ON STABILITY OF A-QUARTIC FUNCTIONAL EQUATIONS IN RANDOM NORMED SPACES

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(Received: August 30, 2022; In format: February 14, 2023; Accepted: March 24, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53120

Abstract

In this paper, we shall prove the generalized Hyers-Ulam stability of the additive-quartic functional equation introduced by C. Muthamilarasi et al. [11] in Random Normed spaces by using direct and fixed-point methods.

2020 Mathematical Sciences Classification: 39B52; 39B82

Keywords and Phrases: Additive-Quartic functional equation; Hyers-Ulam stability; fixed point methods; Random Normed spaces.

1 Introduction

In the field of stability of functional equations, a type of stability named after the Mathematician Ulam [15] is often considered. In 1940, Ulam [15], triggered the study of stability problems for various functional equations. He presented a number of unsolved problems. Since then, this question has attracted the attention of many researchers. In the next year, Hyers [9] gave answer of Ulams question in the case of approximately additive mappings. Thereafter, Hyers result was generalized by Aoki [3] and improved for additive mappings, and subsequently improved by Rassias [[6],[7]] for linear mappings by allowing the Cauchy difference to be unbounded.

Since then, stability of functional equation had been discussed in various spaces by researchers [[2],[4],[5]]. In 1963, Serstnev [13] introduced the theory of random normed spaces (briefly, RN-spaces) which is generalization of deterministic result of normed spaces and also in the study of random operator equations. A number of papers and research monographs have been published on generalizations of the stability of different functional equations in RN- spaces [12]. Recently, in 2017, Abdou et al. [1] discussed the stability of a quintic functional equations in random normed space. In this paper, we shall discuss about the stability of A-Quartic functional equation in random normed space.

To prove our main results, we need some notions and definitions from the literature as follows: A function $F : \mathbb{R} \cup \{-\infty, +\infty\} \rightarrow [0, 1]$ is called a distribution function if it is nondecreasing and left-continuous with $F(0) = 0$ and $F(\infty) = 1$. The class of all probability distribution functions $F$ with $F(0) = 0$ is denoted by $A.D$. $A.D^+$ is a subset of $A$ consisting of all functions $F \in A$ for which $F(\infty) = 1$, where $\lim_{t \to \infty} F(t)$. For any $a \geq 0, \epsilon_a$ is the element of $D^+$, which is defined as

\[
\begin{cases}
0, & \text{if } t \leq 0 \\
1, & \text{otherwise.}
\end{cases}
\]

Definition 1.1 ([14]). A function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous triangular norm (briefly, a $t$-norm) if $T$ satisfies the following conditions:

1. $T$ is commutative and associative,
2. $T$ is continuous,
3. $T(a, 1) = 1$ for all $a[0, 1]$,
4. $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

The examples of continuous $t$-norm are as follows:

$T_M(a, b) = \min\{a, b\}$, $T_P(a, b) = \min\{a, b\}$, $T_L(a, b) = \max\{a + b - 1, 0\}$

Recall that, if $T$ is a $t$-norm and $\{x_n\}$ is a sequence of number in $[0, 1]$, then $T_{i=1}^n x_i$ is defined recurrently by

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\[ T_{i=1}^{1}x_i = x_1 \land T_{i=1}^{n}x_i = T(T_{i=1}^{n-1}x_i, x_n) = T(x_1, x_2, ..., x_n) \] for each \( n \geq 2 \) and \( T_{i=1}^{\infty}x_n \) is defined as \( T_{i=1}^{\infty}x_{n+i} \).

**Definition 1.2 ([13])**. Let \( X \) be a real linear space, \( \mu \) be a mapping from \( X \) into \( D^+ \) (for any \( x \in X \), \( \mu(x) \) is denoted by \( \mu_x \) and \( T \) be a continuous \( t \)-norm. The triple \( (X, \mu, t) \) is called a random normed space (briefly RN-space) if \( \mu \) satisfies the following conditions:

1. \( (RN1) \) \( \mu_x = c_0(t) \) for all \( t > 0 \) if and only if \( x = 0 \);
2. \( (RN2) \) \( \mu_{\alpha x}(t) = \mu_x(t/|\alpha|) \) for all \( x \in X, \alpha \neq 0 \) and all \( t \geq 0 \);
3. \( (RN3) \) \( \mu_x + y(t + s) \geq T(\mu_x(t), \mu_y(s)) \) for all \( x, y \in X \) and all \( t, s > 0 \).

