ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 212-218**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

TRIPLE SERIES EQUATIONS INVOLVING GENERALIZED LAGUERRE POLYNOMIALS

***Omkar Lal Shrivastava[1] , Kuldeep Narain[2] and Sumita Shrivastava[3]**
[1]Department of Mathematics, Government Kamladevi Rathi Girls Postgraduate College, Rajnandgaon, Chhattisgarh, India-491441
[2]Department of Mathematics, Kymore Science College, Kymore, Madhya Pradesh, India-483880
[3]Department of Economics, Government Digvijay Postgraduate College, Rajnandgaon, Chhattisgarh, India-491441
Email: omkarlal@gmail.com, kuldeepnarain2009@gmqail.com, sumitashrivastava9@gmail.com
* Corresponding author email: omkarlal@gmail.com
(Received: November 05, 2022*; In format: November* 10, 2022*' Revised: April* 04, 2023*; Accepted: April* 08, 2023*)*

DOI: https://doi.org/10.58250/jnanabha.2023.53125

Abstract

In this paper, the solutions of two sets of triple series equations involving generalized Laguerre polynomials have been obtained by reducing them to a Fredholm integral equation of second kind. In each case, the problem is reduced to the solution of a Fredholm integral equation of the second kind. We consider certain triple series equations involving generalized Laguerre polynomials which are generalization of those considerd by Lawndes-Srivastava .Connected to this work solutions have been considered by Sneddon, Lowndes and Srivastava ,Dwivedi and Trivedi, Singh *et al.*, Narain, Srivastava - Panda etc.
2020 Mathematical Sciences Classification: 45XX, 45F10, 33C45.
Keywords and Phrases: Triple Series equations, generalized Laguerre polynomials, Fredholm integral equation.

1 Introduction

The problem of dual and triple series equations arises during solving many boundary value problems in Sneddon [17, Chap.5] and Srivastava [20] of Mathematical physics. Earlier Lowndes [5-8] has also obtained solutions for some dual and triple series equations involving Jacobi and Laguerre polynomials. Chandel [3] discussed a problem on Heat conduction employing dual series equation involving Legendre polynomials. Lowndes and Srivastava [9] have shown that a certain class of triple series equations involving the generalized Laguerre polynomials can be reduced to some triple integral equations. Srivastava [18-24] and Srivastava - Panda [25] have investigated the solutions of some dual and triple series equations involving the generalized Laguerre polynomials, Bateman-k functions and the Konhauser biorthogonal polynomials. Ashour , Ismail and Mansour [1] have solved dual and triple series equations involving q-orthogonal polynomials with some examples. Recently, Mudaliar and Narain [11] have solved certain dual and quadruple series equations involving generalized Laguerre polynomials and also Narain [13] has solved triple series equations involving Laguerre polynomials with Matrix Augument. Certain quadruple series equations involving Laguerre polynomials are solved by Shrivastava and Narain [15] recently.Closed-form solutions of triple series equations involving Laguerre polynomials are recently obtained by Singh, Rokne and Dhaliwal [16]. Dwivedi and Trivedi [4] have obtained the solution of triple series equations involving Jacobi and Laguerre polynomials by reducing them to a Fredholm integral equation of second kind. We consider certain triple series equations involving generalized Laguerre polynomials which are generalization of those considered by Sneddon, Lowndes and Srivastava , Dwivedi and Trivedi, Singh *et al.*, Narain, Srivastava - Panda etc. connected to this work. In present paper, the solutions of two sets of triple series equations involving generalized Laguerre polynomials have been obtained. The triple series equations of the first kind

