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(Dedicated to Professor G. C. Sharma on His 85th Birth Anniversary Celebrations)

GROWTH PROPERTIES OF AN ENTIRE FUNCTION OF SEVERAL COMPLEX VARIABLES ON THE BASIS OF RELATIVE ORDER

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Abstract

In this paper, we study some comparative growth properties of composite entire function of several complex variables, on the basis of relative order and relative lower order of an entire function with respect to an entire function. Here we are defining some definitions related to relative order and relative lower order in terms of central index. A Rastogi, Biswas, Pramanik, Somsundaram many authors work on central index.
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1 Introduction

Let $f(z) = \sum_{m_j=0,j=1,2,\cdots,n}^{\infty} a_{m_j} z_j^{m_j}$ where $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, be an entire function of n-complex variables and $M_f(r_1, r_2, \cdots, r_n) := \max_{\{|z_j|=r_j;j=1,2,\cdots,n\}} |f(z_j)|$ be the maximum modulus of f and let $\mu_f(r_1, r_2, \cdots, r_n) := \max_{m_j\geq 0,j=1,2,\cdots,n} |a_{m_j}|r_j^{m_j}$ be the maximum term of f. The central index $\nu_f(r_1, r_2, \cdots, r_n) := \{m_k \mid \mu_f(r_1, r_2, \cdots, r_n) = |a_{m_k}|r^{m_k}\}$ or $|a_{\nu_f(r_1,r_2,\cdots,r_n)}|r^{\nu_f(r_1,r_2,\cdots,r_n)} = \mu_f(r_1, r_2, \cdots, r_n)$. Clearly $\mu_f(r_1, r_2, \cdots, r_n)$ is non decreasing function and $\mu_f(r_1, r_2, \cdots, r_n) \leq M_f(r_1, r_2, \cdots, r_n)$.

Let g be an entire function. Then the ratio $\frac{M_f(r_1,r_2,\cdots,r_n)}{M_g(r_1,r_2,\cdots,r_n)}$, where $r_k \to \infty$, $k = 1, 2, \cdots, n$ is called the growth of f with respect to g in term of maximum moduli.
In fact $\mu_f(r_1, r_2, \cdots, r_n)$ is much weaker than $M_f(r_1, r_2, \cdots, r_n)$ in some sense. So from another angle of view $\frac{\mu_f(r_1,r_2,\cdots,r_n)}{\mu_g(r_1,r_2,\cdots,r_n)}$, where $r_k \to \infty$, $k = 1, 2, \cdots, n$ is called growth of f with respect to g, in term of maximum terms, now in similar way we get growth of f with respect to g in terms of central index, $\frac{\nu_f(r_1,r_2,\cdots,r_n)}{\nu_g(r_1,r_2,\cdots,r_n)}$ where $r_k \to \infty$, $k = 1, 2, \cdots, n$. Rastogi [7], Biswas [1] and Pramanik [10] worked on central index. The details of the notations of maximum modulus, entire functions, growth, maximum term and central index for one variable appear in [1,2,3,4,6,7,8,9,11,13].

To start our paper we just recall the following definitions:

2 Definitions

Definition 2.1 ([12]). *Let f and g be entire functions. The relative order of f with respect to g is defined by*

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \nu_g^{-1}\nu_f(r)}{\log[r]},$$

and relative lower order is defined by

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log \nu_g^{-1}\nu_f(r)}{\log[r]}.$$

Definition 2.2 ([12]). *L order and L lower order for an entire function, where $L \equiv L(r)$ is a positive continuous function such that $L(ar) \sim L(r)$ as $r \to \infty$, for every positive constant a, on the basis of maximum modulus $M(r, f)$. The relative L order of an entire function f with respect to g, in terms of central index is defined by*

$$\rho_g^L(f) = \limsup_{r\to\infty} \frac{\log \nu_g^{-1}\nu_f(r)}{\log[rL(r)]},$$

and relative L lower order is defined by

$$\lambda_g^L(f) = \liminf_{r\to\infty} \frac{\log \nu_g^{-1}\nu_f(r)}{\log[rL(r)]}.$$

In the light of Definition 2.2 and from the concept of several complex variables [5], we would like to introduce the following definitions for several complex variables:

Definition 2.3. *Let f and g be entire functions of n complex variables. The relative order and relative lower order of f with respect to g are defined by*

$$\rho_g(f) = \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_g^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log(r_1 r_2 \cdots, r_n)},$$

and

$$\lambda_g(f) = \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_g^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log(r_1 r_2 \cdots, r_n)},$$

respectively.

