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ON HOMOGENEOUS CUBIC EQUATION WITH FOUR UNKNOWNS $(\mathbf{x^3+y^3})=\mathbf{7zw^2}$

J. Shanthi, S. Vidhyalakshmi and M. A. Gopalan
Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy, Tamil Nadu, India-620002
Email: shanthivishvaa@gmail.com, vidhyasigc@gmail.com and mayilgopalan@gmail.com

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Abstract

This paper concerns with the problem of obtaining non-zero distinct integer solutions to homogeneous cubic equation with four unknowns given by $x^3+y^3=7zw^2$. A few interesting properties among the solutions are presented.

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1 Introduction

The cubic diophantine equations are rich in variety and offer an unlimited field for research [1, 2]. In particular refer [3]- [23] for a few problems on cubic equation with 3 and 4 unknowns. This paper concerns with yet another interesting homogeneous cubic diophantine equation with four unknowns $x^3+y^3=7zw^2$ for determining its infinitely many non-zero distinct integral solutions through employing linear transformations. A few interesting relations among the solutions are presented.

Method of Analysis

The homogeneous cubic equation with four unknowns to be solved is represented by

$$x^3+y^3=7zw^2. \tag{1.1}$$

Introduction of the linear transformations

$$x=u+v, y=u-v, z=2u, \tag{1.2}$$

in (1.1) leads to

$$u^2+3v^2=7w^2. \tag{1.3}$$

Different methods of obtaining the patterns of integer solutions to (1.1) are illustrated below:

2 Patterns

Pattern 2.1. Let

$$w=a^2+3b^2, \tag{2.1}$$

where a and b are non-zero integers.

Write 7 as

$$7=(2+i\sqrt{3})(2-i\sqrt{3}). \tag{2.2}$$

Using (2.1), (2.2) in (1.3) and applying the method of factorization, define

$$(u+i\sqrt{3}v)=(2+i\sqrt{3})(a+i\sqrt{3}b)^2, \tag{2.3}$$

from which, we have

$$\left.\begin{aligned} u&=2a^2-6ab-6b^2\\ v&=a^2+4ab-3b^2 \end{aligned}\right\}. \tag{2.4}$$

Using (2.4) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x&=x(a,b)=3a^2-2ab-9b^2\\ y&=y(a,b)=a^2-10ab-3b^2\\ z&=z(a,b)=4a^2-12ab-12b^2 \end{aligned}\right\}. \tag{2.5}$$

Thus (2.1) and (2.5) represent the non-zero integer solutions to (1.1).

Observation 2.1. 1. $z(a, a+1) - 4y(a, a+1) = 56t_{3,a}$
2. $z(a, 2a-1) - 4y(a, 2a-1) = 28t_{6,a}$
3. $z(a, a) - 4y(a, a) - t_{58,a} \equiv 0 \pmod 3$
4. $z(a, a) - 4y(a, a) - t_{34,a} - t_{26,a} \equiv 0 \pmod{13}$
5. $42[z(a, a) - 4y(a, a)]$ is a nasty number.

Pattern 2.2. Write 7 as

$$7 = \frac{(5+i\sqrt{3})(5-i\sqrt{3})}{4}. \tag{2.6}$$

Using (2.1), (2.6) in (1.3) and applying the method of factorization, define

$$(u + i\sqrt{3}v) = \frac{(5+i\sqrt{3})}{2}(a + i\sqrt{3}b)^2, \tag{2.7}$$

from which, we have

$$\left.\begin{aligned} u &= \frac{1}{2}(5a^2 - 6ab - 15b^2) \\ v &= \frac{1}{2}(a^2 + 10ab - 3b^2) \end{aligned}\right\}, \tag{2.8}$$

Using (2.8) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x = x(a, b) &= 3a^2 + 2ab - 9b^2 \\ y = y(a, b) &= 2a^2 - 8ab - 6b^2 \\ z = z(a, b) &= 5a^2 - 6ab - 15b^2 \end{aligned}\right\}, \tag{2.9}$$

Thus (2.1) and (2.9) represent the non-zero integer solutions to (1.1).