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)} L_n^{(\sigma)}(x) = g_1(x), 0 \le x < a, \tag{1.1}$$

$$\sum_{n=0}^{\infty}\frac{A_n\,(1+H_n)}{\Gamma(\alpha+n+1)}L_n^{(\nu)}(x)=f(x),a<x<b, \tag{1.2}$$

$$\sum_{n=0}^{\infty}\frac{A_n}{(\beta+n+1)}L_n^{(\sigma)}(x)=h_1(x),b<x<\infty, \tag{1.3}$$

and the triple series equations of the second kind

$$\sum_{n=0}^{\infty}\frac{A_n\,(1+H_n)}{\Gamma(\alpha+n+1)}L_n^{(\nu)}(x)=g(x),0\leq x<a, \tag{1.4}$$

$$\sum_{n=0}^{\infty}\frac{A_n}{\Gamma(\beta+n+1)}L_n^{(\sigma)}(x)=f_1(x),a<x<b, \tag{1.5}$$

$$\sum_{n=0}^{\infty}\frac{A_n\,(1+H_n)}{\Gamma(\alpha+n+1)}L_n^{(\nu)}(x)=h(x),b<x<\infty, \tag{1.6}$$

where, A_n is an unknown coefficient, $L_n^{(\alpha)}(x)$ is the generalized Laguerre polynomial $f(x), f_1(x), g(x), g_1(x), h(x)$ and $h_1(x)$ are known functions of x and the parameters $\alpha, \beta, \nu, \sigma$ all are > -1; can be reduced to that of solving a Fredholm integral equation of second kind. It is assumed that the series (1.1) to (1.6) are uniformly convergent and the known functions f, f_1, g, g_1, h, h_1 and their derivatives are continuous bounded and integrable in the interval of their definition.

The analysis throughout is formal and no attempt has been made to justify the various limiting processes.

2 Some Useful Results

Here are some useful results for ready reference:

The orthogonality relation for the Laguerre polynomials is

$$\int_0^{\infty}x^{\alpha}\quad e^{-x}L_m(\alpha;x)L_n(\alpha;x)dx=\frac{\Gamma(\alpha+1+n)}{\Gamma(n+1)}\delta_{m,n},\alpha>-1, \tag{2.1}$$

where $\delta_{m,n}$ is Kronecker delta.

From equations (2.6) and (3. 7) due to Srivastava ([18],p.589 and p.591]) it is easily shown that

$$(\lambda)\Gamma(1-\lambda)S(r,x)=\Gamma(\lambda)\Gamma(1-\lambda)r^{\sigma}x^{\nu}\sum_{n=0}^{\infty}\frac{\Gamma(\beta+n+1)}{\Gamma(\alpha+n+1)}.\frac{\Gamma(n+1)}{\Gamma(\sigma+1+n)}L_n^{(\sigma)}(r)L_n^{(\nu)}(x), \tag{2.2}$$

$$=a_n^{*}\int_0^{t}n(\xi)(r-\xi)^{\lambda-1}(x-\xi)^{\lambda+\nu-\sigma-1}d\xi=a_n^{*}S_t(r,x), \tag{2.3}$$

where $n(\xi)=e^{\xi}.\xi^{\sigma-\lambda}, t=min(r,\ x)$

$\alpha, \beta, \sigma < -1, \lambda+\nu > \sigma$ and

$$a_n^{*}=\frac{\Gamma(1-\lambda)\Gamma(\beta+n+1)\Gamma(\nu+n+1)}{\Gamma(\lambda+\nu-\sigma)\Gamma(\alpha+n+1)\Gamma(\sigma-\lambda+n-1)}.$$

It is further assumed that the parameters $\alpha, \beta, \lambda, \nu$ and σ are so constrained that a_n^{*} is independent of n. This of course is possible when, for instance $\alpha=\nu, \lambda=\sigma-\beta$, the parameter β and σ remains free.