Definition 2.4. *Let f and g be entire functions of n complex variables. The relative L order and relative L lower order of f with respect to g, are defined by*

$$\rho_g^L(f) = \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_g^{-1}\nu_f(r_1, r_2 \cdots, r_n)}{\log[(r_1 r_2 \cdots, r_n)L(r_1 r_2 \cdots, r_n)]},$$

and

$$\lambda_g^L(f) = \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_g^{-1}\nu_f(r_1, r_2 \cdots, r_n)}{\log[(r_1 r_2 \cdots, r_n)L(r_1 r_2 \cdots, r_n)]},$$

respectively. Here idea of L order (respectively, L lower order) of entire function is defined in [10,12].

Definition 2.5. *Let f and g be entire functions of n complex variables. The relative L^* order and relative L^* lower order of f with respect to g is defined by*

$$\rho_g^{L^*}(f) = \limsup_{r_1,r_2,\cdots,r_n\to\infty} \frac{\log \nu_g^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log[(r_1 r_2 \cdots, r_n)\exp^{L(r_1 r_2 \cdots, r_n)}]}.$$

Here idea of L^* order (respectively, L^* lower order) of an entire function where L^* is nothing but a weaker assumption of L [13]. *The relative L^* lower order is defined by*

$$\lambda_g^{L^*}(f) = \liminf_{(r_1,r_2,\cdots,r_n\to\infty)} \frac{\log \nu_g^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log[(r_1 r_2 \cdots, r_n)\exp^{L(r_1 r_2 \cdots, r_n)}]}.$$

3 Results

In this section, we establish some interesting results.

Theorem 3.1. *Let f, g and h be entire functions such that $0 < \lambda_h(f \circ g) \leq \rho_h(f \circ g) < \infty$ and $0 < \lambda_h(f) \leq \rho_h(f) < \infty$.*
Then

$$\frac{\lambda_h(f\circ g)}{\rho_h f} \leq \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq min\left\{\frac{\lambda_h(f\circ g)}{\lambda_h f}, \frac{\rho_h(f\circ g)}{\rho_h{}^{p,q} f}\right\}$$
$$\leq max\left\{\frac{\lambda_h(f\circ g)}{\lambda_h f}, \frac{\rho_h(f\circ g)}{\rho_h f}\right\} \leq \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h{}^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h(f\circ g)}{\lambda_h f}.$$

Proof. From the definition of relative order defined in (2.3), for arbitrary $\epsilon > 0$, we get the following

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \le (\rho_h(f\circ g) + \epsilon)\log(r_1 r_2 \cdots, r_n), \tag{3.1}$$

and

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \ge (\lambda_h(f\circ g) - \epsilon)\log(r_1 r_2 \cdots, r_n), \tag{3.2}$$

when $r_k \to \infty$, where $k = 1, 2, \cdots, n$

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \le (\lambda_h(f\circ g) + \epsilon)\log(r_1 r_2 \cdots, r_n), \tag{3.3}$$

and

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \ge (\rho_h(f\circ g) - \epsilon)\log(r_1 r_2 \cdots, r_n). \tag{3.4}$$

Similarly when we replace $f\circ g$ by f in the above equation, we get the following

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \le (\rho_h(f) + \epsilon)\log(r_1 r_2 \cdots r_n), \tag{3.5}$$

and

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \ge (\lambda_h(f) - \epsilon)\log(r_1 r_2 \cdots r_n). \tag{3.6}$$

When $r_k \to \infty$, where $k = 1, 2, \cdots, n$

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \le (\lambda_h(f) + \epsilon)\log(r_1 r_2 \cdots r_n), \tag{3.7}$$

and

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \ge (\rho_h(f) - \epsilon)\log(r_1 r_2 \cdots r_n). \tag{3.8}$$

