ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 157-164**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

INEQUALITIES VIA MEAN FUNCTIONS USING E - CONVEXITY

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(Received: October 10, 2022; In format: February 14, 2023; Received: February 22, 2023; Accepted: February 28, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53118

Abstract

In this paper, we extend the concept of GG - convexity to GG- E - convexity and then we derived some new integral inequalities for GG- E - convex function using Holder's integral inequalities. Enough examples are given to verify the obtained results.

2020 Mathematical Sciences Classification: 26A51, 26D10, 26D15
Keywords and Phrases: Convex function, E - convex set, E - convex function, E - concave function, GG- E - convex function, GG- E - concave function, Holder integral inequality.

1 Introduction

Convexity contains a broad spectrum of significance in both applied and pure mathematics. Nowadays, the concept of convexity is not confined to convex functions only, but it has also extended to non - convex functions as well as convex programming.Minkowski [9] did the first methodical study of convexity. Convexity plays an important role in convex - optimization. So Many problems of quasi - convex optimization arise in spanning economics [8, 13], industrial organization [14]. Realted to quasi-convex optimization also available In offline case [10, 11, 12, 19]. Convexity is also useful in concept of special means like arithmetic mean, . geometric mean, harmonic mean, logarithmic mean and identric mean Anderson et al.[1] mentioned mean function. Anderson et al. [1] derived similar results for some power series, especially hypergeometric functions. Dragomir [5] gave inequalities at the same time Akdemir et al. [2] gave generalization in sense of convex functions. Then new integral inequalities arise via GG - convexity and GA - convexity [3].

Hanson and Mond [6] extended the class of convex functions to the class of invex functions and showed that programming problems that can be transformed in this way are a strict subset of invex programming problems, then Bector and Singh [4] introduced B - vex functions and discussed differentiable and non differentiable cases. Class of B - vex functions forms a subset of the sets of both semistrictly quasiconvex as well as quasiconvex functions. For the first time Youness [17] provided the concept of well known class of generalised convexity, namely E-convexity. Furthermore, he formulated some results from E - convex functions in programming problems [18]. Yang [15] refined few results of E - convex programming, which were obtained by Younness [17]. Over the past few years, many researchers have focused on the theory of inequalities. Because of the wide range of ideas and applications, the theory of inequalities has become a captivating, engrossing and gripping area for researchers.

In this paper, our aim is to establish some new inequalities for GG - E - convex function. Also we prove that not only the inequalities of the convex function are possible through the mean function, but also the composite functions in which one is convex and the other is non-convex obey the inequalities. The Interesting techniques and the useful ideas of this paper may encourage further research in this dynamic area.

2 Definitions and Preliminaries

Definition 2.1. *The function $f : I \subset R \to R$ is a convex function on I, if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. We say that g is concave if - f is convex [3].

Let $f : I \subset R \to R$ be a convex function where $a, b \in I$ with $a < b$. Then the following double inequality holds :

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

This inequality is well - known in the literature as Hermite - Hadamard inequality [5].

Definition 2.2. *A set $N \subset R^n$ is said to be E - convex iff there is a map $E : R^n \to R^n$ such that*

$$(1-t)E(x) + tE(y) \in N.$$

for each $x, y \in N$ and $0 \leq t \leq 1$[17].

Definition 2.3. *A function $f : R^n \to R^n$ is said to be E - convex on a set $N \subset R^n$, iff there is a map $E : R^n \to R^n$ such that N is an convex set and*

$$f(tE(x) + (1-t)E(y)) \leq tf(E(x)) + (1-t)f(E(y)).$$

for each $x, y \in N$ and $0 \leq t \leq 1$ on the other hand, if

$$f(tE(x) + (1-t)E(y)) \geq tf(E(x)) + (1-t)f(E(y)).$$

then f is called E - concave on N. If the inequality signs in the previous two inequalities are strict, then f is called strictly E convex and strictly E concave, respectively[17].

Definition 2.4. *Let $f : S \to (0, \infty)$ be continuous, where I is subinterval of $(0, \infty)$. Let N and P be any two mean functions, $T \subset R$ and there is a map $E : R \to R$ then we say f is NP -E-convex(concave) on T if*

$$f(N(E(x), E(y)) \leq (\geq)P(f(E(x), f(E(y)).$$

for all $x, y \in S$.