3 Solution of the Equations of First Kind

Let us assume that

$$\sum_{n=0}^{\infty}\frac{A_n}{\Gamma(\beta+n+1)}L_n^{(\sigma)}(x)=\varphi(x),a<x<b, \tag{3.1}$$

where, $\beta > -1$ and $\varphi(x)$ is bounded and integrable in the interval (a, b). On making use of the orthogonality relation (2.1), we find that

$$A_n=\frac{\Gamma(\beta+n+1)\Gamma(n+1)}{\Gamma(\alpha+n+1)}\int_a^{b}r^{\sigma}e^{-r}L_n^{(\sigma)}(r)\phi(r)dr,\ n=0,\ 1,\ 2,\ \ldots; \tag{3.2}$$

provided $\beta > -1$ and $\sigma > -1$. Substituting for A_n in eqn. (1.2) and since the series in eqn. (1.2) is uniformly convergent, we can change the order of the summation and integration and thus we have

$$\int_a^{b}e^{-r}\phi(r)S(r,x)dr\ \ +\int_a^{b}e^{-r}\phi(r)T(r,x)dr=x^{\nu}f(x),a<x<b, \tag{3.3}$$

where, $S(r,x)$ is defined by eqn. (2.2) and

$$T(r,x) = r^{\sigma}x^{v}\sum_{n=0}^{\infty}\frac{\Gamma(\beta+n+1)}{\Gamma(\sigma+1+n)}.\frac{\Gamma(n+1)}{\Gamma(\sigma+1+n)}H_{n}\, L_n^{(\sigma)}(r)L_n^{(\nu)}(x). \tag{3.4}$$

Using the notation of eqn. (2.3) this can be written as:

$$\int_a^x e^{-r}\phi(r)S_r(r,x)dr + \int_x^b e^{-r}\phi(r)S_x(r,x)dr + \int_a^b e^{-r}\phi(r)T(r,x)dr = \frac{x^{\nu}f(x)\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^*}, \quad a<x<b \tag{3.5}$$

provided $\alpha,\beta,\sigma<-1, 0<\lambda<1, \nu+\lambda>\sigma$.

Inverting the order of integration in Carslow [2, eqn. (3.5)], we get

$$\int_a^x \frac{n(\xi)}{(x-\xi)^{1+\sigma-\lambda-\nu}}\overline{\phi}(\xi)d\xi + \int_a^b e^{-r}\phi(r)T(r,x)dr = \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^*}x^{\nu}f(x) - \int_0^a \frac{n(\xi)}{(x-\xi)^{1+\sigma-\lambda-\nu}}d\xi\int_a^b \frac{e^{-r}\phi(r)}{(r-\xi)^{1-\lambda}}dr, a<x<b, \tag{3.6}$$

where

$$\phi(\xi) = \int_\xi^b \frac{e^{-r}\phi(r)}{(r-\xi)^{1-\lambda}}dr, a\le\xi<b, \tag{3.7}$$

provided $\alpha,\beta,\sigma \underline{>} -1, 0<\lambda<1, 0<1-\lambda-\nu+\sigma<1$ and $\phi(r)$ being continuous and integrable in (a,b).

If $\phi(\xi)$ and $\overline{\phi(\xi)}$ are continuous in $a\le\xi\le b$ and $0<\lambda<1$, then (3.7) is an Abel integral equation and its solution is given by

$$e^{-r}\phi(r) = -\frac{\sin(1-\lambda)\pi}{\pi}\frac{d}{dr}\int_r^b \frac{\overline{\phi}(\xi)}{(\xi-r)^{\lambda}}d\xi. \tag{3.8}$$

Similarly, when $\sigma,\beta,\sigma>-1, 0<\lambda<1, 0<1+\sigma-\lambda-\nu<1$ and $f(x), f'(x)$ are continuous in $a\le x\le b$, then from eqn. (3.1) and (3.3), we have

$$n(\xi)\overline{\phi(\xi)} + \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi}\int_a^b e^{-r}\phi(r)dr\frac{d}{d\xi}\int_a^{\xi}\frac{T(r,x)dx}{(\xi-x)^{\lambda+\nu-\sigma}} = F(\xi) - \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi}\int_0^a n(\xi)l(\xi,n)d\eta.\int_a^b \frac{e^{-r}\phi(r)dr}{(r-\eta)^{1-\lambda}}, a<\xi<b, \tag{3.9}$$

where

$$F(\xi) = \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi}.\frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^*}\frac{d}{d\xi}\int_a^{\xi}\frac{x^{\nu}f(x)}{(\xi-x)^{\lambda+\nu-\sigma}}dx, \tag{3.10}$$

is a known function and

$$l(\xi,n) = \frac{d}{d\xi}\int_a^{\xi}\frac{dx}{(\xi-x)^{\lambda+\nu-\sigma}.(x-\eta)^{1+\sigma-\lambda-\nu}}. \tag{3.11}$$

