ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 53(1) (2023), 146-156**
(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

CERTAIN SUMMATION FORMULAE AND RELATIONS DUE TO DOUBLE SERIES ASSOCIATED WITH THE GENERAL HYPERGEOMETRIC TYPE HURWITZ-LERCH ZETA FUNCTIONS

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(Received : February 12, 2023; In format: February 22, 2023; Accepted: February 26, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53117

Abstract

In this paper, we exhibit certain double series associated with general hypergeometric type Hurwitz-Lerch Zeta functions and then derive their summation formulae and relations due to their series and integral identities. We also obtain various known and unknown results in terms of Hurwitz-Lerch Zeta functions and their generating relations.
2020 Mathematical Sciences Classification: 11M35, 33C65.
Keywords and Phrases: Double series associated with general hypergeometric type Hurwitz - Lerch Zeta functions, summation formulae, series and integral identities, Hurwitz-Lerch Zeta generating relations.

1 Introduction and preliminaries

Recently, the authors [10] studied the generalized hypergeometric type Hurwitz-Lerch Zeta function defined by

$$ {}_pH_q\begin{pmatrix}(\alpha)_{1,p}; & \\ (\gamma)_{1,q}; & z,s,a\end{pmatrix} = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\gamma_i)_n n!}\frac{z^n}{(n+a)^s}, \tag{1.1}$$

where $p,q\in\mathbb{N}_0, \alpha_i\in\mathbb{C}, (i=1,2,3,\ldots,p); a,\gamma_i\in\mathbb{C}\backslash\mathbb{Z}_0^-, (i=1,2,3,\ldots,q); s,z\in\mathbb{C}$.

Here in (1.1) the notations denote
$\mathbb{C}=\{z:z=x+iy:x,y\in\mathbb{R}, i=\sqrt{(-1)}\}, \mathbb{Z}_0^-=\{0,-1,-2,\ldots\}$,
$\mathbb{R}=(-\infty,\infty)$, $\mathbb{R}^+=\mathbb{R}\backslash(-\infty,0]$ and $\mathbb{N}_0=\{0,1,2,3,\ldots\}$.

Again for $a\neq 0$, the Pochhammer symbol ([14, p.45] and [21, pp.21-22]) as generalized factorial function is given by

$$(a)_n=\begin{cases}a(a+1)(a+2)\ldots(a+n-1); n\geq 1,\\ 1; n=0,\end{cases}$$

and in general it is defined as

$$(a)_v=\frac{\Gamma(a+v)}{\Gamma(a)} \quad \forall v\in\mathbb{R}.$$

In (1.1) it is also claimed that due to [7,8,10], for fixed and large value of N and with the properties of Gaussian gamma function [21, p.20], we find that the function (1.1) is written as partial sum of hypergeometric type Hurwitz-Lerch Zeta series and the generalized Gaussian hypergeometric series ([14, p. 73] and [21, pp. 42-43]), as

$$ {}_pH_q\begin{pmatrix}(\alpha)_{1,p}; & \\ (\gamma)_{1,q}; & z,s,a\end{pmatrix} = \sum_{n=0}^{N-1}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\gamma_i)_n n!}\frac{z^n}{(n+a)^s} + \frac{\prod_{i=1}^{p}(\alpha_i)_N\Gamma(N+a)z^N}{\prod_{i=1}^{q}(\gamma_i)_N\Gamma(N+s+a)N!}\,{}_{p+2}F_{q+2}\begin{pmatrix}(\alpha+N)_{1,p}, N+a, 1; & \\ (\gamma+N)_{1,q}, N+1, N+s+a; & z\end{pmatrix} \tag{1.2}$$

Since in formula (1.2) for fixed and large N, the first series is finite and the second series is the generalized Gaussian hypergeometric function ${}_pF_q\,(.)$ which follows the convergent conditions given by [21, p.43]

(i) converges for $|z| < \infty$, if $p \leq q$;
(ii) converges for $|z| < 1$, if $p = q + 1$;
(iii) diverges for all $z, z \neq 0$, if $p > q + 1$;
(iv) converges absolutely for $|z| = 1$, if $p = q + 1$ alongwith

$$\Re(\omega) = \Re\left(\sum_{i=1}^{q}\gamma_i + s - \sum_{i=1}^{p}\alpha_i\right) > 0;$$

(v) converges conditionally for $|z| = 1, z \neq 1$, if $p = q + 1$ and $-1 < \Re(\omega) \leq 0$;
(vi) diverges for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) < -1$.

Therefore, the series in (1.1) also satisfies same convergence conditions as given above in (i) to (vi).

In the formula (1.1) taking $p = 2$, $q = 1$, $\alpha_1 = \alpha, \alpha_2 = \beta$, and $\gamma_1 = \gamma$, we convert it specially into the extended hypergeometric type Hurwitz-Lerch Zeta function, used in the probability distributions due to Garg et al. [5], in the form

$${}_2H_1\left(\begin{array}{c}\alpha, \beta;\\ \gamma;\end{array} z, s, a\right) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}\frac{z^n}{(n+a)^s} = \phi_{\alpha,\beta;\gamma}(z, s, a), \tag{1.3}$$

where, $\alpha, \beta, s, z \in \mathbb{C}$ and a, $\gamma \in \mathbb{C}\backslash\mathbb{Z}_0^-$, converges if $\Re(s) > 0$, when $|z| < 1, (z \neq 1)$. But when $z = 1$, for $\Re(\gamma) > \frac{1}{2}\Re(\alpha + \beta + 1) > 0$, the series in (1.3) converges if

$$\Re(s) > \frac{1}{2}\Re(\alpha + \beta) - \frac{1}{2}, (\text{ see [10]}). \tag{1.4}$$

It is remarked that on combining both the conditions of $\Re(\gamma)$ and the $\Re(s)$ given in (1.3) and (1.4), we get

$$\Re(\gamma + s - \alpha - \beta) > 0,$$

which is identical to $\Re(\omega)$ given in (iv) of (1.2) for $p = 2$ and $q = 1$.

