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(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

SOLUTION TO EQUAL SUM OF FIFTH POWER DIOPHANTINE EQUATIONS – A NEW APPROACH

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Abstract

Purpose of writing this paper is to introduce simple parametric solutions to quintic Diophantine equations 5.n.m where integer $n > 2$ and integer $m > 3$. Methodology applied is writing numbers in algebraic form as $a_i x + b_i$ with variable x, then writing fifth power Diophantine equation, in algebraic form with one variable and then transforming it to a linear equation by vanishing its four terms. For achieving this purpose, values to a_i and b_i of algebraic numbers are assigned so as to vanish constant term and coefficient of fifth power of x. Then equating with zero the coefficient of second power and coefficient of x, vanishes other two terms. These operations yield two relations between various a_i and b_i and also a linear equation in x. On putting the value of x obtained from this linear equation in given Diophantine equation, provides solution. Paper provides a single direct parametric solution to all quintic Diophantine equations $5.n.n$ where $\infty > n > 5$ and is simple, easily comprehensible and didactic.

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Keywords and Phrases: Integers, Rational quantity, Linear equation, Diophantine equation of fifth power.

1 Introduction

In this paper, integer solutions to the Diophantine equations 5.4.4, 5.3.4, 5.3.5, 5.5.5, 5.4.5, 5.6.6, 5.5.7, 5.5.6, $5.n.n$ (where n is integer such that $\infty > n > 5$) as detailed below

$$A^5 + B^5 + C^5 + D^5 = E^5 + F^5 + G^5 + H^5, \tag{1.1}$$

$$A^5 + B^5 + C^5 + D^5 = F^5 + G^5 + H^5, \tag{1.2}$$

$$A^5 + B^5 + C^5 + D^5 + E^5 = F^5 + G^5 + H^5, \tag{1.3}$$

$$A^5 + B^5 + C^5 + D^5 + E^5 = F^5 + G^5 + H^5 + I^5 + J^5, \tag{1.4}$$

$$A^5 + B^5 + C^5 + D^5 + E^5 = F^5 + G^5 + H^5 + I^5, \tag{1.5}$$

$$A^5 + B^5 + C^5 + D^5 + E^5 + F^5 = G^5 + H^5 + I^5 + J^5 + K^5 + L^5, \tag{1.6}$$

$$A^5 + B^5 + C^5 + D^5 + E^5 + F^5 + G^5 = H^5 + I^5 + J^5 + K^5 + L^5, \tag{1.7}$$

$$A^5 + B^5 + C^5 + D^5 + E^5 + F^5 = G^5 + H^5 + I^5 + J^5 + K^5, \tag{1.8}$$

and

$$A_1^5 + A_2^5 + A_3^5 + \ldots + A_{n-2}^5 + A_{n-1}^5 + A_n^5 = B_1^5 + B_2^5 + B_3^5 + \ldots + B_{n-2}^5 + B_{n-1}^5 + B_n^5, \tag{1.9}$$

have been determined using numbers in algebraic form. Alphabets $A, B, C, \ldots, L$ used in Equations (1.1) to (1.8) and alphabets $A_1, A_2, A_3, \ldots, A_n$, $B_1, B_2, B_3, \ldots, B_n$ used in equation (1.9) denote integers. W. Eric Weisstein, [8] gave history of solutions to some of these Diophantine equation of fifth power. Swinnerton-Dyer [7] also solved Diophantine equations 5.3.3 using a method of transformation of the equation but the method being presented in the paper is easy, simple, comprehensible and didactic also. In addition to individual parametric solution specific to the equation, we have provided in the paper, there is a single direct parametric solution to all quintic Diophantine equations $5.n.n$, where $\infty > n > 5$. Xeroudakes and Moessner [9], and Gloden [5] determined parametric solutions to solve equation 5.3.4. using two parameters. Rao [6] gave the smallest solution to this equation 5.3.4. Again Xeroudakes and Moessner [9] found several parametric solutions to the equation 5.4.4. Xeroudakes and Moessner [9] and Gloden [5] found the solution to the equation 5.5.6. Chen Shuwen [4] has found the solution to equation 5.6.6. Chaudhry [2, 3] while presenting methods of solution to fifth power Diophantine equation, gave method for representation of every rational number by an algebraic sum of fifth powers of rational numbers. Notwithstanding above works, a geometric approach to solve Diophantine equations of fifth power was also adopted by Bremner [1].

Nomenclature $5.n.m$ of Diophantine equation indicates, it is a fifth degree equation with larger number of terms m and smaller number of terms n. It is already stated in the Abstract, an algebraic equation of power five with variable x has been obtained by assigning algebraic form $(a_i \cdot x + b_i)$ to the integers of Diophantine equation. This equation is then, transformed to a linear equation. Although method of transformation of algebraic equation of fifth power, has been used earlier, the method being presented in the paper is easy, simple, comprehensible and didactic also. We have also provided parametric solutions, followed by exhaustive examples to illustrate and corroborate the results derived. To start with, we express a number, say n as $a \cdot x + b$ where a and b are real rational quantities as assigned by us and x is a real rational variable quantity. For example, a number, say 7, can be written as $3 \cdot x + 1$ assigning values 3 for a and 1 for b where $x = 2$. Similarly, 7 can be written $(3/4)x + 11/2$ where $a = 3/4, b = 11/2$ and $x = 2$. From above, it can be stated, $7 = 3 \cdot 2 + 1 = (3/4) \cdot 2 + 1$. In general,

$$n = a \cdot x + b. \tag{1.10}$$

This proves Lemma 1.1.

Lemma 1.1. *A number n is always expressible in algebraic form as $a \cdot x + b$ where a and b are fixed rational quantities neither zero nor infinity and x is a variable.*

Using Lemma 1.1, equation (1.1) can be written as

$$\begin{aligned}(ax+p)^5 &+ (bx+q)^5 + (cx+r)^5 + (dx+s)^5 \\ &= (ex+t)^5 + (fx+u)^5 + (gx+v)^5 + (hx+w)^5,\end{aligned} \tag{1.11}$$

where $a, b, c, \ldots, h,\ p, q, r, \ldots, v$ are arbitrary rational numbers and x is a real variable. On expanding and rearranging equation (1.11),

$$\begin{aligned}x^5&(a^5+b^5+c^5+d^5-e^5-f^5-g^5-h^5)\\ &+5x^4(a^4p+b^4q+c^4r+d^4s-e^4t-f^4u-g^4v-h^4w)\\ &+10x^3(a^3p^2+b^3q^2+c^3r^2+d^3s^2-e^3t^2-f^3u^2-g^3v^2-h^3w^2)\\ &+10x^2(a^2p^3+b^2q^3+c^2r^3+d^2s^3-e^2t^3-f^2u^3-g^2v^3-h^2w^3)\\ &+5x(ap^4+bq^4+cr^4+ds^4-et^4-fu^4-gv^4-hw^4)\\ &+(p^4+q^4+r^4+s^4-t^4-u^4-v^4-w^4)=0,\end{aligned} \tag{1.12}$$

it is found, resultant equation (1.12) is too tedious and difficult to solve for x. This equation is, therefore, transformed into a linear equation so as to solve it easily. Obviously, constant term and term containing x^4 can be ridden of, if

$$\left(a^5+b^5+c^5+d^5-e^5-f^5-g^5-h^5\right)=0 \tag{1.13}$$

and

$$\left(p^4+q^4+r^4+s^4-t^4-u^4-v^4-w^4\right)=0. \tag{1.14}$$

To achieve this motive, we replace e, f, g, h with a, b, c, d and $t, u, v, w$with p, q, r, s respectively and accordingly, equation (1.1) is written in algebraic form,

$$\begin{aligned}(ax+p)^5 &+ (bx+q)^5 + (cx+r)^5 + (dx+s)^5 \\ &= (ax+q)^5 + (bx+r)^5 + (cx+s)^5 + (dx+p)^5.\end{aligned} \tag{1.15}$$

With this introduction, further steps will be taken to vanish other two terms to transform this equation into a linear equation.

2 Transformation into linear equation

Expansion of equation (1.13) on expansion, yields

$$5x^4\{p(a^4-d^4)+q(b^4-a^4)+r(c^4-b^4)+s(d^4-c^4)\}$$
$$+10x^3\{p^2(a^3-d^3)+q^2(b^3-a^3)+r^2(c^3-b^3)+s^2(d^3-c^3)\}$$
$$+10x^2\{p^3(a^2-d^2)+q^3(b^2-a^2)+r^3(c^2-b^2)+s^3(d^2-c^2)\}$$
$$+5x\{p^4(a-d)+q^4(b-a)+r^4(c-b)+s^4(d-c)\}=0. \tag{2.1}$$

It has solution at $x=0$ but that yields $p^5+q^5+r^5+s^5=p^5+q^5+r^5+s^5$, which is a trivial solution and is ignored. Equation (2.1) can, then be written as

$$x^3\{p(a^4-d^4)+q(b^4-a^4)+r(c^4-b^4)+s(d^4-c^4)\}$$
$$+2x^2\{p^2(a^3-d^3)+q^2(b^3-a^3)+r^2(c^3-b^3)+s^2(d^3-c^3)\}$$
$$+2x\{p^3(a^2-d^2)+q^3(b^2-a^2)+r^3(c^2-b^2)+s^3(d^2-c^2)\}$$
$$+\{p^4(a-d)+q^4(b-a)+r^4(c-b)+s^4(d-c)\}=0. \tag{2.2}$$

It is a cubic equation and to transform it into a linear equation, its constant term and coefficient of x are equated to zero. That is

$$\left\{p^4\left(a-d\right)+q^4\left(b-a\right)+r^4\left(c-b\right)+s^4\left(d-c\right)\right\}=0$$

and

$$2\left\{p^3\left(a^2-d^2\right)+q^3\left(b^2-a^2\right)+r^3\left(c^2-b^2\right)+s^3\left(d^2-c^2\right)\right\}=0.$$

On simplification,

$$a=-\frac{b\left(q^4-r^4\right)+c\left(r^4-s^4\right)+d\left(s^4-p^4\right)}{p^4-q^4}, \tag{2.3}$$

and

$$a^2=-\frac{b^2\left(q^3-r^3\right)+c^2\left(r^3-s^3\right)+d^2\left(s^3-p^3\right)}{p^3-q^3}, \tag{2.4}$$

where $p\neq q$ and also $p\neq -q$.

2.1 Determination of a and b from Equations (2.3) and (2.4)

Equations (2.3) and (2.4) impose certain conditions on values of a and b that make these dependent upon c,d,p,q,r and s. From equations (2.3) and (2.4),

$$\left\{-\frac{b\left(q^4-r^4\right)+c\left(r^4-s^4\right)+d\left(s^4-p^4\right)}{p^4-q^4}\right\}^2=-\frac{b^2\left(q^3-r^3\right)+c^2\left(r^3-s^3\right)+d^2\left(s^3-p^3\right)}{p^3-q^3}. \tag{2.5}$$

For its easy solvability for b, term with coefficient of b^2 in equation (2.5) is eliminated by equating its coefficients to zero and that yields

$$\left(\frac{q^4-r^4}{p^4-q^4}\right)^2=-\left(\frac{q^3-r^3}{p^3-q^3}\right).$$

Considering $q=0$, yields $r=p$ and on simplifying equation (2.5) by substituting r with p and $q=0$, we obtain

$$b=\frac{1}{2}(c-d)\left(1-t^4\right)+\frac{1}{2}(c+d)\frac{\left(1+t+t^2\right)}{\left(1+t^2\right)\left(1+t\right)}, \tag{2.6}$$

where $s/p=t$ and $t\neq -1$. On simplifying equation (2.3) by substituting r with p, $q=0$ and $s/p=t$,

$$a=b-(c-d)\left(1-t^4\right). \tag{2.7}$$

On putting value of b from equation (2.6) in equation (2.7),

$$a=-\frac{1}{2}(c-d)\left(1-t^4\right)+\frac{1}{2}(c+d)\frac{\left(1+t+t^2\right)}{\left(1+t^2\right)\left(1+t\right)}. \tag{2.8}$$

2.2 Determination of value of x and solution to Equation (1.1)

When equation (2.3) and (2.4) are satisfied, then equation (2.2) transforms into

$$x^3\left\{p\left(a^4-d^4\right)+q\left(b^4-a^4\right)+r\left(c^4-b^4\right)+s\left(d^4-c^4\right)\right\}$$
$$+2x^2\left\{p^2\left(a^3-d^3\right)+q^2\left(b^3-a^3\right)+r^2\left(c^3-b^3\right)+s^2\left(d^3-c^3\right)\right\}=0. \quad (2.9)$$

On simplifying after putting $r=p, s/p=t$ and $q=0$, it transforms into linear equation

$$x=-2p(P/Q), \quad (2.10)$$

where

$$P=a^3-b^3+\left(c^3-d^3\right)\left(1-t^2\right) \quad (2.11)$$

and

$$Q=a^4-b^4+\left(c^4-d^4\right)(1-t). \quad (2.12)$$

By putting values of a and b from equations (2.8) and (2.6) in equations (2.10) and (2.11), x is determined by equation (2.9). Substituting this value x in equation (1.13), solution to equation (1.10) is obtained after normalisation. On putting $r=p$ and $q=0$, equation (1.11) takes the form

$(ax+p)^5+(\text{ bx})^5+(cx+p)^5+(dx+s)^5=(\text{ax})^5+(bx+p)^5+(cx+s)^5+(dx+p)^5$. (2.12)

Normalisation, wherever, the word normalisation appears in this paper, it will mean converting-fractions to integers by multiplying these with lowest, common multiplier abbreviated LCM. Based on the method discussed in foregoing paragraphs, some solutions are given in Table 2.1.

