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(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

GENERALIZATION OF CONVEX FUNCTION

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Abstract

In this paper, We established a new class of convex function $(\phi_1, \phi_2) - \beta$-convex, which includes many well-known classes as its subclasses. We defined $(\phi_1, \phi_2) - \beta$-convex function and discussed various properties with non-differentiable and differentiable cases.
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1 Introduction

Convexity has been of great importance in both applied and pure mathematics for the purpose of generalizing existing results in work over the past 60 years. In recent years, several extensions of the concept of convexity of a set and a function have been introduced. There are several inequalities introduced by Minkowski [14], Dragomir [10] and Ardic et al. [4] etc. using convexity. A class of convex functions introduced by Bector and Singh [5] called b-vex functions with differentiable and nondifferentiable cases were presented.

Hanson [11] introduced mathematical Programming Problem for invex functions with inequality constraints. He considered differentiable functions and then proved that instead of assumption of convexity, the objective function and each of the constraints function involved satisfy inequality with respect to the same function. Then Craven [8], inspired by Hanson's work, first systematically introduced the term "invariant convex". After that Craven and Glover [9], Ben Israel and Mond [6] and Martin [13] showed that the class of invex functions is equivalent to the class of functions whose stationary points are points of global minima.

Mishra [15] obtained optimality and duality results by combining the concepts of type I, type II, pseudo-type I, quasi-type I, quasi-pseudo-type I, pseudo-quasi-type I, strictly pseudo-quasi-type I, and univex functions. Mishra and Rueda [16] also introduced and discussed SFJ-univex programming problems then Ojha [18] extended the SFJ-univex programming problems in complex spaces. Antczak [1] introduced several nonlinear programming problems of *(p, r)*-invexity type. Antczak [2] elongated the idea of p-invex set and defined *(p, r)*-pre-invex function (non-differentiable) and *(p, r)*-invex function (differentiable) and obtained optimality conditions for nonlinear programming problem under the idea of those functions. Antczak [3] defined r-preinvexity, r-invexity and obtained optimality criteria and duality relations for these functions in programming problem. Antczak [3] also designed duality theorems for modified r-invex functions based on function η.

Weir, Mond and Craven [25] and also [26, 27] showed that how and where pre-invex functions can replace convex functions in multiple objective optimization. Then Bector - singh [5] and Suneja, Singh and Bector [24] introduced a class of functions, called b-vex functions which forms a subset of the sets of both semi-strictly quasi-convex and quasi-convex functions also.

Pini [19] introduced relations between invexity and generalized properties of convexity also gave a new class of generalized convex sets. Pini and Singh [21, 22] established duality results also defined $(\phi_1, \phi_2)-$

convexity which is an extremely powerful principle for characterization of generalized convexity from an integrated point of view. Where ϕ_1 is a continuous deformation of straight line segments and ϕ_2 identifies generalized convex combinations of values. So after that other type of convexity can be introduced.

$$\phi_1 : D \times D \times [0,1] \to \Re^n, \qquad \phi_1 = \phi_1(x,y,\lambda),$$
$$\phi_1(x,y,0) = y, \;\; \phi_1(x,x,\lambda) = x, \qquad \forall x,y \in D, \;\; \lambda \in [0,1],$$
$$\phi_2 : D \times D \times [0,1] \times F \to Re, \quad \phi_2 = \phi_2(x,y,\lambda,f),$$
$$\phi_2(x,y,0,f) = f(y), \;\; \phi_2(x,x,\lambda,f) = f(x), \quad \forall x,y \in D, \lambda \in [0,1], \;\; f \in F.$$

In this paper, $(\phi_1,\phi_2) - \beta-$ convexity is defined. It is a very powerful new principle for characterizing the generalized convexity of sets and functions from a unified perspective.

In section 2, the definition of $(\phi_1,\phi_2) - \beta-$convex function is given; we show that to appropriate selection of functions ϕ_1 and ϕ_2, some of the well-known classes of generalized convex functions are particular cases of this new class. An example of a $(\phi_1,\phi_2) - \beta-$convex function is also provided that does not belong to any of the known classes. We present some properties of nondifferentiable $(\phi_1,\phi_2) - \beta-$convex functions. In this section, we also examine some properties of the solution of a mathematical programming problem involving.$(\phi_1,\phi_2) - \beta-$convex functions; moreover, we state a senstivity result.

In section 3, we consider the differentible case. Here we state a natural necessary condition for differentiable $(\phi_1,\phi_2) - \beta-$convex functions; in particular, we provided criteria under which the differentiable and the nondifferentiable conditions are equivalent. we state a second order sufficient condition for $(\phi_1,\phi_2) - \beta-$convexity.

2 The Nondifferentiable Case

Let G be a vector space of real valued functions defined on a set $D \subseteq \Re^n$. We are assuming two maps ϕ_1, ϕ_2 which satisfy the following assumptions:

$$\phi_1 : D \times D \times [0,1] \to \Re^n, \qquad \phi_1 = \phi_1(x,y,\lambda),$$
$$\phi_1(p,q,0) = q, \;\; \phi_1(p,p,\lambda) = p, \qquad \forall p,q \in D, \;\; \lambda \in [0,1],$$
$$\phi_2 : D \times D \times [0,1] \times F \to \Re, \quad \phi_2 = \phi_2(p,q,\lambda,g),$$
$$\phi_2(p,q,0,g) = g(y), \;\; \phi_2(p,p,\lambda,g) = g(p), \quad \forall p,q \in D, \lambda \in [0,1], \;\; g \in G,$$
$$\phi_2(p,q,\lambda,g) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta}, \quad if \quad \beta \neq 0,$$
$$\phi_2(p,q,\lambda,g) = \lambda g \phi_1(p,p,\lambda) + (1-\lambda) g \phi_1(p,q,0), \qquad if \quad \beta = 0,$$
$$\phi_2(p,q,\lambda,g) = \lambda g(p) + (1-\lambda) g(q), \qquad if \quad \beta = 0. \tag{2.1}$$

We will also assume that ϕ_1 is continuous with respect to λ. We give the following definitions and preliminaries:

Definition 2.1 (ϕ_1-convex set)**.** *A set D is said to be ϕ_1-convex if $\phi_1(p,q,\lambda) \in D$ for all $p,q \in D$, $\lambda \in [0,1]$. The intersection of ϕ_1-convex sets is also ϕ_1-convex.*
From now onwards, we take D as a ϕ_1-convex set [21].

Definition 2.2 ((ϕ_1,ϕ_2)-convex(concave) function)**.** *A function $g \in G$ is (ϕ_1,ϕ_2)-convex(concave) if*

$$f(\phi_1(p,q,\lambda)) \leq \phi_2(p,q,\lambda,f) \qquad (\geq),$$

for all $p,q \in D$ and $0 \leq \lambda \leq 1$.
If $g = (g_1, g_2, ..., g_k) : D \to \Re^k, g_i \in G$, and g_i is (ϕ_1,ϕ_2)-convex(concave) for i=1,2,...,k, then the vector valued function g is said to be (ϕ_1,ϕ_2)-convex(concave) [21].