Further in the generalized hypergeometric type Hurwitz-Lerch Zeta function (1.1), if we set $q = p - 1, \gamma_1 = \alpha_1, \gamma_2 = \alpha_2, \ldots, \gamma_{p-1} = \alpha_{p-1}, \alpha_p = 1$, it becomes Hurwitz-Lerch Zeta function as

$$\begin{aligned}{}_pH_{p-1}\left(\begin{array}{c}(\alpha)_{1,p-1}, 1;\\ (\alpha)_{1,p-1};\end{array} z, s, a\right) &= \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p-1}(\alpha_i)_n (1)_n}{\prod_{i=1}^{p-1}(\alpha_i)_n n!}\frac{z^n}{(n+a)^s}\\ &= \sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s} = \phi(z, s, a),\end{aligned} \tag{1.5}$$

which converges if $\Re(s) > 0$, when $|z| < 1, (z \neq 1)$, but when $z = 1$, the series (1.5) converges for $\Re(s) > 1$. We also verify it as setting $\gamma = \alpha, \beta = 1$ in remark of Eqn. (1.4) and $\Re(s) > 1$.

In extension of (1.1), we again define a general hypergeometric type Hurwitz -Lerch Zeta function in following form

$${}_pK_q\left(\begin{array}{c}(\alpha)_{1,p};\\ (\gamma)_{1,q};\end{array} A; z, s, a\right) = \sum_{n=0}^{\infty}\frac{A_n\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\gamma_i)_n}\frac{z^n}{(n+a)^s n!}, \tag{1.6}$$

where $p, q \in \mathbb{N}_0, \alpha_i \in \mathbb{C}, (i = 1, 2, 3, \ldots, p); a, \gamma_i \in \mathbb{C}\backslash\mathbb{Z}_0^-, (i = 1, 2, 3, \ldots, q)$; $s, z \in \mathbb{C}$. A symbolizes for a bounded real or complex A_n $\forall n \in \mathbb{N}_0$ and follows certain restrictions.

For a sequence $\langle A_n\rangle = \langle 1\rangle$ $\forall n \in \mathbb{N}_0$, by (1.1) and (1.6), we find an identity

$${}_pK_q\left(\begin{array}{c}(\alpha)_{1,p};\\ (\gamma)_{1,q};\end{array} 1; z, s, a\right) = {}_pH_q\left(\begin{array}{c}(\alpha)_{1,p};\\ (\gamma)_{1,q};\end{array} z, s, a\right).$$

Again, for a sequence $\langle A_n\rangle = \langle(1)_n\rangle$ $\forall n \in \mathbb{N}_0$, we have a relation with (1.1) and (1.6) as

$${}_pK_q\left(\begin{array}{c}(\alpha)_{1,p};\\ (\gamma)_{1,q};\end{array} (1); z, s, a\right) = {}_{p+1}H_q\left(\begin{array}{c}(\alpha)_{1,p}, 1;\\ (\gamma)_{1,q};\end{array} z, s, a\right).$$

It is recalled that Exton [3] obtained some theorems on general hypergeometric generating relations, Srivastava [17] established certain generating relations of Hurwitz-Lerch Zeta functions and recently, Kumar and Chandel [10] derived various relations and identities for double series associated with general Hurwitz-Lerch type Zeta functions. In this motivation, we exhibit these researches for exploring new ideas in the theory of extended generalized hypergeometric type Hurwitz-Lerch Zeta functions and thus consider $x, y, s \in \mathbb{C}; a, 2d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and A_n, a bounded real or complex sequence $\forall n \in \mathbb{N}_0$, which follows certain restrictions to introduce following families of general hypergeometric type Hurwitz-Lerch Zeta functions in the form

$$\phi_1(A, d, d-1/2; 2d; x, y; s, a) = \sum_{m,n=0}^{\infty} \frac{A_n (d)_{m+n} \left(d - \frac{1}{2}\right)_{m+n}}{(2d)_{m+2n}} \frac{x^{m+n} y^n}{(n+a)^s m! n!}, \tag{1.7}$$

$$\phi_2(A, d, d+1/2; 2d; x, y; s, a) = \sum_{m,n=0}^{\infty} \frac{A_n (d)_{m+n} \left(d + \frac{1}{2}\right)_{m+n}}{(2d)_{m+2n}} \frac{x^{m+n} y^n}{(n+a)^s m! n!}. \tag{1.8}$$

Here in left hand sides of (1.7) and (1.8), A stands for bounded real or complex sequence A_n $\forall n \in \mathbb{N}_0$ as in right hand side of their series.

Again, to obtain summation formulae, series and integral identities of the functions defined in (1.7) and (1.8) in terms of (1.6), we make an appeal to following preliminary formulae:

For $z \in \mathbb{C}, |z| \leq 1, 2d \neq 0, -1, -2, \ldots$, (see Erdélyi et al. [2, Vol. I, p. 101], Srivastava and Manocha [21, p. 34])

$$ {}_2F_1\left(\begin{matrix} d, d-\frac{1}{2}; \\ 2d; \end{matrix}\ z\right) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2d}, \tag{1.9}$$

but by (1.9), we immediately have

$$ {}_2F_1\left(\begin{matrix} d, d-\frac{1}{2}; \\ 2d; \end{matrix}\ 1\right) = 2^{2d-1}. \tag{1.10}$$

Also there exists another result

$$ {}_2F_1\left(\begin{matrix} d, d+\frac{1}{2}; \\ 2d; \end{matrix} z\right) = \frac{1}{\sqrt{1-z}}\left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2d}, \tag{1.11}$$

provided that, $z \in \mathbb{C}, |z| < 1, 2d \neq 0, -1, -2, \ldots$.

For all $0 \leq n \leq m$

$$\frac{1}{(m-n)!} = \frac{(-1)^n (-m)_n}{m!}, \tag{1.12}$$

and $\forall n \in \mathbb{N}_0$

$$(\lambda)_{2n} = 2^{2n} \left(\frac{\lambda}{2}\right)_n \left(\frac{\lambda}{2} + \frac{1}{2}\right)_n. \tag{1.13}$$

2 Eulerian Integral representations

In this section, we derive Eulerian integral representations of the general hypergeometric type Hurwitz-Lerch Zeta functions defined in the Eqns. (1.1), (1.6), (1.7) and (1.8) involving known and unknown hypergeometric functions.