***Table* 2.1:** Solution to Diophantine equation $A^5+B^5+C^5+D^5=E^5+F^5+G^5+H^5$

S. N.	Values of c,d,p,s	Calculated a,b,t and x	Normalized and rearranged $A^5+B^5+C^5+D^5=E^5+F^5+G^5+H^5$
1	$c=1,\ d=2,$ $p=1, s=-2$	$a=-\frac{42}{5}, b=\frac{33}{5},$ $t=-2, x=\frac{7160}{12651}$	$(47256)^5+(19811)^5+(60144)^5+(18142)^5$ $=(47493)^5+(59907)^5+(10982)^5$ $+(26971)^5$
2	$c=1,\ d=-2,$ $p=-1, s=2$	$a=\frac{114}{5}, b=-\frac{111}{5},$ $t=-2, x=2040/1223$	$(45289)^5+(817)^5+(5303)^5+(46511)^5$ $=(46512)^5+(4486)^5+(1634)^5+(45288)^5$
3	$c=3,\ d=1,$ $p=1, s=-2$	$a=69/5, b=-81/5,$ $t=-2, x=\frac{7085}{16858}$	$(114777)^5+(26631)^5+(97773)^5+(23943)^5$ $=(114631)^5+(38113)^5+(97919)^5$ $+(12461)^5$
4	$c=3,\ d=-5,$ $p=1, s=-2$	$a=303/5, b=-297/5,$ $t=-2, x=-\frac{112415}{134801}$	$(6677548)^5+(202444)^5+(6812252)^5$ $+(696876)^5$ $=(6677451)^5+(292473)^5+(6812349)^5$ $+(606847)^5$
5	$c=3,\ d=2,$ $p=1, s=-2$	$a=6, b=-9,$ $t=-2, x=\frac{296}{845}$	$(2664)^5+(1098)^5+(1776)^5+(1437)^5$ $=(2621)^5+(1733)^5+(1819)^5+(802)^5$
6	$c=4,\ d=1,$ $p=1, s=-2$	$a=21, b=-24,$ $t=-2, x=\frac{2544}{7585}$	$(61056)^5+(12626)^5+(53424)^5+(10129)^5$ $=(61009)^5+(17761)^5+(53471)^5+(4994)5$
7	$c=0,\ d=1,$ $p=1, s=-2$	$a=-39/5, b=36/5,$ $t=-2, x=640/383$	$(4609)^5+(126)^5+(4991)^5+(1023)^5$ $=(4608)^5+(383)^5+(4992)^5+(766)5$

This proves Lemma 2.1 and Lemma 2.2.

Lemma 2.1. *A Diophantine equation* 5.4.4, $(ax+p)^5 + (\ bx)^5 + (cx+p)^5 + (dx+s)^5 = (ax)^5 + (bx+p)^5 + (cx+s)^5 + (dx+p)^5$ *is always transformable into a linear equation,* $x\left\{a^3 - b^3 + \left(c^3 - d^3\right)\left(1 - t^2\right)\right\} + 2p\{a^3 - b^3 + \left(c^3 - d^3\right)\left(1 - t^2\right)\} = 0$, *where* $a, b, c, d,\ p, q, s$ *are rational quantities,* a, b, t *are given by equations* (2.6), (2.8) *and* $t = s/p$.

Lemma 2.2. *After normalisation, a Diophantine equation* 5.4.4, $(ax+p)^5 + (\ bx)^5 + (cx+p)^5 + (dx+s)^5 = (ax)^5 + (bx+p)^5 + (cx+s)^5 + (dx+p)^5$ *is always true and, in fact, is an identity, when* $a = -\frac{1}{2}(c - d)\left(1 - t^4\right) + \frac{1}{2}(c+d)\frac{(1+t+t^2)}{(1+t^2)(1+t)}, b = \frac{1}{2}(c-d)\left(1 - t^4\right) + \frac{1}{2}(c+d)\frac{(1+t+t^2)}{(1+t^2)(1+t)},\ x = -2p\frac{\left\{a^3-b^3+\left(c^3-d^3\right)\left(1-t^2\right)\right\}}{\{a^4-b^4+(c^4-d^4)(1-t)\}}$, $t = s/p$, c, d,p *are real rational quantities and* $t \neq -1$.

However, it was observed, solution to Diophantine equations 5.4.4 *also yields solutions to Diophantine equation* 5.5.3. *These solutions are given in Table* 2.2.

***Table* 2.2:** Solution to Diophantine equation $A^5 + B^5 + C^5 + D^5 + E^5 = F^5 + G^5 + H^5$

S. N.	Values of c, d, p, s	Calculated a, b, t and x	Normalized and rearranged $A^5 + B^5 + C^5 + D^5 + E^5 = F^5 + G^5 + H^5$
1	$c = 3,\ d = -1,\ p = 1, s = -2$	$a = 147/5,\ b = -153/5,\ t = -2, x = \frac{28115}{33701}$	$(860319)^5 + (95517)^5 + (826581)^5 + (16943)^5 + 5586^5 = (860282)^5 + (118046)^5 + (826618)^5$
2	$c = 4,\ d = -1,\ p = 1, s = -2$	$a = 183/5,\ b = -192/5,\ t = -2, x = \frac{175760}{315951}$	$(6749184)^5 + (807662)^5 + (6432816)^5 + (71138)^5 + (140191)^5 = (6748767)^5 + (1018991)^5 + (6433233)^5$
3	$c = 6,\ d = -1,\ p = 1, s = -2$	$a = 51, b = -54,\ t = -2, x = \frac{13784}{41285}$	$(744336)^5 + (96354)^5 + (702984)^5 + (134)^5 + (27501)^5 = (744269)^5 + (123989)^5 + (703051)^5$
4	$c = 5,\ d = -1,\ p = 1, s = -2$	$a = 219/5,\ b = -231/5,\ t = -2, x = \frac{63285}{151658}$	$(2923767)^5 + (366601)^5 + (2771883)^5 + (13109)^5 + (88373)^5 = (2923541)^5 + (468083)^5 + (2772109)^5$

2.3 Solution to Diophantine Equation 5.4.3, $\mathbf{A^5+B^5+C^5+D^5=F^5+G^5+H^5}$

On putting $a = 0$, in equation (2.12), it transforms into Diophantine equation 5.4.3

$$(p)^5 + (\ \mathrm{bx})^5 + (cx+p)^5 + (dx+s)^5 = (bx+p)^5 + (cx+s)^5 + (dx+p)^5 \tag{2.13}$$

and equation (2.8), (2.6) transform into

$$\frac{c}{d} = \frac{\left(1-t^4\right)^2 + \left(1-t^3\right)}{\left(1-t^4\right)^2 - \left(1-t^3\right)}, \tag{2.14}$$

$$b = 2d \cdot \frac{\left(1-t^4\right)\left(1-t^3\right)}{\left(1-t^4\right)^2 - \left(1-t^3\right)}. \tag{2.15}$$

where b, c, d, p, s are real rational quantities, t is neither equal to one nor equal to zero and also $d \neq 0$. Value of x is, then given by relation

$$x = -2p(P/Q), \tag{2.16}$$

where

$$P = -d^3\left\{2\cdot\frac{\left(1-t^3\right)\left(1-t^4\right)}{\left(1-t^4\right)^2-\left(1-t^3\right)}\right\}^3 + d^3\left(1-t^2\right)\left[\left\{\frac{\left(1-t^4\right)^2+\left(1-t^3\right)}{\left(1-t^4\right)^2-\left(1-t^3\right)}\right\}^3 - 1\right], \tag{2.17}$$

$$Q = -d^4\left\{2\cdot\frac{\left(1-t^3\right)\left(1-t^4\right)}{\left(1-t^4\right)^2-\left(1-t^3\right)}\right\}^4 + d^4\left(1-t\right)\left[\left\{\frac{\left(1-t^4\right)^2+\left(1-t^3\right)}{\left(1-t^4\right)^2-\left(1-t^3\right)}\right\}^4 - 1\right]. \tag{2.18}$$

Based on the method discussed in foregoing paragraphs, some solutions are given in Table 2.3.

***Table* 2.3:** Solution to Diophantine Equation $A^5+B^5+C^5+D^5=F^5+G^5+H^5$

S.N.	Values of t, p, d	Calculated $c, b,$ and x	Normalized And Rearranged $A^5+B^5+C^5+D^5=F^5+G^5+H^5$
1	$t=-2,\ p=1,$ $d=1$	$c=13/12,$ $b=-5/4,$ $x=7872/4525$	$(4525)^5+(13053)^5+(5315)^5+(522)^5$ $=(9840)^5+(1178)^5+(12397)^5$
2	$t=-1/2,$ $p=1,$ $d=1$	$c=-57/7,$ $b=-60/7,$ x $=-11207/29900$	$(26157)^5+(125960)^5+(76307)^5+(18693)^5$ $=(29900)^5+(96060)^5+(121157)^5$

This proves Lemmas 2.3 and 2.4.

Lemma 2.3. *A Diophantine equation* 5.4.3, $(p)^5+(\ bx)^5+(cx+p)^5+(dx+s)^5=(bx+p)^5+(cx+s)^5+(dx+p)^5$ *is always transformable into a linear equation* $x=-2p(P/Q)$ *where* P *and* Q *are given by Equations* (2.17) *and* (2.18)*. Real rational quantities* b, c, *are given by Equations* (2.15), (2.14) *respectively,* d, s, p *are real rational quantities,* $t=s/p$ *which is neither equal to one nor equal to zero and also* $d\neq 0$.

Lemma 2.4. *After normalisation, a Diophantine equation* 5.4.3, $(p)^5+(\ bx)^5+(cx+p)^5+(dx+s)^5=(bx+p)^5+(cx+s)^5+(dx+p)^5$ *is always true and, in fact, is an identity when* $x, P, Q, b,\ c$ *are given by Equations* (2.16), (2.17), (2.18), (2.15), (2.14) *respectively,* $d, s,\ p$ *are real rational quantities,* $t=s/p$ *which is neither equal to one nor equal to zero and also* $d\neq 0$.

