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(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

ON STABILITY OF A-QUARTIC FUNCTIONAL EQUATIONS IN RANDOM NORMED SPACES

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Abstract

In this paper, we shall prove the generalized Hyers-Ulam stability of the additive-quartic functional equation introduced by C. Muthamilarasi et al. [11] in Random Normed spaces by using direct and fixed-point methods.
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1 Introduction

In the field of stability of functional equations, a type of stability named after the Mathematician Ulam [15] is often considered. In 1940, Ulam [15], triggered the study of stability problems for various functional equations. He presented a number of unsolved problems. Since then, this question has attracted the attention of many researchers. In the next year, Hyers [9] gave answer of Ulams question in the case of approximately additive mappings. Thereafter, Hyers result was generalized by Aoki [3] and improved for additive mappings, and subsequently improved by Rassias [[6],[7]] for linear mappings by allowing the Cauchy difference to be unbounded.

Since then, stability of functional equation had been discussed in various spaces by researchers [[2],[4],[5]]. In 1963, Serstnev [13] introduced the theory of random normed spaces (briefly, RN-spaces) which is generalization of deterministic result of normed spaces and also in the study of random operator equations. A number of papers and research monographs have been published on generalizations of the stability of different functional equations in RN- spaces [12]. Recently, in 2017, Abdou et al. [1] discussed the stability of a quintic functional equations in random normed space. In this paper, we shall discuss about the stability of A-Quartic functional equation in random normed space.

To prove our main results, we need some notions and definitions from the literature as follows: A function $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$ is called a distribution function if it is nondecreasing and left -continuous with $F(0) = 0$ and $F(\infty) = 1$. The class of all probability distribution functions F with $F(0) = 0$ is denoted by $A.D^+$ is a subset of A consisting of all functions $F \in A$ for which $F(\infty) = 1$, where $l^-F(x) = \lim_{t\to x^-} F(t)$. For any $a \geq 0, \epsilon_a$ is the element of D^+ , which is defined as

$$\begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{otherwise.} \end{cases}$$

Definition 1.1 ([14]). *A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a t-norm) if T satisfies the following conditions:*

1. T is commutative and associative,
2. T is continuous,
3. $T(a, 1) = 1$ for all $a[0, 1]$,
4. $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

The examples of continuous t-norm are as follows:
$T_M(a, b) = min\{a, b\}$, $T_P(a, b) = minab, T_L(a, b) = max\{a + b - 1, 0\}$
Recall that, if T is a t-norm and $\{x_n\}$ is a sequence of number in $[0, 1]$, then $T^n_{i=1}x_i$ is defined recurrently by

$T^1_{i=1}x_i = x_1 and T^n_{i=1}x_i = T(T^{n-1}_{i=1}x_i, x_n) = T(x_1, x_2, , ..., x_n)$ for each $n \geq 2$ and $T^{\infty}_{i=n}x_n$ is defined as $T^{\infty}_{i=1}x_{n+i}$.

Definition 1.2 ([13]). *Let X be a real linear space, μ be a mapping from X into $D^+(for any x X, \mu(x)$ is denoted by μ_x and T be a continuous t norm. The triple (X, μ, t) is called a random normed space (briefly RN-space) if μ satisfies the following conditions:*

(RN1) $\mu_x = \epsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN2) $\mu_\alpha x(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X, \alpha \neq 0$ and all $t \geq 0$;

(RN3) $\mu_x + y(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s > 0$.

Example 1.3. Every normed space $(X, \|.\|)$ defines a RN-space (X, μ, T_M), where $\mu_x(t) = \frac{t}{t+\|x\|}$, for all $t > 0$ and T_M is the minimum t-norm. This space is called induced random normed space.