Here $\forall n \in \mathbb{N}_0, a, s \in \mathbb{C}, \Re(s) > 0, \Re(a) > 0$, we apply the following Eulerian integral formula [4,11,12,13,18]

$$\frac{1}{\Gamma(s)} \int_0^\infty e^{-(a+n)t} t^{s-1} dt = \frac{1}{(n+a)^s},$$

in the Eqns. (1.1) and (1.6) and obtain their Eulerian integral representations

$$ {}_pH_q\left(\begin{matrix} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{matrix}\ z, s, a\right) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-at} t^{s-1} {}_pF_q\left(\begin{matrix} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{matrix}\ ze^{-t}\right) dt, \tag{2.1}$$

$$ {}_pK_q\left(\begin{matrix} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{matrix}\ A; z, s, a\right) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-at} t^{s-1} {}_pG_q\left(\begin{matrix} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{matrix}\ A; ze^{-t}\right) dt, \tag{2.2}$$

where, ${}_pG_q\left(\begin{matrix} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{matrix}\ A; z\right) = \sum_{n=0}^{\infty} \frac{A_n \prod_{i=1}^p (\alpha_i)_n}{\prod_{i=1}^q (\gamma_i)_n} \frac{z^n}{n!}$ is a general hypergeometric function. A stands for a bounded real or complex sequence A_n $\forall n \in \mathbb{N}_0$. For example if $\langle A_n \rangle = \langle 1 \rangle$, $\forall n \in \mathbb{N}_0$, then there exists a relation

$$ {}_pG_q\left(\begin{matrix} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{matrix}\ 1; z\right) = {}_pF_q\left(\begin{matrix} (\alpha)_{1,p}; \\ (\gamma)_{1,q}; \end{matrix}\ z\right).$$

It is remarked that on specialization of the parameters in (2.1) and (2.2), we may obtain various Hurwitz-Lerch Zeta functions associated with the hypergeometric functions and hypergeometric polynomials found in the literature (for example, Rainville [14], Slater [15], Sneddon [16], Srivastava and Karlsson [20], Srivastava and Manocha [21] and others).

In above motivation of (2.1) and (2.2) $\forall x, y \in \mathbb{C}; 2d \in \mathbb{C}\backslash\mathbb{Z}_0^-; a, s \in \mathbb{C}, \Re(s) > 0, \Re(a) > 0$, we introduce following integral representations of the functions (1.7) and (1.8) defined by

$$\phi_1(A, d, d-1/2; 2d; x, y; s, a) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-at} t^{s-1} \phi_1{}^* \left(A, d, d-1/2; 2d; x, xye^{-t}\right) dt, \tag{2.3}$$

$$\phi_2(A, d, d+1/2; 2d; x, y; s, a) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-at} t^{s-1} \phi_2{}^* \left(A, d, d+1/2; 2d; x, xye^{-t}\right) dt, \tag{2.4}$$

where the general double functions $\phi_1^*(.)$ and $\phi_2^*(.)$ are defined in the double series

$$\phi_1{}^*(A, d, d-1/2; 2d; x, y) = \sum_{m,n=0}^{\infty} \frac{A_n (d)_{m+n} \left(d-\frac{1}{2}\right)_{m+n}}{(2d)_{m+2n}} \frac{x^m y^n}{m! n!}, \tag{2.5}$$

$$\phi_2{}^*(A, d, d+1/2; 2d; x, y) = \sum_{m,n=0}^{\infty} \frac{A_n (d)_{m+n} \left(d+\frac{1}{2}\right)_{m+n}}{(2d)_{m+2n}} \frac{x^m y^n}{m! n!}. \tag{2.6}$$

Here, A denotes for bounded real or complex sequence A_n $\forall n \in \mathbb{N}_0$ and follows certain restrictions.

It is noticed that on specialization of the parameters of (2.5) and (2.6) and making an appeal to the formulae (2.3) and (2.4), we may obtain various Hurwitz-Lerch Zeta functions associated with the hypergeometric functions of two variables like Appell's functions, Kampé de Fériet functions, Humbert functions and others found in the literature (see for example, Bailey [1], Exton [4], Srivastava and Panda [19], Srivastava and Karlsson [20], Srivastava and Manocha [21] and so on).

For example, under the conditions $\sum_{j=1}^{Q} \vartheta_j - \sum_{j=1}^{P} \theta_j > 0$, if we set the sequence $A_n = \frac{\prod_{j=1}^{P} \Gamma(\alpha_j + \theta_j n)}{\prod_{j=1}^{Q} \Gamma(\beta_j + \vartheta_j n)}$, $\forall n \in \mathbb{N}_0, \alpha_j \in \mathbb{C}, \theta_j \in \mathbb{R}^+$, $\forall (j = 1, 2, 3, \ldots, P); \beta_j \in \mathbb{C}, \vartheta_j \in \mathbb{R}^+$, $\forall (j = 1, 2, 3, \ldots, Q)$, then for $2d, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\forall (j = 1, 2, 3, \ldots, Q)$, the functions (2.5) and (2.6) become double Srivastava-Daoust functions [18] in the following form

$$\phi_1{}^*(d, d-1/2, [(\alpha):\theta]; 2d, [(\beta):\vartheta]; x, y) = \frac{2^{2d-1}\Gamma\left(d+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(d-\frac{1}{2}\right)} S^{2:0;P}_{1:0;Q} \left[\begin{matrix} [d:1,1], \left[d-\frac{1}{2}:1,1\right] : -; [(\alpha):\theta]; \\ [2d:1,2] : -; [(\beta):\vartheta]; \end{matrix} \; x, y \right], \tag{2.7}$$

provided that $|x| < \infty, |y| < 1$,

and

$$\phi_2{}^*(d, d+1/2, [(\alpha):\theta]; 2d, [(\beta):\vartheta]; x, y) = \frac{2^{2d-1}}{\Gamma\left(\frac{1}{2}\right)} S^{2:0;P}_{1:0;Q} \left[\begin{matrix} [d:1,1], \left[d+\frac{1}{2}:1,1\right] : -; [(\alpha):\theta]; \\ [2d:1,2] : -; [(\beta):\vartheta]; \end{matrix} \; x, y \right], \tag{2.8}$$

provided that $|x| < \infty, |y| < 1$, respectively.

