ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 202-206**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

COMBINATORIAL PROOFS OF SOME IDENTITIES INVOLVING FIBONACCI AND LUCAS NUMBERS

M. Tamba[1] and Y. S. Valaulikar[2]
[1] School of Physical and Applied Sciences, [2]Ex- Faculty
Goa University, Taleigaon Plateau, Goa, India-403206
Email : tamba@unigoa.ac.in, ysv@unigoa.ac.in ; valaulikarys@gmail.com
(Received: February 21, 2023; In format: March 02, 2023; Accepted: April 02, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53123

Abstract

Using combinatorial methods, we obtain some identities, involving binomial coefficients, for Fibonacci and Lucas numbers. We define a set and show that the cardinality of this set equals Fibonacci number. We discuss some properties of this set. Technique has been extended to obtain results for Lucas numbers.
2020 Mathematical Sciences Classification: 11B39, 11B65, 11B75
Keywords and Phrases: Fibonacci numbers; Lucas numbers; Binomial coefficients; Cardinality of a set

1 Introduction

A recursively defined sequence of positive integers that has been extensively studied is the well-known Fibonacci sequence $\{F_n\}$. Fibonacci sequence has been extended in many directions depending upon its recurrence relation as well as seed values[6, 8]. This sequence has wonderful and amazing properties and has found to be useful in different fields of knowledge[2, 4, 5, 7]. In this paper we look at the following application of Fibonacci numbers in a different manner.

Let us suppose that there are six steps with ground being first step and top being sixth. A person standing on the top (sixth step) wants to come down on the ground (first step) with the restriction that at a time he can take either one or two steps only. *In how many ways he can come to the ground* ? It is known that this can be done in F_6 ways. In [1], this has been established by the method of tilng. We shall arrive at the answer by using a novel approach.

We first introduce some terms and notations to be used.

Terms and Notations.

1.1 For a positive integer n, let Ω_n denotes set of tuples $(u_1, u_2, \cdots, u_k)$ of natural numbers with the property that $u_1 = n, u_k = 1$ and $0 < u_i - u_{i+1} \leq 2, 1 \leq i \leq k-1$.

1.2 Let $|\Omega_n|$ denotes the cardinality of the set Ω_n.

1.3 Let Rank Ω_n denotes the number of tuples $(u_1, u_2, \cdots, u_k)$ in Ω_n such that exactly even number of u_i's are odd.

1.4 For $\lambda = (u_1, u_2, \cdots, u_k) \in \Omega_n$, let Sign $\lambda = (-1)^{(u_1+u_2+\cdots+u_k)}$.

1.5 Let $\wedge_n$ denotes a set of all elements η which is obtained by replacing 1 by 0 in elements of the type $(n, \cdots, 2, 1) \in \Omega_n$.

We illustrate the above defined terms by following example.

Example 1.1 Let $n = 6$. Then
$\Omega_6 = \{(6, 5, 4, 3, 2, 1),\ (6, 5, 4, 3, 1),\ (6, 5, 3, 1),\ (6, 5, 3, 2, 1),\ (6, 4, 2, 1),\ (6, 4, 3, 1),$
$(6, 4, 3, 2, 1),\ (6, 5, 4, 2, 1)\}$. Thus $|\Omega_6| = 8$ and Rank $\Omega_6 = 3$.
For $\lambda = (6, 5, 4, 3, 1) \in \Omega_6$, Sign $\lambda = -1$.
$\wedge_6 = \{(6, 5, 4, 3, 2, 0), (6, 5, 3, 2, 0), (6, 4, 2, 0), (6, 4, 3, 2, 0), (6, 5, 4, 2, 0)\}$;
$|\wedge_6| = 5$, Rank $\wedge_6 = 2$ and for $\eta = (6, 5, 3, 2, 0) \in \wedge_6$, Sign $\eta = 1$.

2 Identities involving Fibonacci numbers

In this section, we shall obtain some identities for Fibonacci numbers. The well-known Fibonacci sequence $\{F_n\}$ is defined by $F_0 = 0,\ F_1 = 1$ and for $n \geq 2,\ F_n = F_{n-1} + F_{n-2}$. F_n is called the n^{th} Fibonacci number. We first have the following proposition.