**Example 1.3.** Every normed space \( (X, \| \cdot \|) \) defines a RN-space \( (X, \mu, T_M) \), where \( \mu_x(t) = \frac{x}{\|x\|} \), for all \( t > 0 \) and \( T_M \) is the minimum \( \mu \)-norm. This space is called induced random normed space.

**Definition 1.3 ([13])**. Let \((X, \mu, T)\) be a RN-space.

1. A sequence \( \{x_n\} \) in \( X \) is said to be convergent to a point \( x \in X \) if, for all \( t > 0 \) and \( \lambda > 0 \) there exists a positive integer \( N \) such that \( \mu(x_n - x)(t) > 1 - \lambda \), whenever \( n \geq N \). In this case, \( x \) is called the limit of the sequence \( \{x_n\} \) and we denote it by \( \lim_{n \to \infty} \mu_{x_n - x} = 1 \).
2. A sequence \( \{x_n\} \) in \( X \) is called a Cauchy sequence if, for all \( t > 0 \) and \( \lambda > 0 \), there exists a positive integer \( N \) such that \( \mu_{x_n - x_m}(t) > 1 - \lambda \), whenever \( n, m \geq N \).
3. The RN-space \((X, \mu, T)\) is said to be complete if every Cauchy sequence in \( X \) is convergent to a point in \( X \).

**Theorem 1.1 ([14])**. If \((X, \mu, T)\) is a RN-space and \( \{x_n\} \) is a sequence of \( X \) such that \( x_n \to x \) then \( \lim_{n \to \infty} \mu_x(t) = \mu_x(t) \) almost everywhere.


\[
\begin{align*}
\begin{align*}
f(ax + a^2y + a^3z) + f(-ax + a^2y + a^3z) &+ f(ax - a^2y + a^3z) + f(ax + a^2y - a^3z) \\
&= 2[f(ax + a^2y) + f(a^2y + a^3z) + f(ax + a^3z) \\
&+ f(ax - a^2y) + f(a^2y - a^3z) + f(a^3z - ax)] \\
&- 2[a^4(f(x) + f(-x)) + a^6(f(y) + f(-y))] \\
&+ a^{12}[(f(z) + f(-z))] - [a(f(x) - f(-x))] \\
&+ a^2(f(y) - f(-y)) + a^3(f(z) - f(-z))].
\end{align*}
\end{align*}
\] (1.1)

for fixed \( a \in Z^+ \) in Banach spaces.

**Lemma 1.1.** Let \( W \) and \( X \) be real vector spaces. If an odd mapping \( f : W \to X \) satisfies (1.1), then \( f \) is additive.

**Lemma 1.2.** Assume that \( W \) and \( X \) are real vector spaces. If an even mapping \( f : W \to X \) satisfies the quartic functional equation

\[
\begin{align*}
f(2w + x) + f(2w - x) = 4f(w + x) + 4f(w - x) + 24f(w) - 6f(x), \text{ if and only if } f : W \to X \text{ satisfies the functional equation (1.1) for all } x, y, z, w \in W. \text{Throughout this paper, let } X \text{ be a real linear space, } (Z, \mu', T_M) \text{ be an RN-space and } (Y, \mu, T_M) \text{ be a complete RN-spaces. For mapping } f : X \to Y, \text{ we define}
\end{align*}
\]

\[
\begin{align*}
Df(x, y, z) &= f(ax + a^2y + a^3z) + f(-ax + a^2y + a^3z) + f(ax - a^2y + a^3z) \\
&+ f(ax + a^2y - a^3z) - 2[f(ax + a^2y) + (a^2y + a^3z) + f(ax + a^3z) \\
&+ f(ax - a^2y) + f(a^2y - a^3z) + f(a^3z - ax)] \\
&+ 2[a^4(f(x) + f(-x)) + a^6(f(y) + f(-y)) + a^{12}(f(z) + f(-z))] \\
&+ [a(f(x) - f(-x)) + a^2(f(y) - f(-y)) + a^3(f(z) - f(-z))],
\end{align*}
\] (1.2)

for all \( x, y, z \in X \).