Definition 2.5. *The GG - E - convex functions are those functions $f : S \subseteq R_+ \to R$ and there is a map $E : R \to R$ such that $x, y \in S$ and*

$$t \in [0,1] \Rightarrow f((E(x))^{1-t}(E(y))^t) \leq (f(E(x))^{1-t}(f(E(y))^t.$$

Definition 2.6. *The GG - E - concave functions are those functions $f : S \subseteq R_+ \to R$ and there is a map $E : R \to R$ such that $x, y \in S$ and*

$$t \in [0,1] \Rightarrow f((E(x))^{1-t}(E(y))^t) \geq (f(E(x))^{1-t}(f(E(y))^t.$$

Lemma 2.1. *Let $g : S \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on S^o and $x, y \in S^o$, where S^o is the interior set of S with $\alpha < \beta$, and there is a map $E : R \to R$. If $g' \in L([E(\alpha)], [E(\beta)])$, then the following identity holds :*

$$\begin{aligned} & E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \textstyle\int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v) \\ = \ & \tfrac{lnE(\beta)-lnE(\alpha)}{2}\left[\textstyle\int_0^1 \left((E(\beta))^t(E(\alpha))^{(2-t)}\right) g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right) dt\right] \\ + \ & \tfrac{lnE(\beta)-lnE(\alpha)}{2}\left[\textstyle\int_0^1 \left((E(\alpha))^t(E(\beta))^{(2-t)}\right) g'\left((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-t)}{2}}\right) dt\right]. \end{aligned}$$

Proof. Let $J_1 = \int_0^1 \left((E(\beta))^t(E(\alpha))^{(2-t)}\right) g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right) dt$ and $J_2 = \int_0^1 \left((E(\alpha))^t(E(\beta))^{(2-t)}\right) g' \Big((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-}{2}}$

Then we notice that

$J_1 = \int_0^1 \left((E(\beta))^t(E(\alpha))^{(2-t)}\right) g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right) dt$

$\frac{2}{lnE(\beta)-lnE(\alpha)} \int_0^1 \left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right) g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right) d\left(E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right).$

Now by the change of variable $E(v) = (E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}})$ and integrating by parts, we have

$J_1 = \frac{2}{lnE(\beta)-lnE(\alpha)}\left[\sqrt{(E(\alpha)E(\beta))}g\sqrt{(E(\alpha)E(\beta))} - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{\sqrt{(E(\alpha)E(\beta))}} g(E(v))dE(v)\right].$

Conformably, we have

$J_2 = \frac{2}{lnE(\beta)-lnE(\alpha)}\left[E(\beta)g(E(\beta)) - \sqrt{(E(\alpha)E(\beta))}g\sqrt{(E(\alpha)E(\beta))} - \int_{\sqrt{(E(\alpha)E(\beta))}}^{E(\beta)} g(E(v))dE(v)\right].$

Multiplying J_1 and J_2 by $\frac{lnE(\beta)-lnE(\alpha)}{2}$ and adding the results we get the appealed identity.

Our first result is given in the following theorem. □

Example 2.1. Let $f : \left[0, \frac{\pi}{2}\right] \to [0, \infty)$ and $n \in \mathbb{N} - \{1\}$ such that

$$f(x) = -\int_0^{\sqrt[n]{x}} ln(cos(t))dt$$

is not $GG - convex$ on $\left(0, \frac{\pi}{2}\right)$ and there is a map $E : \left[0, \sqrt[n]{\frac{\pi}{2}}\right] \to \left[0, \frac{\pi}{2}\right]$ such that $E(x) = x^n$, then the function

$$f(E(x)) = -\int_0^{x} ln(cos(t))dt$$

is GG - E - convex function on $\left(0, \frac{\pi}{2}\right)$.

Example 2.2. Let $f : \left[0, \frac{(4n+1)\pi}{2}\right] \to [0, \infty)$ where $n \in \mathbb{Z}$ such that

$$f(x) = ln(sinx)$$

is not GG - convex on $\left(0, \frac{(4n+1)\pi}{2}\right)$ and there is a map $E : \left[0, \frac{(4n+1)\pi}{2}\right] \to \left[0, \frac{(4n+1)\pi}{2}\right]$ such that $E(x) = \frac{(4n+1)\pi}{2} - x$ then the function

$$f(E(x)) = ln\left(sin\left(\frac{(4n+1)\pi}{2} - x\right)\right)$$

$$f(E(x)) = ln\,(cosx)$$

is GG - E - convex function on $\left(0, \frac{(4n+1)\pi}{2}\right)$.