Next Diophantine equation 5.5.5 *is taken up. Procedure applied to equation* 5.4.4 *will also be used here.*

3 Transformation of Diophantine Equation 5.5.5 $A^5+B^5+C^5+D^5+E^5=F^5+G^5+H^5+I^5+J^5$ into linear equation and its solution

Diophantine equation $A^5+B^5+C^5+D^5+E^5=F^5+G^5+H^5+I^5+J^5$ can be written in algebraic form

$$(ax+p)^5+(bx+q)^5+(cx+r)^5+(dx+s)^5+(ex+u)^5=(ax+q)^5+(bx+r)^5+(cx+s)^5+(dx+u)^5+(ex+p)^5, \tag{3.1}$$

where $a, b, c, d, e, p, q, r, s, u$ are real rational quantities ,$p\neq q$ and also $p\neq -q$. On expansion, it takes the form

$$\begin{aligned}&5x^4\left\{p\left(a^4-e^4\right)+q\left(b^4-a^4\right)+r\left(c^4-b^4\right)+s\left(d^4-c^4\right)+u\left(e^4-d^4\right)\right\}\\&\quad+10x^3\left\{p^2\left(a^3-e^3\right)+q^2\left(b^3-a^3\right)+r^2\left(c^3-b^3\right)+s^2\left(d^3-c^3\right)+u^2\left(e^3-d^3\right)\right\}\\&\quad+10x^2\left\{p^3\left(a^2-e^2\right)+q^3\left(c^2-b^2\right)+r^3\left(c^2-b^2\right)+s^3\left(d^2-c^2\right)+u^3\left(e^2-d^2\right)\right\}\\&\quad+5x\left\{p^4\left(a-e\right)+q^4\left(b-a\right)+r^4\left(c-b\right)+s^4\left(d-c\right)+u^4(e-d)\right\}=0.\end{aligned} \tag{3.2}$$

On equating coefficient of x and x^2 with zero,

$$\left\{p^4\left(a-e\right)+q^4\left(b-a\right)+r^4\left(c-b\right)+s^4\left(d-c\right)+u^4(e-d)\right\}=0 \tag{3.3}$$

and

$$\left\{p^3\left(a^2-e^2\right)+q^3\left(b^2-a^2\right)+r^3\left(c^2-b^2\right)+s^3\left(d^2-c^2\right)+u^3\left(e^2-d^2\right)\right\}=0. \tag{3.4}$$

From Equations (3.3) and (3.4),

$$a=-\frac{b\left(q^4-r^4\right)+c\left(r^4-s^4\right)+d\left(s^4-u^4\right)+e\left(u^4-p^4\right)}{p^4-q^4}, \tag{3.5}$$

$$a^2=-\frac{b^2\left(q^3-r^3\right)+c^2\left(r^3-s^3\right)+d^2\left(s^3-u^3\right)+e^2\left(u^3-p^3\right)}{p^3-q^3}. \tag{3.6}$$

That results in

$$\left\{-\frac{b\left(q^4-r^4\right)+c\left(r^4-s^4\right)+d\left(s^4-u^4\right)+e\left(u^4-p^4\right)}{p^4-q^4}\right\}^2=-\frac{b^2\left(q^3-r^3\right)+c^2\left(r^3-s^3\right)+d^2\left(s^3-u^3\right)+e^2\left(u^3-p^3\right)}{p^3-q^3}. \tag{3.7}$$

For easy solvability, quadratic equation (3.7) in b is transformed into linear equation by equating coefficient of b^2 to zero. That yields

$$\left(\frac{q^4-r^4}{p^4-q^4}\right)^2=-\left(\frac{q^3-r^3}{p^3-q^3}\right).$$

Let $q=0, s=0$, then $r=p$ and equation (3.1) transforms into

$$\begin{aligned}&(ax+p)^5+(\mathrm{bx})^5+(cx+p)^5+(\mathrm{dx})^3+(ex+u)^5\\&\quad=(\mathrm{ax})^5+(bx+p)^5+(\mathrm{cx})^5+(dx+u)^5+(ex+p)^5.\end{aligned} \tag{3.8}$$

Also equation (3.2) transforms into

$$\begin{aligned}&5x^4\left\{p\left(a^4+c^4-b^4-e^4\right)+u\left(e^4-d^4\right)\right\}+10x^3\left\{p^2\left(a^3+c^3-b^3-e^3\right)+u^2\left(e^3-d^3\right)\right\}\\&+10x^2\left\{p^3\left(a^2+c^2-b^2-e^2\right)+u^3\left(e^2-d^2\right)\right\}\\&+5x\left\{p^4\left(a+c-b-e\right)+u^4\left(e-d\right)\right\}=0,\end{aligned} \tag{3.9}$$

and equation (3.3) and (3.4) transform into

$$\left\{p^4\left(a-b+c-e\right)+u^4(e-d)\right\}=0, \tag{3.10}$$

$$\left\{p^3\left(a^2-b^2+c^2-e^2\right)+u^3\left(e^2-d^2\right)\right\}=0. \tag{3.11}$$

From equation (3.10) and (3.11),

$$a=(b+e-c)-t^4\left(e-d\right), \tag{3.12}$$

$$a^2=\left(b^2+e^2-c^2\right)-t^3\left(e^2-d^2\right), \tag{3.13}$$

where $u/p=t$. Therefore,

$$\left\{(b+e-c)-t^4\left(e-d\right)\right\}^2=\left(b^2+e^2-c^2\right)-t^3\left(e^2-d^2\right).$$

On simplification,

$$b=\frac{c^2+\left\{c+t^4(e-d)\right\}^2+t^3\left(e^2-d^2\right)-2e\left\{c+t^4\left(e-d\right)\right\}}{2\left\{c-e+t^4\left(e-d\right)\right\}}, \tag{3.14}$$

where $c-e+t^4(e-d)\neq 0$ and also $c\neq d$. For easy solvability, we assign $t=-1$ and simplify equations (3.14) and (3.15) and obtain

$$b=c-d+\frac{cd-e^2}{c-d} \tag{3.15}$$

and

$$a=\frac{cd-e^2}{c-d}. \tag{3.16}$$

When a and b have values as given by equations (3.15) and (3.16), then equation (3.9) transforms into linear equation

$$x=-2p\left(\frac{a^3-b^3+c^3-d^3}{a^4-b^4+c^4+d^4-2e^4}\right), \tag{3.17}$$

wherec, d, e are arbitrary real rational quantities such that $c\neq d$ and $a^4+c^4+d^4\neq b^4+2e^4$. On putting the value of x given by equation (3.16) in equation (3.8), gives solution to Diophantine Equation 5.5.5.

That also proves Lemmas 3.1 and 3.2. In the Table 3.1, p is neither considered nor assigned any value except one owing to the fact that it does not appear in final equation when value of x is put in equation (3.8).

***Table* 3.1:** Solution to Diophantine equation $A^5+B^5+C^5+D^5+E^5=F^5+G^5+H^5+I^5+J^5$

S .N.	Values of c,d,e	Calculated a,b and x	$(ax+p)^5+(bx)^5+(cx+p)^5+(dx)^3+(ex-p)^5 = (ax)^5+(bx+p)^5+(cx)^5+(dx-p)^5+(ex+p)^5.$
1	$2,3,5$	$19,18,-1/12$	$7^5+18^5+3^5+17^5+7^5=10^5+19^5+6^5+2^5+15^5$
2	$-2,3,5$	$31/5,6/5,-5/4$	$27^5+6^5+15^5+29^5+10^5=14^5+31^5+2^5+19^5+21^5$
3	$3,-4,5$	$-37/7,12/5,-7/8$	$12^5+13^5+43^5+37^5+20^5 = 45^5+28^5+4^5+21^5+27^5$
4	$3,-4,1$	$-13/7,36/7,-7/24$	$36^5+31^5+13^5+4^5+17^5=37^5+3^5+28^5+12^5+21^5$
5	$3,-4,0$	$-12/7,37/7,-21/74$	$111^5+74^5+36^5+10^5+74^5 = 110^5+11^5+84^5+37^5+63^5$
6	$3,4,0$	$-12,1\,3,-3/26$	$10^5+39^5+26^5+9^5+26^5 = 35^5+12^5+36^5+13^5+14^5$
7	$3,4,6$	$24,2\,3\,,-3/46$	$26^5+69^5+12^5+64^5+28^5 = 37^5+72^5+23^5+9^5+58^5$
8	$-4,5,6$	$-56/9,-2\,5/9,27/50$	$75^5+58^5+168^5+85^5+212^5 = 218^5+135^5+112^5+25^5+108^5$
9	$-4,5,0$	$20/9,-61/9,27/122$	$183^5+122^5+60^5+13^5+122^5 = 182^5+14^5+135^5+61^5+108^5$
10	$-4,5,1$	$7/3,-20/3,9/40$	$60^5+31^5+21^5+5^5+49^5=61^5+4^5+45^5+20^5+36^5$

Lemma 3.1. *A Diophantine equation* 5.5.5, $(ax+p)^5+(\ bx)^5+(cx+p)^5+(dx)^5+(ex-p)^5=(ax)^5+(bx+p)^5+(cx)^5+(dx-p)^5+(ex+p)^5$ *is always transformable into a linear equation,* $x=-2p\left(\frac{a^3-b^3+c^3-d^3}{a^4-b^4+c^4+d^4-2e^4}\right)$, *where* a,b *are given by Equations* (3.16), (3.15) *respectively,* c,d,e *andp are arbitrary real rational quantities such that* $c\neq d$ *and* $a^4+c^4+d^4\neq b^4+2e^4$.

Lemma 3.2. *After normalisation, a Diophantine equation* 5.5.5, $(ax+p)^5+(\ bx)^5+(cx+p)^5+(dx)^5+(ex-p)^5=(ax)^5+(bx+p)^5+(cx)^5+(dx-p)^5+(ex+p)^5$ *is always true and, in fact, is an identity, when* $x=-2p\left(\frac{a^3-b^3+c^3-d^3}{a^4-b^4+c^4+d^4-2e^4}\right)$, $a=\frac{cd-e^2}{c-d}$, $b=c-d+\frac{cd-e^2}{c-d}$, c,d,e *andp are arbitrary real rational quantities such that* $c\neq d$ *and* $a^4+c^4+d^4\neq b^4+2e^4$.

4 Solution to Diophantine Equation 5.5.4, $A^5+B^5+C^5+D^5+E^5=G^5+H^5+I^5+J^5$

Procedure of Diophantine equations $A^5+B^5+C^5+D^5+E^5=F^5+G^5+H^5+I^5+J^5$ as determined in foregoing paragraphs will be utilised here by putting $a=0$. In that case, the equation

$$(ax+p)^5+(\ \mathrm{bx})^5+(cx+p)^5+(\mathrm{dx})^5+(ex-p)^5=(\mathrm{ax})^5+(bx+p)^5+(\mathrm{cx})^5+(dx-p)^5+(ex+p)^5$$

transforms into

$$(p)^5+(\ \mathrm{bx})^5+(cx+p)^5+(\mathrm{dx})^5+(ex-p)^5=(bx+p)^5+(\mathrm{cx})^5+(dx-p)^5+(ex+p)^5, \tag{4.1}$$

and equations (3.15), (3.14) and (3.16) transform into

$$d=e^2/c, \tag{4.2}$$

$$b=c-e^2/c, \tag{4.3}$$

$$x=-2p\left(\frac{-b^3+c^3-d^3}{-b^4+c^4+d^4-2e^4}\right), \tag{4.4}$$

respectively where $c\neq 0$. On putting the value of b and d in equation (4.4) and simplifying

$$x=-\frac{3pc}{2\left(c^2-e^2\right)}. \tag{4.5}$$

where p, c and e are arbitrarily assigned real rational quantities such that $c \neq 0$ and also $c^2 = e^2$ Putting the value of x given by equation (4.5) in equation (4.1) gives solutions to Diophantine equation 5.5.4. Based on these equations, some solutions are given in Table 4.1. Here also p is not considered on the basis of explanation already given.

***Table* 4.1:** Solution to Diophantine equation $A^5 + B^5 + C^5 + D^5 + E^5 = G^5 + H^5 + I^5 + J^5$

S.N.	Values of c, e	Calculated value of b, d and x	$A^5 + B^5 + C^5 + D^5 + E^5 = G^5 + H^5 + I + J$
1	3,4	$-7/3, 16/3, 9/14$	$14^5 + 41^5 + 48^5 + 22^5 + 7^5 = 21^5 + 27^5 + 34^5 + 50^5$
2	5, 4	$9/5, 16/5, -5/6$	$6^5 + 3^5 + 25^5 + 22^5 + 14^5 = 9^5 + 19^5 + 16^5 + 26^5$
3	7, 4	$33/7, 16/7, -7/22$	$22^5 + 11^5 + 49^5 + 38^5 + 6^5 = 33^5 + 27^5 + 16^5 + 50^5$
4	3, 5	$-16/3, 25/3, 9/32$	$32^5 + 59^5 + 75^5 + 13^5 + 16^5 = 48^5 + 27^5 + 43^5 + 77^5$
5	5, 8	$-39/5, 64/5, 5/26$	$26^5 + 51^5 + 64^5 + 14^5 + 13^5 = 39^5 + 25^5 + 38^5 + 66^5$

Lemma 4.1. *A Diophantine equation* 5.5.4, $(p)^5 + (\ bx)^5 + (cx + p) + (dx)^5 + (ex - p)^5 = (bx + p)^5 + (cx)^5 + (dx - p)^5 + (ex + p)^5$ *is always transformable into linear equation,* $x = -2p\left(\frac{-b^3+c^3-d^3}{-b^4+c^4+d^4-2e^4}\right)$, *where* b *and* d *are given by equations* (4.3), (4.2), *and* c, e *and* p *are real rational quantities.*

Lemma 4.2. *After normalisation, a Diophantine equation* 5.5.4, $(p)^5 + (bx)^5 + (cx + p)^5 + (dx)^5 + (ex - p)^5 = (bx + p)^5 + (cx)^5 + (dx - p)^5 + (ex + p)^5$ *is always true and, in fact, is an identity, when* $x = -2p\left(\frac{-b^3+c^3-d^3}{-b^4+c^4+d^4-2e^4}\right)$, $d = \frac{e^2}{c}$, $b = c - \frac{e^2}{c}$, *and* c, e *and* p *are arbitrary real rational quantities such that* $c \neq 0$ *and* $c^4 + d^4 \neq b^4 + 2e^4$.