Proposition 2.1. *For $n \geq 1, |\Omega_n| = F_n$.*

Proof. Let $i_n = 1, \forall\ n \geq 0$. For $\lambda = (u_1, u_2, \cdots, u_k) \in \Omega_n$, let i_λ denote the product $i_{u_1} i_{u_2} \cdots i_{u_k}$.Using Fibonacci recurrence relation, we have for $n \geq 2$,

$$F_n = i_{n-1}F_{n-1} + i_{n-2}F_{n-2}. \tag{2.1}$$

So that $i_nF_n = i_n i_{n-1}F_{n-1} + i_n i_{n-2}F_{n-2}$. Using (2.1) with n replaced by $n-1$ and $n-2$ on the right hand side, we get
$i_nF_n = i_n i_{n-1} i_{n-2}F_{n-2} + i_n i_{n-1} i_{n-3}F_{n-3} + i_n i_{n-2} i_{n-3}F_{n-3} + i_n i_{n-2} i_{n-4}F_{n-4}$.

Continuing this way, using (2.1) repeatedly, we get

$$i_nF_n = \sum_{(\lambda \in \Omega_n)} i_\lambda F_1 + \sum_{(\lambda \in \Omega_n)} i_\lambda F_0. \tag{2.2}$$

From definitions and seed values, it follows that $F_n = \sum_{\lambda \in \Omega_n} 1 = |\Omega_n|$.

This completes the proof. □

Remark 2.1 Observe that Ω_6 is the set of all possible ways in which the task, given in our question, can be carried out. Hence the number of ways is equal to $|\Omega_6| = 8 = F_6$.

Let $\binom{n}{r}$ denote the binomial coefficient, that is $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. We give an alternative proof of the following result in ([1, 3, 8]) by using above arguments.

Proposition 2.2. *For $n \geq 1,\ F_n = \sum_{(s=0)}^{[\frac{n-1}{2}]} \binom{n-1-s}{s}$.*

Proof. For $n \geq 1$ and $\lambda = (u_1, u_2, \cdots, u_k) \in \Omega_n,$ let $\epsilon_i = u_i - u_{i+1},\ (1 \leq i \leq k-1)$.

From the construction of Ω_n, it is clear that $\epsilon_i = 1$ or 2 and that

$$n - 1 = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k-1}.$$

First let us consider the case when all ϵ_i's are equal to 1. Here we have

$$n - 1 = 1 + 1 + \cdots + 1, \ \ (n-1 \text{ summands}) \tag{2.3}$$

and there is exactly $1 (= \binom{n-1}{0})$ way to write this. Next suppose exactly one of ϵ_i is 2. Now in this case, we have $n-2$ positions with one 2 and so there are $\binom{n-2}{1}$ ways to choose position of that 2. Next, there will be $(n-3)$ positions with two 2 's. This can be achieved in $\binom{n-3}{2}$ ways.

Proceeding this way we get, in general, that exactly s number of positions will be there with $(n-1-s)$ 2's and is obtained in $\binom{n-1-s}{s}$ ways. Also $\binom{n-1-s}{s}$ will be non zero for $(n-1-s) \geq s$; that is $(n-1) \geq 2s$.

Thus, we have $|\Omega_n| = \sum_{s=0}^{[\frac{n-1}{2}]} \binom{n-1-s}{s}$. Now the result follows from Proposition 2.1. □

Next we have the following result.

Proposition 2.3. *For $n \geq 1,\ |\wedge_n| = F_{n-1}$.*

Proof. For $n \geq 1$ and $\lambda = (u_1, u_2, \cdots, u_k) \in \wedge_n$, let $\epsilon_i = u_i - u_{i+1},\ (1 \leq i \leq k-1)$. From the construction of $\wedge_n$, it is clear that $\epsilon_i = 1$ or 2 and that

$$n - 2 = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k-1}.$$

First consider the case when all ϵ_i's are equal to 1. We shall have

$$n - 2 = 1 + 1 + \cdots + 1, \ \ (n-2 \text{ summands}) \tag{2.4}$$

and there is exactly 1 $(= \binom{n-2}{0})$ way to write this. Next suppose exactly one of ϵ_i is 2. Now in this case we have $n-3$ positions with one 2 and so there are $\binom{n-3}{1}$ ways to choose position of that 2. Next there will be $(n-4)$ positions with two 2's. This can be achieved in $\binom{n-4}{2}$ ways.