Definition 1.3 ([13]). *Let (X, μ, T) be a RN-space.*

1. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$ and $\lambda > 0$ there exists a positive integer N such that $\mu_{(x_n-x)}(t) > 1 - \lambda$, whenever $n \geq N$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} \mu_{x_n-x} = 1$.
2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for all $t > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n-x_m}(t) > 1 - \lambda$, whenever $n \geq m \geq N$.
3. The RN -space (X, μ, T) is said to be complete if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.1 ([14]). *If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence of X such that $x_n \to x$ then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Recently in 2021, Muthamilarasi et al. [11] proved the general solution and generalized Hyers-Ulam stability of additive quartic functional equation.

$$\begin{aligned}
f(ax + a^2y + a^3z) + f(-ax + a^2y + a^3z) &+ f(ax - a^2y + a^3z) + f(ax + a^2y - a^3z) \\
&= 2[f(ax + a^2y) + f(a^2y + a^3z) + f(ax + a^3z) \\
&+ f(ax - a^2y) + f(a^2y - a^3z) + f(a^3z - ax)] \\
&- 2[a^4(f(x) + f(-x)) + a^8(f(y) + f(-y)) \\
&+ a^{12}(f(z) + f(-z))] - [a(f(x) - f(-x)) \\
&+ a^2(f(y) - f(-y)) + a^3(f(z) - f(-z))]. \qquad (1.1)
\end{aligned}$$

for fixed $a \in Z^+$ in Banach spaces.

Lemma 1.1. *Let W and X be real vector spaces. If an odd mapping $f : W \to X$ satisfies* (1.1)*, then f is additive.*

Lemma 1.2. *Assume that W and X are real vector spaces. If an even mapping $f : W \to X$ satisfies the quartic functional equation*
$f(2w + x) + f(2w - x) = 4f(w + x) + 4f(w - x) + 24f(w) - 6f(x)$, if and only if $f : W \to X$ satisfies the functional equation (1.1) for all $x, y, z, w \in W$.Throughout this paper,let X be a real linear space, (Z, μ', T_M) be an RN-space and (Y, μ, T_M) be a complete RN-spaces. For mapping $f : X \to Y$, we define

$$\begin{aligned}
Df(x, y, z) &= f(ax + a^2y + a^3z) + f(-ax + a^2y + a^3z) + f(ax - a^2y + a^3z) \\
&+ f(ax + a^2y - a^3z) - 2[f(ax + a^2y) + (a^2y + a^3z) + f(ax + a^3z) \\
&+ f(ax - a^2y) + f(a^2y - a^3z) + f(a^3z - ax)] \\
&+ 2[a^4(f(x) + f(-x)) + a^8(f(y) + f(-y)) + a^12(f(z) + f(-z))] \\
&+ [a(f(x) - f(-x)) + a^2(f(y) - f(-y)) + a^3(f(z) - f(-z))], \qquad (1.2)
\end{aligned}$$

$$(1.3)$$

for all $x, y, z \in X$.
In this paper, using the direct and fixed-point methods, we investigate the generalised Hyers -Ulam stability of the A-Quartic functional equation (1.1) in random normed spaces under the minimum t-norm.

2 Random stability of the functional equation

In this section, we investigate the generalized Hyers-Ulam stability problem of the A-Quartic functional equation (1.1) in RN-spaces.

Theorem 2.1. *. Let $\phi : X^3 \to Z$ be a function such that, for some $0 < \alpha < a$,*

$$\mu'_{\phi(ax,ay,az)}(t) \geq \mu'_{\alpha(\phi(x,y,z))}(t). \tag{2.1}$$

and $\lim_{n\to\infty} \mu'_{\phi(a^n x,a^n y,a^n z)}(a^n t) = 1$. For all $x, y, z \in X$ and $t > 0$.
If $f : X \to Y$ is an odd mapping with $f(0) = 0$ such that

$$\mu_{Df(x,y,z)}(t) \geq \mu'_{\phi(x,y,z)}(t) \tag{2.2}$$

for all $x, y, z \in X$ and $t > 0$.
Then there exists a unique additive mapping $A : X \to Y$ such that,