By Lowndes ([8],p.276, eqn.26)

$$l(\xi,n) = \frac{(a-\eta)^{\lambda+\nu-\sigma}}{(\xi-\eta)(\xi-a)^{\lambda+\nu-\sigma}},\ 0<1+\sigma-\lambda-\nu<1, \tag{3.12}$$

eqn.(3.9) becomes

$$n(\xi)\overline{\phi(\xi)} + \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi}\int_a^b e^{-r}\phi(r)dr\frac{d}{d\xi}\int_a^{\xi}\frac{T(r,x)dx}{(\xi-x)^{\lambda+\nu-\sigma}} = F(\xi) - \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi(\xi-a)^{\lambda+\nu-\sigma}}\int_0^a \frac{(a-\eta)^{\lambda+\nu-\sigma}\eta(\xi)d\eta}{(\xi-\eta)}.\int_a^b \frac{e^{-r}\phi(r)dr}{(r-\eta)^{1-\lambda}}. \tag{3.13}$$

Using (3.8), we can write

$$\int_a^b \frac{e^{-r}\phi(r)dr}{(r-\eta)^{1-\lambda}} = -\frac{\sin(1-\lambda)\pi}{\pi}\int_a^b \frac{dr}{(r-\eta)^{1-\lambda}}\frac{d}{dr}\int_r^b \frac{\overline{\phi(\xi)}d\xi}{(\xi-r)^{\lambda}}$$

$$= \frac{\sin(1-\lambda)\pi}{\pi} . \frac{1}{(a-\eta)^{1-\lambda}} . \int_a^b \frac{\overline{\phi(\xi)} d\xi}{(\xi - a)^{\lambda}} - (1-\lambda) \int_a^b \frac{dr}{(r-\eta)^{2-\lambda}} . \int_r^b \frac{\overline{\phi(\xi)}}{(\xi - r)^{\lambda}} d\xi. \quad (3.14)$$

Inverting the order of integration in the last term of eqn. (3.14) and using the result of Lowndes ([9], p.276, eqn. 27)

$$\beta \int_a^y \frac{dr}{(r-\xi)^{1+\beta}(y-r)^{1-\beta}} = \frac{(y-a)^{\beta}}{(y-\xi)(a-\xi)^{\beta}}, 0 < \beta < 1, \quad (3.15)$$

we get

$$\int_a^b \frac{e^{-r}\phi(r)dr}{(r-\eta)^{1-\lambda}} = \frac{\sin(1-\lambda)\pi(a-\eta)^{\lambda}}{\pi} \int_a^b \frac{\phi(\xi)d\xi}{(\xi-\eta)(\xi-a)^{\lambda}}, \quad (3.16)$$

provided $0 < \lambda < 1$ and $\overline{\phi(\xi)}$ is bounded and integrable.

Substituting the expression in eqn. (3.13), $\overline{\phi(\xi)}$ is given by

$$n(\xi)\overline{\phi(\xi)} + \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi} \int_a^b e^{-r}\phi(r)dr. \frac{d}{d\xi} \int_a^{\xi} \frac{T(r,x)dx}{(\xi-x)^{\lambda+\nu-\sigma}}$$
$$+ \int_a^b \overline{\phi(\xi)} M(x,\xi)dx = F(\xi), \ a < \xi < b, \quad (3.17)$$

where

$$M(x,\xi) = \frac{\sin(1-\lambda)\pi sin(1+\sigma-\lambda-\nu)\pi}{\pi^2(x-a)^{\lambda}(\xi-a)^{\lambda+\nu-\sigma}} \int_0^a \frac{n(\xi)(a-n)^{\lambda+\nu-\sigma}}{(x-n)} . \frac{(a-n)^{\lambda}}{(\xi-n)} d\eta. \quad (3.18)$$

Eqn. (3.17) is a Fredholm integral equation which determines, $\overline{\phi(\xi)}$. Thus $\varphi(r)$ is then obtained from eqn. (3.8) and the coefficients A_n, which satisfy eqns. (1.1), (1.2) and (1.3) can be found from eqn. (3.2).