5 Solution to Diophantine Equation 5.6.6 $A^5+B^5+C^5+D^5+E^5+F^5=G^5+H^5+I^5+J^5+K^5+L^5$

Equation $A^5 + B^5 + C^5 + D^5 + E^5 + F^5 = G^5 + H^5 + I^5 + J^5 + K^5 + L^5$ can be written in algebraic form as

$$(a_1x + p)^5 + (a_2x)^5 + (a_3x + p)^5 + (a_4x)^5 + (a_5x + p)^5 + (a_6x)^5$$
$$= (a_1x)^5 + (a_2x + p)^5 + (a_3x)^5 + (a_4x + p)^5 + (a_5x)^5 + (a_6x + p)^5 . \quad (5.1)$$

On expansion,

$$(5x^4p\left(a_1^4 + a_3^4 + a_5^4 - a_2^4 - a_4^4 - a_6^4\right) + 10x^3p^2\left(a_1^3 + a_3^3 + a_5^3 - a_2^3 - a_4^3 - a_6^3\right)$$
$$+ 10x^2p^3\left(a_1^2 + a_3^2 + a_5^2 - a_2^2 - a_4^2 - a_6^2\right)$$
$$+ 5xp^4\left(a_1 + a_3 + a_5 - a_2 - a_4 - a_6\right) = 0. \quad (5.2)$$

where $a_1, a_2, a_3, \ldots, a_7$ and p are real rational quantities. Equating coefficients of x and x^2 equal to zero,

$$(a_1 + a_3 + a_5 - a_2 - a_4 - a_6) = 0$$

and

$$\left(a_1^2 + a_3^2 + a_5^2 - a_2^2 - a_4^2 - a_6^2\right) = 0.$$

From aforementioned equations,

$$a_1 = -a_3 - a_5 + a_2 + a_4 + a_6 \quad (5.3)$$

and

$$a_1^2 = -a_3^2 - a_5^2 + a_2^2 + a_4^2 + a_6^2. \quad (5.4)$$

From Equations (5.3) and (5.4)

$$(a_2 - a_3 + a_4 - a_5 + a_6)^2 = -a_3^2 - a_5^2 + a_2^2 + a_4^2 + a_6^2 .$$

On simplification, it yields

$$a_2 = a_3 + a_5 - \frac{a_3a_5 - a_4a_6}{a_3 - a_4 + a_5 - a_6}. \quad (5.5)$$

where $a_3 + a_5 \neq a_4 + a_6$. On putting the value of a_2 in equation (5.3),

$$a_1 = a_4 + a_6 - \frac{a_3a_5 - a_4a_6}{a_3 - a_4 + a_5 - a_6}. \quad (5.6)$$

When equations (5.5) and (5.6) are satisfied, equation (5.2) transforms into

$$x = -2p\frac{(a_1^3 + a_3^3 + a_5^3 - a_2^3 - a_4^3 - a_6^3)}{(a_1^4 + a_3^4 + a_5^4 - a_2^4 - a_4^4 - a_6^4)} = -2p\frac{\left(\sum_{n=1}^{3} a_{2n-1}^3\right) - \left(\sum_{n=1}^{3} a_{2n}^3\right)}{\left(\sum_{n=1}^{3} a_{2n-1}^4\right) - \left(\sum_{n=1}^{3} a_{2n}^4\right)}, \tag{5.7}$$

where $\sum_{n=1}^{3} a_{2n-1}^4 \neq \sum_{n=1}^{3} a_{2n}^4$. Sign$\sum_{n=1}^{3} a_{2n-1}^3$ used in this paper, means summation of terms a_{2n-1}^3 and n varies from 1 to 3. On putting the values of a_1 and a_2 in equation (5.7) and, then putting the value of x in equation (5.1), gives solutions to Diophantine Equation 5.6.6. Based on this method, some solutions are given in the Table 5.1.

***Table* 5.1:** Solution to Diophantine equation $A^5 + B^5 + C^5 + D^5 + E^5 + F^5 = G^5 + H^5 + I^5 + J^5 + K^5 + L^5$

S. N.	Ass. Values of a_3, a_4, a_5 and a_6	Calculated values of a_1, a_2, x	$(a_1x+p)^5 + (a_2x)^5 + (a_3x+p)^5 + (a_4x)^5 + (a_5x+p)^5 + (a_6x)^5 = (a_1x)^5 + (a_2x+p)^5 + (a_3x)^5 + (a_4x+p)^5 + (a_5x)^5 + (a_6x+p)^5$
1	$3, 5, 6, 7$	$19/3, 10/3, -9/92$	$35^5 + 65^5 + 38^5 + 57^5 + 27^5 + 54^5 = 30^5 + 45^5 + 63^5 + 62^5 + 47^5 + 29^5$
2	$2, 5, 6, 7$	$25/4, 9/4, -2/19$	$13^5 + 30^5 + 14^5 + 25^5 + 8^5 + 24^5 = 9^5 + 20^5 + 28^5 + 29^5 + 18^5 + 10^5$
3	$-2, -5, -6, 7$	$67/10, -33/10, 15/13$	$99^5 + 34^5 + 150^5 + 154^5 + 201^5 + 236^5 = 227^5 + 210^5 + 73^5 + 60^5 + 124^5 + 180^5$
4	$-4, -5, -6, 7$	$83/12, -61/12, 18/37$	$183^5 + 70^5 + 180^5 + 142^5 + 249^5 + 326^5 = 323^5 + 252^5 + 109^5 + 144^5 + 106^5 + 216^5$
5	$-3, 5, 6, 7$	$55/9, -26/9, -27/164$	$78^5 + 245^5 + 2^5 + 165^5 + 162^5 + 25^5 = 1^5 + 135^5 + 189^5 + 242^5 + 81^5 + 29^5$
6	$-3, -5, -6, 7$	$75/11, -46/11, 11/16$	$46^5 + 17^5 + 55^5 + 50^5 + 75^5 + 93^5 = 91^5 + 77^5 + 30^5 + 33^5 + 39^5 + 66^5$
7	$-3, -5, 6, 1$	$-15/7, 34/7, -7/4$	$19^5 + 25^5 + 35^5 + 30^5 + 42^5 + 3^5 = 34^5 + 38^5 + 7^5 + 15^5 + 21^5 + 39^5$
8	$-3, 5, 6, 1$	$-5/3, -14/3, -9/8$	$23^5 + 42^5 + 35^5 + 37^5 + 54^5 + 1^5 = 45^5 + 46^5 + 9^5 + 15^5 + 50^5 + 27^5$
9	$3, -5, -6, -1/2$	$27/10, 26/5, 5$	$29^5 + 52^5 + 32^5 + 48^5 + 60^5 + 3^5 = 50^5 + 58^5 + 5^5 + 37^5 + 54^5 + 30^5$

That also proves Lemmas 5.1 and 5.2.

Lemma 5.1. *A Diophantine equation* 5.6.6, $(a_1x+p)^5 + (a_2x)^5 + (a_3x+p)^5 + (a_4x)^5 + (a_5x+p)^5 + (a_6x)^5 = (a_1x)^5 + (a_2x+p)^5 + (a_3x)^5 + (a_4x+p)^5 + (a_5x)^5 + (a_6x+p)^5$ *is always transformable into linear equation,*

$x = -2p\frac{\left(a_1^3+a_3^3+a_5^3-a_2^3-a_4^3-a_6^3\right)}{\left(a_1^4+a_3^4+a_5^4-a_2^4-a_4^4-a_6^4\right)} = -2p\frac{\left(\sum_{n=1}^{3} a_{2n-1}^3\right)-\left(\sum_{n=1}^{3} a_{2n}^3\right)}{\left(\sum_{n=1}^{3} a_{2n-1}^4\right)-\left(\sum_{n=1}^{3} a_{2n}^4\right)}$, *where* a_1 *and* a_2 *are given by equations* (5.5), (5.6) *and* $a_3, a_4,\ a_5, a_6$ *and* p *are arbitrary real rational quantities such that* $\sum_{n=1}^{3} a_{2n-1}^4 \neq \sum_{n=1}^{3} a_{2n}^4$.

Lemma 5.2. *After normalisation, a Diophantine equation* 5.6.6, $(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+(a_4x)^5+(a_5x+p)^5+(a_6x)^5 = (a_1x)^5+(a_2x+p)^5+(a_3x)^5+(a_4x+p)^5+(a_5x)^5+(a_6x+p)^5$ *is always true and, in fact, is an identity, when*

$$x = -2p\frac{\left(a_1^3+a_3^3+a_5^3-a_2^3-a_4^3-a_6^3\right)}{\left(a_1^4+a_3^4+a_5^4-a_2^4-a_4^4-a_6^4\right)} = -2p\frac{\left(\sum_{n=1}^{3} a_{2n-1}^3\right)-\left(\sum_{n=1}^{3} a_{2n}^3\right)}{\left(\sum_{n=1}^{3} a_{2n-1}^4\right)-\left(\sum_{n=1}^{3} a_{2n}^4\right)},$$

$$a_2 = a_3+a_5-\frac{a_3a_5-a_4a_6}{a_3-a_4+a_5-a_6}, a_1 = a_4+a_6-\frac{a_3a_5-a_4a_6}{a_3-a_4+a_5-a_6}$$

and $a_3, a_4,\ a_5, a_6$ *andp are arbitrary real rational quantities such that* $\sum_{n=1}^{3} a_{2n-1}^4 \neq \sum_{n=1}^{3} a_{2n}^4$.

6 Solution to Diophantine Equation 5.7.5 $A^5+B^5+C^5+D^5+E^5+F^5+G^5=H^5+I^5+J^5+K^5+L^5$

While solving Diophantine Equations 5.6.6, solutions to Diophantine Equations 5.7.5 are also obtained and are entered in Table 6.1.

***Table* 6.1:** Solution to Diophantine equation $A^5+B^5+C^5+D^5+E^5+F^5=G^5+H^5+I^5+J^5+K^5+L^5$

S.N.	Assigned values of a_3, a_4, a_5 and a_6	Calculated values of a_1, a_2, x	$(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+(a_4x)^5+(a_5x+p)^5+(a_6x)^5+(-a_1x)^5 = (a_2x+p)^5+(a_3x)^5+(a_4x+p)^5+(a_5x)^5+(a_6x+p)^5$
1	$-2, 5, 6, 7$	$49/8, -15/8, -4/27$	$5^5+15^5+70^5+6^5+49^5+48^5+2^5 = 40^5+56^5+69^5+16^5+14^5$
2	$-2, -5, 6, 7$	$-19/2, -15/2, 3/11$	$10^5+58^5+42^5+57^5+23^5+12^5+8^5 = 35^5+45^5+30^5+36^5+64^5$
3	$3, -5, 6, 7$	$-39/7, 10/7, -7/16$	$10^5+5^5+6^5+49^5+39^5+26^5+51^5 = 55^5+35^5+21^5+42^5+33^5$
4	$3, -5, -6, 7$	$27/5, 2/5, -5/8$	$19^5+2^5+7^5+35^5+6^5+33^5+30^5 = 25^5+38^5+27^5+15^5+27^5$
5	$-3, -5, -6, 1$	$3/5, -22/5, 5/28$	$22^5+25^5+2^5+3^5+6^5+3^5+33^5 = 31^5+13^5+5^5+15^5+30^5$
6	$3, -5, -6, 1/2$	$35/6, 22/3, -9/17$	$71^5+132^5+20^5+9^5+124^5+108^5+25^5 = 90^5+142^5+105^5+98^5+54^5$

7 Solution of Diophantine Equation 5.6.5 $A^5+B^5+C^5+D^5+E^5+F^5=H^5+I^5+J^5+K^5+L^5$

For solutions of Diophantine equation 5.6.5, equations (5.1), (5.5), (5.6) and (5.7) are referred to. When $a_1 = 0$, these equations take the forms after simplification as given below.

$$(p)^5+(a_2x)^5+(a_3x+p)^5+(a_4x)^5+(a_5x+p)^5+(a_6x)^5 = (a_2x+p)^5+(a_3x)^5+(a_4x+p)^5+(a_5x)^5+(a_6x+p)^5, \quad (7.1)$$

$$a_3 = a_4+a_6-\frac{a_4a_6}{(a_4-a_5+a_6)}, \quad (7.2)$$

$$a_2 = a_5-\frac{a_4a_6}{(a_4-a_5+a_6)}, \quad (7.3)$$

$$x = -2p\frac{\left(a_3^3+a_5^3-a_2^3-a_4^3-a_6^3\right)}{\left(a_3^4+a_5^4-a_2^4-a_4^4-a_6^4\right)} = -2p\frac{\left(\sum_{n=2}^{3} a_{2n-1}^3\right)-\left(\sum_{n=1}^{3} a_{2n}^3\right)}{\left(\sum_{n=2}^{3} a_{2n-1}^4\right)-\left(\sum_{n=1}^{3} a_{2n}^4\right)}, \quad (7.4)$$

where a_4, a_5, a_6 and p are arbitrary real quantities such that $\sum_{n=2}^{3} a_{2n-1}^4 \neq \sum_{n=1}^{3} a_{2n}^4$.