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\phi(x,0,0)}(2(a-\alpha)t) \tag{2.3}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = z = 0$ in equation (2.2), we get

$$\mu_{2af(x)-2f(ax)}(t) \geq \mu'_{\phi(x,0,0)}(t). \tag{2.4}$$

$$\mu_{(f(x)-\frac{f(ax)}{a})}(t) \geq \mu_{\phi(x,0,0)}(2at). \tag{2.5}$$

for all $x \in X$ *and* $t > 0$. Replacing x by ax in equation (2.4), we get

$$\begin{aligned}
\mu_{(f(ax)-\frac{f(a^2x)}{a})}(t)\mu'_{\phi(ax,0,0)}(2at) &\geq \mu'_{\phi(x,0,0)}(\frac{2at}{\alpha}),\\
\mu_{(f(ax)-\frac{f(a^2x)}{a})}(t) &\geq \mu'_{\phi(ax,0,0)}(2at)\mu'_{\phi(x,0,0)}(\frac{2at}{\alpha}),\\
\mu_{(f(ax)/a-\frac{f(a^2x)}{a^2})}(t/a) &\geq \mu'_{\phi(x,0,0)}(\frac{2at}{\alpha})),\\
\mu_{(f(ax)/a-\frac{f(a^2x)}{a^2})}(t) &\geq \mu'_{\phi(x,0,0)}(t).
\end{aligned} \tag{2.6}$$

for all $x \in X$ and $t > 0$.
Continuing like this, we have

$$\mu_{(\frac{f(a^n x)}{a^n}-f(\frac{f(a^{n+1}x)}{a^{n+1}}))}(t) \geq \mu'_{\phi(x,0,0)}(\frac{2a^{n+1}t}{a^n}). \tag{2.7}$$

Now, since

$$\begin{aligned}
\frac{f(a^n x)}{a^n} - f(x) &= (\frac{f(a^n x)}{a^n} - \frac{f(a^{n-1}x))}{a^{n-1}})\\
&+ (\frac{(f(a^{n-1}x)}{a^{n-1}} - \frac{f(a^{n-2}x)}{a^{n-2})}\\
&+ ... + (\frac{f(ax)}{a} - f(x)),\\
&= \sum_{j=0}^{n-1}(\frac{f(a^{j+1}x)}{a^{j+1}} - \frac{f(a^j x)}{a^j})\\
\mu_{(\frac{f(a^n x)}{a^n}-f(x))}(\sum_{j=0}^{n-1}\frac{1}{2a}(\frac{\alpha}{a})^j t) &\geq T_M(\mu'_{\phi(x,0,0)}(t),\\
&= \mu'_{\phi(x,0,0)}(t).
\end{aligned} \tag{2.8}$$

Now, replacing x by $a_m x$ in equation (2.8), we get

$$\mu_{(\frac{f(a^{n+m}x)}{a^n})-f(a^m x)}(\sum_{j=0}^{n-1}\frac{1}{2a}(\frac{\alpha}{a})^j t) \geq \mu'_{\phi(a^m x,0,0)}(t),$$

$$\begin{aligned}
\mu_{(\frac{f(a^{n+m}x)}{a^{n+m}}-\frac{f(a^m x)}{a^m})}(\sum_{j=0}^{n-1}\frac{1}{2aa^m}(\frac{\alpha}{a})^j t) &\geq \mu'_{\phi(x,0,0)}(\frac{t}{\alpha^m}),\\
\mu_{(\frac{f(a^{n+m}x)}{a^{n+m}}-\frac{f(a^m x)}{a^m})}(t) &\geq \mu'_{\phi(x,0,0)}(\frac{t}{\alpha^m(\sum_{j=0}^{n-1}\frac{t}{2aa^m}(\frac{\alpha}{a})^j t)}),\\
&= \mu'_{\phi(x,0,0)}\frac{2at}{\sum_{j=0}^{n-1}(\frac{\alpha}{a})^j},\\
&\geq \mu'_{\phi(x,0,0)}\frac{2at}{\sum_{j=m}^{n+m-1}(\frac{\alpha}{a})^{j+m}}. \qquad (2.9)
\end{aligned}$$

for all $x \in X$ and $m, n \in Z$ with $n > m \geq 0$ since $< a$, the sequence $\{\frac{(f(a^n x))}{a^n}\}$ is a Cauchy sequence in the complete RN-spaces (Y, μ, T_M) and so it converges to some point $A(x) \in Y$. Fix $x \in X$ and put $m = 0$ in equation (2.9), we get

$$\mu_{(\frac{(f(a^n x))}{a^n}-f(x))}(t) \geq \mu'_{\phi(x,0,0)}\frac{2at}{\sum_{j=0}^{n-1}(\frac{\alpha}{a})^j},$$