4 Solutions of the Equations of Second Kind

To solve the triple series equations (1.4), (1.5) and (1.6), we put

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)} L_n^{(\sigma)}(x) = \psi_1(x), 0 \le x < a$$
$$= \psi_2(x), b < x < \infty \quad (4.1)$$

where $\psi_1(x)$ and $\psi_2(x)$ are bounded and integrable in the interval of their definitions. Using the orthogonality relation, we get from eqn. (1.5) and eqn. (4.1).

$$A_n = \frac{\Gamma(\beta+n+1)\Gamma(n+1)}{\Gamma(\sigma+n+1)} \left\{ \int_0^a \psi_1(r) \ + \int_b^{\infty} \psi_2(r).r^{\sigma}e^{-r}L_n(\sigma;r)dr, \ n = 0,1,2,\ldots \right. \quad (4.2)$$

provided $\beta > -1, \sigma > -1$.

Substituting A_n in eqns. (1.4) and (1.6) and since these series are uniformly convergent, we get on interchanging the order of summation and integration, that

$$\left\{ \int_b^a \psi_1(r) + \int_b^{\infty} \psi_2(r) \right\} e^{-r} \left\{ S(r,x) + T(r,x) \right\} dr = x^{\nu} g(x).0 \le x < a$$
$$= x^{\nu} h(x), \ b < x < \infty, \quad (4.3)$$

where $S(r,x)$ is given by eqn. (2.3) and $T(r,x)$ is given by eqn. (3.4).

These equations may be written as:

$$\int_0^x e^{-r}\psi_1(r)S_r(r,x)dr + \int_x^a e^{-r}\psi_2(r)S_r(r,x)dr + \int_b^{\infty} e^{-r}\psi_2(r)S_x(r,x)dr$$
$$+ \int_0^a e^{-r}\psi_1(r)T(r,x)dr + \int_b^{\infty} e^{-r}\psi_2(r)T(r,x)dr = \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^x} .x^{\nu}g(x), \ 0 \le x < a, \quad (4.4)$$

$$\int_0^a e^{-r}\psi_1(r)S_r(r,x)dr + \int_b^x e^{-r}\psi_2(r)S_r(r,x)dr + \int_x^{\infty} e^{-r}\psi_2(r)S_x(r,x)dr + \int_0^a e^{-r}\psi_1(r)T(r,x)dr$$
$$+ \int_b^{\infty} e^{-r}\psi_2(r)T(r,x)dr = \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^x} x^{\nu}h(x), \ b < x < \infty, \quad (4.5)$$

provided $\alpha, \beta, \sigma > -1, 0 < \lambda < 1, \lambda + \nu > \sigma$. Since $\psi_1(x)$ and $\psi_2(x)$ are bounded and integrable in their interval of definitions, we get on interchanging the order of integration that

$$\int_a^x \frac{n(\xi)}{(x-\xi)^{1+\sigma-\lambda-\nu}} \left\{ \overline{\psi_1}(\xi) + \int_b^\infty \frac{e^{-r}\psi_2(r)}{(r-\xi)^{1-\lambda}} dr \right\} d\xi + \int_0^a e^{-r}\psi_1(r)T(r,x)dr + \int_b^\infty e^{-r}\psi_2(r)T(r,x)$$
$$= \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a^x} x^\nu g(x), 0 \le x < a, \quad (4.6)$$

