu_w)} 2^{\left[1-\mu-\sum_{q=1}^{w}(1+\delta_q)\nu_q\right]}\prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\prod_{k=1}^{n_3}(a)^{\left[\mu-\lambda_i-\sum_{q=1}^{w}\nu_q\left(\lambda_k^{(q)}-\delta_q\right)\right]}\sum_{s_1=0}^{\left[\frac{l_1}{m_1}\right]}\ldots\sum_{s_v=0}^{\left[\frac{l_v}{m_v}\right]}$$

$$\times\frac{(-l_1)_{m_1 s_1}(A)_{l_1,s_1}}{s_1!}\left(\frac{\xi_1}{2^{\eta_1}a^{\left(\sigma_j^{(1)}-\eta_1\right)}}\right)^{s_1}\ldots\frac{(-l_v)_{m_v s_v}(A)_{l_v,s_v}}{s_v!}\left(\frac{\xi_v}{2^{\eta_v}a^{\left(\sigma_j^{(v)}-\eta_v\right)}}\right)^{s_v}$$

$$\times\frac{\Gamma\left\{1+\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right\}\Gamma\left\{2\mu+2\sum_{p=1}^{v}\eta_p s_p+2\sum_{q=1}^{w}\delta_q\nu_q\right\}}{\Gamma\left\{\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right\}}$$

$$\times\frac{\Gamma\left\{-\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}-\eta_p\right)s_p+\sum_{q=1}^{w}(\lambda_k^{(q)}-\delta_q)\nu_q\right\}}{\Gamma\left\{1+\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}+\eta_p\right)s_p+\sum_{q=1}^{w}(\lambda_k^{(q)}+\delta_q)\nu_q\right\}}$$

$$\times F^{3:0;\ldots;0}_{2:1;\ldots;1}\left[\begin{matrix}\left[-\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}-\eta_p\right)s_p+\sum_{q=1}^{w}(\lambda_k^{(q)}-\delta_q)\nu_q:2(\lambda_k^{(1)}-\delta_1),\right.\\ \left[1+\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}+\eta_p\right)s_p+\sum_{q=1}^{w}(\lambda_k^{(q)}+\delta_q)\nu_q:2(\lambda_k^{(1)}+\delta_1),\right.\end{matrix}\right.$$

$$\begin{matrix}\left.\ldots,2(\lambda_k^{(w)}-\delta_w)\right],\left[1+\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q:2\lambda_k^{(1)},\ldots,2\lambda_k^{(w)}\right],\\ \left.\ldots,2(\lambda_k^{(w)}+\delta_w)\right],\left[\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q:2\lambda_k^{(1)},\ldots,2\lambda_k^{(w)}\right]:\end{matrix}$$

$$\begin{matrix}\left[2\mu+2\sum_{p=1}^{v}\eta_p s_p+2\sum_{q=1}^{w}\delta_q\nu_q:4\delta_1,\ldots,4\delta_w\right]:\_;\ldots;\_;\\ \left[1+\nu_1:1\right],\ldots,\left[1+\nu_w:1\right];\end{matrix}\quad\frac{-\zeta_1^2}{4^{(1+\delta_1)}(a)^{2\left(\lambda_k^{(1)}-\delta_1\right)}},$$

$$\left.\ldots,\frac{-\zeta_w^2}{4^{(1+\delta_w)}(a)^{2\left(\lambda_k^{(w)}-\delta_w\right)}}\right]. \tag{2.1}$$