Observation 2.2. 1. $x(a, a) - z(a, a) - t_{38,a} \equiv 0 \pmod{17}$
2. $y(a, a) - z(a, a) = t_{4,2a}$

Pattern 2.3. Write (1.3) as

$$u^2 + 3v^2 = 7w^2 * 1. \tag{2.10}$$

Write 1 as

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}. \tag{2.11}$$

Using (2.1), (2.2), (2.11) in (1.3) and applying the method of factorization, define

$$(u + i\sqrt{3}v) = (\frac{1+i\sqrt{3}}{2})(2 + i\sqrt{3})(a + i\sqrt{3}b)^2, \tag{2.12}$$

from which, we have

$$\left.\begin{aligned} u &= \frac{1}{2}(-a^2 - 18ab + 3b^2) \\ v &= \frac{1}{2}(3a^2 - 2ab - 9b^2) \end{aligned}\right\}, \tag{2.13}$$

Using (2.13) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x = x(a, b) &= a^2 - 10ab - 3b^2 \\ y = y(a, b) &= -2a^2 - 8ab + 6b^2 \\ z = z(a, b) &= -a^2 - 18ab + 3b^2 \end{aligned}\right\}. \tag{2.14}$$

Thus (2.1) and (2.14) represent the non-zero integer solutions to (1.1).

Observation 2.3. 1. $y(a, a) - x(a, a) - t_{18,a} \equiv 0 \pmod 7$
2. $y(b, b) - z(b, b) - t_{b,22} - t_{b,6} \equiv 0 \pmod 2$

Pattern 2.4. Consider 1 as

$$1 = \frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49}. \tag{2.15}$$

Using (2.1), (2.2), (2.15) in (1.3) and applying the method of factorization, define

$$(u + i\sqrt{3}v) = (\frac{1+i4\sqrt{3}}{7})(2 + i\sqrt{3})(a + i\sqrt{3}b)^2, \tag{2.16}$$

from which, we have

$$\left.\begin{aligned} u &= \frac{1}{7}(-10a^2 - 54ab + 30b^2) \\ v &= \frac{1}{7}(9a^2 - 20ab - 27b^2) \end{aligned}\right\}. \tag{2.17}$$

Using (2.17) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x = x(a,b) &= \frac{1}{7}(-a^2 - 74ab + 3b^2) \\ y = y(a,b) &= \frac{1}{7}(-19a^2 - 34ab + 57b^2) \\ z = z(a,b) &= \frac{1}{7}(-20a^2 - 10ab + 60b^2) \end{aligned}\right\}. \tag{2.18}$$

Since our interest is on finding integer solutions, replacing a by $7A$, b by $7B$ in (2.1) and (2.18), the corresponding integer solutions to to (1.1) are given by

$$\left.\begin{aligned} x = x(A,B) &= 7(-A^2 - 74AB + 3B^2) \\ y = y(A,B) &= 7(-19A^2 - 34AB + 57B^2) \\ z = z(A,B) &= 7(-20A^2 - 10AB + 60B^2) \\ w = w(A,B) &= 49(A^2 + 3B^2) \end{aligned}\right\}. \tag{2.19}$$

Observation 2.4. 1. $x(A,A) - z(A,A) + t_{38,A} + t_{22,A} \equiv 0 \pmod{13}$

2. $y(A,A) - z(A,A) - 7[t_{62,A} + t_{42,A} + t_{26,A} + t_{22,A}] \equiv 0 \pmod 7$

Pattern 2.5. Take 1 as

$$1 = \frac{(1 + i15\sqrt{3})(1 - i15\sqrt{3})}{676}. \tag{2.20}$$

Using (2.1), (2.2), (2.20) in (1.3) and applying the method of factorization, define

$$(u + i\sqrt{3}v) = (\frac{1 + i15\sqrt{3}}{26})(2 + i\sqrt{3})(a + i\sqrt{3}b)^2, \tag{2.21}$$

from which, we have

$$\left.\begin{aligned} u &= \frac{1}{26}(-43a^2 - 186ab + 129b^2) \\ v &= \frac{1}{26}(31a^2 - 86ab - 93b^2) \end{aligned}\right\}. \tag{2.22}$$

Using (2.22) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x = x(a,b) &= \frac{1}{13}(-6a^2 - 136ab + 18b^2) \\ y = y(a,b) &= \frac{1}{13}(-37a^2 - 50ab + 111b^2) \\ z = z(a,b) &= \frac{1}{13}(-43a^2 - 186ab + 129b^2) \end{aligned}\right\}. \tag{2.23}$$