Thus making an appeal to (2.7) and (2.8) in the Eqns. (2.3) and (2.4) respectively, for $a, s \in \mathbb{C}, \Re(s) > 0, \Re(a) > 0; 2d, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\forall (j = 1, 2, 3, \ldots, Q)$, we generate integral representation of the Hurwitz-Lerch double Zeta functions associated with the Srivastava-Daoust double series, given by

$$\begin{aligned} \psi_1 \left(\begin{matrix} d, d-\frac{1}{2} : [(\alpha):\theta]; \\ 2d : [(\beta):\vartheta]; \end{matrix} \; x, y; s, a \right) &= \sum_{m,n=0}^{\infty} \frac{(d)_{m+n} (d-\frac{1}{2})_{m+n}}{(2d)_{m+2n}} \frac{\prod_{j=1}^{P} \Gamma(\alpha_j + \theta_j n)}{\prod_{j=1}^{Q} \Gamma(\beta_j + \vartheta_j n)} \frac{x^{m+n} y^n}{(m+2n+a)^s m! n!} \\ &= \frac{\Gamma(d+\frac{1}{2})}{\sqrt{\pi}\Gamma(d-\frac{1}{2})} \frac{2^{2d-1}}{\Gamma(s)} \int_0^\infty e^{-at} t^{s-1} S^{2:0;P}_{1:0;Q} \left[\begin{matrix} [d:1,1], [d-\frac{1}{2}:1,1] : -; [(\alpha):\theta]; \\ [2d:1,2] : -; [(\beta):\vartheta]; \end{matrix} \; xe^{-t}, xye^{-2t} \right] dt, \end{aligned}$$

provided that $|x| < \infty, |y| < 1$, (2.9)

and

$$\begin{aligned} \psi_2 \left(\begin{matrix} d, d+1/2 : [(\alpha):\theta]; \\ 2d : [(\beta):\vartheta]; \end{matrix} \; ; s, a \right) &= \sum_{m,n=0}^{\infty} \frac{(d)_{m+n} \left(d+\frac{1}{2}\right)_{m+n}}{(2d)_{m+2n}} \frac{\prod_{j=1}^{P} \Gamma(\alpha_j + \theta_j n)}{\prod_{j=1}^{Q} \Gamma(\beta_j + \vartheta_j n)} \frac{x^{m+n} y^n}{(m+2n+a)^s m! n!} \\ &= \frac{2^{2d-1}}{\sqrt{\pi}\Gamma(s)} \int_0^\infty e^{-at} t^{s-1} S^{2:0;P}_{1:0;Q} \left[\begin{matrix} [d:1,1], [d+\frac{1}{2}:1,1] : -; [(\alpha):\theta]; \\ [2d:1,2] : -; [(\beta):\vartheta]; \end{matrix} \; xe^{-t}, xye^{-2t} \right], \end{aligned} \tag{2.10}$$

provided that $|x| < \infty, |y| < 1$, respectively.

Recently, some general Hurwitz-Lerch type Zeta functions associated with the double and multiple Srivastava-Daoust hypergeometric functions are analyzed in [18] which are applied in different scientific problems for example see [6,9]. Therefore importance in further researches, we study analytic continuation properties of the double functions (2.9) and (2.10) through their integral representations.

3 Summation Formulae

In this section, we obtain summation formulae of the general hypergeometric type Hurwitz-Lerch Zeta functions of one and two variables defined by Eqns. (1.1), (1.6), (1.7) and (1.8). Again we show that the functions (1.7) and (1.8) are represented as the sum of functions (1.6).

Lemma 3.1. *If $p, q \in \mathbb{N}_0$, $\alpha_i \in \mathbb{C}, (i = 1, 2, 3, \ldots, p)$; $a, \gamma_i \in \mathbb{C}\backslash\mathbb{Z}_0^-, (i = 1, 2, 3, \ldots, q)$; $s, z \in \mathbb{C}$. Then under the conditions given in* (1.2)*, the summation formula of* (1.1) *exists as*

$$ {}_pH_q\left(\begin{array}{l}(\alpha)_{1,p};\\ (\gamma)_{1,q};\end{array} z, s, a\right) = \exp[-\operatorname{s}\log a] + \frac{\prod_{i=1}^{p}(\alpha_i)}{\prod_{i=1}^{q}(\gamma_i)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r\,{}_pH_q\left(\begin{array}{l}(\alpha+1)_{1,p};\\ (\gamma+1)_{1,q};\end{array} z, s+r+1, 1\right). \quad (3.1)$$

Proof. Considering the formula (1.1) and for $a \neq 0$, using the binomial theorem, we write it as

$$ {}_pH_q\left(\begin{array}{l}(\alpha)_{1,p};\\ (\gamma)_{1,q};\end{array} z, s, a\right) = \frac{1}{a^s} + \sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\gamma_i)_n\, n!}\frac{z^n}{n^{s+r}}. \quad (3.2)$$

The Eqn. (3.2) on aid of (1.1) immediately gives the result (3.1).

Clearly, making an appeal to the formula (3.1), we get following summation formulae in terms of the hyperbolic functions:

$$\frac{\prod_{i=1}^{p}(\alpha_i)}{2\prod_{i=1}^{q}(\gamma_i)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r\,{}_pH_q\left(\begin{array}{l}(\alpha+1)_{1,p};\\ (\gamma+1)_{1,q};\end{array} z, s+r+1, 1\right) - \frac{1}{2}{}_pH_q\left(\begin{array}{l}(\alpha)_{1,p};\\ (\gamma)_{1,q};\end{array} z, s, a\right) + \frac{1}{2}{}_pH_q\left(\begin{array}{l}(\alpha)_{1,p};\\ (\gamma)_{1,q};\end{array} z, s, a^{-1}\right)$$
$$- \frac{\prod_{i=1}^{p}(\alpha_i)}{2\prod_{i=1}^{q}(\gamma_i)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a^{-1})^r\,{}_pH_q\left(\begin{array}{l}(\alpha+1)_{1,p};\\ (\gamma+1)_{1,q};\end{array} z, s+r+1, 1\right)$$
$$= \sinh[s\log a] \quad (3.3)$$

and

$$\frac{1}{2}{}_pH_q\left(\begin{array}{l}(\alpha)_{1,p};\\ (\gamma)_{1,q};\end{array} z, s, a\right) - \frac{\prod_{i=1}^{p}(\alpha_i)}{2\prod_{i=1}^{q}(\gamma_i)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r\,{}_pH_q\left(\begin{array}{l}(\alpha+1)_{1,p};\\ (\gamma+1)_{1,q};\end{array} z, s+r+1, 1\right) -$$
$$+ \frac{1}{2}{}_pH_q\left(\begin{array}{l}(\alpha)_{1,p};\\ (\gamma)_{1,q};\end{array} z, s, a^{-1}\right) - \frac{\prod_{i=1}^{p}(\alpha_i)}{2\prod_{i=1}^{q}(\gamma_i)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a^{-1})^r\,{}_pH_q\left(\begin{array}{l}(\alpha+1)_{1,p};\\ (\gamma+1)_{1,q};\end{array} z, s+r+1, 1\right)$$
$$= \cosh[s\log a]. \quad (3.4)$$