Example 2.3. Let $E : \Re \to \Re$ and $f : \Re \to \Re$ are defined as

$$E(x) = \begin{cases} 2 & x > 0 \\ -x & x \leq 0 \end{cases}$$

And

$$f(x) = x^2$$

Then the function $E(x)$ is not obeying our inequalities and the function $f(x)$ is obeying our inequalities. Also the composition foE will obey our inequalities.

Remark 2.1. Example 2.3 shows that a non convex function $E(x)$ does not obey our inequalities, but a convex function $f(x)$ obey our inequalities. And also an $E-$convex function $foE(x)$ obey our inequalities.

Example 2.4. Let the function $E : \Re \to \Re$ such that $E(x) = lnx$ which is non convex, so this function is not obeying our inequalities.

Example 2.5. Let the function $F : \Re \to \Re$ such that $F(x) = e^x$ which is convex, so this function is obeying our inequalities.

Example 2.6. Let the composition of two functions E, F be defined as $FoE : \Re \to \Re$ such that $F(E(x)) = x$ which is $E-$convex, so this function is obeying our inequalities.

3 Main Results

Theorem 3.1. *Let $g : S \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on S^o and $\alpha, \beta \in S^o$ with $\alpha < \beta$ and $E : R \to R$ is a non decreasing function so $E(\alpha) < E(\beta)$. If $g' \in L[E(\alpha), E(\beta)]$. If $|g'|$ is $GG - E -$ Convex on $[E(\alpha), E(\beta)]$, then the following inequality holds:*

$$|E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v)|$$
$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left(E(\alpha)\sqrt{|g'(E(\alpha))|} + E(\beta)\sqrt{|g'(E(\beta))|}\right) L\left(E(\alpha)\sqrt{|g'(E(\alpha))|}, E(\beta)\sqrt{|g'(E(\beta))|}\right).$$

Proof. From Lemma 2.1, using the property of the modulus and $GG \;\; E$ convexity of $|g'|$, we can write

$$|E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v)|$$
$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\int_0^1 \left((E(\beta))^t (E(\alpha))^{(2-t)}\right)\left|g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right)\right| dt\right]$$
$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\int_0^1 \left((E(\alpha))^t (E(\beta))^{(2-t)}\right)\left|g'\left((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-t)}{2}}\right)\right| dt\right]$$
$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\int_0^1 \left((E(\beta))^t (E(\alpha))^{(2-t)}\right)\left|g'(E(\beta))\right|^{\frac{t}{2}}\left|g'(E(\alpha))\right|^{\frac{(2-t)}{2}} dt\right]$$
$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\int_0^1 \left((E(\alpha))^t (E(\beta))^{(2-t)}\right)\left|g'(E(\alpha))\right|^{\frac{t}{2}}\left|g'(E(\beta))\right|^{\frac{(2-t)}{2}} dt\right]$$
$$= \frac{lnE(\beta)-lnE(\alpha)}{2}\left[(E(\alpha))^2\left|g'(E(\alpha))\right|\int_0^1\left(\frac{E(\beta)\sqrt{|g'(E(\beta))|}}{E(\alpha)\sqrt{|g'(E(\alpha))|}}\right)^t dt\right]$$
$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[(E(\beta))^2\left|g'(E(\beta))\right|\int_0^1\left(\frac{E(\alpha)\sqrt{|g'(E(\alpha))|}}{E(\beta)\sqrt{|g'(E(\beta))|}}\right)^t dt\right].$$

Then we get the desired result. □

Theorem 3.2. *Let $g : S \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on S^o and $\alpha, \beta \in S^o$ with $\alpha < \beta$ and $E : R \to R$ is a non decreasing function so $E(\alpha) < E(\beta)$. If $g' \in L[E(\alpha), E(\beta)]$. If $|g'|^n$ is GG - E - Convex on $[E(\alpha), E(\beta)]$, for all $E(\gamma) \in [E(\alpha), E(\beta)]$,then the following inequality*

$$|E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v)|$$

$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left(E(\alpha)\sqrt{|g'(E(\alpha))|} + E(\beta)\sqrt{|g'(E(\beta))|}\right)\left(L\left((E(\alpha))^m, (E(\beta))^m\right)\right)^{\frac{1}{m}}$$
$$\times\left[\left(L\left(\sqrt{|g'(E(\alpha))|^n}, \sqrt{|g'(E(\beta))|^n}\right)\right)^{\frac{1}{n}}\right]$$

holds, where $n > 1$ and $\frac{1}{m} + \frac{1}{n} = 1$.