The values of a_2, a_3 obtained from equations (7.3), (7.2) and values of a_4, a_5 and a_6 as assigned by us, are then put in equation (7.4), that yields value of x. Putting the value of x so obtained in equation (7.1), gives its solution. Based on this method, some solutions to this equation are given in the Table 7.1.

***Table* 7.1:** Solution to Diophantine equation $A^5 + B^5 + C^5 + D^5 + E^5 + F^5 = H^5 + I^5 + J^5 + K^5 + L^5$

S.N.	Assigned a_4, a_5 and a_6	Calculated7 a_2, a_3, x	$(p)^5 + (a_2x)^5 + (a_3x+p)^5 + (a_4x)^5 + (a_5x+p)^5 + (a_6x)^5 = (a_2x+p)^5 + (a_3x)^5 + (a_4x+p)^5 + (a_5x)^5 + (a_6x+p)^5$
1	$4, 5, 6$	$1/5, 26/5, -5/34$	$1^5 + 20^5 + 30^5 + 33^5 + 14^5 + 4^5 = 34^5 + 8^5 + 9^5 + 26^5 + 25^5$
2	$4, -5, 6$	$-33/5, 42/5, -15/34$	$34^5 + 99^5 + 109^5 + 126^5 + 26^5 + 56^5 = 92^5 + 60^5 + 90^5 + 133^5 + 75^5$
3	$-4, -5, 6$	$-13/7, 38/7, -7/2$	$2^5 + 11^5 + 28^5 + 37^5 + 38^5 + 40^5 = 36^5 + 42^5 + 13^5 + 30^5 + 35^5$
4	$-4, -5, 7$	$-3/2, 13/2, -1$	$2^5 + 3^5 + 8^5 + 12^5 + 13^5 + 12^5 = 11^5 + 14^5 + 5^5 + 10^5 + 10^5$
5	$4, 5, 7$	$1/3, 19/3, -9/68$	$3^5 + 36^5 + 63^5 + 65^5 + 32^5 + 5^5 = 68^5 + 11^5 + 23^5 + 57^5 + 45^5$
6	$-4, 5, -7$	$27/4, -37/4, 6/17$	$34^5 + 81^5 + 94^5 + 111^5 + 14^5 + 50^5 = 77^5 + 48^5 + 84^5 + 115^5 + 60^5$
7	$-4, 5, -3$	$6, -6, 3/2$	$2^5 + 18^5 + 17^5 + 18^5 + 10^5 + 7^5 = 16^5 + 12^5 + 9^5 + 20^5 + 15^5$
8	$-4, 5, 3$	$3, -3, -3/4$	$4^5 + 13^5 + 12^5 + 5^5 + 15^5 + 5^5 = 9^5 + 11^5 + 9^5 + 9^5 + 16^5$

That also proves Lemma 7.1 and 7.2.

Lemma 7.1. *An Diophantine equation* 5.6.5, $(p)^5 + (a_2x)^5 + (a_3x+p)^5 + (a_4x)^5 + (a_5x+p)^5 + (a_6x)^5 = (a_2x+p)^5 + (a_3x)^5 + (a_4x+p)^5 + (a_5x)^5 + (a_6x+p)^5$ *is always transformable into linear equation,* $x = -2p\frac{(a_3^3+a_5^3-a_2^3-a_4^3-a_6^3)}{(a_3^4+a_5^4-a_2^4-a_4^4-a_6^4)} - 2p\frac{(\sum_{n=2}^{3} a_{2n-1}^3)-(\sum_{n=1}^{3} a_{2n}^3)}{(\sum_{n=2}^{3} a_{2n-1}^4)-(\sum_{n=1}^{3} a_{2n}^4)}$, *where* a_2 *and* a_3 *are given by equations* (7.3), (7.2) *and* $a_4,\ a_5, a_6$ *are arbitrarily assigned real rational quantities such that such that* $\sum_{n=2}^{3} a_{2n-1}^4 \neq \sum_{n=1}^{3} a_{2n}^4$.

Lemma 7.2. *After normalisation, a Diophantine equation* 5.6.5, $(p)^5 + (a_2x)^5 + (a_3x+p)^5 + (a_4x)^5 + (a_5x+p)^5 + (a_6x)^5 = (a_2x+p)^5 + (a_3x)^5 + (a_4x+p)^5 + (a_5x)^5 + (a_6x+p)^5$ *is always true and, in fact, is an identity, when* $x = -2p\frac{(a_3^3+a_5^3-a_2^3-a_4^3-a_6^3)}{(a_3^4+a_5^4-a_2^4-a_4^4-a_6^4)} - 2p\frac{(\sum_{n=2}^{3} a_{2n-1}^3)-(\sum_{n=1}^{3} a_{2n}^3)}{(\sum_{n=2}^{3} a_{2n-1}^4)-(\sum_{n=1}^{3} a_{2n}^4)}$, $a_2 = a_5 - \frac{a_4a_6}{(a_4-a_5+a_6)}$, $a_3 = a_4 + a_6 - \frac{a_4a_6}{(a_4-a_5+a_6)}$, *and* $a_4,\ a_5, a_6$

are arbitrarily assigned real rational quantities such that such that $\sum_{n=2}^{3} a_{2n-1}^4 \neq \sum_{n=1}^{3} a_{2n}^4$.

8 Solution to generalised form 5.*n*.*n* of Diophantine Equation 5.*n*.*n* $Y_1^5+Y_2^5+Y_3^5+\ldots+Y_{n-2}^5+Y_{n-1}^5+Y_n^5=Z_1^5+Z_2^5+Z_3^5+\ldots+Z_{n-2}^5+Z_{n-1}^5+Z_n^5$, where n is an integer ≥ 6.

For generalised form of Diophantine equation 5.*n*.*n*, two cases will be taken up, first case will be of the category, when $n = 2k$ (even integers) and second when $n = 2k-1$ (odd integers) where $k \geq 3$.

a. When $n = 2k$,

Diophantine equation 5.8.8 as mentioned below, is taken up.

$$(a_1x+p)^5 + (a_2x)^5 + (a_3x+p)^5 + (a_4x)^5 + (a_5x+p)^5 + (a_6x)^5 + (a_7x+p)^5 + (a_8x)^5$$

$$= (a_1x)^5 + (a_2x+p)^5 + (a_3x)^5 + (a_4x+p)^5 + (a_5x)^5 + (a_6x+p)^5 + (a_7x)^5 + (a_8x+p)^5, \tag{8.1}$$

where $a_1, a_2, a_3, \ldots, a_8$ are arbitrarily assigned real rational quantities. On equating coefficients of x and x^2 equal to zero in equation (8.1) , following equations are obtained

$$a_1 = -(a_3 + a_5 + a_7 - a_2 - a_4 - a_6 - a_8), \tag{8.2}$$

and

$$a_1^2 = -\left(a_3^2 + a_5^2 + a_7^2 - a_2^2 - a_4^2 - a_6^2 - a_8^2\right), \tag{8.3}$$

$$5x^4p\left(a_1^4 + a_3^4 + a_5^4 + a_7^4 - a_2^4 - a_4^4 - a_6^4 - a_8^4\right) + 10x^3p^2\left(a_1^3 + a_3^3 + a_5^3 + a_7^4 - a_2^3 - a_4^3 - a_6^3 - a_8^3\right) = 0. \tag{8.4}$$

Elimination of a_1 from the equations (8.2) and (8.3) yields

$$a_2 = A_8 - \frac{P_8 - Q_8}{A_8 - B_8}, \tag{8.5}$$

and putting this value of a_2 in equation (8.2) results in

$$a_1 = B_8 - \frac{P_8 - Q_8}{A_8 - B_8}. \tag{8.6}$$

where $a_3, a_4, a_5, \ldots, a_8$ are arbitrarily assigned real rational quantities,

$$A_8 = a_3 + a_5 + a_7 = \sum_{i=2}^{4} a_{(2i-1)}, \tag{8.7}$$

$$B_8 = a_4 + a_6 + a_8 = \sum_{i=2}^{4} a_{2i}, \tag{8.8}$$

$$P_8 = a_3(a_5 + a_7) + a_5(a_7) = a_3 \cdot \sum_{i=3}^{4} a_{(2i-1)} + a_5 \cdot a_7, \tag{8.9}$$

$$Q_8 = a_4(a_6 + a_8) + a_6(a_8) = a_4 \cdot \sum_{i=3}^{4} a_{2i} + a_6 \cdot a_8, \tag{8.10}$$

$A_8 \neq B_8$ and also $\sum_{n=2}^{4} a_{2n-1}^4 \neq \sum_{n=1}^{4} a_{2n}^4$. Value of x obtained from equation (8.4) is then given by

$$x = -2p\frac{\left(a_1^3 + a_3^3 + a_5^3 + a_7^3 - a_2^3 - a_4^3 - a_6^3 - a_8^3\right)}{\left(a_1^4 + a_3^4 + a_5^4 + a_7^4 - a_2^4 - a_4^4 - a_6^4 - a_8^4\right)} = -2p\frac{\left(\sum_{n=1}^{4} a_{(2n-1)}^3\right) - \left(\sum_{n=1}^{4} a_{2n}^3\right)}{\left(\sum_{n=1}^{4} a_{(2n-1)}^4\right) - \left(\sum_{n=1}^{4} a_{2n}^4\right)}. \tag{8.11}$$

Substituting this value of x in equation (8.1), will give solutions to Diophantine equation 5.8.8.

Generalising it for$n = 2k$, where $n \geq 3$ for the Diophantine equation

$$(a_1x+p)^5 + (a_2x)^5 + (a_3x+p)^5 + \ldots + \left\{a_{(2k-2)} \cdot x\right\}^5 + \left\{a_{(2k-1)} \cdot x + p\right\}^5 + (a_{2k}x)^5$$
$$= (a_1x)^5 + (a_2x+p)^5 + (a_3x)^5 + \ldots + \left\{a_{(2k-2)} \cdot x + p\right\}^5 + \left\{a_{(2k-1)} \cdot x\right\}^5 + (a_{2k}x+p)^5, \tag{8.12}$$

and following the same procedure as that of Diophantine equation 5.8.8, equations for a_2 and a_1 can be derived as given below

$$a_2 = A_{2k} - \frac{P_{2k} - Q_{2k}}{A_{2k} - B_{2k}}, \tag{8.13}$$

$$a_1 = B_{2k} - \frac{P_{2k} - Q_{2k}}{A_{2k} - B_{2k}}. \tag{8.14}$$

where $a_3, a_4, a_5, \ldots, a_{2k}$ are arbitrarily assigned real rational quantities,

$$A_{2k} = a_3 + a_5 + a_7 + \ldots + a_{2k-5} + a_{2k-3} + a_{2k-1} = \sum_{i=2}^{k} a_{2i-1}, \tag{8.15}$$

$$B_{2k} = a_4 + a_6 + a_8 + \ldots + a_{2k-4} + a_{2k-2} + a_{2k} = \sum_{i=2}^{k} a_{2i}, \tag{8.16}$$

$$P_{2k} = a_3\left(a_5 + a_7 + a_9 \ldots + a_{2k-1}\right) + a_5\left(a_7 + a_9 + a_{11} + \ldots + a_{2k-1}\right) + a_7\left(a_9 + a_{11} + a_{13} + \ldots + a_{2k-1}\right) + \ldots + a_{2k-3}\cdot a_{2k-1} \tag{8.17}$$

and

$$Q_{2k} = a_4\left(a_6 + a_8 + a_{10} \ldots + a_{2k}\right) + a_6\left(a_8 + a_{10} + a_{12} + \ldots + a_{2k}\right) + a_8\left(a_{10} + a_{12} + a_{14} + \ldots + a_{2k-}\right) + \ldots + a_{2k-2}a_{2k}, \tag{8.18}$$

$A_{2k} \neq B_{2k}$ and also $\sum_{n=2}^{4} a_{2n-1}^4 \neq \sum_{n=1}^{4} a_{2n}^4$. In mathematical notations,

$$P_{2k} = a_3\cdot\sum_{i=3}^{k} a_{2i-1} + a_5\cdot\sum_{i=4}^{k} a_{2i-1} + a_7\cdot\sum_{i=5}^{k} a_{2i-1} + \ldots + a_{2k-7}\cdot\sum_{i=k-2}^{k} a_{2i-1} + a_{2k-5}\cdot\sum_{i=k-1}^{k} a_{2i-1} + a_{2k-3}a_{2k-1},$$

$$Q_{2k} = a_4\cdot\sum_{i=3}^{k} a_{2i} + a_6\cdot\sum_{i=4}^{k} a_{2i} + a_8\cdot\sum_{i=5}^{k} a_{2i} + \ldots + a_{2k-6}\cdot\sum_{i=k-2}^{k} a_{2i} + a_{2k-4}\cdot\sum_{i=k-1}^{k} a_{2i} + a_{2k-2}a_{2k}.$$

To avoid repetition , the procedure is not reiterated here for finding the relation of x which is given by

$$x = -2p\frac{\left(a_1^3 + a_3^3 + a_5^3 + \ldots + a_{2k-1}^3\right) - \left(a_2^3 + a_4^3 + a_6^3 + \ldots + a_{2k}^3\right)}{\left(a_1^4 + a_3^4 + a_5^4 + \ldots + a_{2k-1}^4\right) - \left(a_2^4 + a_4^4 + a_6^4 + \ldots + a_{2k}^4\right)} = -2p\frac{\left(\sum_{i=1}^{k} a_{2i-1}^3\right) - \left(\sum_{i=1}^{k} a_{2i}^3\right)}{\left(\sum_{i=1}^{k} a_{2i-1}^4\right) - \left(\sum_{i=1}^{k} a_{2i}^4\right)}. \tag{8.19}$$

where $\sum_{n=1}^{k} a_{2i-1}^4 \neq \sum_{n=1}^{4} a_{2i}^4$. Substitution of this value of x in equation (8.12), will give solution to Diophantine equation $5.n.n$where $n = 2k$ and $k \geq 3$.

b. When $n = 2k - 1$ and $k \geq 3$.