Since our interest is on finding integer solutions, replacing a by $13A$, b by $13B$ in (2.1) and (2.23), the corresponding integer solutions to to (1.1) are given by

$$\left.\begin{aligned} x = x(A,B) &= 13(-6A^2 - 136AB + 18B^2) \\ y = y(A,B) &= 13(-37A^2 - 50AB + 111B^2) \\ z = z(A,B) &= 13(-43A^2 - 186AB + 129B^2) \\ w = w(A,B) &= 169(A^2 + 3B^2) \end{aligned}\right\}. \tag{2.24}$$

Observation 2.5. 1. $z(A,A) - x(A,A) - 13[t_{30,A} + t_{14,A} + t_{10,A}] \equiv 0 \pmod{13}$

2. $x(A,A) - y(A,A) + 15[t_{62,A} + t_{16,A}] = 35A$

Pattern 2.6. Assume 1 as

$$1 = \frac{(1 + i56\sqrt{3})(1 - i56\sqrt{3})}{9409}. \tag{2.25}$$

Using (2.1), (2.2), (2.25) in (1.3) and applying the method of factorization, define

$$(u+i\sqrt{3}v)=(\frac{1+i56\sqrt{3}}{97})(2+i\sqrt{3})(a+i\sqrt{3}b)^2, \tag{2.26}$$

from which, we have

$$\left.\begin{aligned} u&=\frac{1}{97}(-166a^2-678ab+498b^2)\\ v&=\frac{1}{97}(113a^2-332ab-339b^2)\end{aligned}\right\}. \tag{2.27}$$

Using (2.27) and (1.2), the values of x,y and z are given by

$$\left.\begin{aligned} x=x(a,b)&=\frac{1}{97}(-53a^2-1010ab+159b^2)\\ y=y(a,b)&=\frac{1}{97}(-279a^2-346ab+837b^2)\\ z=z(a,b)&=\frac{1}{97}(-332a^2-1356ab+996b^2)\end{aligned}\right\}. \tag{2.28}$$

Since our interest is on finding integer solutions, replacing a by $97A$, b by $97B$ in (2.1) and (2.28), the corresponding integer solutions to to (1.1) are given by

$$\left.\begin{aligned} x=x(A,B)&=97(-53A^2-1010AB+159B^2)\\ y=y(A,B)&=97(-279A^2-346AB+837B^2)\\ z=z(A,B)&=97(-332A^2-1356AB+996B^2)\\ w=w(A,B)&=9409(A^2+3B^2)\end{aligned}\right\}. \tag{2.29}$$

Pattern 2.7. Using (2.1), (2.6), (2.11) in (1.3) and applying the method of factorization, define

$$(u+i\sqrt{3}v)=(\frac{5+i\sqrt{3}}{2})(a+i\sqrt{3}b)^2(\frac{1+i\sqrt{3}}{2}), \tag{2.30}$$

from which, we have

$$\left.\begin{aligned} u&=\frac{1}{2}(a^2-18ab-3b^2)\\ v&=\frac{1}{2}(3a^2+2ab-9b^2)\end{aligned}\right\}. \tag{2.31}$$

Using (2.31) and (1.2), the values of x,y and z are given by

$$\left.\begin{aligned} x=x(a,b)&=2a^2-8ab-6b^2\\ y=y(a,b)&=-a^2-10ab+3b^2\\ z=z(a,b)&=a^2-18ab-3b^2\end{aligned}\right\}. \tag{2.32}$$

Thus (2.1) and (2.32) represent the non-zero integer solutions to to (1.1).

Observation 2.6.

1. $x(b,b)-z(b,b)=2t_{2b,4}$
2. $x(a,a)-y(a,a)+t_{4,2a}=0$

Pattern 2.8. Using (2.1), (2.6), (2.15) in (1.3) and applying the method of factorization, define

$$(u+i\sqrt{3}v)=(\frac{5+i\sqrt{3}}{2})(a+i\sqrt{3}b)^2(\frac{1+i4\sqrt{3}}{7}), \tag{2.33}$$

from which, we have

$$\left.\begin{aligned} u&=\frac{1}{2}(-a^2-18ab+3b^2)\\ v&=\frac{1}{2}(3a^2-2ab-9b^2)\end{aligned}\right\}. \tag{2.34}$$

Using (2.34) and (1.2), the values of x,y and z are given by

$$\left.\begin{aligned} x=x(a,b)&=a^2-10ab-3b^2\\ y=y(a,b)&=-2a^2-8ab+6b^2\\ z=z(a,b)&=-a^2-18ab+3b^2\end{aligned}\right\}. \tag{2.35}$$

Thus (2.1) and (2.35) represent the non-zero integer solutions to to (1.1).