Proof. From Lemma 2.1, using the property of the modulus, $GG \;\; E$ convexity of $|g'|^n$ and Holder integral inequality, we can write

$$|E(\beta)g(E(\beta)) - E(\alpha)g(E(\alpha)) - \int_{E(\alpha)}^{E(\beta)} g(E(v))dE(v)|$$

$$= \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\int_0^1 \left((E(\beta))^t (E(\alpha))^{(2-t)}\right)\left|g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right)\right| dt\right]$$

$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\int_0^1 \left((E(\alpha))^t (E(\beta))^{(2-t)}\right)\left|g'\left((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-t)}{2}}\right)\right| dt\right]$$

$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_{E(\alpha)}^{E(\beta)} (E(\beta))^{tm}(E(\alpha))^{(2-t)m} dt\right)^{\frac{1}{m}}\left(\int_0^1\left|g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right)\right|^n\right)^{\frac{1}{n}}\right]$$

$$+ \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_{E(\alpha)}^{E(\beta)}(E(\alpha))^{tm}(E(\beta))^{(2-t)m}dt\right)^{\frac{1}{m}}\left(\int_0^1\left|g'\left((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-t)}{2}}\right)\right|^n\right)^{\frac{1}{n}}\right]$$

$$\leq \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[(E(\alpha))^2\left(\int_0^1\left(\frac{(E(\beta))^m}{(E(\alpha))^m}\right)^t dt\right)^{\frac{1}{m}}\left(\int_0^1\left|g'(E(\beta))\right|^{\frac{tn}{2}}\left|g'(E(\alpha))\right|^{\frac{(2-t)n}{2}}dt\right)^{\frac{1}{n}}\right]$$

$$+ \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[(E(\beta))^2\left(\int_0^1\left(\frac{(E(\alpha))^m}{(E(\beta))^m}\right)^t dt\right)^{\frac{1}{m}}\left(\int_0^1\left|g'(E(\alpha))\right|^{\frac{tn}{2}}\left|g'(E(\beta))\right|^{\frac{(2-t)n}{2}}dt\right)^{\frac{1}{n}}\right].$$

Then we get the desired result. □

Theorem 3.3. *Under the assumptions of Theorem* 3.2*, the following inequality holds :*

$$|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)|$$

$$\leq \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left(L\left(\sqrt{|g'(E(\alpha))|^n},\sqrt{|g'(E(\beta))|^n}\right)\right)^{\frac{1}{n}}$$

$$\times \quad \left[\left(\frac{(E(\beta))^{m+1}-mE(\beta)-(E(\beta))}{m+1}\right)^{\frac{1}{m}}(E(\alpha))^2\sqrt{|g'(E(\alpha))|}+\left(\frac{(E(\alpha))^{m+1}-mE(\alpha)-(E(\alpha))}{m+1}\right)^{\frac{1}{m}}(E(\beta))^2\sqrt{|g'(E(\beta))|}\right].$$

Proof. From Lemma 2.1, using the property of the modulus, $GG \quad E$ convexity of $|g'|^n$ and Holder integral inequality, we can write

$$|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)|$$

$$\leq \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1(E(\beta))^{tm}dt\right)^{\frac{1}{m}}\left(\int_0^1(E(\beta))^{(2-t)n}\left|g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right)\right|^n dt\right)^{\frac{1}{n}}\right]$$

$$+ \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1(E(\alpha))^{tm}dt\right)^{\frac{1}{m}}\left(\int_0^1(E(\alpha))^{(2-t)n}\left|g'\left((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-t)}{2}}\right)\right|^n dt\right)^{\frac{1}{n}}\right]$$

$$\leq \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1(E(\beta))^{tm}dt\right)^{\frac{1}{m}}\left(\int_0^1(E(\beta))^{(2-t)n}\left|g'(E(\beta))\right|^{\frac{t}{2}}\left|g'(E(\alpha))\right|^{\frac{(2-t)}{2}}dt\right)^{\frac{1}{n}}\right]$$

$$+ \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1(E(\alpha))^{tm}dt\right)^{\frac{1}{m}}\left(\int_0^1(E(\beta))^{(2-t)n}\left|g'(E(\alpha))\right|^{\frac{t}{2}}\left|g'(E(\beta))\right|^{\frac{(2-t)}{2}}dt\right)^{\frac{1}{n}}\right].$$