From (3.2) and (3.5) it follows for sufficiently large value of $(r_1, r_2, \cdots, r_n)$ that

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \ge \frac{\lambda_h(f\circ g) - \epsilon}{\rho_h(f) + \epsilon}.$$

Since ϵ is arbitrary

$$\liminf_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \ge \frac{\lambda_h(f\circ g)}{\rho_h(f)}. \tag{3.9}$$

From (3.3) and (3.6), we obtain

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{\lambda_h(f\circ g) + \epsilon}{\lambda_h(f) - \epsilon}.$$

Since ϵ is arbitrary

$$\liminf_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{\lambda_h(f\circ g)}{\lambda_h(f)}, \tag{3.10}$$

from (3.1) and (3.8)

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{\rho_h(f\circ g) + \epsilon}{\rho_h(f) - \epsilon}.$$

Since ϵ is arbitrary

$$\liminf_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{\rho_h(f\circ g)}{\rho_h(f)}. \tag{3.11}$$

From (3.9), (3.10) and (3.11), we obtain

$$\frac{\lambda_h(f\circ g)}{\rho_h(f)} \le \liminf_{(r_1, r_2, \cdots, r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le min\left\{\frac{\lambda_h(f\circ g)}{\lambda_h(f)}, \frac{\rho_h(f\circ g)}{\rho_h(f)}\right\}. \tag{3.12}$$

From (3.2) and (3.7) for $r_k \to \infty$, where $k = 1, 2, \cdots, n$, we get

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \ge \frac{\lambda_h(f\circ g) - \epsilon}{\lambda_h(f) + \epsilon}.$$

As $\epsilon > 0$ is arbitrary, we obtain

$$\limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \geq \frac{\lambda_h(f\circ g))}{\lambda_h(f))}. \tag{3.13}$$

From (3.1) and (3.6), we obtain the following

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq \frac{\rho_h(f\circ g)+\epsilon}{\lambda_h(f)-\epsilon}.$$

As $\epsilon > 0$, we obtain

$$\limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq \frac{\rho_h(f\circ g)}{\lambda_h(f)}. \tag{3.14}$$

Similarly from (3.4) and (3.5)

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \geq \frac{\rho_h(f\circ g)-\epsilon}{\rho_h(f)+\epsilon}.$$

As ϵ is arbitrary

$$\limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \geq \frac{\rho_h(f\circ g)}{\rho_h(f)}. \tag{3.15}$$

Combining (3.13), (3.14) and (3.15), we obtain

$$max\left\{\frac{\lambda_h{}^{p,q}(f\circ g)}{\lambda_h f}, \frac{\rho_h(f\circ g)}{\rho_h f}\right\} \leq \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu f(r_1,r_2,\cdots,r_n)} \leq \frac{\rho_h(f\circ g)}{\lambda_h(f)}. \tag{3.16}$$

Hence, from (3.12) and (3.16), we obtain

$$\frac{\lambda_h(f\circ g)}{\rho_h(f)} \leq \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq min\left\{\frac{\lambda_h(f\circ g)}{\lambda_h f}, \frac{\rho_h(f\circ g)}{\rho_h f}\right\}$$

$$\leq max\left\{\frac{\lambda_h(f\circ g)}{\lambda_h f}, \frac{\rho_h(f\circ g)}{\rho_h f}\right\} \leq \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h{}^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq \frac{\rho_h(f\circ g)}{\lambda_h(f)}.$$

□

Theorem 3.2. *Let f, g and h be entire functions such that $0 < \lambda_h(f\circ g) \leq \rho_h(f\circ g) < \infty$ and $0 < \lambda_h(f) \leq \rho_h(f) < \infty$.*
Then

$$\frac{\lambda_h(f\circ g)}{\rho_h(f)} \leq \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq \frac{\lambda_h(f\circ g)}{\lambda_h f}$$

$$\leq \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h{}^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq \frac{\rho_h(f\circ g)}{\lambda_h(f)}.$$