Proceeding this way we get, in general, that exactly s number of positions will be there with $(n-2-s)$ 2's and is obtained in $\binom{n-2-s}{s}$ ways. Also $\binom{n-2-s}{s}$ will be non zero for $(n-2-s) \geq s$; that is $(n-2) \geq 2s$.

Thus we have $|\wedge_n| = \sum_{s=0}^{[\frac{n-2}{2}]} \binom{n-2-s}{s} = F_{n-1}$ (by Proposition 2.3). □

3 Identities involving Lucas numbers

In this section, we shall obtain some identities involving Lucas numbers. Lucas sequence $\{L_n\}$ is defined by $L_0 = 2, L_1 = 1$ and for $n \geq 2, L_n = L_{n-1} + L_{n-2}$. L_n is called the n^{th} Lucas number. We first give the following result proved alternatively in [6, 8].

Proposition 3.1. *For* $n \geq 1,\ L_n = F_n + 2F_{n-1}$.

Proof. Let $i_n = 1$ for all $n \geq 0$. For $\lambda = (u_1, u_2, \cdots, u_k) \in \Omega_n$ or $\wedge_n$, let i denote the product $i_{u_1} i_{u_2} \cdots i_{u_k}$. Now we have $L_n = L_{n-1} + L_{n-2},\ (n \geq 2)$, which may be written as

$$L_n = i_{n-1} L_{n-1} + (n-2) L_{n-2}, \tag{3.1}$$

so that, using (3.1) with n replaced by $n-1$ and $n-2$, we get

$$\begin{aligned} i_n L_n &= i_n i_{n-1} L_{n-1} + i_n i_{n-2} L_{n-2} \\ &= i_n i_{n-1} i_{n-2} L_{n-2} + i_n i_{n-1} i_{n-3} L_{n-3} + i_n i_{n-2} i_{n-3} L_{n-3} + i_n i_{n-2} i_{n-4} L_{n-4}. \end{aligned}$$

Continuing this way, using (3.1) repeatedly, we get

$$i_n L_n = \sum_{\lambda \in \Omega_n} i_\lambda L_1 + \sum_{\lambda \in \wedge_n} i_\lambda L_0, \tag{3.2}$$

Using seed values for Lucas sequence, we get

$$\begin{aligned} i_n L_n &= \sum_{\lambda \in \Omega_n} 1 + 2 \sum_{\lambda \in \wedge_n} 1, \\ &= |\Omega_n| + 2| \wedge_n | \\ &= F_n + 2F_{n-1} \text{ (Using Propositions 2.1 and 2.4).} \end{aligned} \tag{3.3}$$

Hence the result. □

Next if G_n is the nth generalized Fibonacci or Gibonacci number satisfying the relation $G_n = G_{n-1} + G_{n-2},\ (n \geq 2)$ with $G_0 = a$ and $G_1 = b$, then arguing as in Proposition 3.1, we get

Proposition 3.2. *For* $n \geq 1,\ G_n = bF_n + aF_{n-1}$.

4 Some Properties of Ω_n and $\wedge_n$

In this section we discuss some properties of Ω_n and $\wedge_n$. First we define a Fibonacci type sequence $\{S_n\}$.

Let $j_n = (-1)^n,\ \forall\ n \geq 0$. For $\lambda = (u_1, u_2, \cdots, u_k) \in \Omega_n$ or $\wedge_n$, let j_λ denote the product $j_{u_1} j_{u_2} \cdots j_{u_k}$. Define a sequence

$$S_n = j_{n-1} S_{n-1} + j_{n-2} S_{n-2},\ (n \geq 2) \text{ with } S_0 = 2 \text{ and } S_1 = 1, \tag{4.1}$$

which implies

$$\begin{aligned} j_n S_n &= j_n j_{n-1} S_{n-1} + j_n j_{n-2} S_{n-2} \\ &= j_n j_{n-1} j_{n-2} S_{n-2} + j_n j_{n-1} j_{n-3} S_{n-3} + j_n j_{n-2} j_{n-3} S_{n-3} + j_n j_{n-2} j_{n-4} S_{n-4}, \end{aligned}$$

where last expression is obtained by using (4.1) with n replaced by $n-1$ and $n-2$.

