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(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

ON THE DIOPHANTINE EQUATIONS $x^2 + 139^m = y^n$ AND $x^2 + 499^m = y^n$

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Abstract

In this paper, we consider the Diophantine equations $x^2 + 139^m = y^n$ and $x^2 + 499^m = y^n$ $n \geq 3$, $m > 0$ and determine solutions of the equations.

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1 Introduction

The problem of solving the equation $x^2+7 = 2^n$ was proposed by Ramanujan [13] in 1913. This equation was solved perfectly by Nagell [11] in 1960 using techniques from algebraic number theory. In the generalized form, this equation is called generalized Ramanujan- Nagell equation $x^2 + k = y^n$, k, x, y, n belongs to integers, $n \geq 3$, a kind of exponent type equation. This equation has been studied extensively. When $n = 3$, it is an elliptic curve. Mordell studied this type of equation carefully and collected most of the important results in his book [10]. However, when $n \geq 3$, it is a hyperelliptic curve which seems to be more difficult to study, but there is now a vast body of literature on it also.

For some small positive integers k, the solutions have been determined. Lebesgue [8] and Nagell [12] showed that there are no non-trivial solutions when $k = 1$ and $k = 3, 5$, respectively.

Ljungrren [7] proved in the case of $k = 2$ that the equation has only one positive solution. Several special case of the Diophantine equation $x^2 + q^m = y^n$ where q is a prime and m, n, x and y are positive integers have been studied in the last few years. When $q = 2$ and m is an odd integer, it was proved by Cohn [5] that this equation has exactly three families of solutions. When $q = 3$, and m is an odd integer, the equation has three families of solution as proved by Arif and Abu Muriefah [1]. It was shown by Luca [9] that there exists only one family of solution when $q = 3$ and m is an even integer. Tao [14] solved the equation when $q = 5$ and showed that there is no solution. J. H. E. Cohn [6] refined the earlier elementary approaches and solved the equation for 77 values of q under 100. Using advanced methods, Bugeaud et al. [4] solved this kind of equation for $1 \leq k \leq 100$.

In this short communication, we consider the Diophantine equations $x^2+139^m = y^n$ and $x^2+499^m = y^n$, $n \geq 3$, $m > 0$ and determine solutions of the equations.

2 Main Results

Theorem 2.1. *Let m be odd. Then the Diophantine equation*

$$x^2 + 139^m = y^n, \tag{2.1}$$

has only one solution in positive integers x, y, m and the unique solution is given by $x = 322$, $y = 47$, $m = 1$ and $n = 3$.

We start by stating the following lemma which will be used further.

Lemma 2.1. *The equation $139x^2 + 1 = y^n$ where n is an odd integer ≥ 3 has no solution in integers x and y for y odd and ≥ 1.*

The proof of the Theorem 2.1 is divided into two cases $(139, x) = 1$ and $139|x$. It is sufficient to consider x a positive integer.

Proof. Suppose $m = 2k+1$, $k \geq 0$. Then equation (2.1) becomes

$$x^2 + 139^{2k+1} = y^n, \quad n \geq 3. \tag{2.2}$$

If x is odd and y is even, then $x^2 + 139^{2k+1} \equiv 4(mod\ 8)$, but $y^n \equiv 0(mod\ 8)$, which is not possible. Thus x is even and y is odd. □

Case (i) Let $(139, x) = 1$. Let n be odd , then there is no loss of generality in considering $n = p$, an odd prime.Then from [Theorem 6, [6]] we have only two possibilities and they are

$$x + 139^k\sqrt{-139} = (s + t\sqrt{-139})^p, \tag{2.3}$$

where $y = s^2 + 139t^2$, for some rational integers s and t and

$$x + 139^k\sqrt{-139} = (\frac{s + t\sqrt{-139}}{2})^3, \tag{2.4}$$

because $139 \equiv 3(mod\ 8)$, $s \equiv t \equiv 1(mod\ 2)$ where $y = (s^2 + 139t^2)/4$ for some rational integers s and t and $x = |\frac{s^3 - 417st^2}{8}|$.

In (2.3), since $y = s^2 + 139t^2$ and y is odd and so only one of s or t is odd and other is even. Equating imaginary parts of 2.3, we get

$$139^k = t\sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r+1} s^{p-2r-1}(-139t^2)^r. \tag{2.5}$$

So t is odd and s is even. Since 139 does not divide the term inside summation, we get $t = \pm 139^k$.

$$\pm 1 = \sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r+1} s^{p-2r-1}(-139^{2k+1})^r. \tag{2.6}$$

This is equation (1) in [6] and Lemmas 4 and 5 in [6] show that both the signs are impossible. Hence (2.3) gives rise to no solution.