Definition 2.3 (($\phi_1, \phi_2) - \beta$-convex(concave) function). *A function $g \in G$ is said to be $(\phi_1, \phi_2) - \beta$-convex(concave) if*

$$
\begin{aligned}
g(\phi_1(p,q,\lambda)) \leqq \phi_2(p,q,\lambda,g) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta} \\
if \quad \beta \neq 0 \quad (\geq), \\
f(\phi_1(p,q,\lambda)) \leqq \phi_2(p,q,\lambda,g) = \lambda g(p) + (1-\lambda)g(q) \\
if \quad \beta = 0 \quad (\geq),
\end{aligned}
\tag{2.2}
$$

for all $p, q \in D$ and $0 \leq \lambda \leq 1$.
If $g = (g_1, g_2, ..., g_k) : D \to \Re^k, g_i \in G$, and g_i is $(\phi_1, \phi_2) - \beta$-convex(concave) for i=1,2,...,k, then the vector valued function g is said to be $(\phi_1, \phi_2) - \beta$-convex(concave).

Definition 2.4 (ϕ_1-quasi-convex function). *A function $g \in G$ is said to be ϕ_1-quasi-convex [21] on D if for every $p, q \in D, \lambda \in [0,1]$*

$$g(\phi_1(p,q,\lambda)) \leqq max\{g(p), g(q)\}.$$

Definition 2.5 ($\phi_1 - \beta$-quasi-convex function). *A function $g \in G$ is said to be $\phi_1 - \beta$-quasi-convex on D if for every $p, q \in D, \lambda \in [0,1]$*

$$
\begin{aligned}
g(\phi_1(p,q,\lambda)) \leqq \phi_2(p,q,\lambda,g) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta} \\
\leqq max\{g(p), g(p)\} \quad if \quad \beta \neq 0,
\end{aligned}
$$

$$g(\phi_1(p,q,\lambda)) \leqq \phi_2(p,q,\lambda,g) = \lambda g(p) + (1-\lambda)g(q) \leqq max\{g(p), g(p)\} \quad if \quad \beta = 0. \tag{2.3}$$

Remark 2.1. We say that this definition is independent on the vector or topological structure on D; In fact, D can be any set.

Remark 2.2. If ϕ_1, ϕ_2 satisfy the assumptions of (2.1), then every $(\phi_1, \phi_2) - \beta$- convex function is $\phi_1 - \beta$-quasi-convex. We give some examples below.

Example 2.1. Let D be a convex subset of $\Re^n$, and define $\phi_1(p,q,\lambda) = \lambda p + (1-\lambda)q$, $\phi_2(p,q,\lambda,g) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta} \quad if \quad \beta \neq 0$ and $\phi_2(p,q,\lambda,g) = \lambda g(p) + (1-\lambda)g(q) \quad if \quad \beta = 0$, then the convex function on D is $(\phi_1, \phi_2) - \beta$-convex.

Example 2.2. If $\eta : \Re^n \times \Re^n \to \Re^n$, D is a pre-invex set with respect to η, then an η-pre-invex function $g : D \to \Re$ is $(\phi_1, \phi_2) - \beta$-convex with $\phi_1(p,q,\lambda) = \lambda\eta(p,q) + q$ and $\phi_2(p,q,\lambda,g) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta} \quad if \quad \beta \neq 0$ also $\phi_2(p,q,\lambda,g) = \lambda g(p) + (1-\lambda)g(q) \quad if \quad \beta = 0$, where $\eta(p,q) = p - q$ [27].

Example 2.3. Let $D \subseteq N$ where N is an Euclidean manifold and D is geodesically convex. A geodesically convex function on D is $(\phi_1, \phi_2) - \beta$-convex, with $\phi_1(p,q,\lambda) = \gamma_{p,q}(\lambda)$ and $\phi_2(p,q,\lambda,g) \leq ln(\lambda e^{\beta g(\gamma_{p,p}(\lambda)} + (1-\lambda)e^{\beta g(\gamma_{p,q}(0)})^{1/\beta} \quad if \quad \beta \neq 0$ also $\phi_2(p,q,\lambda,g) = \lambda g(\gamma_{p,p}(\lambda)) + (1-\lambda)g(\gamma_{p,q}(0)) \quad if \quad \beta = 0$, where $\gamma_{p,q}$ is the geodesic from q to p [23].

Example 2.4. Let D be a convex subset of $\Re^n$, $\phi_1(p,q,\lambda) = \lambda\eta(p,q) + q$ and
$\phi_2(p,q,\lambda,g) \leq ln(a_1(p,q,\lambda)e^{\beta g(\phi_1(p,p,a_1(p,q,\lambda)} + (1-a_1(p,q,\lambda))e^{\beta g(\phi_1(p,q,0)})^{1/\beta} \quad if \quad \beta \neq 0$
also
$\phi_2(p,q,\lambda,g) = a_1(p,q,\lambda)g(p) + (1-a_1(p,q,\lambda))g(q) \quad if \quad \beta = 0.$
Then every B-vex function on D (with respect to a_1) is $(\phi_1, \phi_2) - \beta$-convex [5, 24].

Example 2.5. Let I be a one - to - one mapping from $D \subseteq \Re^n$ to $\Re^n$, and Φ a strictly monotone increasing function mapping a subset $\sum$ of $\Re$ onto $\Re$. A function $g : D \to \sum$ is called $(I, \Phi) - \beta$-convex if, for any $p, q \in D$ and $\lambda \in [0, 1]$

$$f(N_I([p, q], \lambda)) \leqq n_\Phi[(g(p), g(q)), \lambda],$$

provided that $rangeg \subset dom\Phi$. Here

$$N_I([p, q], \lambda) \leq I^{-1}(ln(\lambda e^{\beta I(\phi(p,p,\lambda)} + (1-\lambda)e^{\beta I(\phi(p,q,0)})^{1/\beta} \quad if \quad \beta \neq 0,$$

also

$$N_I([p, q], \lambda) = \lambda I(p) + (1-\lambda)I(q) \quad if \quad \beta = 0,$$

$$n_\Phi[(g(p), g(q)), \lambda] \leq \Phi^{-1}(ln(\lambda e^{\beta\Phi(g(\phi(p,p,\lambda)} + (1-\lambda)e^{\beta\Phi(g(\phi(p,q,0)})^{1/\beta} \quad if \quad \beta \neq 0,$$

$$n_\Phi[(g(p), g(q)), \lambda] = \Phi^{-1}(\lambda\Phi(g(p) + (1-\lambda)\Phi(g(q)) \quad if \quad \beta = 0.$$

Choosing $\phi_1(p, q, \lambda) = N_I([p, q], \lambda)$ and $\phi_2(p, q, \lambda, g) = n_\Phi[(g(p), g(q)), \lambda]$, we see that an $(I, \Phi) - \beta$-convex function is a particular $(\phi_1, \phi_2) - \beta$-convex function [7].

Remark 2.3. The functions ϕ_1, ϕ_2 of the Examples 1 - 5 satisfy (2.1).