So, for any $\delta > 0$,

$$\begin{aligned}
\mu_{(A(x)-f(x))}(\delta+t) &\geq T_M(\mu_{A(x)-\frac{f(a^n x)}{a^n}}(\delta), \mu_{\frac{f(a^n x)}{a^n}-f(x)}(t) \qquad (2.10)\\
&\geq T_M(\mu_{A(x)-\frac{f(a^n x)}{a^n}}(\delta), \mu'_{\phi(x,0,0)}\frac{2at}{\sum_{j=0}^{n-1}(\frac{\alpha}{a})^j}).
\end{aligned}$$

for all $x \in X$ and $t > 0$.
Taking the limit in (2.10) as $n \to \infty$, we get

$$\mu_{(A(x)-f(x))}(\delta+t) \geq \mu'_{\phi(x,0,0)}(\frac{2at}{\frac{1}{1-\frac{\alpha}{a}}}) = \mu'_{\phi(x,0,0)}(2t(a-\alpha)) \qquad (2.11)$$

Since δ is arbitrary, by taking $\delta \to 0$ in equation (2.11), we have

$$\mu_{(A(x)-f(x))}(t) \geq \mu'_{\phi(x,0,0)}(2(a-\alpha)(t), \qquad (2.12)$$

for all $x \in X$ and $t > 0$.
Therefore, we conclude that the condition of equation (2.3) holds.
Also, by replacing x, y and z by $a^n x, a^n y$ and $a^n z$ in equation (2.2), we have

$$\mu_{\frac{Df(a^n x, a^n y, a^n z}{a^n}}(t) \geq \mu_{\phi(a^n x, a^n y, a^n z)}(a^n)(t) = \mu'_{\phi(x,y,z)}(\frac{a}{\alpha})^n(t),$$

for all $x, y, z \in X$ and $t > 0$.
It follows from $\lim_{n\to\infty}\mu'_{\phi(a^n x, a^n y, a^n z)}(a^n t) = 1$, that A satisfies the equation (1.1), which implies that A is an additive mapping.
To prove the uniqueness of the quartic mapping A, let us assume that there exists another mapping $A'X \to Y$ which satisfies equation (2.3). Fix $x \in X$, then $A(a^n x) = a^n A(x)$ and $A'(a^n x) = a^n A'(x)$ for all $n \in Z^+$. Thus it follows from the equation (2.3) that

$$\begin{aligned}
\mu_{(A(x)-A'(x))}(t) &= \mu_{(\frac{A(a^n x)}{a^n}-\frac{A'(a^n x)}{a^n})}(t)\\
&\geq T_M(\mu_{\frac{A(a^n x)}{a^n}-\frac{f(a^n x)}{a^n})}(\frac{t}{2}), \mu_{\frac{f(a^n x)}{a^n}-\frac{A'(a^n x)}{a^n})}(\frac{t}{2}) \qquad (2.13)\\
&\geq \mu'_{\phi(x,0,0)}((a-\alpha)(\frac{a}{\alpha})^n t).
\end{aligned}$$

Since, $\lim_{n\to\infty}(a-\alpha)(\frac{a}{\alpha})^n t = \infty$, we have $\mu_{(A(x)-A'(x))}(t) = 1$ for all $t > 0$.
Thus the additive mapping is unique.
This completes the proof. □

Theorem 2.2. *Let $\phi : X^3 \to Z$ be a function such that, for some $0 < \alpha < a^4$,*

$$\mu'_{(\phi(ax,ay,az))}(t) \geq \mu_{\alpha\phi(x,y,z)}(t) \qquad (2.14)$$

and $\lim_{n\to\infty}\mu'_{a^n\phi(a^n x, a^n y, a^n z)}(t) = 1$ for all $x, y, z \in X$ and $t > 0$. If $f : X \to Y$ is an even mapping with $f(0) = 0$ which satisfies equation (2.2), then there exists a unique additive mapping $Q : X \to Y$ such that

$$\mu_{(f(x)-A(x))}(t) \geq \mu'_{\phi(x,0,0)}(4(a^4-\alpha)t), \qquad (2.15)$$

for all $x \in X$ and $t > 0$.