Such $5.n.n$ equations can be written as

$$\left(a_1x + p\right)^5 + \left(a_2x\right)^5 + \left(a_3x + p\right)^5 + \ldots + \left(a_{2k-3}x + p\right)^5 + \left(a_{2k-2}x\right)^5 + \left(a_{2k-1}x - p\right)^5 = \left(a_1x\right)^5 + \left(a_2x + p\right)^5 + \left(a_3x\right)^5 + \ldots + \left(a_{2k-3}x\right)^5 + \left(a_{2k-2}x - p\right)^5 + \left(a_{2k-1}x + p\right)^5. \tag{8.20}$$

Kindly note in case of Diophantine equations where $n = 2k$, signs of p were always positive since number of terms containing p appearing in Left Hand Side was equal to those appearing in Right Hand Side, thus constant term containing p^5 vanished. But when n is odd, number of terms containing p in LHS is more by one than the correspondent terms in RHS, therefore, last term of LHS is taken as $-p$ and last but one term of RHS is also taken as $-p$. With this arrangement, constant term of equation (8.20) expansion vanishes. Assuming $n = 7$,Diophantine equation 5.7.7 can be written as

$$\left(a_1x + p\right)^5 + \left(a_2x\right)^5 + \left(a_3x + p\right)^5 + \left(a_4x\right)^5 + \left(a_5x + p\right)^5 + \left(a_6x\right)^5 + \left(a_7x - p\right)^5 = \left(a_1x\right)^5 + \left(a_2x + p\right)^5 + \left(a_3x\right)^5 + \left(a_4x + p\right)^5 + \left(a_5x\right)^5 + \left(a_6x - p\right)^5 + \left(a_7x + p\right)^5. \tag{8.21}$$

Equating coefficients of x and x^2 with zero yields,

$$a_1 = -a_3 - a_5 + a_2 + a_4 + a_6, \tag{8.22}$$

$$a_1^2 = -a_3^2 - a_5^2 + 2a_7^2 + a_2^2 + a_4^2 - a_6^2, \tag{8.23}$$

and

$$x = -2p\frac{\left(a_1^3 + a_3^3 + a_5^3 - a_2^3 - a_4^3 - a_6^3\right)}{\left(a_1^4 + a_3^4 + a_5^4 - 2a_7^4 - a_2^4 - a_4^4 + a_6^4\right)} = -2p\frac{\left(\sum_{i=1}^{3} a_{2i-1}^3\right) - \left(\sum_{i=1}^{3} a_{2i}^3\right)}{\left(\sum_{i=1}^{3} a_{2i-1}^4\right) - 2\left(a_7^4 - a_6^4\right) - \left(\sum_{i=1}^{3} a_{2i}^4\right)}, \tag{8.24}$$

where $\sum_{n=1}^{3} a_{2i-1}^4 - \sum_{n=1}^{4} a_{2i}^4 \neq 2\left(a_7^4 - a_6^4\right)$. From Equations (8.23) and (8.24),

$$a_1 = B_7 - \frac{P_7 - Q_7 + a_7^2 - a_6^2}{A_7 - B_7}, \tag{8.25}$$

$$a_2 = A_7 - \frac{P_7 - Q_7 + a_7^2 - a_6^2}{A_7 - B_7}, \tag{8.26}$$

where

$$A_7 = a_3 + a_5, \tag{8.27}$$

$$B_7 = a_4 + a_6, \tag{8.28}$$

$$P_7 = a_3\left(a_5\right), \tag{8.29}$$

$$Q_7 = a_4\left(a_6\right). \tag{8.30}$$

Generalising it for Diophantine equation (8.21) where $n = 2k-1$, the equations, then can be written

$$a_2 = A_{2k-1} - \frac{P_{2k-1} - Q_{2k-1} + a_{2k-1}^2 - a_{2k-2}^2}{A_{2k-1} - B_{2k-1}}, \tag{8.31}$$

$$a_1 = B_{2k-1} - \frac{P_{2k-1} - Q_{2k-1} + a_{2k-1}^2 - a_{2k-2}^2}{A_{2k-1} - B_{2k-1}}, \tag{8.32}$$

where

$$A_{2k-1} = a_3 + a_5 + a_7 + \ldots a_{2k-3} = \sum_{i=2}^{k-1} a_{2i-1}, \tag{8.33}$$

$$B_{2k-1} = a_4 + a_6 + a_8 + \ldots a_{2k-2} = \sum_{i=2}^{k-1} a_{2i}, \tag{8.34}$$

$$\begin{aligned} P_{2k-1} &= a_3\left(a_5 + a_7 + a_9 + \ldots + a_{2k-3}\right) + a_5\left(a_7 + a_9 + a_{11} + \ldots + a_{2k-3}\right) \\ &\quad + a_7\left(a_9 + a_{11} + a_{13} + \ldots + a_{2k-3}\right) + \ldots + a_{2k-5}\left(a_{2k-3}\right), \end{aligned} \tag{8.35}$$

$$\begin{aligned} Q_{2k-1} &= a_4\left(a_6 + a_8 + a_{10} + \ldots + a_{2k-2}\right) + a_6\left(a_8 + a_{10} + a_{12} + \ldots + a_{2k-2}\right) \\ &\quad + a_8\left(a_{10} + a_{12} + a_{14} + \ldots + a_{2k-2}\right) + \ldots + a_{2k-4}\left(a_{2k-2}\right), \end{aligned} \tag{8.36}$$

and

$$x = -2p\frac{\left(a_1^3 + a_3^3 + a_5^3 + \ldots a_{2k-3}^3\right) - \left(a_2^3 + a_4^3 + a_6^3 + \ldots + a_{2k-2}^3\right)}{\left(a_1^4 + a_3^4 + a_5^4 + \ldots + a_{2k-3}^4 - 2a_{2k-1}^4\right) - \left(a_2^4 + a_4^4 + a_6^4 + \ldots + a_{2k-4}^4 - a_{2k-2}^4\right)}, \tag{8.37}$$

where $\sum_{n=1}^{3} a_{2i-1}^{k-1} - \sum_{n=1}^{k-1} a_{2i}^4 \neq 2\left(a_{2k-1}^4 - a_{2k-2}^4\right)$. In mathematical notations,

$$P_{2k-1} = a_3 \cdot \sum_{i=3}^{k-1} a_{2i-1} + a_5 \cdot \sum_{i=4}^{k-1} a_{2i-1} + a_7 \cdot \sum_{i=5}^{k-1} a_{2i-1} + \ldots + a_{2k-7} \cdot \sum_{i=k-2}^{k-1} a_{2i-1} + a_{2k-5} \cdot a_{2k-3},$$

$$Q_{2k-1} = a_4 \cdot \sum_{i=3}^{k-1} a_{2i} + a_6 \cdot \sum_{i=4}^{k-1} a_{2i} + a_8 \cdot \sum_{i=5}^{k-1} a_{2i} + \ldots + a_{2k-6} \cdot \sum_{i=k-2}^{k-1} a_{2i} + a_{2k-4} \cdot a_{2k-2},$$

and

$$x = -2p\frac{\left(\sum_{i=1}^{k-1} a_{2i-1}^3\right) - \left(\sum_{i=1}^{k-1} a_{2i}^3\right)}{\left(\sum_{i=1}^{k-1} a_{2i-1}^4\right) - 2\left(a_{2k-1}^4 - a_{2k-2}^4\right) - \left(\sum_{i=1}^{k-1} a_{2i}^4\right)}.$$

where $\sum_{n=1}^{3} a_{2i-1}^{k-1} - \sum_{n=1}^{k-1} a_{2i}^4 \neq 2\left(a_{2k-1}^4 - a_{2k-2}^4\right)$ and $a_1, a_2, a_3, \ldots, a_{2k-1}$ are arbitrarily assigned real rational quantities. Based on these equations, some Diophantine equations have been solved. Kindly refer to Table 8.1..

This proves Lemma 8.1, Lemma 8.2, Lemma 8.3 and Lemma 8.4. From figures mentioned in Table 8,1, it is observed that while solving Diophantine equations $5.n.n$, solutions to Diophantine equation $5.n-1.n.$ were also obtained.

Lemma 8.1. *A Diophantine equation* $5.n.n$ *where* $n = 2k$ *and* $k \geq 3$, *then* $\left(a_1x + p\right)^5 + \left(a_2x\right)^5 + \left(a_3x + p\right)^5 + \ldots + \left(a_{2k-2}x\right)^5 + \left(a_{2k-1}x + p\right)^5 + \left(a_{2k}x\right)^5 = \left(a_1x\right)^5 + \left(a_2x + p\right)^5 + \left(a_3x\right)^5 + \ldots + \left(a_{2k-2}x + p\right)^5 + \left(a_{2k-1}x\right)^5 + \left(a_{2k}x + p\right)^5$ *is always transformable into linear equation* $x = -2p\frac{\left(a_1^3 + a_3^3 + a_5^3 + \ldots + a_{2k-1}^3\right) - \left(a_2^3 + a_4^3 + a_6^3 + \ldots + a_{2k}^3\right)}{\left(a_1^4 + a_3^4 + a_5^4 + \ldots + a_{2k-1}^4\right) - \left(a_2^4 + a_4^4 + a_6^4 + \ldots + a_{2k}^4\right)}$, *where* $a_{1,}a_2, A_{2k}, B_{2k}, P_{2k}$ *and* Q_{2k} *are given by equations* (8.14), (8.13), (8.15). (8.16), (8.17) *and* (8.18) *respectively and* $a_{1,}a_2, a_{3,} \ldots, a_{2k}$ *are all real rational quantities such that* $\sum_{n=1}^{k} a_{2i-1}^4 \neq \sum_{n=1}^{4} a_{2i}^4$.

***Table* 8.1:** Solution to Diophantine equation $Y_1^5+Y_2^5+Y_3^5+\ldots+Y_{n-2}^5+Y_{n-1}^5+Y_n^5=Z_1^5+Z_2^5+Z_3^5+\ldots+Z_{n-2}^5+Z_{n-1}^5+Z_n^5$

D.E.	Assigned a_3, a_4. $a_5 \ldots a_n$	Calculated A_n, B_n, P_n, Q_n	Calculated a_1, a_2, x	$(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+\cdots+(a_nx)^5 = (a_1x)^5+(a_2x+p)^5+(a_3x)^5+\cdots+(a_nx+p)^5$
5.11.11	$1, -1, 2, -2, 3, -3, 4, -4, 6,$	$10, -10$ $35, 35$	$-11, 9$ $31/50$	$291^5+31^5+62^5+93^5+124^5+329^5+31^5+19^5+62^5+93^5+124^5+236^5 = 279^5+81^5+112^5+143^5+174^5+136^5+341^5+12^5+43^5+174^5$
5.12.12	$1, -1, 2, -2, 3, -3, 4, -4, 5, -6$	$15, -16, 85, 95$	$-\frac{486}{31}$, [illegible] $\frac{527}{176}$	$8086^5+527^5+1054^5+1581^5+2108^5+3162^5+8251^5+527^5+1054^5+1581^5+2108^5+2635^5 = 8075^5+703^5+1230^5+1757^5+2284^5+2811^5+8262^5+351^5+878^5+1405^5+1932^5+2986^5$
5.15.15	$1, -1, 2, -2, 3, -3, 4, -4, 5, -5, 6, -6, 8$	$21, -21, 175, 175$	$-\frac{65}{3}$, [illegible] $\frac{2844}{6523}$	$58097^5+2844^5+5688^5+8532^5+11376^5+14220^5+17064^5+61351^5+2844^5+679^5+5688^5+8532^5+11376^5+14220^5+17064^5+26275^5 = 57828^5+6367^5+9211^5+12055^5+14899^5+17743^5+20587^5+19229^5+61620^5+2165^5+5009^5+7853^5+10697^5+20587^5$

Lemma 8.2. *After normalisation, a Diophantine equation* 5.$n.n$, $(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+\ldots+(a_{2k-2}x)^5+(a_{2k-1}x+p)^5+(a_{2k}x)^5=(a_1x)^5+(a_2x+p)^5+(a_3x)^5+\ldots+(a_{2k-2}x+p)^5+(a_{2k-1}x)^5+(a_{2k}x+p)^5$ *is always true and, in fact, is an identity where where* $n=2k$ $k\geq 3$, $x=-2p\frac{\left(a_1^3+a_3^3+a_5^3+\ldots+a_{2k-1}^3\right)-\left(a_2^3+a_4^3+a_6^3+\ldots+a_{2k}^3\right)}{\left(a_1^4+a_3^4+a_5^4+\ldots+a_{2k-1}^4\right)-\left(a_2^4+a_4^4+a_6^4+\ldots+a_{2k}^4\right)}$, $a_1, a_2, A_{2k}, B_{2k}, P_{2k}$ *and* Q_{2k} *are given by equations* (8.14), (8.13), (8.15). (8.16), (8.17) *and* (8.18) *respectively and* $a_1, a_2, a_3, \ldots, a_{2k}$ *are all real rational quantities such that* $\sum_{n=1}^{k} a_{2i-1}^4 \neq \sum_{n=1}^{4} a_{2i}^4$.