Example 2.6. A function $g : \Re^n \to \bar{\Re}^1 = \Re \cup \{-\infty\}$ is called G-convex on the convex set D if, for every $p, q \in D$, $p \neq q$, $\lambda \in (0, 1)$,

$$g((1-\lambda)q + \lambda p) \leqq G(g(p), g(q), \|p - q\|, \lambda),$$

where $G(m_1, m_2, \delta, \alpha) : \bar{\Re}^1 \times \bar{\Re}^1 \times \Re_+ \times \Re_+ \to \bar{\Re}^1$ is continuous and non-decreasing in (m_1, m_2) and $\|.\|$ is an arbitrary norm on $\Re^n$. If we take $\phi_1(p, q, \lambda) = \lambda p + (1-\lambda)q$ and $\phi_2(p, q, \lambda, g) = G(g(p), g(q), \|p - q\|, \lambda) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta}$, we get that a G-convex function is an example of $(\phi_1, \phi_2) - \beta$-convex function [12].

Now we will give some examples of $(\phi_1, \phi_2) - \beta$-convex function that justify our results.

Example 2.7. Let $D \subset \Re$ be the set $D = (-\infty, \infty)$, and $g : D \to \Re$ be the function defined as follows:

$$g(p) = \begin{cases} 4p, & if p > 0 \\ p^2 - p + 1, & if p < 0 \end{cases}.$$

Define the functions $\phi_1 : D \times D \times [0, 1]$ and $\phi_2 : D \times D \times [0, 1] \times G$ as follows:

$$\phi_1(p, q, \lambda) = \begin{cases} (1-\lambda)q + \lambda p, & if pq > 0 \\ q, & if pq < 0 \end{cases},$$

$$\phi_2(p, q, \lambda, g) = \begin{cases} g(q), & if \lambda = 0 \\ max\{g(p), g(q)\}, & if 0 < \lambda \leq 1 \end{cases}.$$

This function is $(\phi_1, \phi_2) - \beta-$ convex and also justify our results.

Example 2.8. Let $D \subset \Re$ be the set $D = (-\infty, -1) \cup (1, \infty)$, and $g : D \to \Re$ be the function defined as follows:

$$g(p) = \begin{cases} |p| - 1, & if |p| < 1 \\ 1, & if |p| > 1 \end{cases}.$$

Define the functions $\phi_1 : D \times D \times [0,1]$ and $\phi_2 : D \times D \times [0,1] \times G$ as follows:

$$\phi_1(p, q, \lambda) = \begin{cases} (1-\lambda)q + \lambda p, & if pq > 0 \\ q, & if pq < 0 \end{cases},$$

$$\phi_2(p, q, \lambda, g) = \begin{cases} g(q), & if \lambda = 0 \\ max\{g(p), g(q)\}, & if 0 < \lambda \leq 1 \end{cases}.$$

This function is $(\phi_1, \phi_2) - \beta-$ convex and also justify our results.

Now for suitable assumptions on ϕ_1 and/or ϕ_2, we will discuss some properties of the class of $(\phi_1, \phi_2) - \beta$-convex functions.

Observation (a). We are assuming that, ϕ_2 issuperlinear with respect to $g \in G$, that is ϕ_2 is superadditive and positively homogeneous. Then the class of $(\phi_1, \phi_2) - \beta$-convex functions is a convex cone. (Practically, if g,h are $(\phi_1, \phi_2) - \beta$-convex, and $\alpha > 0$.

$$(g+h)(\phi_1(p,q,\lambda)) \leqq \phi_2(p,q,\lambda,g) + \phi_2(p,q,\lambda,h) \leqq \phi_2(p,q,\lambda,g+h)$$

$$(g+h)(\phi_1(p,q,\lambda)) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta}$$

$$+ln(\lambda e^{\beta h(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta h(\phi_1(p,q,0)})^{1/\beta}$$

$$\leq ln(\lambda e^{\beta(g+h)(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta(g+h)(\phi_1(p,q,0)})^{1/\beta} \quad if \quad \beta \neq 0$$

$$(g+h)(\phi_1(p,q,\lambda)) \leq \phi_2(p,q,\lambda,g) + \phi_2(p,q,\lambda,h)$$

$$= \lambda g(p) + (1-\lambda)h(q) + \lambda g(p) + (1-\lambda)h(q) = \lambda(g+h)(p) + (1-\lambda)(g+h)(q)$$

$$if \quad \beta = 0$$

$$(\alpha g)(\phi_1(p,q,\lambda)) = \alpha(g(\phi_1(p,q,\lambda)) \leqq \alpha\phi_2(p,q,\lambda,g) = \phi_2(p,q,\lambda,\alpha g)$$

$$\leq ln(\lambda e^{\beta\alpha g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta\alpha g(\phi_1(p,q,0)})^{1/\beta} \quad if \quad \beta \neq 0$$

$$(\alpha g)(\phi_1(p,q,\lambda)) = \alpha(g(\phi_1(p,q,\lambda)) \leqq \alpha\phi_2(p,q,\lambda,g)$$

$$= \phi_2(p,q,\lambda,\alpha g) = \lambda(\alpha g)(p) + (1-\lambda)(\alpha g)(q) \quad if \quad \beta = 0.$$

Observation (b). We are also assuming that, $g : D \to \Re$ is $(\phi_1, \phi_2) - \beta$-convex, $h : \Re \to \Re$ is an increasing function and $(\phi_3, \phi_4) - \beta$-convex and $hog \in G$. Then, if

$$\phi_2(p,q,\lambda,g) \leqq \phi_3(g(p), g(q), \lambda),$$

$$\phi_4(g(p), g(q), \lambda, h) \leqq \phi_2(p,q,\lambda,hog),$$

the function hog is $(\phi_1, \phi_2) - \beta$-pre convex.(Practically, we have that)

$$(hog)(\phi_1(p,q,\lambda)) \leqq h(\phi_2(p,q,\lambda,g))$$

$$\leqq h(\phi_3(g(p), g(q), \lambda))$$

$$\leqq \phi_4(g(p), g(q), \lambda, h)$$

$$\leqq \phi_2(p,q,\lambda,hog)$$

$$\leq ln(\lambda e^{\beta\alpha(hog)(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta\alpha(hog)(\phi_1(p,q,0)})^{1/\beta} \quad if \quad \beta \neq 0$$

$$= \lambda(hog)(p) + (1-\lambda)(hog)(q) \quad if \quad \beta = 0.$$

Remark 2.4. In Examples 2.1 - 2.4, we see that ϕ_2 is linear with respect to g. In Example 2.5, if Φ is superlinear then ϕ_2 will be superlinear.

Now we will consider a scalar value optimization problem, which can be expressed as

$$(P) \ \ ming(p) \ \ s.t. \ \ h(p) \leq 0,$$

where $g : D \to \Re$, $h : D \to \Re^k$ Denote the feasible set by D_0, where

$$D_0 = \{p \in D : h(p) \leq 0\}.$$

Then the following holds:

Proposition 2.1. *Suppose that*

(i) $h = (h_1, h_2, ..., h_k)$ *is* (ϕ_1, ϕ_2) *-* β*- convex(see Definition* (2.3)*:*

(ii) g *is* (ϕ_1, ϕ_2) *-* β*- convex.*

Then the set of solutions of problem (P) will be ϕ_1- β- convex.