If we calculate the integral above, we get the desired result. □

Theorem 3.4. *Under the assumptions of Theorem* 3.2*, the following inequality holds :*

$$|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)|$$

$$\leq \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left(E(\alpha)\sqrt{|g'(E(\alpha))|}+E(\beta)\sqrt{|g'(E(\beta))|}\right)$$

$$\times\left(L\left((E(\alpha))^n\sqrt{|g'(E(\alpha))|^n},(E(\beta))^n\sqrt{|g'(E(\beta))|^n}\right)\right)^{\frac{1}{n}}.$$

Proof. From Lemma 2.1, using the property of the modulus, $GG \quad E$ - convexity of $|g'|^n$ and Power mean integral inequality, we can write

$$\left|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)\right|$$

$$\leq \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1 dt\right)^{1-\frac{1}{n}}\left(\int_0^1(E(\beta))^{tn}(E(\alpha))^{(2-t)n}\left|g'\left((E(\beta))^{\frac{t}{2}}(E(\alpha))^{\frac{(2-t)}{2}}\right)\right|^n dt\right)^{\frac{1}{n}}\right]$$

$$+ \quad \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1 dt\right)^{1-\frac{1}{n}}\left(\int_0^1(E(\alpha))^{tn}(E(\beta))^{(2-t)n}\left|g'\left((E(\alpha))^{\frac{t}{2}}(E(\beta))^{\frac{(2-t)}{2}}\right)\right|^n dt\right)^{\frac{1}{n}}\right]$$

$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[(E(\alpha))^2\left|g^{'}(E(\alpha))\right|\left(\int_0^1\left(\frac{(E(\beta))^n|g^{'}(E(\beta))|^{\frac{n}{2}}}{(E(\alpha))^n|g^{'}(E(\alpha))|^{\frac{n}{2}}}\right)^t dt\right)^{\frac{1}{n}}\right]$$

$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[(E(\beta))^2\left|g^{'}(E(\beta))\right|\left(\int_0^1\left(\frac{(E(\alpha))^n|g^{'}(E(\alpha))|^{\frac{n}{2}}}{(E(\beta))^n|g^{'}(E(\beta))|^{\frac{n}{2}}}\right)^t dt\right)^{\frac{1}{n}}\right].$$

Then we get the desired result. □

Theorem 3.5. *Under the assumptions of Theorem 3.2, the following inequality holds :*

$$|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)|$$

$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\frac{E(\beta)-1}{ln(E(\beta))}\right)^{\frac{1}{m}}E(\alpha)\sqrt{|g^{'}(E(\alpha))|}L^{\frac{1}{n}}\left((E(\alpha))^n\sqrt{|g^{'}(E(\alpha))|^n},(E(\beta))\sqrt{|g^{'}(E(\beta))|^n}\right)\right]$$

$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\frac{E(\alpha)-1}{ln(E(\alpha))}\right)^{\frac{1}{m}}E(\beta)\sqrt{|g^{'}(E(\beta))|}L^{\frac{1}{n}}\left((E(\alpha))^n\sqrt{|g^{'}(E(\alpha))|},(E(\beta))^n\sqrt{|g^{'}(E(\beta))|^n}\right)\right].$$

Proof. From Lemma 2.1, using the property of the modulus, $GG\ \ E$ - convexity of $|g'|^n$ and Power mean integral inequality, we can write

$$|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)|$$

$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1(E(\beta))^t dt\right)^{\frac{1}{m}}\left(\int_0^1(E(\beta))^t(E(\alpha))^{(2-t)n}\left|g^{'}(E(\beta))\right|^{\frac{tn}{2}}\left|g^{'}(E(\alpha))\right|^{(1-\frac{t}{2})n}dt\right)^{\frac{1}{n}}\right]$$