Lemma 8.3. *An equation* 5.$n.n$ *where* $n=2k-1$ *and* $k>3$, *then* $(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+\ldots+(a_{2k-2}x)^5+(a_{2k-1}x-p)^5=(a_1x)^5+(a_2x+p)^5+(a_3x)^5+\ldots+(a_{2k-2}x-p)^5+(a_{2k-1}x+p)^5$ *is always transformable into linear equation* $x=-2p\frac{\left(a_1^3+a_3^3+a_5^3+\ldots a_{2k-3}^3\right)-\left(a_2^3+a_4^3+a_6^3+\ldots+a_{2k-2}^3\right)}{\left(a_1^4+a_3^4+a_5^4+\ldots+a_{2k-3}^4\right)-2\left(a_{2k-1}^4-a_{2k-2}^4\right)-\left(a_2^4+a_4^4+a_6^4+\ldots+a_{2k-2}^4\right)}$, *where* $a_1, a_2, A_{2k-1}, B_{2k-1}, P_{2k-1}$ *and* Q_{2k-1} *are given by equations* (8.26), (8.25), (8.27), (8.28), (8.29) *and* (8.30) *respectively and* $a_1, a_2, a_3, \ldots, a_{2k-1}$ *are all real rational quantities such that* $\sum_{n=1}^{3} a_{2i-1}^{k-1}-\sum_{n=1}^{k-1} a_{2i}^4 \neq 2\left(a_{2k-1}^4-a_{2k-2}^4\right)$.

Lemma 8.4. *After normalisation, a Diophantine equation* 5.$n.n$, $(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+\ldots+(a_{2k-2}x)^5+(a_{2k-1}x-p)^5=(a_1x)^5+(a_2x+p)^5+(a_3x)^5+\ldots+(a_{2k-2}x-p)^5+(a_{2k-1}x+p)^5$ *is always true and in fact is an identity where* $n=2k$ *and* $k\geq 3, x=-2p\frac{\left(a_1^3+a_3^3+a_5^3+\ldots a_{2k-3}^3\right)-\left(a_2^3+a_4^3+a_6^3+\ldots+a_{2k-2}^3\right)}{\left(a_1^4+a_3^4+a_5^4+\ldots+a_{2k-3}^4\right)-2\left(a_{2k-1}^4-a_{2k-2}^4\right)-\left(a_2^4+a_4^4+a_6^4+\ldots+a_{2k-2}^4\right)}$, $a_1, a_2, A_{2k-1}, B_{2k-1}, P_{2k-1}$ *and* Q_{2k-1} *are given by equations* (8.26), (8.25), (8.27). (8.28), (8.29) *and* (8.30) *respectively and* $a_1, a_2, a_3, \ldots, a_{2k-1}$ *are* *all real rational quantities such that* $\sum_{n=1}^{3} a_{2i-1}^{k-1} - \sum_{n=1}^{k-1} a_{2i}^{4} \neq 2\left(a_{2k-1}^{4} - a_{2k-2}^{4}\right)$.

9 Parametrisation

9.1 Parametric solution to Diophantine Equation 5.4.4

For parametric solution to Diophantine equation 5.4.4, referring to equation (2.9), equation (2.12) can be written as

$$(2a \cdot P - Q)^5 + (2b \cdot P)^5 + (2c \cdot P - Q)^5 + (2d \cdot P - t \cdot Q)^3$$
$$= (2a \cdot P)^5 + (2b \cdot P - Q)^5 + (2c \cdot P - t \cdot Q)^5 + (2d \cdot P - Q)^5, \tag{9.1}$$

where a, b, P and Q are given by Equations (2.8), (2.6), (2.10) and (2.11) respectively and $t = s/p$. For the sake of brevity, these are not reiterated here. Evidently a and b are dependent upon c, d and t meaning thereby that by changing the value of any one out of three, will give new values of a and b, since P and Q are dependent upon upon a, b and t, therefore, varying c, d or t will give a new set of solution. Kindly peruse Table 2.1, where t is kept equal to -2 and values of c and d have been changed. Alternatively, all $c, d$and t can be varied, therefore, equation (9.1) can be transformed into two variables or one variable by keeping two or one parameter constant. That proves there can be infinite parametric solutions to Diophantine Equation 5.4.4.

9.2 Parametric solution to Diophantine Equation 5.4.3

For parametric solution to Diophantine equation 5.4.3, a is equated with zero then equation (9.1) transforms into

$$(-Q)^5 + (2b \cdot P)^5 + (2c \cdot P - Q)^5 + (2d \cdot P - tQ)^3 = (2b \cdot P - Q)^5 + (2c \cdot P - t \cdot Q)^5 + (2d \cdot P - Q)^5. \tag{9.2}$$

a in Diophantine equation 5.4.4 was dependent upon $c, d$and t and now a has been equated with zero, therefore, c now depends upon d and t by equation (2.14) meaning thereby that now there are two variable d and t. Value of b is dependent upon t by equation (2.15). Values of P and Q now are given by equation (2.17) and (2.18). By fixing one variable, say t, then parametric solution will be in one variable. This also has infinite parametric solutions by fixing t or d at different values. Based on this parametrisation, some solutions are given in Table 2.3.

9.3 Parametric solution to Diophantine Equation 5.5.5

For parametric solution of Diophantine equation 5.5.5, equations (3.16) and (3.15) can be written as $a = \frac{cd-e^2}{c-d} = c\left(\frac{z-y^2}{1-z}\right)$ and $b = c\left\{1 - z + \left(\frac{z-y^2}{1-z}\right)\right\}$ where $\frac{e}{c} = y$, $\frac{d}{c} = z$. Considering $z = -1$, these equations take the form, $a = -\frac{c}{2}\left(1 + y^2\right)$ and $b = c\left\{2 - \frac{1}{2}\left(1 + y^2\right)\right\}$. Putting these values of a and b in equation (3.17) and simplifying

$$x = \frac{3p}{c}\left(\frac{1}{y^2 - 3}\right), \tag{9.3}$$

where $c \neq 0$. Also Diophantine equation (3.8) takes the form

$$(ax + p)^5 + (\text{ bx})^5 + (cx + p)^5 + (-cx)^3 + (ex - p)^5$$
$$= (\text{ax})^5 + (bx + p)^5 + (\text{cx})^5 + (-cx - p)^5 + (ex + p)^5. \tag{9.4}$$

On putting the value of $x$given by equation (9.3) in equation (9.4) and simplifying,

$$\left\{-(y^2 + 9)\right\}^5 + \left\{3\left(3 - y^2\right)\right\}^5 + \left\{2y^2\right\}^5 + \left\{-6\right\}^5 + \left\{-2y^2 + 6y + 6\right\}^5$$
$$= \left\{-3\left(y^2 + 1\right)\right\}^5 + \left\{-y^2 + 3\right\}^5 + \left\{6\right\}^5 + \left\{-2y^2\right\}^5 + \left\{2y^2 + 6y - 6\right\}^5. \tag{9.5}$$

This is a parametric solution with one variable, however, by changing the value of z and t, an infinite parametric solutions can be had. On the basis of parametric solution given by equation (9.5), some solutions to Diophantine Equation 5.5.5 are given in Table 9.1. Since solutions to Diophantine equation 5.5.5 also give solution to equation 5.6.4 both are given in the Table 9.1. These solutions prove veracity of parametric solutions given by equation (9.5).

9.4 Parametric solution to Diophantine Equation 5.5.4

For parametric solution to Diophantine equation 5.5.4, d is equated with zero and equation (9.4) takes the form

$$(ax + p)^5 + (\text{ bx})^5 + (cx + p)^5 + (ex - p)^5 + p^5$$
$$= (\text{ax})^5 + (bx + p)^5 + (\text{cx})^5 + (ex + p)^5 \tag{9.6}$$

***Table* 9.1:** Solution to Diophantine equation $A^5 + B^5 + C^5 + D^5 + E^5 = F^5 + G^5 + H^5 + I^5 + J^5$ and $A^5 + B^5 + C^5 + D^5 + E^5 + F^5 = G^5 + H^5 + I^5 + J^5$

S.N.	y	$A^5 + B^5 + C^5 + D^5 + E^5 = F^5 + G^5 + H^5 + I^5 + J^5$ and $A^5 + B^5 + C^5 + D^5 + E^5 + F^5 = G^5 + H^5 + I^5 + J^5$
1	2	$13^5 + 3^5 + 6^5 + 14^5 + 6^5 = 8^5 + 10^5 + 15^5 + 1^5 + 8^5$
2	4	$25^5 + 39^5 + 6^5 + 2^5 + 6^5 + 50^5 = 32^5 + 51^5 + 13^5 + 32^5$
3	5	$34^5 + 66^5 + 6^5 + 14^5 + 6^5 + 74^5 = 50^5 + 78^5 + 22^5 + 50^5$
4	6	$45^5 + 99^5 + 6^5 + 30^5 + 6^5 + 102^5 = 72^5 + 111^5 + 33^5 + 72^5$
5	7	$58^5 + 138^5 + 6^5 + 50^5 + 6^5 + 134^5 = 98^5 + 150^5 + 46^5 + 98^5$
6	8	$73^5 + 183^5 + 6^5 + 74^5 + 6^5 + 170^5 = 128^5 + 195^5 + 61^5 + 128^5$
7	9	$90^5 + 234^5 + 6^5 + 102^5 + 6^5 + 210^5 = 162^5 + 246^5 + 78^5 + 162^5$
8	10	$109^5 + 291^5 + 6^5 + 134^5 + 6^5 + 254^5 = 200^5 + 303^5 + 254^5 + 200^5$

and Equation (3.15) transforms into $b = c - e^2/c$ and equation (3.16) transforms to $a = -e^2/c$. On putting these values of a, $b$and d in Equation (3.17) and then after simplification,

$$x = \frac{3p}{2c}\left\{\frac{1}{(y^2-1)}\right\}, \tag{9.7}$$

where $c \neq 0$. and $y \neq 1$.Putting this value x in in equation (9.6) and after simplification,

$$\left[-(y^2+2)\right]^5 + \left[3\left(1-y^2\right)\right]^5 + \left[2y^2+1\right]^5 + \left[3y-2y^2+2\right]^5 + \left[2y^2-2\right]^5$$
$$= \left[-3y^2\right]^5 + \left[1-y^2\right]^5 + [3]^5 + \left[2y^2+3y-2\right]^5. \tag{9.8}$$

This is a parametric solution with one variable, however, by changing the value t in equation (3.14), infinite parametric solutions can be had. On the basis of parametric solution given by equation (9.8), some solutions of Diophantine Equation 5.5.4 are given in Table 9.2**.** These solutions prove veracity of parametric solutions given by this equation.