Similarly by the formula (1.6), we get

$$ {}_pK_q\left(\begin{array}{l}(\alpha)_{1,p};\\ (\gamma)_{1,q};\end{array} A; z, s, a\right) = A_0\exp[-s\log a] + \frac{\prod_{i=1}^{p}(\alpha_i)}{\prod_{i=1}^{q}(\gamma_i)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r\,{}_pK_q\left(\begin{array}{l}(\alpha+1)_{1,p};\\ (\gamma+1)_{1,q};\end{array} A^+; z, s+r+1, 1\right), \quad (3.5)$$

where $p, q \in \mathbb{N}_0, \alpha_i \in \mathbb{C}, (i = 1, 2, 3, \ldots, p); a, \gamma_i \in \mathbb{C}\backslash\mathbb{Z}_0^-, (i = 1, 2, 3, \ldots, q); s, z \in \mathbb{C}, A$ stands for a sequence A_n, a bounded real or complex sequences $\forall n \in \mathbb{N}_0$. Also A^+stands for a sequence A_{n+1}, $\forall n \in \mathbb{N}_0$. □

Theorem 3.1. *If all $x, y, s \in \mathbb{C}; a, 2d \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and A_n be bounded real or complex sequences $\forall n \in \mathbb{N}_0$, then under the conditions $|x| \leq 1$, the double series* (1.7) *follows a summation formula*

$$\phi_1(A, d, d-1/2; 2d; x, y; s, a)$$
$$= \frac{A_0}{a^s}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d} + \frac{xy(2d-1)}{4(2d+1)}\left(\frac{1+\sqrt{1-x}}{2}\right)^{-2d-1}$$
$$\times\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r\,{}_1K_1\left(\begin{array}{l}d+\frac{1}{2};\\ d+\frac{3}{2};\end{array} A^+; \frac{xy}{(1+\sqrt{1-x})^2}, s+r+1, 1\right), \quad (3.6)$$

where, ${}_1K_1$ (.) is a general hypergeometric type Hurwitz -Lerch Zeta function (1.6) *and A^+stands for the sequence A_{n+1}, a bounded real or complex sequences $\forall n \in \mathbb{N}_0$ that follows certain restrictions.*

Proof. We consider the double series (1.7) in the form

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{A_n(d)_{m+n}\left(d-\frac{1}{2}\right)_{m+n}}{(2d)_{m+2n}}\frac{x^m(xy)^n}{(n+a)^s m!n!},$$

and apply series rearrangement techniques to derive hypergeometric function

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\sum_{n=0}^{\infty}\frac{A_n(d)_n\left(d-\frac{1}{2}\right)_n}{(2d)_{2n}}\frac{(xy)^n}{(n+a)^s n!}{}_2F_1\left(\begin{array}{c}d+n,d+n-\frac{1}{2};\\ 2d+2n;\end{array}x\right). \tag{3.7}$$

Now in (3.7) under the conditions $|x|\leq 1$, using the formulae (1.9) and (1.13), we get

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}\sum_{n=0}^{\infty}\frac{A_n\left(d-\frac{1}{2}\right)_n}{\left(d+\frac{1}{2}\right)_n}\frac{\left(\frac{xy}{(1+\sqrt{1-x})^2}\right)^n}{(n+a)^s n!}. \tag{3.8}$$

In (3.8) applying the formula (1.6), we obtain the result

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}{}_1K_1\left(\begin{array}{c}d-\frac{1}{2};\\ d+\frac{1}{2};\end{array}A;\frac{xy}{(1+\sqrt{1-x})^2},s,a\right),$$

in which making an appeal to the techniques of Lemma 3.1, we obtain the summation formula (3.6). Hence the Theorem 3.1 is proved. □

Corollary 3.1. *If in the Theorem 3.1 put $x=1$ and all $y,s\in\mathbb{C};a,2d\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and A_n be bounded real or complex sequences $\forall n\in\mathbb{N}_0$, then following summation formula exists*

$$2^{1-2d}\phi_1(A,d,d-1/2;2d;1,y;s,a)=\frac{A_0}{a^s}+\frac{y(2d-1)}{(2d+1)}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r{}_1K_1\left(\begin{array}{c}d+\frac{1}{2};\\ d+\frac{3}{2};\end{array}A^+;y,s+r+1,1\right). \tag{3.9}$$

Also there exists an identity

$$2^{1-2d}\phi_1(A,d,d-1/2;2d;1,y;s,a)={}_1K_1\left(\begin{array}{c}d-\frac{1}{2};\\ d+\frac{1}{2};\end{array}A;y,s,a\right). \tag{3.10}$$

Proof. Considering the Eqn. (3.6) and putting $x=1$, we obtain the summation formula (3.9). Further making same process with an appeal to the Eqns. (1.6), (1.10) and (3.8), we find an identity (3.10). □

Corollary 3.2. *If $\Re(s)>0,\Re(a)>0$, then under the conditions of the Theorem 3.1 an Eulerian integral representation of the double series* (1.7) *exists in the following form*

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\frac{1}{\Gamma(s)}\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}\int_0^{\infty}e^{-at}t^{s-1}{}_1G_1\left(\begin{array}{c}d-\frac{1}{2};\\ d+\frac{1}{2};\end{array}A;\frac{xye^{-t}}{(1+\sqrt{1-x})^2}\right)dt \tag{3.11}$$

Proof. Consider the formula (3.8) and then here under the conditions of the Theorem 3.1, use an Eulerian integral formula and thus apply the techniques of Section 2, we get the formula