Now let us consider equation (2.4). By equating imaginary parts , we obtain,

$$8 \cdot 139^k = t(3s^2 - 139t^2). \tag{2.7}$$

If $t = \pm 1$ in (2.7), we have

$$\pm 8 \cdot 139^k = 3s^2 - 139. \tag{2.8}$$

When we consider $k = 0$, then $\pm 8 = 3s^2 - 139$.

First we consider negative sign,

$$-8 = 3s^2 - 139.$$

Then

$$3s^2 = 131,$$

which is not possible.

Now we consider the positive sign,

$$8 = 3s^2 - 139. \tag{2.9}$$

This implies that

$$3s^2 = 147$$

or,

$$s = \pm 7.$$

This equation has only solution when

$$t = \pm 1,\ s = \pm 7,\ k = 0 \text{ and } y = \frac{s^2 + 139t^2}{4} = 47.$$

Hence from (2.4), we have $x = |\frac{s^3-417st^2}{8}| = 322$.

Finally if $t = \pm 139^k$, then from equation (2.7), we have

$$\pm 8 = 3s^2 - 139^{2k+1}, \tag{2.10}$$

where $k > 0$, which is impossible modulo 139. Hence there is no solution of this equation.

Now if n is even, then it is sufficient to consider $n = 4$, hence the equation (2.2) becomes

$$x^2 + 139^{2k+1} = y^4$$

or,

$$y^4 - x^2 = 139^{2k+1}$$

or,

$$(y^2 - x)(y^2 + x) = 139^{2k+1}.$$

Since $(139, x) = 1$, we have

$$y^2 + x = 139^{2k+1} \tag{2.11}$$

and

$$y^2 - x = 1. \tag{2.12}$$

Eliminating x from equations (2.11) and (2.12), we get

$$2y^2 = 139^{2k+1} + 1.$$

Then $2y^2 \equiv 4 \ (mod\ 8)$ i.e.$y^2 \equiv 2 \ (mod\ 4)$ as y is odd, which is impossible.

Case (ii) Suppose that $139|x$, then $x = 139^u \cdot X$; so that, $139|y$, then $y = 139^v \cdot Y$, where $u > 0, v > 0$ and $(139, X) = (139, Y) = 1$. Then

$$139^{2u}X^2 + 139^{2k+1} = 139^{nv}Y^n.$$

There are following possibilities for solving this equation as discussed below:

1) $2u = min(2u, 2k+1, nv)$. Then by cancelling 139^{2u}, we get

$$X^2 + 139^{2(k-u)+1} = 139^{nv-2u}Y^n.$$

If $nv - 2u = 0$, then we get $X^2 + 139^{2(k-u)+1} = Y^n$ with $(139, X) = 1$. If $k - u = 0$, this equation has the only solution $x = 322$ and $n = 3$. If $k - u > 0$, then it has no solution.

2) $2k + 1 = min(2u, 2k+1, nv)$. Then $139^{2u-2k-1} \cdot X^2 + 1 = 139^{nv-2k-1}Y^n$ and considering this equation modulo 139, which is not possible. Hence this equation has no solution.

3) $nv = min(2u, 2k+1, nv)$. Then $139^{2u-nv} \cdot X^2 + 139^{2k+1-nv} = Y^n$. This is possible modulo 139 only if $2u - nv = 0$ or $2k + 1 - nv = 0$ and both cases are not possible. This completes the proof of the theorem.

Theorem 2.2. *The equation*

$$x^2 + 499^m = y^n, \qquad n \geq 3,\ m > 0 \tag{2.13}$$

has only one solution in positive integers (x, y, m) and the solution is given by

$$x = 2158,\ y = 167,\ m = 1,\ n = 3.$$

Lemma 2.2. *The equation $499x^2 + 1 = y^n$ where n is an odd integer ≥ 3 has no solution in integers x and y for y odd and ≥ 1*

The proof of the theorem is divided into two cases $(499, x) = 1$ and $499|x$. It is sufficient to consider x a positive integer.

Proof. Let us suppose that $m = 2k + 1$. We shall assume that $k > 0$, $n > 3$.