Proof. The feasible set D_0 is ϕ_1- β-convex; Practically, if $p_1, p_2 \in D_0$, from (i) and (2.1) we have

$$\begin{aligned} h_i(\phi_1(p_1, p_2, \lambda)) \leq \phi_2(p_1, p_2, \lambda, h_i) &\leq ln(\lambda e^{\beta h_i(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta h_i(\phi_1(p,q,0)})^{1/\beta} \\ &\leq max\{h_i(\phi_1(p_1, p_2, \lambda)), h_i(\phi_1(p_1, p_2, 0))\} \leq 0 \end{aligned}$$

for any $i = 1, 2, ..., k$. Next, let $\min_{p \in D_0} g(p)$ be attained at p_1^0 and p_2^0. By the hypothesis (ii) and (2.1)

$$\begin{aligned} f(\phi_1(p_1^0, p_2^0, \lambda)) \leq \phi_2(p_1^0, p_2^0, \lambda, g) &\leq ln(\lambda e^{\beta g(\phi_1(p_1^0, p_1^0, \lambda)} + (1-\lambda)e^{\beta g(\phi_1(p_1^0, p_2^0, 0))})^{1/\beta} \\ &\leq max\{g(\phi_1(p_1^0, p_1^0, \lambda)), g(\phi_1(p_1^0, p_2^0, 0))\} = g(p_1^0) \end{aligned}$$

But $g(p_1^0) = g(p_2^0) = min_{p \in D_0}$ g(p), hence $g(\phi_1(p_1^0, p_1^0, \lambda)) = g(p_1^0)$ which completes the proof. ☐

Definition 2.6 (($(\phi_1, \phi_2) - \beta$ - pre-strictly convex(concave) function)**.** *Let* $p^0 \in D$*. We say that g is* $(\phi_1, \phi_2) - \beta$ *strictly convex(concave) at* p_0 *if*

$$\begin{aligned} g(\phi_1(q, p_0, \lambda)) < \phi_2(q, p_0, \lambda, g) \qquad\qquad (>) \\ \leq ln(\lambda e^{\beta g(\phi_1(q,q,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(q,p_0,0)})^{1/\beta} \qquad (>), \end{aligned} \tag{2.4}$$

we say that g is weakly $(\phi_1, \phi_2) - \beta$ *strictly convex(concave) at* p_0 *if* (4) *holds for some* $\lambda \in (0, 1)$*. If* (4) *is satisfied at any* $p_0 \in D$*, then g is* $(\phi_1, \phi_2) - \beta$ *strictly convex(concave) on D.*

Proposition 2.2. *Suppose that* D_0 *is a* $\phi_1 - \beta$ *convex set, and*

(i) g *is* $(\phi_1, \phi_2) - \beta$ *strictly convex at* $p_0 \in D_0$*.*

(ii) p_0 *is a solution of problem (P).*

Then p_0 *is the unique solution of problem (P).*

Proof. Let p^* be another solution of (P). $p^* \neq p_0$. Then, for all $\lambda \in (0, 1)$

$$\begin{aligned} g(\phi_1(p^*, p_0, \lambda)) \leq \phi_2(p^*, p_0, \lambda, g) &\leq ln(\lambda e^{\beta g(\phi_1(p^*, p^*, \lambda)} + (1-\lambda)e^{\beta g(\phi_1(p^*, p_0, 0)})^{1/\beta} \\ &\leq max\{g(\phi_1(p^*, p^*, \lambda)), g(\phi_1(p^*, p_0, 0))\} = g(p_0) \end{aligned}$$

which contradicts hypothesis(ii).

In case of $(\phi_1, \phi_2) - \beta$ concave functions(see Definition *3*), we have the following: ☐

Theorem 2.1. *Suppose that*

(i) g *is* $(\phi_1, \phi_2) - \beta$ *strictly concave in D;*

(ii) $\forall p_0 \in int(D_0) \exists p, q \in D_0, p \neq q, \lambda \in (0,1]$ *such that* $\phi_1(p,q,\lambda) = p_0$;

(iii) D_0 *is* $\phi_1 - \beta$ *convex;*

(iv) $\phi_2(p,q,\lambda,g) \geq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta} \geq min\{g(\phi_1(p,p,\lambda)), g(\phi_1(p,q,0))\}$ *for every* $p, q \in D_0$, $\lambda \in [0,1]$.

Then there are no interior points of D_0 *which are solution of (P), i.e. if* p_0 *is a solution of (P), then* p_0 *is a boundary point of* D_0.

Proof. If the solution set of (P) is empty, or int(D_0) is empty, there is nothing to prove. Let p_0 is a solution of (P), and $p_0 \in int(D_0)$. Then by (ii) there exist $p, q \in D_0, p \neq q$ and $\lambda \in (0,1]$ such that $p_0 = \phi_1(p,q,\lambda)$. By (i) we have that

$$g(p_0) = g(\phi_1(p,q,\lambda)) > \phi_2(p,q,\lambda,g)$$
$$\geq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta} \geq min\{g(\phi_1(p,p,\lambda)), g(\phi_1(p,q,0))\} \geq g(p_0).$$

Contradiction, so it is concluded that p_0 is not a solution of (P). Let $\mu_\delta(p_0)$ denote a neighbourhood of p_0 of radius δ. □

Theorem 2.2. *Suppose that*

(i) g *is* $(\phi_1, \phi_2) - \beta$ *strictly convex;*

(ii) $p_0 \in D_0$ *is a local minimum of (P);*

(iii) $\forall \delta_1 > 0$, *and* $\forall p \in D_0, \exists \lambda \in (0,1]$ *such that* $\phi_1(p_0,p,\lambda) \in \mu_{\delta_1}(p_0)$;

(iv) D_0 *is* $\phi_1 - \beta$ *convex;*

Then p_0 *is a strict global minimum of (P).*

Proof. By hypothesis (iv), for every $p \in D_0$, and for every $\lambda \in [0,1]$, $\phi_1(p_0,p,\lambda) \in D_0$. Since p_0 is a local minimum of (P), there exists $\mu_{\delta_2}(p_0)$ such that for every $p \in \mu_{\delta_2}(p_0) \cap D_0$, $g(p_0) \leq g(p)$. Now let $p \in D_0$, $p \neq p_0$. Then, by hypothesis (ii) and (iii), with $\delta_1 = \delta_2$ we have that $g(p_0) \leq \phi_1(p_0,p,\lambda))$ for some $\lambda \in (0,1]$. Therefore, using (i) and (2.1), we have

$$g(p_0) \leq g(\phi_1(p_0,p,\lambda)) < \phi_2(p_0,p,\lambda,g)$$
$$\leq ln(\lambda e^{\beta g(\phi_1(p_0,p_0,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p_0,p,0)})^{1/\beta} \leq max\{g(\phi_1(p_0,p_0,\lambda)), g(\phi_1(p_0,p,0))\}$$

Obviously, $max\{g(\phi_1(p_0,p_0,\lambda)), g(\phi_1(p_0,p,0))\} \neq g(p_0)$ since $g(p_0) \not< g(p_0)$. Therefore $g(p_0) < g(p)$. Since p is an arbitrary member of D_0, the proof is complete.