Proof. To prove Theorem 2.1, we first express Srivastava polynomials and Bessel functions in series forms given by (1.1) and (1.5) respectively, we have

$$\text{L.H.S. of (2.1)} = \int_0^\infty x^{\mu-1}\left\{\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_1}\ldots\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_{n_1}}\right\}$$

$$\times\sum_{s_1=0}^{\left[\frac{l_1}{m_1}\right]}\ldots\sum_{s_v=0}^{\left[\frac{l_v}{m_v}\right]}\frac{(-l_1)_{m_1 s_1}(A)_{l_1,s_1}}{s_1!}\left(\xi_1 x^{\eta_1}\right)^{s_1}\left\{\left(x+a+\sqrt{x^2+2ax}\right)^{-\sigma_1^{(1)}s_1}\ldots\right.$$

$$\left.\left(x+a+\sqrt{x^2+2ax}\right)^{-\sigma_{n_2}^{(1)}s_1}\right\}\ldots\frac{(-l_v)_{m_v s_v}(A)_{l_v,s_v}}{s_v!}\left(\xi_v x^{\eta_v}\right)^{s_v}\left\{\left(x+a+\right.\right.$$

$$\left.\left.\sqrt{x^2+2ax}\right)^{-\sigma_1^{(v)}s_v}\ldots\left(x+a+\sqrt{x^2+2ax}\right)^{-\sigma_{n_2}^{(v)}s_v}\right\}\sum_{r_1=0}^{\infty}\ldots\sum_{r_w=0}^{\infty}$$

$$\frac{(-1)^{r_1}}{r_1!\Gamma(1+\nu_1+r_1)}\left(\frac{\zeta_1 x^{\delta_1}}{2}\right)^{\nu_1+2r_1}\left\{\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_1^{(1)}(\nu_1+2r_1)}\ldots\right.$$

$$\left.\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_{n_3}^{(1)}(\nu_1+2r_1)}\right\}\ldots\frac{(-1)^{r_w}}{r_w!\Gamma(1+\nu_w+r_w)}\left(\frac{\zeta_w x^{\delta_w}}{2}\right)^{\nu_w+2r_w}$$

$$\left\{\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_1^{(w)}(\nu_w+2r_w)} \ldots \left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_{n_3}^{(w)}(\nu_w+2r_w)}\right\} dx.$$

Now, we interchange the order of summations and integration (permissible with the uniform convergence of the series forms under the given conditions), we obtain

$$= \frac{\left(\frac{\zeta_1}{2}\right)^{\nu_1} \ldots \left(\frac{\zeta_w}{2}\right)^{\nu_w}}{\Gamma(1+\nu_1)\ldots\Gamma(1+\nu_w)} \sum_{s_1=0}^{\left[\frac{l_1}{m_1}\right]} \ldots \sum_{s_v=0}^{\left[\frac{l_v}{m_v}\right]} \frac{(-l_1)_{m_1 s_1} A_{l_1,s_1}}{s_1!}\xi_1^{s_1} \ldots \frac{(-l_v)_{m_v s_v} A_{l_v,s_v}}{s_v!}\xi_v^{s_v}$$

$$\times \sum_{r_1=0}^{\infty} \ldots \sum_{r_w=0}^{\infty} \frac{\left(\frac{-\zeta_1^2}{4}\right)^{r_1}}{r_1!(1+\nu_1)_{r_1}} \ldots \frac{\left(\frac{-\zeta_w^2}{4}\right)^{r_w}}{r_w!(1+\nu_w)_{r_w}} \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\prod_{k=1}^{n_3} \int_0^{\infty} x^{\left[\mu+\sum_{p=1}^{v}\eta_p s_p+\sum_{q=1}^{w}\delta_q(\nu_q+2r_q)-1\right]}$$

$$\times \left(x+a+\sqrt{x^2+2ax}\right)^{-\left[\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}(\nu_q+2r_q)\right]} dx.$$