Observation 2.7. 1. $x(b,b) - z(b,b) - t_{b,10} \equiv 0 \pmod 3$
2. $y(b,b) - z(b,b) - 2t_{b,14} \equiv 0 \pmod 5$

Pattern 2.9. Using (2.1), (2.6), (2.20) in (1.3) and applying the method of factorization, define

$$(u + i\sqrt{3}v) = (\frac{5 + i\sqrt{3}}{2})(a + i\sqrt{3}b)^2(\frac{1 + i15\sqrt{3}}{26}), \tag{2.36}$$

from which, we have

$$\left.\begin{aligned} u &= \frac{1}{13}(-10a^2 - 114ab + 30b^2) \\ v &= \frac{1}{13}(19a^2 - 20ab - 57b^2) \end{aligned}\right\}. \tag{2.37}$$

Using (2.37) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x = x(a,b) &= \frac{1}{13}(9a^2 - 134ab - 27b^2) \\ y = y(a,b) &= \frac{1}{13}(-29a^2 - 94ab + 87b^2) \\ z = z(a,b) &= \frac{1}{13}(-20a^2 - 228ab + 60b^2) \end{aligned}\right\}. \tag{2.38}$$

Since our interest is on finding integer solutions, replacing a by $13A$, b by $13B$ in (2.1) and (2.38), the corresponding integer solutions to (1.1) are given by

$$\left.\begin{aligned} x = x(A,B) &= 13(9A^2 - 134AB - 27B^2) \\ y = y(A,B) &= 13(-29A^2 - 94AB + 87B^2) \\ z = z(A,B) &= 13(-20A^2 - 228AB + 60B^2) \\ w = w(A,B) &= 169(A^2 + 3B^2) \end{aligned}\right\}. \tag{2.39}$$

Pattern 2.10. Using (2.1), (2.6), (2.25) in (1.3) and applying the method of factorization, define

$$(u + i\sqrt{3}v) = (\frac{5 + i\sqrt{3}}{2})(a + i\sqrt{3}b)^2(\frac{1 + i56\sqrt{3}}{97}), \tag{2.40}$$

from which, we have

$$\left.\begin{aligned} u &= \frac{1}{194}(-163a^2 - 1686ab + 489b^2) \\ v &= \frac{1}{194}(281a^2 - 326ab - 843b^2) \end{aligned}\right\}. \tag{2.41}$$

Using (2.41) and (1.2), the values of x, y and z are given by

$$\left.\begin{aligned} x = x(a,b) &= \frac{1}{97}(59a^2 - 1006ab - 177b^2) \\ y = y(a,b) &= \frac{1}{97}(-222a^2 - 680ab + 666b^2) \\ z = z(a,b) &= \frac{1}{97}(-163a^2 - 1686ab + 489b^2) \end{aligned}\right\}. \tag{2.42}$$

Since our interest is on finding integer solutions, replacing a by $97A$, b by $97B$ in (2.1) and (2.42), the corresponding integer solutions to (1.1) are given by

$$\left.\begin{aligned} x = x(A,B) &= 97(59A^2 - 1006AB - 177B^2) \\ y = y(A,B) &= 97(-222A^2 - 680AB + 666B^2) \\ z = z(A,B) &= 97(-163A^2 - 1686AB + 489B^2) \\ w = w(A,B) &= 9409(A^2 + 3B^2) \end{aligned}\right\}. \tag{2.43}$$

Pattern 2.11. (1.3) is rewritten as

$$u^2 = 7w^2 - 3v^2. \tag{2.44}$$

In (2.44), taking

$$\left.\begin{aligned} w &= X + 3T \\ v &= X + 7T \\ u &= 2U \end{aligned}\right\}, \tag{2.45}$$

it leads to

$$X^2 - U^2 = 21T^2, \tag{2.46}$$

which is written as the system of double equations as shown in Table 2.1:

***Table* 2.1:** System of Double Equations

$SYSTEM$	1.1	1.2	1.3	1.4
$X+U$	T^2	$3T^2$	$7T^2$	$7T$
$X-U$	21	7	3	$3T$

Consider system (1.1) in the Table 2.1: Solving the pair of equations, note that

$$X = \frac{T^2+21}{2}$$
$$U = \frac{T^2-21}{2}.$$

The choice

$$T = 2k+1, \tag{2.47}$$

gives

$$\left.\begin{aligned} X &= 2k^2+2k+11 \\ U &= 2k^2+2k-10 \end{aligned}\right\}. \tag{2.48}$$

The substitution of (2.47) and (2.48) in (2.45) gives

$$\left.\begin{aligned} u &= 4k^2+4k-20 \\ v &= 2k^2+16k+18 \\ w &= 2k^2+8k+14 \end{aligned}\right\}. \tag{2.49}$$

In view of (1.2), one obtains

$$\left.\begin{aligned} x &= 6k^2+20k-2 \\ y &= 2k^2-12k-38 \\ z &= 8k^2+8k-40 \end{aligned}\right\}. \tag{2.50}$$

Thus (2.49) and (2.50) represent the non-zero integer solutions to (1.1).