Proof. The above theorem follows from (3.9), (3.10), (3.13) and (3.16). □

Theorem 3.3. *Let f, g and h be entire functions such that $0 < \lambda_h(f\circ g) \leq \rho_h(f\circ g) < \infty$ and $0 < \lambda_h(f) \leq \rho_h(f) < \infty$.*
Then

$$\liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \leq \frac{\rho_h(f\circ g)}{\rho_h f}$$

$$\leq \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h{}^{-1}\nu_f(r_1,r_2,\cdots,r_n)}.$$

Proof. From the definition of relative order of f for $\epsilon > 0$ and $r_k \to \infty$, where $k = 1,2,\cdots,n$

$$\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n) \geq (\rho_h(f)-\epsilon)\log(r_1r_2\cdots r_n), \tag{3.17}$$

and

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n) \leq (\rho_h(f\circ g)+\epsilon)\log(r_1r_2\cdots r_n). \tag{3.18}$$

From (3.17) and (3.18) we obtain

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{\rho_h(f\circ g)+\epsilon}{\rho_h f - \epsilon}.$$

As $\epsilon > 0$

$$\liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{\rho_h(f\circ g)}{\rho_h f}. \tag{3.19}$$

Since $r_k \to \infty$, where $k = 1, 2, \cdots, n$

$$\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \ge (\rho_h^{p,q}(f\circ g) - \epsilon)\log^{[q]}(r_1 r_2 . \cdots, r_n). \tag{3.20}$$

Now combining form of (3.5) and (3.20) is the following:

$$\frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \ge \frac{\rho_h(f\circ g)+\epsilon}{\rho_h f - \epsilon}.$$

As $\epsilon > 0$ is arbitrary

$$\limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \ge \frac{\rho_h(f\circ g)}{\rho_h(f)}. \tag{3.21}$$

Hence, from (3.21) and (3.19), we obtain

$$\liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{\rho_h(f\circ g)}{\rho_h f}$$
$$\le \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_{f\circ g}^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)}.$$

□

Theorem 3.4. *Let f, g and h be entire functions such that $0 < {\lambda_h}^L(f\circ g) \le {\rho_h}^L(f\circ g) < \infty$ and $0 < {\lambda_h}^L(f) \le {\rho_h}^L(f) < \infty$.*

Then

$$\frac{{\lambda_h}^L(f\circ g)}{{\rho_h}^L(f)} \le \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le min\left\{\frac{{\lambda_h}^L(f\circ g)}{{\lambda_h}^L f}, \frac{{\rho_h}^L(f\circ g)}{{\rho_h}^L f}\right\}$$
$$\le max\left\{\frac{{\lambda_h}^L(f\circ g)}{{\lambda_h}^L f}, \frac{{\rho_h}^L(f\circ g)}{{\rho_h}^L f}\right\} \le \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log {\nu_h}^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{{\rho_h}^L(f\circ g)}{{\lambda_h}^L(f)}.$$

Proof. From the definition (2.4) and arbitrary $\epsilon > 0$

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \le (\rho_h^L(f\circ g) + \epsilon)\log[(r_1 r_2 \cdots r_n)L(r_1 r_2 \cdots r_n)], \tag{3.22}$$

and

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \ge (\lambda_h^L(f\circ g) - \epsilon)\log[(r_1 r_2 \cdots r_n)L(r_1 r_2 \cdots r_n)]. \tag{3.23}$$

Since $r_k \to \infty$, where $k = 1, 2, \cdots, n$

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \le (\lambda_h^L(f\circ g) + \epsilon)\log[(r_1 r_2 \cdots r_n)L(r_1 r_2 \cdots r_n)], \tag{3.24}$$

and

$$\log \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \ge (\rho_h^L(f\circ g) - \epsilon)\log[(r_1 r_2 \cdots r_n)L(r_1 r_2 \cdots r_n)]. \tag{3.25}$$

Similarly for the function f, we obtain

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \le (\rho_h^L(f) + \epsilon)\log[(r_1 r_2 \cdots r_n)L(r_1 r_2 \cdots r_n)], \tag{3.26}$$

and

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \ldots, r_n) \ge (\lambda_h^L(f) - \epsilon)\log[(r_1 r_2 \cdots r_n)L(r_1 r_2 \cdots r_n)]. \tag{3.27}$$