Now, using the Oberhettinger's integral Eq.(1.7) formula, we get

$$= \frac{\left(\frac{\zeta_1}{2}\right)^{\nu_1} \ldots \left(\frac{\zeta_w}{2}\right)^{\nu_w}}{\Gamma(1+\nu_1)\ldots\Gamma(1+\nu_w)} \sum_{s_1=0}^{\left[\frac{l_1}{m_1}\right]} \ldots \sum_{s_v=0}^{\left[\frac{l_v}{m_v}\right]} \frac{(-l_1)_{m_1 s_1} (A)_{l_1,s_1}}{s_1!}\xi_1^{s_1} \ldots \frac{(-l_v)_{m_v s_v} (A)_{l_v,s_v}}{s_v!}\xi_v^{s_v}$$

$$\times \sum_{r_1=0}^{\infty} \ldots \sum_{r_w=0}^{\infty} \frac{\left(\frac{-\zeta_1^2}{4}\right)^{r_1}}{r_1!(1+\nu_1)_{r_1}} \ldots \frac{\left(\frac{-\zeta_w^2}{4}\right)^{r_w}}{r_w!(1+\nu_w)_{r_w}} 2^{\left[1-\mu-\sum_{p=1}^{v}\eta_p s_p-\sum_{q=1}^{w}\delta_q(\nu_q+2r_q)\right]}$$

$$\times \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\prod_{k=1}^{n_3} (a)^{\left[\mu+\sum_{p=1}^{v}\eta_p s_p+\sum_{q=1}^{w}\delta_q(\nu_q+2r_q)-\lambda_i-\sum_{p=1}^{v}\sigma_j^{(p)}s_p-\sum_{q=1}^{w}\lambda_k^{(q)}(\nu_q+2r_q)\right]}$$

$$\left[\frac{\Gamma\left\{1+\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}(\nu_q+2r_q)\right\}\Gamma\left\{2\sum_{q=1}^{w}\delta_q(\nu_q+2r_q)\right.}{\Gamma\left\{\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}(\nu_q+2r_q)\right\}}\right.$$

$$\frac{\left.+2\mu+2\sum_{p=1}^{v}\eta_p s_p\right\}\Gamma\left\{\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}-\eta_p\right)s_p-\sum_{q=1}^{w}\delta_q(\nu_q+2r_q)\right.}{\Gamma\left\{1+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}+\eta_p\right)s_p+\sum_{q=1}^{w}\delta_q(\nu_q+2r_q)\right.}$$

$$\left.\frac{\left.-\mu+\sum_{q=1}^{w}\lambda_k^{(q)}(\nu_q+2r_q)\right\}}{\left.+\mu+\sum_{q=1}^{w}\lambda_k^{(q)}(\nu_q+2r_q)\right\}}\right].$$

$$= \frac{\zeta_1^{\nu_1}\ldots\zeta_w^{\nu_w}}{\Gamma(1+\nu_1)\ldots\Gamma(1+\nu_w)} 2^{\left[1-\mu-\sum_{q=1}^{w}(1+\delta_q)\nu_q\right]} \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\prod_{k=1}^{n_3} (a)^{\left[\mu-\lambda_i-\sum_{q=1}^{w}\nu_q\left(\lambda_k^{(q)}-\delta_q\right)\right]}$$

$$\times \sum_{s_1=0}^{\left[\frac{l_1}{m_1}\right]} \ldots \sum_{s_v=0}^{\left[\frac{l_v}{m_v}\right]} \frac{(-l_1)_{m_1 s_1}(A)_{l_1,s_1}}{s_1!}\left(\frac{\xi_1}{2^{\eta_1}a^{\left(\sigma_j^{(1)}-\eta_1\right)}}\right)^{s_1} \ldots \frac{(-l_v)_{m_v s_v}(A)_{l_v,s_v}}{s_v!}\left(\frac{\xi_v}{2^{\eta_v}a^{\left(\sigma_j^{(v)}-\eta_v\right)}}\right)^{s_v}$$

$$\times \sum_{r_1=0}^{\infty} \ldots \sum_{r_w=0}^{\infty} \frac{1}{r_1!(1+\nu_1)_{r_1}}\left(\frac{-\zeta_1^2}{4^{(1+\delta_1)}a^{2\left(\lambda_k^{(1)}-\delta_1\right)}}\right)^{r_1} \ldots \frac{1}{r_w!(1+\nu_w)_{r_w}}\left(\frac{-\zeta_w^2}{4^{(1+\delta_w)}a^{2\left(\lambda_k^{(w)}-\delta_w\right)}}\right)^{r_w}$$