Replace x, y, z by $x, 0, 0$ respectively in equation (2.14), we obtain

$$\mu_{(4f(ax)-4a^4f(x))}(t) \geq \mu'_{\phi(x,0,0)}(t),$$
$$\mu_{4a^4(\frac{f(ax)}{a^4}-f(x))}(t) \geq \mu'_{\phi(x,0,0)}(t),$$
$$\mu_{(\frac{f(ax)}{a^4}-f(x))}(\frac{t}{4a^4}) \geq \mu'_{\phi(x,0,0)}(t),$$
$$\mu_{\frac{f(ax)}{a^4}-f(x)}(t) \geq \mu'_{\phi(x,0,0)}(4a^4t). \tag{2.16}$$

for all $x \in X$ and $t > 0$.
Replacing x by ax in equation (2.16), we get

$$\mu_{\frac{f(a^2x)}{a^4}-f(ax)}(t) \geq \mu'_{\phi(ax,0,0)}(4a^4t),$$
$$\geq \mu'_{\phi(x,0,0)}(\frac{4a^4t}{a}),$$
$$\mu_{\frac{f(a^2x)}{a^8}-\frac{f(ax)}{a^4}}(t) \geq \mu'_{\phi(x,0,0)}(\frac{4a^4t}{a}),$$
$$\mu_{(\frac{f(a^2x)}{a^8})-(\frac{f(ax)}{a^4})}(t) \geq \mu'_{\phi}(x,0,0)(\frac{4a^8t}{a}), \tag{2.17}$$

for all $x \in X$ and $t > 0$.
Now again, replacing x by ax in equation (2.17), we have

$$\mu_{\frac{f(a^3x)}{a^8}-\frac{f(a^2x)}{a^4}}(t) \geq \mu_{\phi(ax,0,0)}(\frac{4a^8t}{a}),$$
$$\mu_{\frac{f(a^3x)}{a^{12}}-\frac{f(a^2x)}{a^8}}(t) \geq \mu'_{\phi(x,0,0)}(\frac{4a^{12}t}{2}),$$

Continuing this process, we get

$$\mu_{\frac{f(a^nx)}{a^{4n}}-\frac{f(a^{n-1}x)}{a^{4(n-1)}}}(t) \geq \mu'_{\phi(x,0,0)}\frac{(4a^{4n}t)}{(n-1)},$$

Now, since

$$\frac{f(a^nx)}{a^{4n}} - f(x) = \sum_{j=0}^{n-1} \frac{f(a^{j+1}x)}{a^{4(j+1)}} - \frac{f(a^jx)}{a^{4j}},$$

Now,

$$\mu_{\frac{f(a^nx)}{a^{4n}}-f(x)}(\sum_{j=0}^{n-1} \frac{1}{(4a^4)}(\frac{\alpha}{a^4})^j t) \geq T_M(\mu'_{\phi(x,0,0)}(t))$$
$$= \mu'_{\phi(x,0,0)}(t). \tag{2.18}$$