Since $r_k \to \infty$, where $k = 1, 2, \cdots, n$

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \le (\lambda_h^L(f) + \epsilon)\log[(r_1 r_2 \cdots r_n)L(r_1 r_2 \cdots r_n)], \tag{3.28}$$

and

$$\log \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \ge (\rho_h^L(f) - \epsilon)\log[(r_1 r_2 \cdots r_n)L(r_1 r_2 \cdots r_n)]. \tag{3.29}$$

From (3.23) and (3.26) it follows for sufficiently large value of $(r_1, r_2, \cdots, r_n)$

$$\frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\lambda_h^L(f \circ g) - \epsilon}{\rho_h^L(f) + \epsilon}.$$

Since $\epsilon > 0$ is arbitrary

$$\liminf_{(r_1, r_2, \cdots, r_n) \to \infty} \frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\lambda_h^L(f \circ g)}{\rho_h^L(f)}. \tag{3.30}$$

From (3.24) and (3.27), we obtain

$$\frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \ldots r_n)}{\log \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\lambda_h^L(f \circ g) + \epsilon}{\lambda_h^L(f) - \epsilon}.$$

Since $\epsilon > 0$ is arbitrary

$$\liminf_{r_1, r_2, \cdots, r_n \to \infty} \frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\lambda_h^L(f \circ g)}{\lambda_h^L(f)}. \tag{3.31}$$

From (3.22) and (3.29), we obtain

$$\frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h^L(f \circ g) + \epsilon}{\rho_h^L(f) - \epsilon}.$$

As $\epsilon > 0$

$$\liminf_{(r_1, r_2, \cdots, r_n) \to \infty} \frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h^L(f \circ g)}{\rho_h^L(f)}. \tag{3.32}$$

From (3.30), (3.31) and (3.32), we get the following

$$\frac{\lambda_h^L(f \circ g)}{\rho_h^L(f)} \leq \liminf_{r_1, r_2, \cdots, r_n \to \infty} \frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \leq min\left\{\frac{\lambda_h^L(f \circ g)}{\lambda_h^L(f)}, \frac{\rho_h^L(f \circ g)}{\rho_h^L(f)}\right\}. \tag{3.33}$$

From (3.23) and (3.28) for $r_k \to \infty$, where $k = 1, 2, \cdots, n$, then

$$\frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\lambda_h^L(f \circ g) - \epsilon}{\lambda_h^L(f) + \epsilon}.$$

As $\epsilon > 0$ is arbitrary

$$\limsup_{(r_1, r_2, \cdots, r_n) \to \infty} \frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1} \nu_f(r_1, r_2, \ldots, r_n)} \geq \frac{\lambda_h^L(f \circ g)}{\lambda_h^L(f)}. \tag{3.34}$$

From (3.22) and (3.27)

$$\frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h^L(f \circ g) + \epsilon}{\lambda_h^L(f) - \epsilon}.$$

As $\epsilon > 0$ is arbitrary

$$\limsup_{(r_1, r_2, \cdots, r_n) \to \infty} \frac{\log^{[p]} \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h^L(f \circ g)}{\lambda_h^L(f)}. \tag{3.35}$$

Similarly from (3.25) and (3.26), we get the following:

$$\frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\rho_h^L(f \circ g) - \epsilon}{\rho_h^L(f) + \epsilon}.$$

As $\epsilon > 0$ is arbitrary

$$\limsup_{(r_1, r_2, \cdots, r_n) \to \infty} \frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\rho_h^L(f \circ g)}{\rho_h^L(f)}. \tag{3.36}$$

From (3.34), (3.35) and (3.36) we obtain

$$max\left\{\frac{\lambda_h{}^L(f \circ g)}{\lambda_h{}^L f}, \frac{\rho_h{}^L(f \circ g)}{\rho_h{}^L f}\right\} \leq \limsup_{(r_1, r_2, \cdots, r_n) \to \infty} \frac{\log \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1} \nu f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h{}^L(f \circ g)}{\lambda_h{}^L(f)}. \tag{3.37}$$