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}\sum_{n=0}^{\infty}\frac{A_n\left(d-\frac{1}{2}\right)_n}{\left(d+\frac{1}{2}\right)_n n!}\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-at}t^{s-1}\left(\frac{xye^{-t}}{(1+\sqrt{1-x})^2}\right)^n dt. \tag{3.12}$$

Now in right hand side of (3.12) using the function (2.2), we obtain the result (3.11). □

Theorem 3.2. *If all $x,y,s\in\mathbb{C};a,2d\in\mathbb{C}\backslash\mathbb{Z}_0^-$ such that If $\Re(s)>0,\Re(a)>0$ and A_n be bounded real or complex sequences $\forall n\in\mathbb{N}_0$, then by the function* (1.7) *under the conditions $|x|\leq 1$, following summation formula of* (1.7) *also exists*

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\sum_{n=0}^{\infty}\frac{(d)_n\left(d-\frac{1}{2}\right)_n}{(2d)_n n!}{}_1K_1\left(\begin{array}{c}-n;\\ 2d+n;\end{array}A;-y,s,a\right)x^n, \tag{3.13}$$

where, the partial sum of the extended general hypergeometric type Hurwitz -Lerch Zeta function (1.6) *is defined by*

$${}_1K_1\left(\begin{array}{c}-n;\\ 2d+n;\end{array}A;-y,s,a\right)=\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-at}t^{s-1}{}_1G_1\left(\begin{array}{c}-n;\\ 2d+n;\end{array}A;-ye^{-t}\right)dt,\Re(s)>0,\Re(a)>0,\ \forall n=0,1,2,3,\ldots, \tag{3.14}$$

${}_1G_1(\cdot)$ is defined in (2.2).

Proof. Considering the function (1.7) and applying the series rearrangement techniques, we find that

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{A_n(d)_m\left(d-\frac{1}{2}\right)_m}{(2d)_{m+n}}\frac{x^m y^n}{(n+a)^s(m-n)!n!}.$$

Now using the formula (1.12) and making an appeal to the Eulerian integral formula given in the Section 2 we find that

$$\phi_1(A,d,d-1/2;2d;x,y;s,a)=\sum_{n=0}^{\infty}\frac{(d)_n\left(d-\frac{1}{2}\right)_n x^n}{(2d)_n n!}\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-at}t^{s-1}\left\{\sum_{m=0}^{n}\frac{A_m(-n)_m}{(2d+n)_m}\frac{(-ye^{-t})^m}{m!}\right\}dt. \tag{3.15}$$

In right hand side of the Eqn. (3.15) making an appeal to the formula (2.2), we derive (3.14) and from which, we finally obtain the result (3.13).

In the similar manner, we obtain following results: □

Theorem 3.3. *If all $x,y,s\in\mathbb{C};a,2d\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and A_n be bounded real or complex sequences $\forall n\in\mathbb{N}_0$, then due to the function* (1.8) *under the conditions $|x|<1$, following summation formula exists*

$$\begin{aligned}\phi_2(A,d,d+1/2;2d;x,y;s,a)&=\frac{A_0}{(a)^s}\frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d}\\&+\frac{xy}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{-1-2d}\sum_{r=0}^{\infty}\frac{(s)_r}{r!}(-a)^r{}_0K_0\left(\begin{matrix}-;\\-;\end{matrix}A^+;\frac{xy}{(1+\sqrt{1-x})^2},s+r+1,1\right)\end{aligned} \tag{3.16}$$

Here, A^+ stands for A_{n+1}, a bounded real or complex sequences $\forall n\in\mathbb{N}_0$, follows certain restrictions.

Proof. Under the conditions given in the Theorem 3.3, for the double series (1.8), we write

$$\phi_2(A,d,d+1/2;2d;x,y;s,a)=\sum_{n=0}^{\infty}\frac{A_n(d)_n\left(d+\frac{1}{2}\right)_n(xy)^n}{(2d)_{2n}n!(n+a)^s}{}_2F_1\left[\begin{matrix}d+n,d+n+\frac{1}{2};\\2d+2n;\end{matrix}\;x\right]. \tag{3.17}$$

Now in the Eqn. (3.17) using of the formulae (1.11)-(1.13) for $|x|<1$, we obtain

$$\phi_2(A,d,d+1/2;2d;x,y;s,a)=\frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d}\sum_{n=0}^{\infty}\frac{A_n\left(\frac{xy}{(1+\sqrt{1-x})^2}\right)^n}{n!(n+a)^s}. \tag{3.18}$$

Now in Eqn. (3.18) making an appeal to formula (1.6) and the theory given in Theorem 3.1, we derive the result (3.16). □

Corollary 3.3. *If $\Re(s)>0,\Re(a)>0$, then due to the function* (1.8) *following formula holds*

$$\phi_2(A,d,d+1/2;2d;x,y;s,a)=\frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d}\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-at}t^{s-1}{}_0K_0\left(\begin{matrix}-;\\-;\end{matrix}A;\frac{xye^{-t}}{(1+\sqrt{1-x})^2}\right)dt. \tag{3.19}$$

Proof. In the Eqn. (3.18) of the Theorem 3.3 applying the Eulerian formula given in (2.1), we derive

$$\phi_2(A,d,d+1/2;2d;x,y;s,a)=\frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d}\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-at}t^{s-1}\left\{\sum_{n=0}^{\infty}\frac{A_n}{n!}\left(\frac{xye^{-t}}{(1+\sqrt{1-x})^2}\right)^n\right\}dt. \tag{3.20}$$

Now in (3.20), applying the formula (1.6) and same technique of proof of the Theorem 3.1, we obtain the formula (3.19). □

Theorem 3.4. *If all $x,y,s\in\mathbb{C};a,2d\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and A_n be bounded real or complex sequences $\forall n\in\mathbb{N}_0$, then due to the function* (1.8) *under the conditions $|x|<1$, following summation formula exists*

$$\varphi_2(A,d,d+\frac{1}{2};2d;x,y;s,a)=\sum_{n=0}^{\infty}\frac{(d)_n(d+\frac{1}{2})_n}{(2d)_n n!}{}_1K_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}A;-y,s,a\right)x^n, \tag{3.21}$$

where, the function ${}_1K_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}A;-y,s,a;\right)$ is defined by (3.14).