Now replacing x by a^mx in equation (2.18), we get

$$\mu_{\frac{f(a^{n+m}x)}{a^{4n}}}(\sum_{j=0}^{n-1} \frac{1}{4a^4}(\frac{\alpha}{a^4})^j t) \geq \mu'_{\phi(a^mx,0,0)}(t),$$
$$\mu_{\frac{f(a^{n+m}x)}{a^{4n+4m}}-\frac{f(a^mx)}{a^{4m}}}(\sum_{j=0}^{n-1} \frac{1}{4a^4a^{4m}}(\frac{\alpha}{a^4})^j t) \geq \mu'_{\phi(x,0,0)}(\frac{t}{\alpha^m}),$$
$$\mu_{\frac{f(a^{n+m}x)}{a^{4(n+m)}}-\frac{f(a^mx)}{a^4m}}(t) \geq \mu'_{\phi(x,0,0)(\frac{4a^4t}{\sum_{j=0}^{n-1}(\frac{\alpha}{a^4})^{j+m}})} \tag{2.19}$$

for all x and $m, n \in Z^+$ with $n > m \geq 0$. Since $< a^4$, the sequence $(\frac{f(a^n x)}{a^{4n}})$ is a Cauchy sequence in the complete RN-space (Y, μ, T_M) and it converge to a point $Q(x) \in Y$.
Fix $x \in X$ and $m = 0$ in equation (2.19), we get

$$\mu_{\frac{f(a^n x)}{a^{4n}} - f(x)}(t) \geq \mu'_{\phi(x,0,0)} \frac{2a^4 t}{\sum_{j=0}^{n+m-1} (\frac{\alpha}{a^4})^j},$$

and so, for any $\delta > 0$,

$$\begin{aligned} \mu_{(Q(x)-f(x))}(\delta + t) &\geq T_M \mu_{(Q(x) - \frac{f(a^n x)}{a^{4n}})}(\delta), \mu_{(\frac{f(a^n x)}{a^{4n}} - f(x))}(t), \\ &\geq T_M \mu_{(Q(x) - \frac{f(a^n x)}{a^{4n}})}(\delta), \mu'_{\phi(x,0,0)}(\frac{4a^4 t}{\sum_{j=0}^{n+m-1} (\frac{\alpha}{a^4})^j}), \end{aligned} \tag{2.20}$$

for all $x \in X$ and $t > 0$. Taking the limit $n \to \infty$ in equation (2.20), we get

$$\begin{aligned} \mu_{(Q(x)-f(x))}(\delta + t) &= \geq \mu'_{\phi(x,0,0)}(\frac{4a^4 t}{\frac{1}{1-\frac{\alpha}{a^4}}}) \\ &= \mu'_{\phi(x,0,0)}(4t(a^4 - \alpha)). \end{aligned} \tag{2.21}$$

Since δ is arbitrary, by taking $\delta \to 0$ in equation (2.21), we have

$$\mu_{(Q(x)-f(x))}(t) \geq \mu'_{\phi(x,0,0)}(4t(a^4 - \alpha)). \tag{2.22}$$

for all $x \in X, t > 0$.
Therefore, we conclude that the condition of equation (2.15) holds.
Also replacing x, y, z by $a^n x, a^n y, a^z$ respectively in equation (2.15), we have

$$\begin{aligned} \mu_{\frac{a^n x, a^n y, a^z}{a^n}}(t) &\geq \mu'_{\phi(a^n x, a^n y, a^z)}(a^n t), \\ &\geq \mu'_{\phi(x,y,z)}((\frac{a^4}{\alpha})^n t). \end{aligned}$$

It follows from $\lim_{n\to\infty} \mu'_{\phi(a^n x, a^n y, a^z)}(a^{4n} t) = 1$ that Q satisfies the equation (1.1), which implies Q is a quartic mapping.

Lemma 2.1 ([8]). *Suppose that (ω, d) is a complete generalized metric space and $J : \omega \to \omega$ is astrictly contractive mapping with Lipschitz constant $L < 1$. Then for each $x \in \omega$, either $d(J^n x, J^{n+1} x) = \infty$. for all non negative integers $n \geq 0$ or there exists a natural numbaer n_0 such that*

1. $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
2. The sequence $J^n x$ is convergent to a fixed point $y*$ og J;
3. $y*$ is the unique fixed point of J in the set $A = \{y \in \omega : d(J^{n_0} x, y) < \infty\}$;
4. $d(y, y*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in A$.