On the basis of Theorem 2.2, the following results can be obtained. □

Theorem 2.3. *Suppose that*

(i) g *is* $(\phi_1, \phi_2) - \beta$ *convex;*

(ii) $p_0 \in D_0$ *is a strict local minimum of (P);*

(iii) $\forall \delta_1 > 0$, *and* $\forall p \in D_0, \exists \lambda \in (0,1]$ *such that* $\phi_1(p_0,p,\lambda) \in \mu_{\delta_1}(p_0) \setminus \{p_0\}$;

(iv) D_0 *is* $\phi_1 - \beta$ *convex;*

Then p_0 *is a strict global minimum of (P).*

Theorem 2.4. *Suppose that*

(i) g *is* $(\phi_1, \phi_2) - \beta$ *convex;*

(ii) $p_0 \in D_0$ *is a local minimum of (P);*

(iii) $\forall \delta_1 > 0$, *and* $\forall p \in D_0, \exists \lambda \in (0,1]$ *such that* $\phi_1(p_0,p,\lambda) \in \mu_{\delta_1}(p_0)$;

(iv) D_0 *is* $\phi_1 - \beta$ *convex;*

(v) $\phi_2(p_0, p, \lambda, g) \leq ln(\lambda e^{\beta g(\phi_1(p_0,p_0,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p_0,p,0))})^{1/\beta} \leqq max\{g(\phi_1(p_0, p_0, \lambda)), g(\phi_1(p_0, p, 0))\}$ *for every* $p \in D_0$ *with* $g(p) \neq g(p_0)$*, and for all* $\lambda \in (0, 1)$.

Then p_0 *is a global minimum of (P).*

Example 2.9. Let $D \subset \Re$ be the set $D = (-\infty, -3) \cup (3, \infty)$, and $g : D \to \Re$ be the function defined as follows:

$$g(p) = \begin{cases} |p| - 3, & if|p| < 3 \\ 1, & if|p| \geqq 3 \end{cases}.$$

Define the functions $\phi_1 : D \times D \times [0, 1]$ and $\phi_2 : D \times D \times [0, 1] \times G$ as follows:

$$\phi_1(p, q, \lambda) = \begin{cases} (1-\lambda)q + \lambda p, & ifpq > 0 \\ q, & ifpq < 0 \end{cases},$$

$$\phi_2(p, q, \lambda, g) = \begin{cases} g(q), & if\lambda = 0 \\ max\{g(p), g(q)\}, & if 0 < \lambda \leq 1 \end{cases}.$$

This function is $(\phi_1, \phi_2) - \beta-$ convex which verifies our results.

Now we will study a regularity property of the product of $(\phi_1, \phi_2) - \beta$ convex functions (i=2,3). First, we say the following

Lemma 2.1. *Suppose that g,h are satisfying the conditions and also real valued functions defined on D,*

(i) $g(p) \geqq 0$, $h(p) \geqq 0$

(ii) $g(p) - g(q))(h(p) - h(q)) \geqq 0 \ \forall \ p, q \in D.$

Then for every $p, q \in D$*, either*

$$g(p)h(p) \geqq g(q)h(p) \ \ and \ \ g(p)h(p) \geqq g(p)h(q)$$

or

$$g(q)h(q) \geqq g(p)h(q) \ \ and \ \ g(q)h(q) \geqq g(q)h(p).$$

Proof. Since, by (ii),

$$g(p) - g(q))(h(p) - h(q)) \geqq 0 \ \ \forall p, q \in D$$

it follows that either

$$g(p) \geqq g(q) \ \ and \ \ h(p) \geqq h(q)$$

or

$$g(q) \geqq g(p) \ \ and \ \ h(q) \geqq h(p)$$

which further implies(in view of (i)) that either

$$g(p)h(p) \geqq g(q)h(p) \ \ and \ \ g(p)h(p) \geqq g(p)h(q)$$

or

$$g(q)h(q) \geqq g(p)h(q) \ \ and \ \ g(q)h(q) \geqq g(q)h(p).$$

□

Proposition 2.3. *Suppose that*

(i) g,h are nonnegative functions defined on D and satisfying the inequality

$$(g(p) - g(q))(h(p) - h(q)) \geqq 0, \ \ \forall p, q \in D;$$

(ii) g *is* $(\phi_1, \phi_2) - \beta$ *convex,* h *is* $(\phi_1, \phi_2) - \beta$ *convex.*

Then pq is $\phi_1 - \beta$ quasi-convex.

Proof. For any $p, q \in D$ and $\lambda \in [0,1]$,

$$(gh)(\phi_1(p,q,\lambda)) = g(\phi_1(p,q,\lambda))h(\phi_1(p,q,\lambda))$$
$$\leqq \phi_2(p,q,\lambda,g)\phi_2(p,q,\lambda,h)$$
$$\leq \{ln(\lambda e^{\beta f(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta f(\phi_1(p,q,0)})^{1/\beta}\}$$
$$\times\{ln(\lambda e^{\beta h(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta h(\phi_1(p,q,0)})^{1/\beta}\}$$
$$\leqq max\{g(p), g(q)\}.max\{h(p), h(q)\}.$$

Now $max\{g(p), g(q)\}.max\{h(p), h(q)\}$, in view of lemma, is less than or equal to $max\{g(p), g(q)\}.max\{h(p), h(q)\}$; hence it follows that

$$(gh)(\phi_1(p,q,\lambda)) \leq ln(\lambda e^{\beta(gh)(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta(gh)(\phi_1(p,q,0)})^{1/\beta}$$
$$\leq max\{g(p)h(p), g(q)h(q)\}$$
$$= max\{(gh)(p), (gh)(q)\}.$$

Therefore gh is $\phi_1 - \beta$-quasi-convex. □

Now we will consider the following family of problems:

$$\min g(p) \ \ s.t. \ \ h(p) \leqq \epsilon,$$

where $g : \Re^n \to \Re$, $h : \Re^n \to \Re^k$, $\epsilon \in \Re^k$. Denote by $g^*(\epsilon)$ the function

$$g^* : \Re^k \to \Re, \ \ g^*(\epsilon) = inf\{g(p) : h(p) \leqq \epsilon\} \ ([25]).$$

Assume that g is $(\phi_1, \phi_2) - \beta$ convex, where $\phi_2(p_1, p_2, \lambda, g) = \phi_4(g(p_1), g(p_2), \lambda)$ and the vector function h is $(\phi_1, \phi_2) - \beta$ convex, where

$$\bar{\phi_2} : \Re^n \times \Re^n \times [0,1] \times G^k \to \Re^k, \ \ \bar{\phi_2}(p,q,\lambda,h) = \phi_3(h(p), h(q), \lambda),$$

and $\phi_3(b_1, b_2, \lambda)$ is nondecreasing in (b_1, b_2) with respect to the component wise order (if $b_1^i \leqq c_1^i$ and $b_2^j \leqq c_2^j$, $\forall i, j$, then $\phi_3(b_1, b_2, \lambda) \leqq \phi_3(c_1, c_2, \lambda)$, for every $\lambda \in [0,1]$)
We have the following