Continuing this way, using (4.1) repeatedly, we get

$$j_n S_n = \sum_{\lambda \in \Omega_n} j_\lambda S_1 + \sum_{\lambda \in \wedge_n} j_\lambda S_0. \tag{4.2}$$

Using seed values, we get

$$j_n S_n = \sum_{\lambda \in \Omega_n} (Sign\ \lambda) + 2 \sum_{\lambda \in \wedge_n} (Sign\ \lambda). \tag{4.3}$$

In view of (3.3), this gives the following:

Proposition 4.1. *For* $n \geq 1,\ L_n + (-1)^n S_n = 2(Rank\ \Omega_n) + 4(Rank\ \wedge_n)$.

Next we have ,

Proposition 4.2. *For* $m \geq 0,\ S_{2m+1} = S_{2m+4}$.

Proof. Note that equation (4.1) can be rewritten as

$$S_n = (-1)^n(S_{n-2} - S_{n-1}). \tag{4.4}$$

So that $S_{2m+4} = S_{2m+2} - S_{2m+3}$ and $S_{2m+3} = -S_{2m+1} + S_{2m+2}$, which in turn gives $S_{2m+1} = S_{2m+4}$. □

Proposition 4.3. *For $m \geq 0$, $S_{2m+1} = (-1)^m F_{m-1}$ and $S_{2m+4} = (-1)^m F_{m-1}$.*

Proof. First note that if $S_{2m+1} = (-1)^m\ F_{m-1}$ is true then, by Proposition 4.2, $S_{2m+4} = (-1)^m F_{m-1}$.

For $m = 0$, since $F_{-1} = 1$, $S_1 = 1$ which is true.

Suppose $S_{2m+1} = (-1)^m F_{m-1}, \forall\ m < n$.Then

$$\begin{aligned} S_{2n+1} &= S_{2n} - S_{2n-1} \\ &= S_{2(n-2)+4} - S_{2(n-1)+1} \\ &= (-1)^{n-2}F_{n-3} - (-1)^{n-1}F_{n-2} \\ &= (-1)^n[F_{n-3} + F_{n-2}] = (-1)^n F_{n-1}. \end{aligned}$$

This completes the proof. □

5 Computation of Rank Ω_n and Rank $\wedge_n$

In this section, we shall obtain some recurrence relations for Rank Ω_n and Rank $\wedge_n$.

Proposition 5.1. *For $m \geq 2$,*
(a) Rank Ω_{2m} = Rank Ω_{2m-1}+ Rank Ω_{2m-2}.
(b) Rank $\Omega_{2m-1} = F_{2m-1} -$ (Rank Ω_{2m-2}+ Rank Ω_{2m-3}).

Proof. Define $A_n = \{(u_1, u_2, \cdots, u_k) \in \Omega_n \mid u_1 = n \text{ and } u_2 = n-1\}$ and $B_n = \{(u_1, u_2, \cdots, u_k) \in \Omega_n \mid u_1 = n \text{ and } u_2 = n-2\}$.
Note that Ω_n is a disjoint union of A_n and B_n.

(a) If $n = 2m$, then Rank A_n = Rank Ω_{n-1} and Rank B_n = Rank Ω_{n-2}.
Hence Rank Ω_n = Rank A_n+ Rank B_n = Rank Ω_{n-1}+ Rank Ω_{n-2} as required.

(b) If $n = 2m-1$, then Rank $A_n = |\Omega_{n-1}|-$ Rank Ω_{n-1} and Rank $B_n = |\Omega_{n-2}|-$ Rank Ω_{n-2}. Then

$$\begin{aligned} \text{Rank } |\Omega_n| &= \text{Rank } A_n + \text{Rank } B_n \\ &= (F_{n-1} - \text{Rank } \Omega_{n-1} + (F_{n-2} - \text{Rank } \Omega_{n-2}) \\ &= F_n - (\text{Rank } \Omega_{n-1} + \text{Rank } \Omega_{n-2}). \end{aligned}$$

as required. □

Proceeding in the same way as above, we can prove the following relations for Rank $\wedge_n$.