Proof. Making an appeal to the function (1.8), we get

$$\phi_2(A,d,d+1/2;2d;x,y;s,a) = \sum_{n=0}^{\infty} \frac{(d)_n \left(d+\frac{1}{2}\right)_n x^n}{(2d)_n n!} \sum_{m=0}^{n} \frac{A_m(-n)_m}{(2d+n)_m} \frac{(-y)^m}{(m+a)^{s s} m}$$

$$= \sum_{n=0}^{\infty} \frac{(d)_n \left(d+\frac{1}{2}\right)_n x^n}{(2d)_n n!} \frac{1}{\Gamma(\mathrm{s})} \int_0^{\infty} e^{-at} t^{s-1} \left\{ \sum_{m=0}^{n} \frac{A_m(-n)_m}{(2d+n)_m} \frac{(-ye^{-t})^m}{m!} \right\} dt. \quad (3.22)$$

Now in the second series of (3.22) making an appeal to the function (3.14), we obtain the summation formula (3.21).

We present following applications of our results derived in the Sections 2 and 3 : □

4 Applications

In this section, we make an application of the Theorems presented in the previous Sections 2 and 3 . Then we obtain generating relations and the integral identities.

Application 4.1. If all conditions of the Theorems 3.1 and 3.2 are satisfied, then there exists a generating relation of the extended general hypergeometric type Hurwitz -Lerch Zeta function

$$\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1} {}_1K_1\left(\begin{matrix} d-\frac{1}{2}; \\ d+\frac{1}{2}; \end{matrix} \; A; \frac{xy}{(1+\sqrt{1-x})^2}, s, a\right) = \sum_{n=0}^{\infty} \frac{(d)_n \left(d-\frac{1}{2}\right)_n x^n}{(2d)_n n!} {}_1K_1\left(\begin{matrix} -n; \\ 2d+n; \end{matrix} \; A; -y, s, a\right). \quad (4.1)$$

Solution. Considering the Eqn. (3.8) of the Theorem 3.1 and applying (3.15) of the Theorem 3.2, we derive the equality given by

$$\phi_1(A,d,d-1/2;2d;x,y;s,a) = \left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1} \sum_{n=0}^{\infty} \frac{A_n \left(d-\frac{1}{2}\right)_n}{\left(d+\frac{1}{2}\right)_n} \frac{\left(\frac{xy}{(1+\sqrt{1-x})^2}\right)^n}{(n+a)^s n!}$$

$$= \sum_{n=0}^{\infty} \frac{(d)_n \left(d-\frac{1}{2}\right)_n x^n}{(2d)_n n!} \sum_{m=0}^{n} \frac{A_m(-n)_m}{(2d+n)_m} \frac{(-y)^m}{(m+a)^s m!}. \quad (4.2)$$

Now in the relation (4.2) making an appeal to the extended general hypergeometric type Hurwitz -Lerch Zeta function (1.6) in the last two equalities, we obtain the generating relation (4.1).

Application 4.2. If all conditions of the Theorems 3.3 and 3.4 are satisfied, then there exists a generating relation of the extended general hypergeometric type Hurwitz -Lerch Zeta function as

$$\frac{1}{\sqrt{1-x}} \left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1} {}_0K_0\left(\begin{matrix} -; \\ -; \end{matrix} \; A; \frac{xy}{(1+\sqrt{1-x})^2} z, s, a\right) = \sum_{n=0}^{\infty} \frac{(d)_n \left(d+\frac{1}{2}\right)_n}{(2d)_n n!} {}_1K_1\left(\begin{matrix} -n; \\ 2d+n; \end{matrix} \; A; -y, s, a\right) x^n. \quad (4.3)$$

Proof. Making an appeal to the Theorems 3.3 and 3.4, we get the equalities

$$\phi_2(A,d,d+1/2;2d;x,y;s,a) = \frac{1}{\sqrt{1-x}} \left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d} \sum_{n=0}^{\infty} \frac{A_n \left(\frac{xy}{(1+\sqrt{1-x})^2}\right)^n}{n!(n+a)^s}$$

$$= \sum_{n=0}^{\infty} \frac{(d)_n \left(d+\frac{1}{2}\right)_n x^n}{(2d)_n n!} \sum_{m=0}^{n} \frac{A_m(-n)_m}{(2d+n)_m} \frac{(-y)^m}{(m+a)^s m!}. \quad (4.4)$$

Then in the relation (4.4) making an appeal to the extended general hypergeometric type Hurwitz-Lerch Zeta function (1.6) in the last two equalities, we obtain the generating relation (4.3).

Application 4.3. If all conditions of the Theorems 3.1 and 3.2 are satisfied, then there exists an Eulerian integral identity for the extended general hypergeometric type Hurwitz -Lerch Zeta function (1.7), given by

$$\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1} \int_0^{\infty} e^{-at} t^{s-1} {}_1G_1\left(\begin{matrix} d-\frac{1}{2}; \\ d+\frac{1}{2}; \end{matrix} \quad A; \frac{xy}{(1+\sqrt{1-x})^2} e^{-t}\right) dt$$

$$= \sum_{n=0}^{\infty} \frac{(d)_n (d-\frac{1}{2})_n x^n}{(2d)_n n!} \int_0^{\infty} e^{-at} t^{s-1} {}_1G_1\left(\begin{matrix} -n; \\ 2d+n; \end{matrix} \quad A; -ye^{-t}\right) dt. \quad (4.5)$$

Solution. Making an appeal to the methods given in the Eqn. (2.1) and the formula (2.2) in the relation (4.1), we derive the Eulerian integral identity (4.5).

Application 4.4. If all conditions of the Theorems 3.3 and 3.4 are satisfied, then there exists an Eulerian integral identity for the extended general hypergeometric type Hurwitz -Lerch Zeta function (1.8), as

$$\frac{1}{\sqrt{1-x}}\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}\int_0^\infty e^{-at}t^{s-1}{}_0G_0\left(\begin{matrix}-;\\-;\end{matrix}\quad A;\frac{xy}{(1+\sqrt{1-x})^2}e^{-t}\right)dt$$
$$=\sum_{n=0}^{\infty}\frac{(d)_n(d+\frac{1}{2})_n x^n}{(2d)_n n!}\int_0^\infty e^{-at}t^{s-1}{}_1G_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}\quad A;-ye^{-t}\right)dt \tag{4.6}$$

Solution. Making an appeal to the same techniques given in the Eqn. (2.1) and the formula (2.2) in the relation (4.3), we derive the Eulerian integral identity (4.6).