Theorem 2.5. *The function g^* is a $(\phi_3, \phi_4) - \beta$convex on $\Re^k$ (i.e. $g^*(\phi_3(\epsilon_1, \epsilon_2, \lambda)) \leqq \phi_4(g^*(\epsilon_1), g^*(\epsilon_2), \lambda)$).*

Proof. Notice that if $h(p_1) \leqq \epsilon_1$, $h(p_2) \leqq \epsilon_2$ then

$$h(\phi_1(p_1, p_2, \lambda)) \leqq \phi_3(h(p_1), h(p_2), \lambda) \leqq \phi_3(\epsilon_1, \epsilon_2, \lambda);$$

in particular,

$$\{(p_1, p_2) : h(p_1) \leqq \epsilon_1, h(p_2) \leqq \epsilon_2\} \subseteq \{(p_1, p_2) : h(\phi_1(p_1, p_2, \lambda)) \leqq \phi_3(\epsilon_1, \epsilon_2, \lambda)\}$$

Hence

$g^*(\phi_3(\epsilon_1, \epsilon_2, \lambda)) = inf\{g(p) : h(p) \leqq \phi_3(\epsilon_1, \epsilon_2, \lambda)\}$
$\leqq inf\{g(\phi_1(p_1, p_2, \lambda)) : h(\phi_1(p_1, p_2, \lambda)) \leqq \phi_3(\epsilon_1, \epsilon_2, \lambda)\}$
$\leqq inf\{\phi_2(p_1, p_2, \lambda, g) : h(p_1) \leqq \epsilon_1, h(p_2) \leqq \epsilon_2\}$ $(from(2.6))$
$\leq inf\{ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta} : h(p_1) \leqq \epsilon_1, h(p_2) \leqq \epsilon_2\}$
$= \phi_4(g^*(\epsilon_1), g^*(\epsilon_2), \lambda)$. □

3 The Differentiable Case

Let us assume that ϕ_1, ϕ_2 have right partial derivative with respect to λ at $\lambda = 0$, for all $p, q \in D$, for all $g \in G$. If we consider a differentiable $(\phi_1, \phi_2) - \beta$ convex function g, defined on $D \subseteq \Re^n$, taking into account (1), for $p, q \in D$ and $\lambda \in (0, 1]$ we get that

$g(\phi_1(p, q, \lambda) \leqq \phi_2(p, q, \lambda, g)$

$\Longrightarrow\ g(\phi_1(p, q, \lambda) - g(q) \leqq \phi_2(p, q, \lambda, g) - g(q)$

$\Longrightarrow\ g(\phi_1(p, q, \lambda) - g(\phi_1(p, q, 0) \leqq \phi_2(p, q, \lambda, g) - \phi_2(p, q, 0, g)$

$\Longrightarrow\ \frac{1}{\lambda}(g(\phi_1(p, q, \lambda) - g(\phi_1(p, q, 0)) \leqq \frac{1}{\lambda}(\phi_2(p, q, \lambda, g) - \phi_2(p, q, 0, g))$

$\Longrightarrow\ \frac{1}{\lambda}(g(\phi_1(p, q, \lambda) - g(\phi_1(p, q, 0))$

$\Longrightarrow \leqq \frac{1}{\lambda}(ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta} g(\phi_1(p, q, 0))^{1/\beta} - \phi_2(p, q, 0, g))$

and, taking the limit of both sided for $\lambda \to 0^+$ (and since $\phi_1(p, q, 0) = q$), we have

$$\nabla_q g(\phi_1(p, q, 0)) \frac{\partial^+ \phi_1}{\partial \lambda}(p, q, \lambda)\Big|_{\lambda=0} \leqq \frac{\partial^+ \phi_2}{\partial \lambda}(p, q, \lambda, g)\Big|_{\lambda=0} \nabla_q g(\phi_1(p, q, 0)) \frac{\partial^+ \phi_1}{\partial \lambda}(p, q, \lambda)\Big|_{\lambda=0}$$

$$\leqq \frac{\partial^+ (ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta})}{\partial \lambda}\Big|_{\lambda=0}$$

We therefore have the following.

Proposition 3.1. *We are assuming that ϕ_1, ϕ_2 have right partial derivative with respect to λ at $\lambda = 0$. Then a differentiable $(\phi_1, \phi_2) - \beta$ convex function g satisfies the inequality*

$$\phi_2(p, q, g) \geqq \nabla_q g(q) \phi_1(p, q),$$

for every $p, q \in D$, where

$$\phi_1(p, q) = \frac{\partial^+ \phi_1}{\partial \lambda}(p, q, \lambda)\Big|_{\lambda=0}, \phi_2(p, q, f) = \frac{\partial^+ \phi_2}{\partial \lambda}(p, q, \lambda, g)\Big|_{\lambda=0} \phi_1(p, q) = \frac{\partial^+ \phi_1}{\partial \lambda}(p, q, \lambda)\Big|_{\lambda=0},$$

$$\phi_2(p, q, g) = \frac{\partial^+ (ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1-\lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta})}{\partial \lambda}\Big|_{\lambda=0}.$$

Remark 3.1. The same result holds in a more general setting, where D is a subset of a Riemannian manifold, and the r.h.s. of (2.4) is defined as $dg_q(\phi_1(p, q))$.

It is easy to verify that a $\phi_1 - \beta$ quasi-convex function h satisfies the condition

$$h(p) \leqq h(q) \implies \nabla_q h(q) \phi_1(p, q) \leqq 0,$$

for every $p, q \in D$.

Definition 3.1. *Let $\psi : D \times D \to D$. We say that ψ is skew-symmetric on $D \times D$ if $\psi(p, q) = -\psi(q, p)$ for every (p,q) $\in D \times D$.*

Corollary 3.1. *(To Proposition* 3.1*) Suppose that g is differentiable and $(\phi_1, \phi_2) - \beta$ convex; if ϕ_1, ϕ_2 are related to ϕ_1, ϕ_2 as in (Proposition* 3.1*), and skew-symmetric for any (p,q) $\in D \times D$, then ∇g is $\phi_1 - \beta -$monotone on D, i.e.*

$$(\nabla_p g(p) - \nabla_q g(q)) \phi_1(p, q) \geqq 0 \quad \forall (p, q) \in D \times D.$$

Proof. By (Proposition 3.1) we have that

$$\phi_2(p, q, g) \geqq \nabla_q g(q) \phi_1(p, q) \quad \phi_2(q, p, g) \geqq \nabla_p g(p) \phi_1(q, p)$$

and the conclusion follows from the skew-symmetry.