Consider system (1.2) in the Table 2.1: Solving the pair of equations, note that

$$X = \frac{3T^2+7}{2}$$
$$U = \frac{3T^2-7}{2}.$$

Using (2.47) the above equation become

$$\left.\begin{aligned} X &= 6k^2+6k+5 \\ U &= 6k^2+6k-2 \end{aligned}\right\}. \tag{2.51}$$

The substitution of (2.47) and (2.51) in (2.45) gives

$$\left.\begin{aligned} u &= 12k^2+12k-4 \\ v &= 6k^2+20k+12 \\ w &= 6k^2+12k+8 \end{aligned}\right\}. \tag{2.52}$$

In view of (1.2), one obtains

$$\left.\begin{aligned} x &= 18k^2+32k+8 \\ y &= 6k^2-8k-16 \\ z &= 24k^2+24k-8 \end{aligned}\right\}. \tag{2.53}$$

Thus (2.52) and (2.53) represent the non-zero integer solutions to (1.1).

Consider system (1.3) in the Table 2.1: Solving the pair of equations, note that

$$X = \frac{7T^2 + 3}{2}$$
$$U = \frac{7T^2 - 3}{2}$$

Using (2.47) the above equation become

$$\left.\begin{aligned} X &= 14k^2 + 14k + 5 \\ U &= 14k^2 + 14k + 2 \end{aligned}\right\}. \tag{2.54}$$

The substitution of (2.47) and (2.54) in (2.45) gives

$$\left.\begin{aligned} u &= 28k^2 + 28k + 4 \\ v &= 14k^2 + 28k + 12 \\ w &= 14k^2 + 20k + 8 \end{aligned}\right\}. \tag{2.55}$$

In view of (1.2), one obtains

$$\left.\begin{aligned} x &= 42k^2 + 56k + 16 \\ y &= 14k^2 - 8 \\ z &= 56k^2 + 56k + 8 \end{aligned}\right\}. \tag{2.56}$$

Thus (2.55) and (2.56) represent the non-zero integer solutions to (1.1).

Consider system (1.4) in the Table 2.1: On solving, it is seen that $X = 5T, U = 2T$.
In view of (2.45), we have

$$\left.\begin{aligned} u &= 4T \\ v &= 12T \\ w &= 8T \end{aligned}\right\}. \tag{2.57}$$

Substituting the above values of u and v in (1.2), we get

$$\left.\begin{aligned} x &= 16T \\ y &= -8T \\ z &= 8T \end{aligned}\right\}. \tag{2.58}$$

Thus (2.57) and (2.58) represent the non-zero integer solutions to (1.1).

Pattern 2.12. It is seen that (2.46) is satisfied by

$$\left.\begin{aligned} T &= 2rs \\ U &= 21r^2 - s^2 \\ X &= 21r^2 + s \end{aligned}\right\}. \tag{2.59}$$

Substituting the values of T, U and X in (2.53), we get

$$\left.\begin{aligned} u &= 42r^2 - 2s^2 \\ v &= 21r^2 + 14rs + s^2 \\ w &= 21r^2 + 6rs + s^2 \end{aligned}\right\}. \tag{2.60}$$

Substituting the above values of u and v in (1.2), the non-zero distinct integral values of x, y and z are given by

$$\left.\begin{aligned} x &= x(r,s) = 63r^2 + 14rs - s^2 \\ y &= y(r,s) = 21r^2 - 14rs - 3s^2 \\ z &= z(r,s) = 84r^2 - 4s^2 \end{aligned}\right\}. \tag{2.61}$$

Thus (2.60) and (2.61) represent the non-zero integer solutions to (1.1).

3 Conclusion

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutins to the homogeneous cubic equation with four unknowns. As the cubic equations are rich in variety, one may search for other forms of cubic equations with multivariables to obtain their corresponding solutions.

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