***Table* 9.2:** Solution to Diophantine equation $A^5 + B^5 + C^5 + D^5 + E^5 = G^5 + H^5 + I^5 + J^5$

S.N.	y	$A^5 + B^5 + C^5 + D^5 + E^5 = F^5 + G^5 + H^5 + I^5$
1	3	$11^5 + 24^5 + 7^5 + 3^5 + 25^5 = 27^5 + 8^5 + 19^5 + 16^5$
2	4	$18^5 + 45^5 + 18^5 + 3^5 + 42^5 = 33^5 + 20^5 + 48^5 + 15^5$
3	5	$27^5 + 72^5 + 33^5 + 3^5 + 63^5 = 51^5 + 48^5 + 75^5 + 24^5$
4	6	$38^5 + 105^5 + 52^5 + 3^5 + 88^5 = 73^5 + 70^5 + 108^5 + 35^5$
5	7	$51^5 + 144^5 + 75^5 + 3^5 + 117^5 = 99^5 + 96^5 + 147^5 + 48^5$
6	8	$66^5 + 189^5 + 102^5 + 3^5 + 150^5 = 129^5 + 126^5 + 192^5 + 63^5$
7	9	$83^5 + 240^5 + 133^5 + 3^5 + 187^5 = 163^5 + 160^5 + 243^5 + 80^5$

9.5 Parametric solution to Diophantine Equations 5.6.6 and 5.6.5

Equation (5.7) can be written as $x = -2p(P/Q)$ where $P = a_1^3 + a_3^3 + a_5^3 - a_2^3 - a_4^3 - a_6^3$ and $Q = a_1^4 + a_3^4 + a_5^4 - a_2^4 - a_4^4 - a_6^4$. Putting this value of x in equation (5.1), it takes the form

$$(-2a_1P+Q)^5 + (-2a_2P)^5 + (-2a_3P+Q)^5 + (-2a_4P)^5 + (-2a_5P+Q)^5 + (-2a_6P)^5$$
$$= (-2a_1P)^5 + (-2a_2P+Q)^5 + (-2a_3P)^5 + (-2a_4P+Q)^5 + (-2a_5P)^5 + (-2a_6P+Q)^5, \tag{9.9}$$

where a_1 and a_2 are given by equation (5.6) and (5.5) and a_3, a_4, a_5 and a_6 are real rational quantities. By fixing the value of one variable, say a_3, and assigning different real rational values to a_4, a_5 and a_6, infinite numbers of parametric solutions to Diophantine equation 5.6.6 are obtained.

Following the same procedure and equating $a_1 = 0$, parametric solutions to Diophantine Equation 5.6.5 are obtained as follow.

$$((Q)^5 + (-2a_2P)^5 + (-2a_3P+Q)^5 + (-2a_4P)^5 + (-2a_5P+Q)^5 + (-2a_6P)^5 = (-2a_2P+Q)^5 + (-2a_3P)^5 + (-2a_4P+Q)^5 + (-2a_5P)^5 + (-2a_6P+Q)^5, \quad (9.10)$$

where$P = a_3^3+a_5^3-a_2^3-a_4^3-a_6^3$,$Q = a_3^4+a_5^4-a_2^4-a_4^4-a_6^4$. ,$a_3$ and a_4 are given by Equations (7.2), (7.3) and a_4, a_5 and a_6 are real rational quantities. By fixing the value of one variable say a_4 and assigning different real rational values to a_5 and a_6, infinite numbers of parametric solutions to Diophantine equation 5.6.5 are obtained. Based on this parametrisation, some solutions are given in Table 7.1. and may be perused.

9.6 Parametric solution to Diophantine Equation 5.$n.n$ where $n = 2k$ and $k \geq 3$

Equation (8.19) can be written as $x = -2p(P/Q)$ where $P = \left(a_1^3+a_3^3+a_5^3\ldots+a_{2k-1}^3\right) - \left(a_2^3+a_4^3+a_6^3+\ldots+a_{2k}^3\right)$ and $Q = \left(a_1^4+a_3^4+\ldots+a_{2k-1}^4\right) - \left(a_2^4+a_4^4+a_6^4+\ldots+a_{2k}^4\right)$. Putting this value of x in equation (8.1), it takes the form

$$(-2a_1P+Q)^5 + (-2a_2P)^5 + (-2a_3P+Q)^5 + \ldots + (-2a_{2k-2}P)^5 + (-2a_{2k-1}P+Q)^5 + (-2a_{2k}P)^5 = (-2a_1P)^5 + (-2a_2P+Q)^5 + (-2a_3P)^5 + \ldots + (-a_{2k-2}P+Q)^5 + (-2a_{2k-1}P)^5 + (-2a_{2k}P+Q)^5, \quad (9.11)$$

where a_1 and a_2 are given by equations (8.14), (8.13),A_{2k} and B_{2k} are given by Equations (8.15) and (8.16), P_{2k} and Q_{2k} are given by equations (8.17) and (8.18), a_3, a_4 and a_5 are real rational quantities. By fixing the value of one variable say a_3 and assigning different real rational values to a_4, a_5, $a_6, \ldots a_{2k}$infinite numbers of parametric solutions to Diophantine equation 5.$n.n$ where $n = 2k$ are obtained. Based on this parametrisation, some solutions are given in Table 8.1 and may be perused.

9.7 Parametric solution to Diophantine Equation5.n.n where n = 2k − 1 and $\infty >$k > 3

Equation (8.31) can be written as $x = -2p(P/Q)$ where $P = \left(a_1^3+a_3^3+a_5^3\ldots+a_{2k-3}^3\right) - \left(a_2^3+a_4^3+a_6^3+\ldots+a_{2k-2}^3\right)$ and $Q = \left(a_1^4+a_3^4+\ldots+a_{2k-3}^4-a_{2k-1}^4\right) - \left(a_2^4+a_4^4+a_6^4+\ldots+a_{2k-4}^4-a_{2k-2}^4\right)$. Putting this value of x in equation (8.21), it takes the form

$$(-2a_1P+Q)^5 + (-2a_2P)^5 + (-2a_3P+Q)^5 + \ldots + (-2a_{2k-2}P)^5 + (-2a_{2k-1}P-Q)^5 = (-2a_1P)^5 + (-2a_2P+Q)^5 + (-2a_3P)^5 + \ldots + (-2a_{2k-2}P-Q)^5 + (-a_{2k-1}P+Q)^5. \quad (9.12)$$

where a_1 and a_2 are given by equations (8.32), (8.31), A_{2k-1} and B_{2k-1} are given by Equations (8.33) and (8.34),P_{2k-1} and Q_{2k-1} are given by equations (8.35) and (8.36), $a_3, a_4a_5, \ldots a_{2k-1}$ are real rational quantities. By fixing the value of one variable say a_3 and assigning different real rational values to a_4, a_6, $\ldots a_{2k-1}$ infinite numbers of parametric solutions to Diophantine equation 5.$n.n$where $n = 2k-1$ are obtained. Based on this parametrisation, some solutions are given in Table 8.1 and may be perused.

10 Results and conclusions

On overviewing what have been derived in this paper, it can be concluded that a real rational number say n can be expressed in algebraic form as $a \cdot x + b$ where a and$b$ are real rational quantities as assigned and x is a real rational quantity which is a variable. On the basis of this representation, a Diophantine equation say 5.$n.n$

$$Y_1^5 + Y_2^5 + Y_3^5 + \ldots + Y_{n-2}^5 + Y_{n-1}^5 + Y_n^5 = Z_1^5 + Z_2^5 + Z_3^5 + \ldots + Z_{n-2}^5 + Z_{n-1}^5 + Z_n^5$$

where integer $n > 3$ can be written as algebraic equation

$$(a_1x+A_1)^5 + (a_2x+A_2)^5 + (a_3x+A_3)^5 + .. + (a_nx+A_n)^5 = (b_1x+B_1)^5 + (b_2x+B_2)^5 + (b_3x+B_3)^5 + .. + (b_nx+B_n)^5$$

where $a_1, a_2, a_3, \ldots a_n, b_1, b_2, b_3, \ldots b_n, A_1, A_2, A_3, \ldots A_n$ and $B_1, B_2, B_3, \ldots B_n$ are real rational quantities. Obviously, this is a fifth power equation, if R_1, R_2, R_3, R_4 and R_5 are its rational roots then substituting R_1, R_2, R_3, R_4 and R_5 in above would be its solutions. Stumbling block for these solutions is determination of roots R_1, R_2, R_3, R_4 and R_5 of fifth power algebraic equation. To tide over this difficulty, fifth degree equation is transformed into a linear equation. In the preliminary stage, coefficient of x^5 is equated with zero so that $a_1^5+a_2^5+a_3^5+\ldots+a_{n-2}^5+a_{n-1}^5+a_n^5 = b_1^5+b_2^5+b_3^5+\ldots+b_{n-2}^5+b_{n-1}^5+b_n^5$. This was achieved by assigning values so that $a_i = b_i$, where i varies from 1 to n. Next task was to get rid off constant term. That required

$$A_1^5 + A_2^5 + A_3^5 + \ldots + A_{n-2}^5 + A_{n-1}^5 + A_n^5 = B_1^5 + B_2^5 + B_3^5 + \ldots + B_{n-2}^5 + B_{n-1}^5 + B_n^5.$$

For that equation $5.n.n$ was written in the following manner
$(a_1x+A_1)^5+(a_2x+A_2)^5+(a_3x+A_3)^5+\ldots+(a_nx+A_n)^5++(a_nx+A_n)^5=(a_1x+A_2)^5+(a_2x+A_3)^5+(a_3x+A_4)^5+\ldots+(a_{n-1}x+A_n)^5+(a_nx+A_1)^5$.

To further simplify it, the equation with $n=2k$ was written as
$(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+\ldots+(a_{2k-1}x+p)^5+(a_{2k}x)^5=(a_1x)^5+(a_2x+p)^5+(a_3x)^5+\ldots+(a_{2k-1}x)^5+(a_{2k}x+p)^5$
by putting $A_2=A_4=A_6=\ldots=A_n=0$, and $A_1=A_3=A_5=\ldots=A_{n-1}=p$, where p is a rational number. This equation, in fact, is a cubic equation
$x^3\left\{a_1^4+a_3^4+a_5^4+\ldots+a_{2k-1}^4-a_2^4-a_4^4+a_6^4+\ldots+a_{2k}^4\right\}$
$+2x^2p\left\{a_1^3+a_3^3+a_5^3+\ldots+a_{2k-1}^3-a_2^3-a_4^3-a_6^3-\ldots-a_{2k}^3\right\}$
$+2xp^2\left\{a_1^2+a_3^2+a_5^2+\ldots+a_{np2k-1}^2-a_2^2-a_4^2-a_6^2-\ldots-a_{2k}^2\right\}$
$+p^3\left\{a_1+a_3+a_5-\ldots+a_{2k-1}-a_2-a_4-a_6-\ldots-a_{2k}\right\}=0.$

By equating coefficients of x and constant term to zero, this equation transforms into

$$x=-2p\frac{(a_1^3+a_3^3+a_5^3+\ldots+a_{2k-1}^3)-(a_2^3+a_4^3+a_6^3+\ldots+a_{2k}^3)}{(a_1^4+a_3^4+a_5^4+\ldots+a_{2k-1}^4)-(a_2^4+a_4^4+a_6^4+\ldots+a_{2k}^4)},$$

where $a_2, a_1, A_{2k}, B_{2k}, P_{2k}$ and Q_{2k} are given by Equations 8.13, 8.14, 8.15. 8.16, 8.17 and 8.18 respectively.

When $n=2k-1$ and $k>3$, Diophantine equation is written as
$(a_1x+p)^5+(a_2x)^5+(a_3x+p)^5+\ldots+(a_{2n-2}x)^5+(a_{2n-1}x-p)^5=(a_1x)^5+(a_2x+p)^5+(a_3x)^5+\ldots+(a_{2n-2}x-p)^5+(a_{2n-1}x+p)^5$.
when a_1 and a_2 are given by Equations (8.32), (8.31), A_{2k-1} and B_{2k-1} are given by Equations (8.33) and (8.34), P_{2k-1} and Q_{2k-1} are given by Equations (8.35) and (8.36),$a_3, a_4a_5,\ldots a_{2k-1}$ are real rational quantities, above said equation transforms into linear equation (8.31). For Diophantine equations, 5.6.6 and 5.5.5, above said methods were adopted. For Diophantine equations $5.m.n$ where $m<n$,terms $n-m$ in numbers can be eliminated by equating $a_1, a_3\ldots$ equal to zero.

Highlights of the paper are

a) Write summing numbers of Diophantine Equation in algebraic form as a_ix+b_i choosing a_i and b_i so that constant term and coefficient of fifth power of x vanishes. Put these algebraic numbers in Diophantine Equation and expand it.
b) Equate to zero coefficients of power two and power one of x and obtain two relations between various a_i and b_i.
c) Satisfy above two relations and obtain transformed linear equation. Substitute value of x obtained from transformed linear equation in algebraic numbers.
d) Note value of algebraic numbers of the form a_ix+b_i after multiplication with lowest common multiplier.

Data Availability Statement

All data generated or analysed during this study are included in this published article.

Conflict of interest

There is no conflict of interest of any nature involved in this paper.

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