The local condition expressed by (Proposition 3.1) is usually not sufficient to guarantee the $(\phi_1, \phi_2) - \beta -$ convexity of g, unless we specify some more restrictive and global properties of the functions ϕ_1 and ϕ_2. Indeed, consider $\phi_1(p, q, \lambda) = q + \lambda \eta(p, q)$, $\phi_2(p, q, \lambda, g) = (1 - \lambda) g(q) + \lambda g(p)$ □

In Mohan and Neogy provided a counterexample, showing that the condition

$$f(x) - f(y) \geqq \nabla_y f(y)\eta(x, y)$$

does not imply in general that

$$g(q + \lambda\eta(p, q)) \leqq (1 - \lambda)g(q) + \lambda g(p) \quad \forall\lambda \in [0, 1].$$

We will assume that the function g is differentiable on D. The following results relate the necessary condition for a differentiable $\phi_1, \phi_2)-\beta$-convex function g, and the definition of $\phi_1, \phi_2)-\beta-$convexity. In the first result we assume that a "regularity condition" is satisfied by ϕ_1, whereas ϕ_2 is the usual r.h.s. of the definition of convexity, providing a slight extension of the ordinary convex case.

Proposition 3.2. *Assume that ϕ_1 is differentiable with respect to λ in $[0, 1]$: if the following are satisfied*

(i) $\phi_1(p, q, 0) = q, \phi_1(p, q, 1) = p$;

(ii) $\frac{\partial\phi_1}{\partial v}(p, q, v)(t' - v) = \phi_1(\phi_1(p, q, t'), (\phi_1(p, q, v))$;

(iii) $\phi_2(p, q, \lambda, g) = (1 - \lambda)g(q) + \lambda g(p)$

for every $p, q \in D$, $v, t', \lambda \in [0, 1]$, then a function f satisfying (Proposition 3.1) is $\phi_1, \phi_2 - \beta$-convex.

Proof. By Proposition 3.1 and condition (i), it follows that $g(p) - g(q) \geqq \nabla_q g(q)\phi_1(p, q)$, and for every $p, q \in D$, we get that the function $h(w) = f(\phi_1(p, q, w))$ is convex ; indeed

$$\begin{aligned} h(t') - h(v) &= g(\phi_1(p, q, t')) - g(\phi_1(p, q, v)) \\ &\geqq \nabla_{\phi_1} g(\phi_1(p, q, v))\phi_1(\phi_1(p, q, t'), (\phi_1(p, q, v)) \text{ (by Proposition 3.1)} \\ &= \nabla_{\phi_1} g(\phi_1(p, q, v))\frac{\partial\phi_1}{\partial v}(p, q, v)(t' - v) \text{ (by (ii))} \\ &\quad h'(v)(t' - v). \end{aligned}$$

It follows that h is convex. Hence, $h(\lambda) \leqq (1 - \lambda)h(0) + \lambda h(1)$. Now by hypotheses (i) and (ii) we get that

$$g(\phi_1(p, q, \lambda) \leqq (1 - \lambda)g(q) + \lambda g(p) = \phi_2(p, q, \lambda, g),$$

(see [26], where a special case of the Proposition 3.2 is proved).
More generally, the following result relating Proposition 3.1 and $\phi_1, \phi_2 - \beta-$convexity holds. □

Theorem 3.1. *We are assuming that, g is a differentiable function on D, where D is a ϕ_1- convex subset of $\Re^n$. Let $\phi_i(i = 1, 2)$ be the function associated with ϕ_i as in (Proposition 3.1). Then we are assuming that there exists a function $H' : \Re \times \Re \times [0, 1] \to \Re$, $H' = H'(w, t', \lambda)$, and the following conditions are satisfied:*

(i) $H'(\phi_2(p, \phi_1(p, q, \lambda), g), \phi_2(q, \phi_1(p, q, \lambda), g), \lambda) \leqq \phi_2(p, q, \lambda, g) - g(\phi_1(p, q, \lambda))$;

(ii) H' is non decreasing in (w,t'), for every λ fixed (if $w_1 \leqq w_2, t'_1 \leqq t'_2$. we have that $H'(w_1, w_1, \lambda) \leqq H'(w_2, t'_2, \lambda)$);

(iii) $H'(\nabla_{\phi_1}(g(\phi_1(p, q, \lambda))\phi_1(p, \phi_1(p, q, \lambda)), \nabla_{\phi_1}(g(\phi_1(p, q, \lambda))\phi_1(q, \phi_1(p, q, \lambda)) = 0$ *for every* $\lambda \in [0, 1]$, $g \in G$, $p, q \in D$;

(iv) $\phi_2(p, r, g) \geqq \nabla_r g(r)\phi_1(p, r)$, $\forall p, r \in D$.

Then g is $(\phi_1, \phi_2) - \beta-$convex on D.

Proof. From (iv), with $r = \phi_1(p, q, \lambda)$ we have that

$$\phi_2(p, \phi_1(p, q, \lambda), g) \geqq \nabla_{\phi_1}(g(\phi_1(p, q, \lambda))\phi_1(p, \phi_1(p, q, \lambda)),$$

$$\phi_2(q, \phi_1(p, q, \lambda), g) \geqq \nabla_{\phi_1}(g(\phi_1(p, q, \lambda))\phi_1(q, \phi_1(p, q, \lambda)).$$

Let $w = \phi_2(p, \phi_1(p, q, \lambda), g), t' = \phi_2(q, \phi_1(p, q, \lambda), g)$; from (ii) and (iii), we get that

$$H'(w, t', \lambda) \geqq H'(\nabla_{\phi_1} g(\phi_1(p, q, \lambda))\phi_1(p, \phi_1(p, q, \lambda)),$$

$$\nabla_{\phi_1} g(\phi_1(p,q,\lambda))\phi_1(q,\phi_1(p,q,\lambda)),\lambda) = 0,$$

Finally, by (i), we have that

$$\phi_2(p,q,\lambda,g) - g(\phi_1(p,q,\lambda)) \geqq 0, \qquad \forall p,q \in D, \lambda \in [0,1],$$

that is g is $(\phi_1,\phi_2)-\beta-$convex. Notice that Condition C in [27] is a particular case of Theorem 3.1 where $\phi_1(p,q,\lambda) = q + \lambda\eta(p,q), \phi_2(p,q,\lambda,g) = (1-\lambda)g(q) + \lambda g(p)$, and $H'(w,t',\lambda) = \lambda w + (1-\lambda)t'$. $\square$

Example 3.1. Let $D \subset \Re$ be the set $D = [0,2]$, and $f : D \to [0,1]$ be the function defined as follows:

$$f(x) = \begin{cases} \sqrt{x}, & if x \in [0,1] \\ \sqrt{2-x}, & if x \in (1,2] \end{cases}.$$

Define the functions $\phi_1 : D \times D \times [0,1]$ and $\phi_2 : D \times D \times [0,1] \times F$ as follows:

$$\phi_1(x,y,\lambda) = \begin{cases} (1-\lambda)(2(2-\sqrt{y})) + \lambda(2-\sqrt{x}), & if xy > 0 \\ 2(2-\sqrt{y}), & if xy < 0 \end{cases},$$

$$\phi_2(x,y,\lambda,f) = \begin{cases} f(2(2-\sqrt{y})), & if \lambda = 0 \\ max\{f(2-\sqrt{x}), f(2(2-\sqrt{y}))\}, & if 0 < \lambda \leq 1 \end{cases}.$$

This function is $(\phi_1,\phi_2)-\beta-$ convex which verifies our results.