In this paper, using the direct and fixed-point methods, we investigate the generalised Hyers-Ulam stability of the A-Quartic functional equation (1.1) in random normed spaces under the minimum \( T \)-norm.
Let $\phi : X^3 \to Z$ be a function such that, for some $0 < \alpha < a$, 
\[ \mu'_{\phi(ax,ay,az)}(t) \geq \mu'_{\phi(x,y,z)}(t). \] 
(2.1)

and $\lim_{n \to \infty} \mu'_{\phi(a^n,x,a^n,y,a^n,z)}(a^n t) = 1$. For all $x, y, z \in X$ and $t > 0$. 
If $f : X \to Y$ is an odd mapping with $f(0) = 0$ such that 
\[ \mu_{Df(x,y,z)}(t) \geq \mu'_{\phi(x,y,z)}(t) \] 
(2.2)

for all $x, y, z \in X$ and $t > 0$.

Then there exists a unique additive mapping $A : X \to Y$ such that, 
\[ \mu_{f(x)-A(x)}(t) \geq \mu'_{\phi(x,0,0)}(2(a - \alpha)t) \] 
(2.3)

for all $x \in X$ and $t > 0$.

Proof. Putting $y = z = 0$ in equation (2.2), we get 
\[ \mu_{2af(x)-2f(ax)}(t) \geq \mu'_{\phi(x,0,0)}(t). \] 
(2.4)

\[ \mu_{f(x)-f(ax)}(t) \geq \mu'_{\phi(x,0,0)}(2at). \] 
(2.5)

for all $x \in X$ and $t > 0$. Replacing $x$ by $ax$ in equation (2.4), we get
\[ \mu_{f(ax)-f(ax)}(t) \geq \mu'_{\phi(x,0,0)}(2at). \]
(2.6)

for all $x \in X$ and $t > 0$.

Continuing like this, we have
\[ \mu_{f(a^n x)-f(a^n x)}(t) \geq \mu'_{\phi(x,0,0)}\left(\frac{2a^{n+1}t}{a^n}\right). \]
(2.7)

Now, since
\[ \frac{f(a^n x)}{a^n} - f(x) = \frac{f(a^n x)}{a^n} - \frac{f(a^{n-1} x)}{a^{n-1}} + \frac{f(a^{n-1} x)}{a^{n-1}} - \frac{f(a^{n-2} x)}{a^{n-2}} + \ldots + \left(\frac{f(ax)}{a} - f(x)\right), \]
\[ = \sum_{j=0}^{n-1}\frac{f(a^{j+1} x)}{a^{j+1}} - \frac{f(a^j x)}{a^j}, \]

\[ \mu_{f(a^n x)-f(x)}\left(\sum_{j=0}^{n-1} \frac{1}{2a} \left(\frac{\alpha}{a}\right)^j t\right) \geq T_M(\mu'_{\phi(x,0,0)}(t), \]
\[ = \mu'_{\phi(x,0,0)}(t). \]
(2.8)

Now, replacing $x$ by $a_m x$ in equation (2.8), we get
\[ \mu_{f(a^{n+m} x)-f(a^m x)}\left(\sum_{j=0}^{n-1} \frac{1}{2a} \left(\frac{\alpha}{a}\right)^j t\right) \geq \mu'_{\phi(a^m x,0,0)}(t), \]
for all $x \in X$ and $m, n \in Z$ with $n > m \geq 0$ since $<a$, the sequence $\{ \frac{f(\alpha^n x)}{a^n} \}$ is a Cauchy sequence in the complete $RN$-spaces $(Y, \mu, T_M)$ and so it converges to some point $A(x) \in Y$. Fix $x \in X$ and put $m = 0$ in equation (2.9), we get

$$\mu(\frac{f(\alpha^n x)}{a^n} - f(x))(\delta + t) \geq T_M(\mu(A(x)-f(x))(\delta), \mu(\frac{f(\alpha^n x)}{a^n} - f(x))(t)) \geq T_M(\mu(A(x)-f(x))(\delta), \mu'(\phi(x,0,0))2t(a - \alpha)).$$

(2.10)

for all $x \in X$ and $t > 0$.

Taking the limit in (2.10) as $n \to \infty$, we get

$$\mu(A(x)-f(x))(\delta + t) \geq \mu'(\phi(x,0,0))(2a - \alpha)(t),$$

(2.12)

for all $x \in X$ and $t > 0$.

Therefore, we conclude that the condition of equation (2.3) holds.