Proposition 5.2. *For $m \geq 2$,*
(a) Rank $\wedge_{2m}$ = Rank $\wedge_{2m-1}$+ Rank $\wedge_{2m-2}$.
(b) Rank $\wedge_{2m-1} = F_{2m-2} -$ (Rank $\wedge_{2m-2}$+ Rank $\wedge_{2m-3}$).

Next we have following representation for Rank Ω_n.

Proposition 5.3. *For $m \geq 2$,*
(a) Rank $\Omega_{2m} = \sum_{s=0}^{[\frac{(2m-1)}{4}]} \binom{2m-2-2s}{2s+1}$.
(b) Rank $\Omega_{2m-1} = \sum_{s=0}^{[\frac{(2m-2)}{4}]} \binom{2m-1-2s}{2s}$.

Proof. (a) If $m \geq 2$, and $\lambda = (u_1, u_2, \cdots, u_k) \in \Omega_{2m}$, let $\epsilon_i = u_i - u_{i+1}$, $(1 \leq i \leq k-1)$. From the construction of Ω_{2m} it is clear that $\epsilon_i = 1$ or 2 and that

$$2m - 1 = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k-1}.$$

First consider the case when all ϵ_i's are equal to 1. In this case we will have

$$2m - 1 = 1 + 1 + \cdots + 1, \ (n - 2 \text{ summands}) \tag{5.1}$$

and there is exactly 1 $(= \binom{2m-1}{0})$ way to write this. In this case there are odd number of odd entries. So we do not count this case. Next suppose exactly one of ϵ_i is 2. Now in this case we have $n - 2$ positions with one 2 and so there are $\binom{n-2}{1}$ ways to choose position of that 2. Here there are even number of odd entries. Counting this we have the required result.
Similarly we can prove (b). □

Arguing as in above proposition, we can prove the following:

Proposition 5.4. *For $m \geq 2$,*
(a) Rank $\wedge_{2m} = \sum_{s=0}^{[\frac{(2m-2)}{4}]} \binom{2m-3-2s}{2s+1}$.
(b) Rank $\wedge_{2m-1} = \sum_{s=0}^{[\frac{(2m-3)}{4}]} \binom{2m-2-2s}{2s}$.

6 Conclusion

In this paper we have used simple combinatorial arguments to prove some known results. For this purpose we have defined two sets and some properties of these sets are discussed. The technique can be extended to other Fibonacci like numbers to obtain the known results in a simple way.

Acknowledgement. Authors are thankful to the Editor for suggestions made to improve the presentation of the paper.

References

[1] A. T. Benjamin and Jennifer J. Quinn, *Proofs that Really Counts: The Art of Combinatorial Proof*, The Mathematical Association of America, Washington D. C., 2003.

[2] V. R. Gend, The Fibonacci sequence and the golden ratio in music, *Notes on Number Theory and Discrete Mathematics*, **20**(1) (2014), 72-77.

[3] H. H. Gulec and N. Taskara, On the properties of Fibonacci numbers with binomial coefficients, *International Journal of Contemporary Mathematical Sciences*, **4**(25) (2009), 1251-1256.

[4] V. E. Hoggatt, *Fibonacci and Lucas Numbers.* A publication of the Fibonacci Association, Santa Clara, 1969.

[5] F. T. Howard, Applications of Fibonacci Numbers, **9**, *Proceedings of the Tenth International Research Conference on Fibonacci Numbers and their Applications*, Springer, New York, 2004.

[6] T. Koshy, T. *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, INC., New York, 2001.

[7] J. A. Raphael and V. Sundaram, Secured communication through Fibonacci numbers and unicode symbols, *International Journal of Scientific and Engineering Research*, **3**(4) (2012), 1-5.

[8] S. Vajda, *Fibonacci and Lucas numbers and the Golden section: Theory and Applications*, Dover Publications, New York, 2008.