Application 4.5. If all conditions of the Theorems 3.1 and 3.2 are satisfied, then for the extended general hypergeometric type Hurwitz -Lerch Zeta function (1.7), there exists a hypergeometric generating relation

$$\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}{}_1G_1\left(\begin{matrix}d-\frac{1}{2};\\d+\frac{1}{2};\end{matrix}\ A;\frac{xy}{(1+\sqrt{1-x})^2}e^{-t}\right)=\sum_{n=0}^{\infty}\frac{(d)_n\left(d-\frac{1}{2}\right)_n x^n}{(2d)_n n!}{}_1G_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}\ A;-ye^{-t}\right). \tag{4.7}$$

Solution. By the Eulerian integral identity (4.5), we find that

$$\int_0^\infty e^{-at}t^{s-1}R^{(1)}_{d,A}(x,y;t)dt=0, \tag{4.8}$$

where,

$$R_{d,A}{}^{(1)}(x,y;t)=\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}{}_1G_1\left(\begin{matrix}d-\frac{1}{2};\\d+\frac{1}{2};\end{matrix}\ A;\frac{xy}{(1+\sqrt{1-x})^2}e^{-t}\right)$$
$$-\sum_{n=0}^{\infty}\frac{(d)_n\left(d-\frac{1}{2}\right)_n x^n}{(2d)_n n!}{}_1G_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}\ A;-ye^{-t}\right).$$

Then, equating both sides of Eqn. (4.8), we obtain the result (4.7).

Application 4.6. If all conditions of the Theorems 3.3 and 3.4 are satisfied, then for the extended general hypergeometric type Hurwitz -Lerch Zeta function (1.8), there exists the hypergeometric generating relation

$$\frac{1}{\sqrt{1-x}}\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}{}_0G_0\left(\begin{matrix}-;\\-;\end{matrix}\ A;\frac{xy}{(1+\sqrt{1-x})^2}e^{-t}\right)=\sum_{n=0}^{\infty}\frac{(d)_n\left(d+\frac{1}{2}\right)_n x^n}{(2d)_n n!}{}_1G_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}\ A;-ye^{-t}\right). \tag{4.9}$$

Solution. By the Eulerian integral identity (4.6), we find that

$$\int_0^\infty e^{-at}t^{s-1}R_{d,A}{}^{(2)}(x,y;t)dt=0, \tag{4.10}$$

where,

$$R^{(2)}_{d,A}(x,y;t)=\frac{1}{\sqrt{1-x}}\left(\frac{2}{(1+\sqrt{1-x})}\right)^{2d-1}{}_0G_0\left(\begin{matrix}-;\\-;\end{matrix}\quad A;\frac{xy}{(1+\sqrt{1-x})^2}e^{-t}\right)$$
$$-\sum_{n=0}^{\infty}\frac{(d)_n\left(d+\frac{1}{2}\right)_n x^n}{(2d)_n n!}{}_1G_1\left(\begin{matrix}-n;\\2d+n;\end{matrix}\ A;-ye^{-t}\right).$$

Finally, equating both sides of Eqn. (4.10), we obtain the result (4.9).

In concluding remarks, we derive interesting summation formulae from our above obtained results.

5 Interesting Results as Special Cases

Particularly, in Eqn. (4.2) set $A_n=(2)_n\ \forall n\in\mathbb{N}_0, d=\frac{3}{2}$, there exists an interesting summation formula in terms of Hurwitz-Lerch Zeta function

$$\phi_1\left(2,\frac{3}{2},1;3;x,y;s,a\right)=\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n x^n}{(3)_n}\sum_{m=0}^{n}\frac{(2)_n(-n)_m}{(3+n)_m}\frac{(-y)^m}{(m+a)^s m!}$$
$$=\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n x^n}{(3)_n}{}_2H_1\left(\begin{matrix}2,-n;\\3+n;\end{matrix}\ -y,s,a\right)=\left(\frac{4}{(1+\sqrt{1-x})^2}\right)\phi\left(\frac{xy}{(1+\sqrt{1-x})^2},s,a\right), \tag{5.1}$$

where, all $x, y, s \in \mathbb{C}, |x| \leq 1; a \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

In the result (5.1) for $x = 1$, we obtain the following identical formulae for Hurwitz-Lerch Zeta function

$$\frac{1}{4}\phi_1\left(2, \frac{3}{2}, 1; 3; 1, y; s, a\right) = \frac{1}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n}{(3)_n}\sum_{m=0}^{n}\frac{(2)_n(-n)_m}{(3+n)_m}\frac{(-y)^m}{(m+a)^s m!}$$
$$= \frac{1}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n}{(3)_n}{}_2H_1\left(\begin{matrix}2, -n; \\ 3+n;\end{matrix}\; -y, s, a\right) = \phi(y, s, a), \quad (5.2)$$

where, all $y, s \in \mathbb{C}; a \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Again, in Eqn. (4.4) choosing $A_n = (1)_n \ \forall n \in \mathbb{N}_0, 2d \neq 0, -1, -2, \ldots$, there exists another interesting summation formula in terms of Hurwitz-Lerch Zeta function

$$\phi_2(1, d, d+1/2; 2d; x, y; s, a) = \sum_{n=0}^{\infty}\frac{(d)_n\left(d+\frac{1}{2}\right)_n x^n}{(2d)_n n!}\sum_{m=0}^{n}\frac{(-n)_m}{(2d+n)_m}\frac{(-y)^m}{(m+a)^s}$$
$$= \sum_{n=0}^{\infty}\frac{(d)_n(d+\frac{1}{2})_n x^n}{(2d)_n n!}{}_1H_1\left(\begin{matrix}-n; \\ 2d+n;\end{matrix}\; -y, s, a\right)$$
$$= \frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2d}\varphi\left(\frac{xy}{(1+\sqrt{1-x})^2}, s, a\right). \quad (5.3)$$

where, $x, y, s \in \mathbb{C}, |x| < 1;\ a, 2d \in Z_0^-$.

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