$$\times \frac{\Gamma\left\{1+\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right\}\Gamma\left\{2\mu+2\sum_{p=1}^{v}\eta_p s_p+2\sum_{q=1}^{w}\delta_q\nu_q\right\}}{\Gamma\left\{\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right\}}$$

$$\times \frac{\Gamma\left\{-\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}-\eta_p\right)s_p+\sum_{q=1}^{w}(\lambda_k^{(q)}-\delta_q)\nu_q\right\}}{\Gamma\left\{1+\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}+\eta_p\right)s_p+\sum_{q=1}^{w}(\lambda_k^{(q)}+\delta_q)\nu_q\right\}}$$

$$\left[\frac{\left(1+\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right)_{2\sum_{q=1}^{w}\lambda_k^{(q)}r_q}\left(2\sum_{p=1}^{v}\eta_p s_p+2\sum_{q=1}^{w}\delta_q\nu_q\right.}{\left(\lambda_i+\sum_{p=1}^{v}\sigma_j^{(p)}s_p+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right)_{2\sum_{q=1}^{w}\lambda_k^{(q)}r_q}}\right.$$

$$\left.\frac{\left.+2\mu\right)_{4\sum_{q=1}^{w}\delta_q r_q}\left(-\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}-\eta_p\right)s_p+\sum_{q=1}^{w}\left(\lambda_k^{(q)}-\delta_q\right)\nu_q\right)_{2\sum_{q=1}^{w}\left(\lambda_k^{(q)}-\delta_q\right)r_q}}{\left(1+\mu+\lambda_i+\sum_{p=1}^{v}\left(\sigma_j^{(p)}+\eta_p\right)s_p+\sum_{q=1}^{w}\left(\lambda_k^{(q)}+\delta_q\right)\nu_q\right)_{2\sum_{q=1}^{w}\left(\lambda_k^{(q)}+\delta_q\right)r_q}}\right].$$

Now, using the Srivastava and Daoust function Eq.(1.6), we arrive at the desire form given in *RHS* of (2.1).

On substituting $l_p = 0$ (for $p = 1, 2, ..., v$) in Theorem 2.1 the Srivastava polynomial $S_{l_p}^{m_p}(x)$ reduces to unity, i.e., $S_0^{m_p}(x) = 1$ and we can deduce the following Corollary 2.1 based on the main integral presented in Theorem 2.1.

Corollary 2.1. *For* $\mu, \lambda_i, \nu_q \in \mathbb{C}, Re(\nu_q) > -1$ *and* $Re\left(\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q+\lambda_i\right)$ $> Re\left(\sum_{q=1}^{w}\delta_q\nu_q+\mu\right) > 0$ $(i = 1, 2, ..., n_1,\ k = 1, 2, ..., n_3,\ q = 1, 2, ..., w)$, *the following integral formula holds true:*

$$\int_0^\infty x^{\mu-1}\prod_{i=1}^{n_1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_i}\prod_{q=1}^{w}J_{\nu_q}\left[\zeta_q x^{\delta_q}\prod_{k=1}^{n_3}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_k^{(q)}}\right]dx$$

$$=\frac{\zeta_1^{\nu_1}\ldots\zeta_w^{\nu_w}}{\Gamma(1+\nu_1)\ldots\Gamma(1+\nu_w)}2^{\left[1-\mu-\sum_{q=1}^{w}(1+\delta_q)\nu_q\right]}\prod_{i=1}^{n_1}\prod_{k=1}^{n_3}(a)^{\left[\mu-\lambda_i+\sum_{q=1}^{w}\nu_q\left(\delta_q-\lambda_k^{(q)}\right)\right]}$$

$$\times\frac{\Gamma\left\{2\mu+2\sum_{q=1}^{w}\delta_q\nu_q\right\}\Gamma\left\{1+\lambda_i+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right\}\Gamma\left\{-\mu+\lambda_i+\sum_{q=1}^{w}(\lambda_k^{(q)}-\delta_q)\nu_q\right\}}{\Gamma\left\{\lambda_i+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q\right\}\Gamma\left\{1+\mu+\lambda_i+\sum_{q=1}^{w}(\lambda_k^{(q)}+\delta_q)\nu_q\right\}}$$