From (3.33) and (3.37)

$$\frac{\lambda_h{}^L(f \circ g)}{\rho_h{}^L(f)} \leq \liminf_{(r_1, r_2, \cdots, r_n) \to \infty} \frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log \nu_h^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \leq min\left\{\frac{\lambda_h{}^L(f \circ g)}{\lambda_h{}^L f}, \frac{\rho_h{}^L(f \circ g)}{\rho_h{}^L f}\right\}$$

$$\leq max\left\{\frac{\lambda_h{}^L(f \circ g)}{\lambda_h{}^L f}, \frac{\rho_h{}^L(f \circ g)}{\rho_h{}^L f}\right\} \leq \limsup_{(r_1, r_2, \cdots, r_n) \to \infty} \frac{\log \nu_h^{-1} \nu_{f \circ g}(r_1, r_2, \ldots, r_n)}{\log \nu_h{}^{-1} \nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h{}^L(f \circ g)}{\lambda_h{}^L(f)}.$$

□

From the above theorems we can obtain the following corollaries:

Corollary 3.1. *Let f, g and h be entire functions such that $0 < {\lambda_h}^L(f \circ g) \le {\rho_h}^L(f \circ g) < \infty$ and $0 < {\lambda_h}^L(f) \le {\rho_h}^L(f) < \infty$.}*

$$\frac{{\lambda_h}^L(f \circ g)}{{\rho_h}^L(f)} \le \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \le \frac{{\lambda_h}^L(f \circ g)}{{\lambda_h}^L f}$$
$$\le \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log {\nu_h}^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \le \frac{{\rho_h}^L(f \circ g)}{{\lambda_h}^L(f)}.$$

Proof. When we take L-lower order and L-order in Theorem 3.2 then we get the corollary. □

Corollary 3.2. *Let f, g and h be entire functions such that $0 < {\lambda_h}^L(f \circ g) \le {\rho_h}^L(f \circ g) < \infty$ and $0 < {\lambda_h}^L(f) \le {\rho_h}^L(f) < \infty$.*

Then

$$\liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \le \frac{{\rho_h}^L(f \circ g)}{{\rho_h}^L f}$$
$$\le \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log {\nu_h}^{-1}\nu_f(r_1,r_2,\cdots,r_n)}.$$

Proof. When we take L-order and L- lower order in Theorem 3.3 we can get the result. □

Corollary 3.3. *Let f, g and h be entire functions such that $0 < {\lambda_h}^{L^*}(f \circ g) \le {\rho_h}^{L^*}(f \circ g) < \infty$ and $0 < {\lambda_h}^{L^*}(f) \le {\rho_h}^{L^*}(f) < \infty$.*

Then

$$\frac{{\lambda_h}^{L^*}(f \circ g)}{{\rho_h}^{L^*}(f)} \le \liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \le \frac{{\lambda_h}^{L^*}(f \circ g)}{{\lambda_h}^{L^*} f}$$
$$\le \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log {\nu_h}^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \le \frac{{\rho_h}^{L^*}(f \circ g)}{{\lambda_h}^{L^*}(f)}.$$

Proof. When we take L^* order and L^* lower order in Theorem 3.2 we can get the result. □

Corollary 3.4. *Let f, g and h be entire functions such that $0 < {\lambda_h}^{L^*}(f \circ g) \le {\rho_h}^{L^*}(f \circ g) < \infty$ and $0 < {\lambda_h}^{L^*}(f) \le {\rho_h}^{L^*}(f) < \infty$.*

Then

$$\liminf_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \le \frac{{\rho_h}^{L^*}(f \circ g)}{{\rho_h}^{L^*} f}$$
$$\le \limsup_{(r_1,r_2,\cdots,r_n)\to\infty} \frac{\log \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log {\nu_h}^{-1}\nu_f(r_1,r_2,\ldots,r_n)}.$$

Proof. When we take L^* order and L^* lower order in Theorem 3.3 we can get the result. □

4 Conclusion

In this paper we have established some inequalities between relative order and relative lower order of entire functions of several complex variables in terms of central index. Further we have obtained some corollaries of the above theorems.

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