$$\int_b^x \frac{n(\xi)\overline{\psi_2}(\xi)d\xi}{(x-\xi)^{1+\sigma-\lambda-\nu}} + \int_0^a e^{-r}\psi_1(r)T(r,x)dr + \int_b^\infty e^{-r}\psi_2(r)T(r,x)dr = \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^x}, x^\nu h(x)$$
$$- \int_0^a \frac{n(\xi)\overline{\psi_1(\xi)}d\xi}{(x-\xi)^{1+\sigma-\lambda-\nu}} - \int_0^b \frac{n(\xi)d\xi}{(x-\xi)^{1+\sigma-\lambda-\nu}} . \int_b^\infty \frac{e^{-r}\psi_2(r)dr}{(r-\xi)^{1-\lambda}}, \; b < x < \infty, \quad (4.7)$$

where,

$$\begin{cases} (i)\ \overline{\psi}_1(\xi) = \int_\xi^a \frac{e^{-r}\psi_1(r)dr}{(r-\xi)^{1-\lambda}} \\ (\text{ii }) \overline{\psi}_2(\xi) = \int_\xi^\infty \frac{e^{-r}\psi_2(r)dr}{(r-\xi)^{1-\lambda}} \end{cases} \quad (4.8)$$

provided $\alpha, \beta, \sigma > -1, 0 < \lambda < 1, 0 < 1-\lambda-\nu+\sigma < 1$, when $0 < 1+\sigma-\lambda-\nu < 1$. On using equations (10) to (14) of Lowndes ([8],p.168) with the help of eqns. (4.6), (4.7), (4.8) in a similar manner as to obtain eqns. (3.8) and (3.13) , we find that

$$n(\xi)\overline{\psi_1(\xi)} + \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi} \frac{d}{d\Psi} \int_a^\xi \left\{ \int_0^a e^{-r}\psi_1(r)dr \int_b^\infty e^{-r}\psi_2(r)dr \right\} T(r,x) \frac{dx}{(\xi-x)^{\lambda+\nu-\sigma}}$$
$$= \quad G(\xi) - n(\xi) \int_b^\infty \frac{e^{-r}\psi_2(r)dr}{(r-\xi)^{1-\lambda}}, \quad (4.9)$$

$$n(\xi)\psi_2(\xi) + \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi} \frac{d}{d\xi} \int_a^\xi \left\{ \int_0^a e^{-r}\psi_1(r)dr + \int_b^\infty e^{-r}\psi_2(r)dr \right\} T(r,x) \frac{dx}{(\xi-x)^{\lambda+\nu-\sigma}}$$
$$= H(\xi) - \frac{\sin(1+\sigma-\lambda-\nu)}{\pi(\xi-b)^{\lambda+\nu-\sigma}} \int_0^a (b-\eta)^{\lambda+\nu-\sigma} n(\eta)\overline{\psi_1}(\xi)d\eta - \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi(\xi-b)^{\lambda+\nu-\sigma}}$$
$$\times \int_a^b \frac{(b-\eta)^{\lambda+\nu-\sigma}}{(\xi-\eta)} .n(\eta)d\eta \int_0^\infty \frac{e^{-r}\psi_2(r)dr}{(\xi-n)^{1-\lambda}}, \quad (4.10)$$

$$e^{-r}\overline{\psi_1}(r) = -\frac{\sin(1-\lambda)}{\pi} \frac{d}{dr} \int_r^a \frac{\overline{\psi_1}(\xi)d\xi}{(\xi-r)^\lambda}, \; 0 < r < a, \quad (4.11)$$

$$e^{-r}\overline{\psi_2}(r) = -\frac{\sin(1-\lambda)}{\pi} \frac{d}{dr} \int_r^\infty \frac{\overline{\psi_2}(\xi)}{(\xi-r)^\lambda}, b < r < \infty, \quad (4.12)$$

where, $G(\xi)$ and $H(\xi)$ are known functions, defined as

$$G(\xi) = \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi} . \left\{ \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{a_n^x} \right\} . \frac{d}{d\xi} \int_0^\xi \frac{x^v g(x)dx}{(\xi-x)^{\lambda+\nu-\sigma}}, 0 < \xi < a, \quad (4.13)$$

$$H(\xi) = \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi} . \frac{\{\Gamma(\lambda)\Gamma(1-\lambda)\}}{a_n^x} . \frac{d}{d\xi} \int_b^\xi \frac{x^v h(x)dx}{(\xi-x)^{\lambda+\nu-\sigma}}, b < \xi < \infty, \quad (4.14)$$