$$\times F_{2:1;\ldots;1}^{3:0;\ldots;0}\left[\begin{array}{l}\left[-\mu+\lambda_i+\sum_{q=1}^{w}(\lambda_k^{(q)}-\delta_q)\nu_q:2(\lambda_k^{(1)}-\delta_1),\ldots,2(\lambda_k^{(w)}-\delta_w)\right],\\ \left[1+\mu+\lambda_i+\sum_{q=1}^{w}(\lambda_k^{(q)}+\delta_q)\nu_q:2(\lambda_k^{(1)}+\delta_1),\ldots,2(\lambda_k^{(w)}+\delta_w)\right],\end{array}\right.$$

$$\begin{array}{l}\left[1+\lambda_i+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q:2\lambda_k^{(1)},\ldots,2\lambda_k^{(w)}\right],\left[2\mu+2\sum_{q=1}^{w}\delta_q\nu_q:\right.\\ \left[\lambda_i+\sum_{q=1}^{w}\lambda_k^{(q)}\nu_q:2\lambda_k^{(1)},\ldots,2\lambda_k^{(w)}\right]:\left[1+\nu_1:1\right],\ldots,\end{array}$$

$$\left.\begin{array}{l}\left.4\delta_1,\ldots,4\delta_v\right]:\_;\ldots;\_;\\ \left[1+\nu_w:1\right];\end{array}\quad\frac{-\zeta_1^2}{4^{(1+\delta_1)}(a)^{2\left(\lambda_k^{(1)}-\delta_1\right)}},\ldots,\frac{-\zeta_w^2}{4^{(1+\delta_w)}(a)^{2\left(\lambda_k^{(w)}-\delta_w\right)}}\right]. \tag{2.2}$$

On setting $n_1 = 1, n_2 = 1, n_3 = 1, v = 1, w = 1$ in Theorem 2.1, we arrive at the following Corollary 2.2, where the outcome is computed in the form of generalized Wright function.

Corollary 2.2. *For* $\eta,\ \lambda_1,\ \nu_1 \in \mathbb{C},\ l_1,\ s_1 \geq 0,\ Re(\nu_1) > -1,\ Re(\sigma_1^{(1)}) > Re(\eta_1) > 0,$ *and* $Re\left(\lambda_1+\lambda_1^{(1)}\nu_1\right) >$ $Re\left(\mu+\delta_1\nu_1\right) > 0$, *the following integral formula holds true:*

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_1}S_{l_1}^{m_1}\left[\xi_1 x^{\eta_1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\sigma_1^{(1)}}\right]$$

$$\times J_{\nu_1}\left[\zeta_1 x^{\delta_1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda_1^{(1)}}\right]dx=\frac{\zeta_1^{\nu_1}}{\Gamma(1+\nu_1)}2^{[1-\mu-(1+\delta_1)\nu_1]}$$

$$\times (a)^{\left[\mu-\lambda_1+\nu_1\left(\delta_1-\lambda_1^{(1)}\right)\right]} \sum_{s_1=0}^{\left[\frac{l_1}{m_1}\right]} \frac{(-l_1)_{m_1 s_1} A_{l_1,s_1}}{s_1!} \left(\frac{\xi_1}{2^{\eta_1} a^{\left(\sigma_1^{(1)}-\eta_1\right)}}\right)^{s_1}$$

$$\times {}_3\Psi_3 \left[\begin{array}{l} \left[-\mu+\lambda_1+(\sigma_1^{(1)}-\eta_1)s_1+(\lambda_1^{(1)}-\delta_1)\nu_1, 2(\lambda_1^{(1)}-\delta_1)\right], \\ \left[1+\mu+\lambda_1+(\sigma_1^{(1)}+\eta_1)s_1+(\lambda_1^{(1)}+\delta_1)\nu_1, 2(\lambda_1^{(1)}+\delta_1)\right], \end{array}\right.$$