$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1(E(\alpha))^t dt\right)^{\frac{1}{m}}\left(\int_0^1(E(\alpha))^t(E(\beta))^{(2-t)n}\left|g^{'}(E(\alpha))\right|^{\frac{tn}{2}}\left|g^{'}(E(\beta))\right|^{(1-\frac{t}{2})n}dt\right)^{\frac{1}{n}}\right].$$

Then we get the desired result. □

Theorem 3.6. *Under the assumptions of Theorem 3.2, the following inequality holds :*

$$|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)|$$

$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left((E(\alpha))^{\frac{m}{n}}\sqrt{g^{'}(E(\alpha))}+((E(\beta))^{\frac{m}{n}}\sqrt{g^{'}(E(\beta))}\right)$$

$$\times\left(L\left((E(\alpha))^{\frac{n-m}{n-1}},(E(\beta))^{\frac{n-m}{n-1}}\right)\right)^{1-\frac{1}{n}}\left(L\left((E(\alpha))^m\sqrt{|g^{'}(E(\alpha))|^n}+(E(\beta))^m\sqrt{|g^{'}(E(\beta))|^n}\right)\right)^{\frac{1}{n}}.$$

Proof. From Lemma 2.1, using the property of the modulus, $GG\ \ E$ convexity of $|g'|^n$ and Power mean integral inequality, we can write

$$|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)|$$

$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1\left((E(\beta))^t(E(\alpha))^{(2-t)}\right)^{\frac{n-m}{n-1}}dt\right)^{1-\frac{1}{n}}\right]$$

$$\times\left[\left(\int_0^1\left((E(\beta))^t(E(\alpha))^{(2-t)}\right)^m\left|g^{'}(E(\alpha))\right|^{(\frac{2-t}{2})n}\left|g^{'}(E(\beta))\right|^{\frac{tn}{2}}dt\right)^{\frac{1}{n}}\right]$$

$$+ \frac{lnE(\beta)-lnE(\alpha)}{2}\left[\left(\int_0^1\left((E(\alpha))^t(E(\beta))^{(2-t)}\right)^{\frac{n-m}{n-1}}dt\right)^{1-\frac{1}{n}}\right]$$

$$\times\left[\left(\int_0^1\left((E(\alpha))^t(E(\beta))^{(2-t)}\right)^m\left|g^{'}(E(\beta))\right|^{(\frac{2-t}{2})n}\left|g^{'}(E(\alpha))\right|^{\frac{tn}{2}}dt\right)^{\frac{1}{n}}\right].$$

Then we get the desired result. □

Theorem 3.7. *Let $g:S\subseteq R_+=(0,\infty)\to R$ be a differentiable function on S^o and $\alpha,\beta\in S^o$ with $\alpha<\beta$ and $E:R\to R$ is a non decreasing function so $E(\alpha)<E(\beta)$. If $g^{'}\in L[E(\alpha),E(\beta)]$. If $|g'|$ is $GG-E-$ Convex on $[E(\alpha),E(\beta)]$, then the following inequality holds:*

$$|E(\beta)g(E(\beta))-E(\alpha)g(E(\alpha))-\int_{E(\alpha)}^{E(\beta)}g(E(v))dE(v)|$$

$$\leq \frac{lnE(\beta)-lnE(\alpha)}{2}\left(E(\alpha)\sqrt{|g^{'}(E(\alpha))|}+E(\beta)\sqrt{|g^{'}(E(\beta))|}\right)L\left(E(\alpha)\sqrt{|g^{'}(E(\alpha))|},E(\beta)\sqrt{|g^{'}(E(\beta))|}\right).$$

4 Application Area

The concept of E - convexity is the generalizations of convex sets and convex functions, in a respective manner. E - convexity is also used in the study of E-convex programming. We believe that our new class of functions will have a very profound research in this entrancing domain of inequalities and also in the pure and applied sciences. The interesting inequalities and successful ideas in this article can be extended to other mean functions like harmonic mean, Arithmetic mean etc. As we move forward, we aim to continue that research to find inequalities for non-convex functions as well.

5 Conclusions

In this paper, we derived the inequalities for GG - E - Convex function using Holder integral inequality. If we take $E(x)$ as an identity function then it shows the result of [16].

Acknowledgement. Authors are very much thankful to the Editor and Reviewer for their valuable suggestions to bring the paper in its present form.

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