Proposition 3.3. *If $g : D \to \Re$ is differentiable, and Φ_1 satisfies assumptions (ii) and (iii) in Theorem* 3.1*, then g is Φ_1-quasi-convex if and only if (Remark* 3.1*) holds.*

Proof. Similar to the proof given in [27].

Under appropriate assumptions on Φ_2, a differentiable $(\phi_1,\phi_2)-\beta-$convex function, turns out to be invex, and we can guarantee that a stationary point is a global minimum. Here is a sufficient condition. Assume that ϕ_2 satisfies the inequality

$$c(p,q,\lambda,g)\phi_2(p,q,\lambda,g) \leqq (1-\lambda)g(q) + \lambda g(p), \tag{3.1}$$

for all $p,q \in D, \lambda \in [0,1], g \in G$, and for some function $c = c(p,q,\lambda,g) : D \times D \times [0,1] \times G \to \Re$, with $c(p,q,0,g) = 1$, $\frac{\partial c}{\partial \lambda}(p,q,\lambda,g)\big|_{\lambda=0} = 0$. Then we have the following. $\square$

Proposition 3.4. *Let g be a differentiable $(\phi_1,\phi_2)-\beta-$convex function, where ϕ_1 and ϕ_2 are differentiable with respect to λ at $\lambda = 0$, for every $p,q \in D$. Assume that condition* (3.1) *holds. Then g is invex with respect to $\eta(p,q) = \phi_1(p,q)$. In particular, every stationary point of g is a global minimum.*

Proof. From (3.1), we have that

$$(1-\lambda)g(q) + \lambda g(p) - g(q) \geqq c(p,q,\lambda,g)\phi_2(p,q,\lambda,g) - c(p,q,0,g)\phi_2(p,q,0,g).$$

Adding and subtracting $c(p,q,0,g)\phi_2(p,q,\lambda,g)$ to the right hand side of the above inequality and then dividing both sides by λ and taking the limit $\lambda \to 0^+$, we get

$$g(p) - g(q) \geqq \frac{\partial c}{\partial \lambda}(p,q,\lambda,t')\bigg|_{\lambda=0} \phi_2(p,q,0,g) + c(p,q,0,g)\phi_2(p,q,g) = \phi_2(p,q,g)$$

Since, by Proposition 3.1, $\phi_2(p,q,g) \geqq \nabla_q g(q)\phi_1(p,q)$, we have that

$$g(p) - g(q) \geqq \phi_2(p,q,g) \geqq \nabla_q g(q)\phi_1(p,q).$$

This proves that g is $\phi_1(p,q)-$ invex and hence every stationary point is a global minimum point.

Now we will assume that $g : D \to R$ and $\phi_1 : D \times D \times [0,1] \to D$ satisfy the assumptions

(i) $g \in C^2(D)$;

(ii) $\phi_1(p,q) \in C^2([0,1])$.

After that we have the following sufficient condition for $(\phi_1, \phi_2) - \beta-$ convexity: $\square$

Proposition 3.5. *In our previous assumptions, g is $(\phi_1, \phi_2) - \beta-$ convex for every $\phi_2(p, q, t', g) = \int_0^{t'} g(p, q, w)dw + g(q)$, where h is any solution of the differential inequality*

$$\frac{\partial h}{\partial t'}(p, q, t') \geqq \left(\frac{\partial \phi_1}{\partial t'}\right)^{T'}(p, q, t')H'_{\phi_1}g(\phi_1(p, q, t'))\frac{\partial \phi_1}{\partial t'}(p, q, t')$$
$$+\nabla_{\phi_1}g(\phi_1(p, q, t'))\frac{\partial^2 \phi_1}{\partial t'}(p, q, t')$$
$$h(p, q, 0) = \nabla_q g(q)\phi_1(p, q) \tag{3.2}$$

$\left(H'_{\phi_1}\right.$ *denotes the Hessian of the function ϕ_1, and* $\left(\frac{\partial\phi_1}{\partial t'}\right)^{T'}$ *the transpose of* $\left.\left(\frac{\partial\phi_1}{\partial t'}\right)\right)$

Proof. Consider, for every $p, q \in D$,

$$s(t') = g(\phi_1(p, q, t')) - \phi_2(p, q, t', g),$$

where $\phi_2(p, q, t', g) = \int_0^{t'} h(p, q, w)dw + g(q)$, and h satisfies (Proposition 3.4). We prove that $s(t') \leqq 0$ for every $t' \in [0, 1]$. We have that

$$s(0) = g(q) - g(q) = 0,$$
$$s^{'}(0) = \nabla_q g(q)\phi_1(p, q) - \nabla_q g(q)\phi_1(p, q) = 0,$$
$$s^{''}(t') = \left(\frac{\partial \phi_1}{\partial t'}\right)^{T'}(p, q, t')H'_{\phi_1}f(\phi_1(p, q, t'))\frac{\partial \phi_1}{\partial t'}$$
$$+\nabla_{\phi_1}g(\phi_1(p, q, t'))\frac{\partial^2 \phi_1}{\partial t'}(p, q, t') - h^{'}(t') \leqq 0.$$

Therefore, $s(t) \leqq 0$ for every $t' \in [0, 1]$, and $g(\phi_1(p, q, t') \leqq \phi_2(p, q, t', g)$ for every $p, q \in D$, $t' \in [0, 1]$. $\square$

4 Conclusions

In this paper, we established a new class of convexity named $(\phi_1, \phi_2) - \beta$-convexity. Our new class is a super class of many well-known classes.

- When we take $\phi_1(p, q, \lambda) = \lambda\eta(p, q) + q$ and $\phi_2(p, q, \lambda, g) = \lambda g(p) + (1 - \lambda)g(q)$ then it shows the result of [16]
- When we take $\phi_1(p, q, \lambda) = \gamma_{p,q}(\lambda)$ and $\phi_2(p, q, \lambda, g) = \lambda g(\gamma_{p,p}(\lambda)) + (1 - \lambda)g(\gamma_{p,q}(0))$ then it shows the result of [23]
- If $a_1(p, q, \lambda) = \lambda$ then it shows the result of [4, 19]
- When we take $\phi_1(p, q, \lambda) = N_I([p, q], \lambda)$ and $\phi_2(p, q, \lambda, g) = n_\Phi[(g(p), g(q)), \lambda]$, we see that an $(I, \Phi)-\beta$-convex function is a particular $(\phi_1, \phi_2) - \beta$-convex function, which shows the result of [24]
- If we take $\phi_1(p, q, \lambda) = \lambda p + (1 - \lambda)q$ and $\phi_2(p, q, \lambda, g) = G(g(p), g(q), \|p - q\|, \lambda) \leq ln(\lambda e^{\beta g(\phi_1(p,p,\lambda)} + (1 - \lambda)e^{\beta g(\phi_1(p,q,0)})^{1/\beta}$, we get that a G-convex function is an example of $(\phi_1, \phi_2) - \beta$-convex function, which shows the result of [25]
- If we take $\beta = 0$ then this function is convert into (ϕ_1, ϕ_2)-convex function, which shows the result of [22]

We can extend results of our paper for interval-valued function under the assumptions of $(\phi_1, \phi_2) - \beta$-convexity.

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