$$\left.\begin{array}{l} \left[1+\lambda_1+\sigma_1^{(1)}s_1+\lambda_1^{(1)}\nu_1, 2\lambda_1^{(1)}\right], \left[2\mu+2\eta_1 s_1+2\delta_1\nu_1, 4\delta_1\right]; \\ \left[\lambda_1+\sigma_1^{(1)}s_1+\lambda_1^{(1)}\nu_1, 2\lambda_1^{(1)}\right], \left[1+\nu_1, 1\right]; \end{array} \frac{-\zeta_1^2}{4^{(1+\delta_1)} a^{2\left(\lambda_1^{(1)}-\delta_1\right)}}\right]. \tag{2.3}$$

3. Special Cases

In this section, we present some of the well-known and interesting special cases which can be determined by specializing the parameters of the Corollaries 2.1 and 2.2.

(i) For $n_1 = 1, n_3 = 1, \delta_q = 0$ and $\lambda_1^{(q)} = 1$ $(q = 1, 2, ..., w)$, Corollary 2.1 reduces to an interesting result given by Choi and Agarwal [7, Theorem 1, p. 671, (2.1)].

(ii) Assuming $n_1 = 1, \delta_q = 1$ and $\lambda_1^{(q)} = 1$ $(q = 1, 2, ..., w)$, Corollary 2.1 produces another known result of Choi and Agarwal [7, Theorem 2, p. 671, (2.2)].

(iii) Substituting $l_1 = 0, S_{l_1}^{m_1}[x] = 1, \delta_1 = 0$ and $\lambda_1^{(1)} = 1$, Corollary 2.2 reduces in to [8, Theorem 1, p. 3, (2.1)] investigated by Choi and Agarwal.

(iv) Taking $l_1 = 0, S_{l_1}^{m_1}[x] = 1, \delta_1 = 1$ and $\lambda_1^{(1)} = 1$, Corollary 2.2 reduces to another result due to Choi and Agarwal [8, Theorem 2, p. 3, (2.2)].

(v) Also, for $\eta_1 = 0, \delta_1 = 0, \xi_1 = \frac{\xi}{2}, m_1 = 1, \lambda_1^{(1)} = 1, \sigma_1^{(1)} = 1, A_{l_1,s_1} = \binom{l_1+\tau}{l_1}\frac{(l_1+\tau+\varsigma+1)_{s_1}}{(\tau+1)_{s_1}}, S_{l_1}^1(x) = P_{l_1}^{(\tau,\varsigma)}(1-2x)$ and using equation (1.3) and (1.2), the Corollary 2.2 reduces to [13, Theorem 1, p. 341, (2.1)] and [13, Theorem 2, p. 343, (2.5)] respectively, presented by Khan et al.

Remark: For the appropriate settings of parameters in the proposed integral, i.e., Theorem 2.1, one can derive a number of interesting integrals. Further, an interesting form involving the product of several Bessel functions established by Exton and Srivastava [12, *p.* 4, (2.8)] can be deduced as a particular case of the Theorem 2.1. Furthermore, one can also obtain the legendary and classical integral formulas investigated by Bailey [1, *p.* 38, (1.2)] and Rice [21, *p.* 60, (2.6), (2.8)] for proper choice of parameter in the Theorem 2.1.

4. Conclusion

By the use of Oberhettinger type integral formula, in the current investigation by the applications of the Oberhettinger integral formula we have establised some of the results involving the Bessel functions and general class of polynomial whose outcomes in terms of the Srivastava and Daoust functions and also the Bessel function of the first kind is a special case of Fox H-function [8, *p.* 9, (4.1)]. Consequently, all the result of this paper can easily converted in terms of the Fox H-function for the appropriate settings of parameters. We can find some other results in terms of the Srivastava and Daoust functions for the proper settings of parameters in the general class of polynomial.

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