Theorem 2.3. *Let $\phi : X^3 \to D^+$ be a function such that, for some $0 < \alpha < a^4$,*

$$\mu'_{\phi(x,y,z)}(t) \leq \mu'_{\phi(ax,ay,az)}(\alpha t) \tag{2.23}$$

for all $x, y, z \in X$ and $t > 0$. If $f : X \to Y$ is an even mapping with $f(0) = 0$ such that

$$\mu_{D(x,y,z)}(t) \geq \mu'_{\phi(x,y,z)}(t). \tag{2.24}$$

for all $x, y, z \in X$ and $t > 0$.
Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\mu_{(f(x)-Q(x))}(t) \geq \mu'_{\phi(x,y,z)}(2(a^4 - \alpha)t), \tag{2.25}$$

for all $x \in X, t > 0$.

Proof. It follows from equation (2.24) that

$$\mu_{(f(x) - \frac{f(ax)}{a^4})}(t) \geq \mu'_{\phi(x,0,0)}(4a^4 t), \tag{2.26}$$

for all $x \in X, t > 0$.
Let $\omega = \{g : X \to Y, g(x) = 0\}$ and mapping d defined on ω by

$$d(g, h) = \inf\{c \in [0, \infty) : \mu_{g(x)-h(x)}\}(ct) \geq \mu'_{\phi(x,0)}(t), \forall x \in X\}$$

where as usual $\inf \phi = -\infty$. Then (ω, d) is a generalized complete metric space. Now let us consider the mapping $J : \omega \to \omega$ defined by
$Jg(x) = \frac{1}{a^4} g(ax)$, for all $g \in \omega$ and $x \in X$.
Let $g, h \in \omega$ and $c \in [0, \infty)$ be any arbitrary constant with $d(g, h) < c$.
Then $\mu_{(g(x)-h(x))}(ct) \geq \mu'_{\phi(x,0,0)}$ for all $x \in X, t > 0$ and so,

$$\mu_{(Jg(x)-Jh(x))}(\frac{\alpha ct}{a^4}) = \mu_{g(ax)-h(ax)}(\alpha ct) \geq \mu'_{\phi(x,0,0)}(t) = \mu'_{\phi(\alpha x,0,0)}, \tag{2.27}$$

for all $x \in X, t > 0$. Hence we have $d(Jg, Jh) \leq \frac{\alpha c}{a^4} \leq \frac{\alpha c}{a^4} d(g, h)$.
for all $g, h \in \omega$.
Then J is a contractive mapping on ω with the Lipschitz constant $L = \frac{\alpha}{a^4} < 1$.
Thus it follows from Lemma 2.1, that there exists a mapping $Q : X \to Y$ which is a unique fixed point of J in the set $\omega_1 = \{g \in \omega : d(g, h) < \infty\}$, such that
$Q(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^4 n}$ for all $x \in X$ since $\lim_{n\to\infty} d(J^f, Q) = 0$. Also, using $\mu_{(f(x)-\frac{f(ax)}{a^4})}(t) \geq \mu'_{\phi(x,0,0)}(4(a^4 - \alpha)t)$, we have $d(f, Jf) \leq \frac{1}{4(a^4-\alpha)}$.
Therefore using Lemma 2.1, we get
$d(f, Q) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{4(a^4-\alpha)}$.
This means that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}(4(a^4 - \alpha)t),$$

for all $x \in X, t > 0$. Also by replacing x, y, z by $2^n x, 2^n y, 2^n z$ in equation (2.4) respectively, we have

$$\mu_{DQ(x,y,z)}(t) \geq \lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y, 2^z)}(a^{4n} t) = \lim_{n\to\infty} \mu'_{\phi(x,y,z)}((\frac{a^4}{\alpha})^n t) = 1,$$

for all $x, y, z \in X$ and $t > 0$. By (RN1), the mapping is quartic.
To prove the uniqueness let us assume that there exists a quartic mapping $Q' : X \to Y$, which satisfies equation (2.25). Then Q' is a fixed point of J in ω_1.
However it follows from the Lemma 2.3, that J has only one fixed point in ω_1.
Hence $Q = Q'$. □

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