Also, by replacing $x, y$ and $z$ by $a^n x, a^n y$ and $a^n z$ in equation (2.2), we have

$$\mu(\alpha^n x, a^n y, a^n z)(a^n x)(t) = \mu'(\phi(x,y,z))(\alpha^n x)(t),$$

for all $x, y, z \in X$ and $t > 0$.

It follows from $\lim_{n \to \infty} \mu'(\phi(x,y,z))(a^n t) = 1$, that $A$ satisfies the equation (1.1), which implies that $A$ is an additive mapping.

To prove the uniqueness of the quartic mapping $A$, let us assume that there exists another mapping $A' : X \to Y$ which satisfies equation (2.3). Fix $x \in X$, then $A(a^n x) = a^n A(x)$ and $A'(a^n x) = a^n A'(x)$ for all $n \in Z^+$. Thus it follows from the equation (2.3) that

$$\mu(A(x)-A'(x))(t) = \mu(\frac{A(a^n x)}{a^n} - \frac{A'(a^n x)}{a^n})(t) \geq T_M\left(\mu(\frac{A(a^n x)}{a^n} - \frac{f(a^n x)}{a^n})(\frac{t}{2}), \mu(\frac{A(a^n x)}{a^n} - \frac{A'(a^n x)}{a^n})(\frac{t}{2})\right) \geq \mu'(\phi(x,0,0))(a - \alpha)(\alpha^n t).$$

(2.13)

Since, $\lim_{n \to \infty}(a - \alpha)(\frac{t}{2})^n t = \infty$, we have $\mu(A(x)-A'(x))(t) = 1$ for all $t > 0$.

Thus the additive mapping is unique.

This completes the proof.

Theorem 2.2. Let $\phi : X^3 \to Z$ be a function such that, for some $0 < \alpha < a^4$,

$$\mu'(\phi(x,y,z))(t) \geq \mu(\phi(x,y,z))(t)$$

(2.14)

and $\lim_{n \to \infty} \mu'(\phi(a^n x, a^n y, a^n z))(t) = 1$ for all $x, y, z \in X$ and $t > 0$. If $f : X \to Y$ is an even mapping with $f(0) = 0$ which satisfies equation (2.2), then there exists a unique additive mapping $Q : X \to Y$ such that

$$\mu(f(x)-A(x))(t) \geq \mu'(\phi(x,0,0))(A(a^4 - \alpha) t),$$

(2.15)

for all $x \in X$ and $t > 0$. 

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Replace \(x, y, z\) by \(x, 0, 0\) respectively in equation (2.14), we obtain

\[
\mu(x) - 4a^t f(x) \geq \mu^\prime(x, 0, 0)(t),
\]

\[
\mu(x) - 4a^t f(x) \geq \mu^\prime(x, 0, 0)(t),
\]

\[
\mu(x) - 4a^t f(x) \geq \mu^\prime(x, 0, 0)(t),
\]

\[
\mu(x) - 4a^t f(x) \geq \mu^\prime(x, 0, 0)(4a^4 t),
\]

(2.16)

for all \(x \in X\) and \(t > 0\).

Replacing \(x\) by \(ax\) in equation (2.16), we get

\[
\mu(x) - f(ax) \geq \mu^\prime(x, 0, 0)(4a^4 t),
\]

\[
\mu(x) - f(ax) \geq \mu^\prime(x, 0, 0)(4a^4 t),
\]

\[
\mu(x) - f(ax) \geq \mu^\prime(x, 0, 0)(4a^4 t),
\]

\[
\mu(x) - f(ax) \geq \mu^\prime(x, 0, 0)(4a^4 t),
\]

(2.17)

for all \(x \in X\) and \(t > 0\).

Now again, replacing \(x\) by \(ax\) in equation (2.17), we have

\[
\mu(ax) - f(ax) \geq \mu^\prime(x, 0, 0)(\frac{4a^8 t}{a}),
\]

\[
\mu(ax) - f(ax) \geq \mu^\prime(x, 0, 0)(\frac{4a^8 t}{a}),
\]

\[
\mu(ax) - f(ax) \geq \mu^\prime(x, 0, 0)(\frac{4a^8 t}{a}),
\]

Continuing this process, we get

\[
\mu(ax) - f(ax) \geq \mu^\prime(x, 0, 0)(\frac{4a^4 n t}{n-1}),
\]

Now, since

\[
f(ax) = \sum_{j=0}^{n-1} f(a^{j+1} x) - f(a^j x)
\]