By a method similar to that used to obtain eqn. (3.16), we can show that

$$\int_b^\infty \frac{e^{-r}\psi_2(r)dr}{(r-\xi)^{1-\lambda}} = \frac{\sin(1-\lambda)\pi}{\pi(b-\xi)^{-\lambda}} \int_b^\infty \frac{(\eta-b)^{-\lambda}\psi_2(\eta)d\eta}{(\eta-\xi)}. \quad (4.15)$$

Using this result and eqn. (4.9), it can be shown after some manipulation, that eqn. (4.10) can be written as

$$n(\xi)\psi_2(\xi) + \int_b^\infty \psi_2(x)N(x,\xi)\; dx = H(\xi) - \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi(\xi-b)^{\sigma+\nu-\lambda}} \int_0^a \frac{(b-\eta)^{\lambda+\nu-\sigma}}{(\xi-\eta)} G(\eta)d\eta, \quad (4.16)$$

where $N(x,\xi)$ is the kernel

$$N(x,\xi) = \frac{\sin(1+\sigma-\lambda-\nu)\pi.sin(1-\lambda)\pi}{\pi^2(x-a)^\lambda(\xi-a)^{\lambda+\nu-\sigma}} . \int_a^b \frac{n(\eta)(a-\eta)^\lambda(a-\eta)^{\lambda+\nu-\sigma}}{(x-\eta)(\xi-\eta)} d\eta, \; b < \xi < \infty, \quad (4.17)$$

provided $\alpha, \beta, \sigma < 1, 0 < \lambda < 1$ and $0 < 1-\lambda - \nu + \sigma < 1$.

Equation (4.16) is a Fredhelm integral equation of second kind which determines $\psi_2(\xi)$, $\psi_2(r)$, can be found from eqn. (4.12) and $\psi_1(r)$ from

$$e^{-r}\psi(r) = -\frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi}\frac{d}{dr}\int_r^a \frac{G(\xi)d\xi}{n(\xi)(\xi-r)^\lambda} + \frac{\sin(1+\sigma-\lambda-\nu)\pi}{\pi(a-r)^\lambda} \times \int_b^\infty \frac{e^{-n}(n-a)^\lambda\psi_2(\xi)}{(r-\eta)}d\xi, 0 < r < a. \quad (4.18)$$

Finally the coefficients A_n which satisfy the triple series equations of second kind when $\alpha, \beta, \sigma > -1, 0 < \lambda < 1, 0 < 1-\lambda - \nu + \sigma < 1$ are given by eqn. (4.2).

5 Conclusion

The generalized Laguerre polynomials have been applied by many authors like Lowndes [7,8], Srivastava [18,19,21], Srivastava- Panda[25] and Mudaliar-Narain [11] to solve dual, triple and quadruple series equations. The solutions presented in this paper are obtained by employing the techniques of Sneddon[17], Lowndes[8,9]and Srivastava[19]. Method of this paper, involving different boundary conditions, has a distinct advantage over that by the multiplying factor technique. These solutions are useful in Mathematical Physics, Mixed Boundary Problems in Potential Theory, Quantum Physics etc.We have obtained the solution of two sets of triple series equations involving generalized Laguerre polynomials by reducing them to the solution of a Fredholm integral equation of the second kind.

Acknowledgement. The authors express their sincere gratitude to the editors and referees for carefully reading the manuscript and for their valuable comments and suggestions which greatly improved this paper.

Conflict of interest: We declare that authors have no conflict of interest.

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