If x is odd and y even, we get $x^2 + 499^{2k+1} \equiv 4 \ (mod\ 8)$, but $y^n \equiv 0 \ (mod\ 8)$. Hence, we suppose that x is even and y is odd. □

Case (i) Let $(499, x) = 1$. Let n be odd , then there is no loss of generality in considering $n = p$, an odd prime.Then from [[6], Theorem 6] we have only two possibilities and they are

$$x + 499^k\sqrt{-499} = (a + b\sqrt{-499})^p, \tag{2.14}$$

where

$$y = a^2 + 499b^2,$$

and

$$x + 499^k\sqrt{-499} = (a + b\sqrt{-499}/2)^3, \tag{2.15}$$

because $499 \equiv 3(mod\ 8)$, $a \equiv b \equiv 1(mod\ 2)$, where $y = \frac{a^2+499b^2}{4}$ for some rational integers a and b and $x = |\frac{a^3-1497ab^2}{8}|$.

In (2.14), since $y = a^2 + 499b^2$ and y is odd and so only one of a or b is odd and other is even. Equating imaginary parts, we get

$$499^k = b\sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r+1} a^{p-2r-1}(-499b^2)^r. \tag{2.16}$$

So b is odd and a is even. Since 499 does not divide the term inside summation, we get $b = \pm 499^k$.

$$\pm 1 = \sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r+1} a^{p-2r-1}(-499b^{2k+1})^r. \tag{2.17}$$

This is Cohn [6, eqn (1)]. Therefore, Lemmas 4 and 5 due to Cohn [6] show that both the signs are impossible. Hence (2.14) gives rise no solution.

Now let us consider equation (2.15). By equating imaginary parts, we obtain,

$$8 \cdot 499^k = b(3a^2 - 499b^2). \tag{2.18}$$

If $b = \pm 1$ in (2.18), we have

$$\pm 8 \cdot 499^k = 3a^2 - 499. \tag{2.19}$$

When we consider $k = 0$, we get $\pm 8 = 3a^2 - 499$. We consider negative sign

$$-8 = 3a^2 - 499$$

or,

$$3a^2 = -8 + 499$$

or,

$$3a^2 = 491,$$

which is not possible.

Now we consider positive sign

$$8 = 3a^2 - 499 \tag{2.20}$$

or,

$$3a^2 = 507$$

or,

$$a = \pm 13.$$

This equation has only solution when $b = \pm 1$, $a = \pm 13$, $k = 0$ and $y = \frac{a^2+499b^2}{4} = 167$. Hence from (2.15), we have $x = |\frac{a^3-1497ab^2}{8}| = 2158$. Hence $x = 2158$.

Finally if $b = \pm 499^k$, then we have

$$\pm 8 = 3a^2 - 499^{2k+1}, \tag{2.21}$$

where $k > 0$, which is impossible modulo 499. Hence there is no solution of this equation.

Now if x is even, then from the equation (2.13), it is sufficient to consider $n = 4$, hence

$$(y^2 + x)(y^2 - x) = 499^{2k+1}.$$

Since $(499, x) = 1$, we have

$$y^2 + x = 499^{2k+1}, \tag{2.22}$$

and

$$y^2 - x = 1. \tag{2.23}$$

Eliminating x from both equations (2.22) and (2.23), we get

$$2y^2 = 499^{2k+1} + 1.$$

Then $2y^2 \equiv 4 \ (mod\ 8)$ i.e. $y^2 \equiv 2 \ (mod\ 4)$, which is impossible.

Case (ii) Let $499|x$. Then, of course, $499|y$. Suppose that $x = 499^u \cdot X$, $y = 499^v \cdot Y$, where $u > 0, v > 0$ and $(499, X) = (499, Y) = 1$. Then

$$499^{2u} \cdot X^2 + 499^{2k+1} = 499^{nv} \cdot Y^n,$$

i) $2u = min(2u, 2k+1, nv)$. Then by cancelling 499^{2u}, we get

$$X^2 + 499^{2k+1-2u} = 499^{nv-2u} Y^n$$

If $nv - 2u = 0$, then we get $X^2 + 499^{2(k-u)+1} = Y^n$, with $(499, X) = 1$. If $k - u = 0$, this equation has only solution $x = 2158$ and $n = 3$. If $k - u > 0$, then it has no solution.

ii) $2k + 1 = min(2u, 2k+1, nv)$. Then $499^{2u-2k-1} \cdot X^2 + 1 = 499^{nv-2k-1} Y^n$ and considering this equation modulo 499, we get $nv - 2k - 1 = 0$, so n is odd, $499(499^{k-u-1}X)^2 + 1 = Y^n$. By the Lemma **2.2** this equation has no solution.

iii $nv = min(2u, 2k+1, nv)$. Then $499^{2u-nv} \cdot X^2 + 499^{2k+1-nv} = Y^n$. This is possible modulo 499 only if $2u - nv = 0$ or $2k + 1 - nv = 0$ and both cases are not possible. Hence this completes the proof of the ***Theorem*** **2.2**.

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