Now,

\[
\mu(ax) - f(ax) \geq T_M(\mu^\prime(x, 0, 0)(t)) = \mu^\prime(x, 0, 0)(t),
\]

(2.18)

Now replacing \(x\) by \(a^m x\) in equation (2.18), we get

\[
\mu(ax) - f(ax) \geq \mu^\prime(x, 0, 0)(\frac{4a^4 m t}{a^m}),
\]

\[
\mu(ax) - f(ax) \geq \mu^\prime(x, 0, 0)(\frac{4a^4 m t}{a^m}),
\]

\[
\mu(ax) - f(ax) \geq \mu^\prime(x, 0, 0)(\frac{4a^4 m t}{a^m}),
\]

\[
\mu(ax) - f(ax) \geq \mu^\prime(x, 0, 0)(\frac{4a^4 m t}{a^m}),
\]

(2.19)
for all $x$ and $m, n \in \mathbb{Z}^+$ with $n > m \geq 0$. Since $< a^4$, the sequence $(\frac{f(a^n x)}{a^m})$ is a Cauchy sequence in the complete $R\mathbb{N}$-space $(Y, \mu, T_M)$ and it converge to a point $Q(x) \in Y$.

Fix $x \in X$ and $m = 0$ in equation (2.19), we get

$$
\mu \left( \frac{f(a^n x)}{a^m} - f(x) \right)(t) \geq \mu'_{\phi(x,0,0)} \frac{2a^4t}{\sum_{j=0}^{n+m-1} (\frac{a}{a^4})^j},
$$

and so, for any $\delta > 0$,

$$
\mu(Q(x) - f(x))(\delta + t) \geq T_M \mu(Q(x) - \frac{f(a^n x)}{a^m})(\delta), \mu(\frac{f(a^n x)}{a^m} - f(x))(t),
$$

$$
\geq T_M \mu(Q(x) - f(x))(\delta), \mu'_{\phi(x,0,0)} \frac{4a^4t}{\sum_{j=0}^{n+m-1} (\frac{a}{a^4})^j},
$$

(2.20)

for all $x \in X$ and $t > 0$. Taking the limit $n \to \infty$ in equation (2.20), we get

$$
\mu(Q(x) - f(x))(\delta + t) \geq \mu'_{\phi(x,0,0)}(\frac{4a^4t}{\sum_{j=0}^{\infty} (\frac{a}{a^4})^j}) = \mu'_{\phi(x,0,0)}(4t(a^4 - \alpha)).
$$

(2.21)

Since $\delta$ is arbitrary, by taking $\delta \to 0$ in equation (2.21), we have

$$
\mu(Q(x) - f(x))(t) \geq \mu'_{\phi(x,0,0)}(4t(a^4 - \alpha)).
$$

(2.22)

for all $x \in X, t > 0$.

Therefore, we conclude that the condition of equation (2.15) holds.

Also replacing $x, y, z$ by $a^n x, a^n y, a^z$ respectively in equation (2.15), we have

$$
\mu'_{\phi(a^n x, a^n y, a^z)}(a^n t) \geq \mu'_{\phi(x,y,z)}((\frac{a}{\alpha})^n t).
$$

It follows from $\lim_{n \to \infty} \mu'_{\phi(a^n x, a^n y, a^z)}(a^n t) = 1$ that $Q$ satisfies the equation (1.1), which implies $Q$ is a quartic mapping.

**Lemma 2.1** ([8]). Suppose that $(\omega, d)$ is a complete generalized metric space and $J : \omega \to \omega$ is a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each $x \in \omega$, either $d(J^n x, J^{n+1} x) = \infty$. for all non negative integers $n \geq 0$ or there exists a natural number $n_0$ such that

1. $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
2. The sequence $J^n x$ is convergent to a fixed point $y$ of $J$;
3. $y$ is the unique fixed point of $J$ in the set $A = \{ y \in \omega : d(J^n x, y) < \infty \}$;
4. $d(y, y') \leq \frac{1}{1-L} d(y, J y')$ for all $y, y' \in A$.

**Theorem 2.3.** Let $\phi : X^3 \to D^+$ be a function such that, for some $0 < \alpha < a^4$,

$$
\mu'_{\phi(x,y,z)}(t) \leq \mu'_{\phi(a x, a y, a z)}(\alpha t)
$$

(2.23)

for all $x, y, z \in X$ and $t > 0$. If $f : X \to Y$ is an even mapping with $f(0) = 0$ such that

$$
\mu(D(x,y,z))(t) \geq \mu'_{\phi(x,y,z)}(t),
$$

(2.24)

for all $x, y, z \in X$ and $t > 0$.

Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$
\mu(f(x) - Q(x))(t) \geq \mu'_{\phi(x,y,z)}(2(a^4 - \alpha)t),
$$

(2.25)

for all $x \in X, t > 0$.

**Proof.** It follows from equation (2.24) that

$$
\mu(f(x) - \frac{f(ax)}{a}) (t) \geq \mu'_{\phi(x,0,0)}(4a^4 t),
$$

(2.26)

for all $x \in X, t > 0$.

Let $\omega = \{ g : X \to Y, g(x) = 0 \}$ and mapping $d$ defined on $\omega$ by

$$
d(g, h) = \inf \{ c \in [0, \infty) : \mu(g(x) - h(x))(ct) \geq \mu'_{\phi(x,0,0)}(t), \forall x \in X \}
$$
Then as usual $\inf \phi = -\infty$. Then $(\omega, d)$ is a generalized complete metric space. Now let us consider the mapping $J : \omega \to \omega$ defined by 
\[ Jg(x) = \frac{1}{4} g(ax), \] 
for all $g \in \omega$ and $x \in X$.

Let $g, h \in \omega$ and $c \in [0, \infty)$ be any arbitrary constant with $d(g, h) < c$.

Then $\mu(g(x) - h(x))(ct) \geq \mu'_{\phi(x, 0, 0)}(ct)$ for all $x \in X, t > 0$ and so,

\[ \mu(Jg(x) - Jh(x))(\frac{\alpha c t}{4}) = \mu(\frac{1}{4} g(ax) - \frac{1}{4} h(ax))(\alpha c t) \geq \mu'_{\phi(x, 0, 0)}(t) = \mu'_{\phi(\frac{t}{4}, \frac{t}{4})}, \] 
(2.27)

for all $x \in X, t > 0$. Hence we have $d(Jg, Jh) \leq \frac{c}{4t} \leq \frac{c}{4t} d(g, h)$.

for all $g, h \in \omega$.

Then $J$ is a contractive mapping on $\omega$ with the Lipschitz constant $L = \frac{c}{4t} < 1$.

Thus it follows from Lemma 2.1, that there exists a mapping $Q : X \to Y$ which is a unique fixed point of $J$ in the set $\omega_1 = \{ g \in \omega : d(g, h) < \infty \}$, such that

\[ Q(x) = \lim_{n \to \infty} f(\frac{\alpha^nx}{a^n}) \] 
for all $x \in X$ since $\lim_{n \to \infty} d(Jf, Q) = 0$. Also, using $\mu(\frac{f(x) - f(ax)}{\alpha c t})(t) \geq \mu'_{\phi(x, 0, 0)}(4(a^4 - \alpha)t)$, we have $d(f, Jf) \leq \frac{1}{4(a^4 - \alpha)}$.

Therefore using Lemma 2.1, we get 
\[ d(f, Q) \leq \frac{1}{4(a^4 - \alpha)} d(f, Jf) \leq \frac{1}{4(a^4 - \alpha)}. \] 
This means that 
\[ \mu_f(x - Q(x))(t) \geq \mu'_{\phi(x, 0, 0)}(4(a^4 - \alpha)t), \]

for all $x \in X, t > 0$. Also by replacing $x, y, z$ by $2^nx, 2^ny, 2^nz$ in equation (2.4) respectively, we have 
\[ \mu_{DQ(x, y, z)}(t) \geq \lim_{n \to \infty} \mu'_{\phi(2^n x, 2^n y, 2^n z)}(\alpha^n t) = \lim_{n \to \infty} \mu'_{\phi(x, y, z)}(\alpha^n t) = 1, \]
for all $x, y, z \in X$ and $t > 0$. By (RN1), the mapping is quartic.

To prove the uniqueness let us assume that there exists a quartic mapping $Q' : X \to Y$, which satisfies equation (2.25). Then $Q'$ is a fixed point of $J$ in $\omega_1$.

However it follows from the Lemma 2.3, that $J$ has only one fixed point in $\omega_1$.

Hence $Q = Q'$.

\[ \square \]

**Acknowledgement.** Authors are very much thankful to the Editor and Reviewer for their valuable suggestions to bring the paper in its present form.

**References**


